

Hypergeometric series and Hodge cycles of four dimensional cubic hypersurfaces ¹

H. MOVASATI

IMPA - Instituto de Matematica Pura e Aplicada
 Estrada Dona Castorina 110
 Jardim Botânico 22460-320
 Rio de Janeiro - RJ Brasil
 Email: hossein@impa.br

S. REITER

Technische Universität Darmstadt, Fachbereich Mathematik
 Schloßgartenstr. 7, 64289 Darmstadt, Germany
 Email: reiter@mathematik.tu-darmstadt.de

Abstract

In this article we find connections between the values of Gauss hypergeometric functions and the dimension of the vector space of Hodge cycles of four dimensional cubic hypersurfaces. Since the Hodge conjecture is well-known for those varieties we calculate values of Hypergeometric series on certain *CM* points. Our methods is based on the calculation of the Picard-Fuchs equations in higher dimensions, reducing them to the Gauss equation and then applying the Abelian Subvariety Theorem to the corresponding hypergeometric abelian varieties.

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1 Introduction

Recently, there have been attempts to relate algebraic values of hypergeometric functions in algebraic points to certain phenomenons in Arithmetic Algebraic Geometry. In this direction one can mention F. Beukers article [3] in which he calculates explicitly the values of (elliptic) hypergeometric functions in some *CM* points. Such a hypergeometric function can be rewritten as an elliptic integral and so the word *CM* comes from the corresponding *CM* elliptic curves. H. Shiga and J. Wolfart in [20] (see also [17] and [16]) have studied the algebraic values of Schwarz functions (certain quotient of two hypergeometric functions) in algebraic points. Their geometric tool is Abelian Subvariety Theorem and its consequence

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on the dimension of the periods of an abelian variety. Again, the idea behind algebraicity of a value of the Schwarz function is that the corresponding abelian variety has more automorphism than the neighboring varieties. For a general survey about algebraic and transcendental periods see Waldschmidt's expository article [18].

The idea of this article took place in our mind when we observed that many abelian integrals of affine hypersurfaces satisfy second order Picard-Fuchs differential equations and they can be written in terms of hypergeometric functions. A consequence of the Hodge conjecture (see Proposition 1) gives special values of such integrals on the so called Hodge cycles. This motivated us to use the mentioned geometric phenomenon and obtain special values of hypergeometric functions. Fortunately, the Hodge conjecture is known in all cases which we use. We concentrate mainly on cubic hypersurfaces and we do not yet know whether one can classify the hypergeometric functions appearing in this way. Below is the summary of the results of this article.

Let $n \geq 2$ be an even number. We consider a family of n -dimensional varieties $M := \{M_t\}_{t \in \mathbb{P}^1}$ and a meromorphic differential n -form ω whose restriction to each fiber has no residues around poles. We assume that:

1. For all $t \in \mathbb{P}^1$ except a finite number of them, the fiber M_t is smooth.
2. The Hodge decomposition of $H^n(M_t, \mathbb{C})$ is of the form

$$H^n(M_t, \mathbb{C}) = H^{\frac{n}{2}-1, \frac{n}{2}+1} \oplus H^{\frac{n}{2}, \frac{n}{2}} \oplus H^{\frac{n}{2}+1, \frac{n}{2}-1},$$

$$\dim_{\mathbb{C}}(H^{\frac{n}{2}-1, \frac{n}{2}+1}) = \dim_{\mathbb{C}}(H^{\frac{n}{2}+1, \frac{n}{2}-1}) = 1$$

and $\omega|_{M_t}$ form a basis of $H^{\frac{n}{2}-1, \frac{n}{2}+1}$.

3. The Picard-Fuchs equation of $\int_{\delta_t} \omega$, where $\delta_t \in H_n(M_t, \mathbb{Z})$ is a continuous family of cycles, is a pull-back of the Gauss equation

$$(1) \quad z(1-z)y'' + (c - (a+b+1)z)y' - aby = 0.$$

For a number field k , the vector space of Hodge cycles with coefficients in k is defined as follows:

$$(2) \quad \mathbf{H}_t(k) := H^{\frac{n}{2}, \frac{n}{2}} \cap H^n(M_t, k).$$

The usual definition of a Hodge cycles is with $k = \mathbb{Q}$. Since the hypergeometric function

$$F(a, b, c|z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad c \notin \{0, -1, -2, -3, \dots\},$$

where $(a)_n := a(a+1)(a+2) \cdots (a+n-1)$, solves the Gauss equation we find certain relations between the values of F -functions and Hodge cycles of the fibers M_t . For instance, the family

$$(3) \quad M_t : f(x) := x_1^3 + x_2^3 + \cdots + x_5^3 - x_1 - x_2 = t,$$

satisfies the conditions 1,2 and 3 and has three critical fibers corresponding to $t = 0, \pm 2(\frac{2}{3})^{\frac{3}{2}}$. Here $\omega = \nabla \frac{dx}{df}$, where dx is the wedge product of all dx_i 's, $\frac{dx}{df}$ is the Gelfand-Leary form of dx and ∇ is the Gauss-Manin connection with respect to the parameter t (see §3.1). Let $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ be the beta function. For two numbers $r, s \in \mathbb{C}$ we say that $r \sim s$ if $\frac{r}{s} \in \bar{\mathbb{Q}}$. We prove that:

Theorem 1. *Let k be a number field with $\mathbb{Q}(\zeta_3) \subset k$,*

$$(4) \quad E_t : y^2 = x^3 - 3x + 2 - \frac{27}{4}t^2, \quad t \in \bar{\mathbb{Q}}$$

and $z = \frac{27}{16}t^2$.

1. *If E_t is not a CM-elliptic curve then the k -vector space $H_t(k)$ has codimension 2 ;*
2. *For those values of $t \in \bar{\mathbb{Q}}$ such that the elliptic curve E_t is CM, the following value of the Schwarz function*

$$(5) \quad D(0, 0, 1|z) := -e^{-\pi i \frac{5}{6}} \frac{F(\frac{5}{6}, \frac{1}{6}, 1|z)}{F(\frac{5}{6}, \frac{1}{6}, 1|1-z)}$$

is algebraic and if it belongs to k then the k -vector space $H_t(k)$ has codimension one.

3. *The number (5) belongs to $\mathbb{Q}(\zeta_3)$ for some $t \in \bar{\mathbb{Q}}$ if and only if*

$$F\left(\frac{5}{6}, \frac{1}{6}, 1|z\right) \sim \frac{1}{\pi^2} \Gamma\left(\frac{1}{3}\right)^3, \quad F\left(\frac{5}{6}, \frac{1}{6}, 1|1-z\right) \sim \frac{1}{\pi^2} \Gamma\left(\frac{1}{3}\right)^3.$$

We have $j(E_t) = \frac{-2^4 3^3}{z(z-1)}$ and, for instance, if $j(E_t) = 2^4 3^3 5^3, -2^{15} 3 \cdot 5^3$ then the condition of the third part of the above theorem is satisfied (see [15] p.483). The proof of the first and second part uses a simple consequence of Abelian Subvariety Theorem on the periods of an elliptic curve. The proof of the third part is based on the Hodge conjecture for the variety M_t . We note that this conjecture is proved for cubic hypersurfaces of dimension 4 by C. Clemens, J. P. Murre and S. Zucker (see [21] and its references).

Let us now explain the content of each section. In section 2.1 we introduce the notion of a Hodge cycle by means of vanishing of abelian integrals. §2.2 is devoted to a consequence of the Hodge conjecture on the values of integrals over Hodge cycles. In §2.3 we recall the notion of Gauss-Manin connection associated to a fibration and state a conjecture which has been verified for some examples. §2.4 is devoted to the examples of Hodge structures whose Hodge numbers is of the type $1, x, 1$. In §2.5 we use Abelian Subvariety theorem and derive the fact that a simple abelian variety A is CM if and only if the periods of a differential form of the first kind on A generate a $\bar{\mathbb{Q}}$ -vector space of dimension one. In §3.1 we fix up the notations related to the Gauss-equation and hypergeometric functions. Following the ideas of [16], [20] in §3.2 we introduce a one dimensional integral representation of hypergeometric functions. §4 is devoted to the calculations related to the example (4). In this section we prove Theorem 1. In the last section we give many other examples of cubic hypersurfaces which can be analyzed by the methods of this article and hence may result to some algebraic relations between the values of hyper geometric functions.

2 Families of varieties and Hodge cycles

The classical definition of a Hodge cycle over $k = \mathbb{Q}$ is given in (2). This definition can be formulated by means of n -dimensional abelian integrals. Since such abelian integrals satisfy Picard-Fuchs equations, our problem of studying Hodge cycles reduces to the study of Picard-Fuchs equation. For d an integer we set

$$G_d := \{\zeta_d^i \mid i = 0, 1, \dots, d-1\}, \quad \zeta_d = e^{\frac{2\pi i}{d}}$$

2.1 Hodge cycles

In the following sections we will take a family $\{M_t\}_{t \in \mathbb{P}^1}$ of n dimensional varieties and compute certain Picard-Fuchs equations. In all examples we take f a polynomial of degree d in \mathbb{C}^{n+1} and set $L_t := \{f = t\}$, $M_t := \overline{L_t}$, where the closure is taken in \mathbb{P}^{n+1} . We assume that M_t is smooth for all t except a finite number of them and denote by k a number field.

A cycle $\delta \in H_n(M_t, k)$ is called Hodge if

$$\int_{\delta} \omega = 0, \quad \omega \in F^{\frac{n}{2}+1} H_{\text{dR}}(M_t),$$

where F^\bullet denotes the Hodge filtration of $H_{\text{dR}}^n(M_t)$. By Poincaré duality this definition of a Hodge cycle translate into the definition (2) in the Introduction. If the support of δ is in L_t then we can reformulate this definition using the mixed Hodge structure of L_t , i.e. $\delta \in H_n(L_t, k)$ is Hodge if

$$\int_{\delta} \omega = 0, \quad \omega \in \text{Gr}_F^i \text{Gr}_n^W H_{\text{dR}}^n(L_t), \quad i \geq \frac{n}{2} + 1,$$

where

$$(6) \quad 0 = F^{n+1} \subset F^n \subset \dots \subset F^1 \subset F^0 = H_{\text{dR}}^n(L_t),$$

$$0 = W_{n-1} \subset W_n \subset W_{n+1} = H_{\text{dR}}^n(L_t)$$

are the Hodge and weight filtrations of the Mixed Hodge structure of $H_{\text{dR}}^n(L_t)$. We denote by $\mathbf{H}_t(k)$ the k -vector space of Hodge cycles in $H_n(M_t, k)$. For more details see ([13]).

Remark 1. For two number fields $k_1 \subset k_2$ we do not have necessarily $\mathbf{H}_t(k_1) \otimes_{k_1} k_2 = \mathbf{H}_t(k_2)$. This can be seen in the next section.

2.2 Hodge cycles and algebraic values of abelian integrals

We follow the notations of the previous section. Let $\delta \in H_n(L_t, \mathbb{Q})$ be a Hodge cycle. The Hodge conjecture claims that there is an algebraic cycle $Z = \sum_{i=1}^k r_i Z_i$, $\dim_{\mathbb{C}} Z_i = \frac{n}{2}$ such that the topological class $[Z] := \sum_{i=1}^k r_i [Z_i] \in H_n(M_t, \mathbb{Q})$ of Z is equal to the image of δ in $H_n(M_t, \mathbb{Q})$. In other words, every Hodge cycle is an algebraic cycle.

Proposition 1. *If $[Z] \in H_n(L_t, \mathbb{Q})$ is an algebraic cycle and L_t is defined over $\overline{\mathbb{Q}}$ then for any differential form ω on L_t which does not have residues at infinity and is defined over $\overline{\mathbb{Q}}$ we have*

$$\int_{[Z]} \omega \in (2\pi i)^{\frac{n}{2}} \overline{\mathbb{Q}}.$$

See Deligne's lecture [5] Proposition 1.5 for a proof. Note that if M_t is defined over $\overline{\mathbb{Q}}$ and $[Z]$ is an algebraic cycle we can assume that Z is defined over $\overline{\mathbb{Q}}$. Deligne has applied the above proposition to Fermat hypersurfaces and has obtained algebraic relations between the values of Γ function on rational points (see Theorem 7.15 [5]). Our approach to the algebraic relations between the values of F -functions follows the same method used by Deligne.

2.3 Gauss-Manin connection and Picard-Fuchs equations

In this section we consider a family of algebraic varieties \mathcal{M} over $\mathbb{C}(t)$ and a differential 1-form $\omega \in H_{dR}^n(\mathcal{M})$. We denote by M_t the specialization of \mathcal{M} to a fixed $t \in \mathbb{C}$ and we call them fibers. We use also the notation ω for its specialization to the fiber M_t ; being clear in the text what we mean. Let $S \subset \mathbb{C}$ be the locus of singular fibers. The multi valued functions

$$I_{\gamma_t}(t) := \int_{\gamma_t} \omega$$

spans the solution space of a Picard-Fuchs equation

$$(7) \quad p_m(t)y^{(m)} + p_{m-1}(t)y^{(m-1)} + \cdots + p_1(t)y' + p_0(t)y = 0, \quad m \leq 2g, \quad p_i \in \mathbb{C}[t],$$

where $2g = \dim H_{dR}^n(\mathcal{M})$ and $\gamma_t \in H_n(M_t, \mathbb{Z})$ is a continuous family of cycles which can be defined in any simply connected region in $\mathbb{C} \setminus S$. Moreover, if the degree m of (7) is equal to $2g$ then a basis $\gamma_{i,t} \in H_n(M_t, \mathbb{Z})$, $i = 1, 2, \dots, 2g$ gives a basis $I_{\gamma_{i,t}}$, $i = 1, 2, \dots, 2g$ of (7). One obtains (7) in the following way: The Gauss-Manin connection ∇ is defined in the cohomology $H_{dR}^n(\mathcal{M})$ and it has poles in S . Its main property is

$$(8) \quad \frac{d}{dt} \int_{\gamma_t} \omega = \int_{\gamma_t} \nabla \omega.$$

Since $H_{dR}^n(\mathcal{M})$ is a $\mathbb{C}(t)$ -vector space of dimension $2g$, there must be $\mathbb{C}[t]$ -linear relation between

$$\omega, \nabla \omega, \dots, \nabla^{2g} \omega,$$

integrating such linear equation and using (8) we get (7). The set $S' := \{t \in \mathbb{C} \mid p_m(t) = 0\}$ is called the singular set of (7) and we know that $S \subset S'$. The points in $S' \setminus S$ are called apparent singularities (see [1]). Let

$$(9) \quad \mathcal{P}(\mathcal{M}, \omega) := \left\{ \int_{\gamma_t} \omega \mid \gamma_t \in H_n(M_t, \mathbb{Q}) \right\}.$$

In the next sections we will need to verify the following conjecture for our example.

Conjecture: Let $(\mathcal{M}_i, \omega_i)$, $i = 1, 2$ be two pairs as above with $\dim \mathcal{M}_1 \geq \dim \mathcal{M}_2$. If the associated period space $\mathcal{P}(\mathcal{M}_i, \omega_i)$, $i = 1, 2$ span the solution space of the same Picard-Fuchs equation then there is a non-zero complex number γ and a number field k such that $\mathcal{P}(\mathcal{M}_1, \omega_1)$ is generated over \mathbb{Q} by $\gamma \cdot k \cdot \mathcal{P}(\mathcal{M}_2, \omega_2)$.

One may expect that the number γ and the number field k depend only on some numerical invariants of the pairs $(\mathcal{M}_i, \omega_i)$, $i = 1, 2$.

In the examples which we have verified this conjecture, the number γ is obtained by the values of the beta function on rational points.

2.4 Four dimensional cubic hypersurfaces

In this section we consider the polynomial $f = x_1^d + x_2^d + \cdots + x_{n+1}^d - g(x)$, $\deg(g) < d$ and the associated family of varieties

$$M_t : \overline{\{x \in \mathbb{C}^{n+1} \mid f(x) = t\}}.$$

In this article we only deal with the following cases:

$$d = 3, \quad n = 4, \quad h^{0,4} = h^{4,0} = 0, \quad h^{1,3} = h^{3,1} = 1, \quad h^{2,2} = 20,$$

where $h^{i,j}$'s are the Hodge numbers of a smooth M_t in the corresponding case (see for instance [10]). The 5-form $\omega = Pdx$, where $dx := dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n+1}$ and $P \in \mathbb{C}[x]$, can be interpreted as a section of the cohomology bundle of the fibration $\{M_t\}_{t \in \mathbb{P}^1}$ by considering the Gelfand-Leray form $\frac{\omega}{df}$ (see for instance [2]). A way of constructing such section is as follows: We take the residue of $\frac{\omega}{f-t}$ on M_t , which is an element in $H^n(M_t, \mathbb{C})$.

Theorem 2. *If $n = 4, d = 3$ then the differential form $\nabla(\frac{dx}{df})$ is a basis of $H^{3,1}$, where $H_{dR}^4(M_t) = H^{3,1} \oplus H^{2,2} \oplus H^{1,3}$ is the Hodge decomposition.*

The above theorem for the residue of $\frac{dx}{f-t}$ in M_t is a particular case of Griffiths theorem for the Hodge filtration of hypersurfaces (see [8], [10] and also [13] Theorem 2). The fact that the mentioned residue can be expressed by the Gauss-Manin connection of $\frac{dx}{df}$ is explained in Lemma 2 [13]. See also this article for a generalization of the above theorem.

2.5 Abelian Subvariety Theorem

In this section we use Abelian Subvariety Theorem and calculate the dimension of the $\bar{\mathbb{Q}}$ -vector spaces spanned by the periods of a differential 1-form of the first kind on a CM abelian variety.

Let A be an abelian variety defined over $\bar{\mathbb{Q}}$ and t_A be the tangent space at the origin $0 \in A$. The exponential map

$$\exp : t_A(\mathbb{C}) \rightarrow A(\mathbb{C})$$

is well-defined and set $\Lambda := \exp^{-1}(0)$. Here, by $t_A(\mathbb{C})$ we mean the complexification of t_A . Let W be a proper linear subspace of t_A (defined over $\bar{\mathbb{Q}}$) such that

$$(10) \quad 0 \neq \gamma \in \Lambda \cap W(\mathbb{C}).$$

Theorem 3. *Under the above conditions, there exists a proper connected abelian subvariety $B \subset A$ (defined over $\bar{\mathbb{Q}}$) such that*

$$t_B \subset W, \quad \text{and } \gamma \in t_B(\mathbb{C}).$$

For this version of Abelian Subvariety Theorem see for instance Lemma 1 of [17].

Corollary 1. *For a simple abelian variety A and a differential form of the first kind ω , both defined over $\bar{\mathbb{Q}}$, the $\bar{\mathbb{Q}}$ -vector space $\{\int_{\delta} \omega \mid \delta \in H_1(A(\mathbb{C}), \bar{\mathbb{Q}})\}$ is of dimension one if and only if A is CM and ω is an eigendifferential under the action of the CM field.*

Proof. This lemma is Proposition 3 of [17]. Let g be the dimension of A . We take a \mathbb{Q} -basis $\delta_i, i = 1, 2, \dots, 2g$ of the homology $H_1(A(\mathbb{C}), \mathbb{Q})$. Without losing generality, we assume that $\int_{\delta_1} \omega \neq 0$. The non trivial part of the theorem is to prove that if there are constants $a_i \in \bar{\mathbb{Q}}, i = 2, 3, \dots, 2g$ such that

$$(11) \quad \int_{\delta_i} \omega + a_i \int_{\delta_1} \omega = 0, \quad i = 2, 3, \dots, 2g$$

then A is CM. To prove this we define

$$B = A \times A, \quad \omega_i = \omega \oplus (a_i \omega) \in \Omega_B, \quad \delta = \delta_i \oplus \delta_1 \in H_1(B(\mathbb{C}), \mathbb{Q})$$

for a fixed i , where the the sums are well-defined using the isomorphisms $\Omega_B \cong \Omega_A \oplus \Omega_A$ and $H_1(B, \mathbb{Q}) \cong H_1(A, \mathbb{Q}) \oplus H_1(A, \mathbb{Q})$. Now the condition (11) implies that

$$\int_{\delta} \omega' = 0.$$

Looking ω'_0 as a linear map from t_A to $\bar{\mathbb{Q}}$ this implies that $\delta \in W := \ker(\omega'_0)$. Therefore, by Abelian Subvariety Theorem there exists an abelian subvariety $B' \subset B$ such that δ is supported in $H_1(B', \mathbb{Q})$ and the restriction of ω' to B' is identically zero. Since A is simple, B' is of dimension g and the projections $\pi_i, i = 1, 2$ are isogeny between B' and A and $\pi_2 \circ \pi_1^{-1}$ gives a non trivial endomorphism of A because it sends δ_1 to $n\delta_i$ for some $n \in \mathbb{Z}$. We have proved that $H_1(A(\mathbb{C}), \mathbb{Q})$ as $\text{End}_0(A)$ -module is of dimension one which implies that A is CM . \square

3 Hypergeometric functions

Recall the definition of Pochhammer cycles and the corresponding Kummer solutions form [9] and [16] §5. The Pochhammer cycle associated to the points $a, b \in \mathbb{C}$ and a path $s : [0, 1] \rightarrow \mathbb{C}$ connecting a to b , is the commutator

$$[\gamma_b, \gamma_a] = \gamma_b^{-1} \cdot \gamma_a^{-1} \cdot \gamma_b \cdot \gamma_a,$$

where γ_b is a loop along s starting and ending at the point $b = s(\frac{1}{2})$ which encircles b once anticlockwise, and γ_a is a similar loop with respect to a . We will need the following simple proposition:

Proposition 2. *Let f be a holomorphic multi-valued function in \mathbb{C} and $\mu, \alpha \notin \mathbb{Z}$ be the exponents of f at $z = a$ (resp. $z = b$), i.e. in a neighborhood of a we can write $f = g \cdot (z - a)^\mu$, g a holomorphic function around a . Then for a path s connecting a to b outside the branching points of f we have:*

$$\int_{[\gamma_b, \gamma_a]} f(x) dx = (1 - e^{2\pi i \alpha})(1 - e^{2\pi i \mu}) \int_s f(x) dx.$$

3.1 Gaussian System

To the Gauss-equation (1) we can associate the system

$$(12) \quad Y' = \left(\frac{1}{z} \begin{pmatrix} c-1 & -b \\ 0 & 0 \end{pmatrix} + \frac{1}{z-1} \begin{pmatrix} 0 & 0 \\ a & c-a-b-1 \end{pmatrix} \right) Y,$$

with the fundamental system

$$\begin{aligned} Y &= \begin{pmatrix} \int_{[\gamma_0, \gamma_z]} \varphi(x, z) \frac{dx}{x} & \int_{[\gamma_1, \gamma_z]} \varphi(x, z) \frac{dx}{x} \\ \int_{[\gamma_0, \gamma_z]} \varphi(x, z) \frac{dx}{1-x} & \int_{[\gamma_1, \gamma_z]} \varphi(x, z) \frac{dx}{1-x} \end{pmatrix} \\ &= \begin{pmatrix} \int_0^z \varphi(x, z) \frac{dx}{x} & \int_1^z \varphi(x, z) \frac{dx}{x} \\ \int_0^z \varphi(x, z) \frac{dx}{1-x} & \int_1^z \varphi(x, z) \frac{dx}{1-x} \end{pmatrix} C = \end{aligned}$$

$$(13) \quad \begin{pmatrix} F(a, b, c|z) & (1-z)F(1-a, 1-b, 1+c-a-b|1-z) \\ \frac{a}{c} z F(a+1, b+1, c+1|z) & \frac{c-a-b}{-b} F(-a, -b, c-a-b|1-z) \end{pmatrix}$$

$$\cdot \begin{pmatrix} B(a, c-a)z^{c-1} & 0 \\ 0 & -e^{-\pi i(1+a-c)}B(1-b, c-a)(1-z)^{c-a-b-1} \end{pmatrix} C,$$

where

$$\varphi(x, z) := x^a(1-x)^{-b}(z-x)^{c-a-1}$$

and

$$C := \begin{pmatrix} (1-e^{2\pi ia})(1-e^{2\pi i(c-a-1)}) & 0 \\ 0 & (1-e^{-2\pi ib})(1-e^{2\pi i(c-a-1)}) \end{pmatrix}.$$

The first equality is derived from Proposition 2. Other equalities are obtained from Theorem 4.4.3 p. 99 together with chapter 1.3. of [9] (see also [16] Theorem 5.3 for corrected ones). In particular the last equality is well-defined when $c, c-a-b \notin \{0, -1, -2, -3, \dots\}$. Our abc -notations is different form those in [16]. Let a', b', c' be those in [16] p. 648. Then

$$a = a' - c' + 1, \quad b = b' - c' + 1, \quad c = 2 - c' \quad \text{and} \quad a' = a - c + 1, \quad b' = b - c + 1, \quad c' = 2 - c$$

The corresponding angular parameters are given by

$$(14) \quad \nu_0 = c - 1, \quad \nu_1 = c - a - b, \quad \nu_\infty = a - b$$

Let us remark that the above fundamental system can also be written as

$$Y = \begin{pmatrix} z^{c-1} & 0 \\ 0 & -b^{-1}z^c \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{pmatrix},$$

where (y_1, y_2) is a basis of the solution space of (1). For a differential system $Y' = AY$ the determinant of the fundamental system satisfies $\det(Y)' = \text{trace}(A) \cdot \det(Y)$ and so in our case we have

$$(15) \quad \det(Y) \sim \pi \frac{B(a, c-a)}{B(b, c-b)} z^{c-1} (z-1)^{c-a-b-1}.$$

The constant in the above relation is obtained by the formula

$$F(a, b, c | 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

and substituting $z = 1$ in (13).

The system (12) has monodromy

$$A_0 = \begin{pmatrix} e^{2\pi ic} & e^{-2\pi ib} - 1 \\ 0 & 1 \end{pmatrix}, \quad A_1 := \begin{pmatrix} 1 & 0 \\ e^{2\pi i(c-a)}(e^{-2\pi ia} - 1) & e^{2\pi i(c-a-b)} \end{pmatrix}, \quad A_\infty := A_0 A_1,$$

at $0, 1, \infty$ respectively. The Schwarz function associated to the Gauss equation (1) is

$$(16) \quad D(\nu_0, \nu_1, \nu_\infty | z) = \frac{\int_{[\gamma_0, \gamma_z]} \varphi(x, z) \frac{dx}{x}}{\int_{[\gamma_1, \gamma_z]} \varphi(x, z) \frac{dx}{x}} = \frac{(1-e^{2\pi ia})}{-e^{\pi i(1+a-c)}(1-e^{-2\pi ib})} \frac{B(a, c-a)}{B(1-b, c-a)} \frac{z^{c-1} F(a, b, c | z)}{F(1-a, 1-b, c-a-b+1 | 1-z)}.$$

In particular, we get for $1 = c = a + b$

$$D(0, 0, 2a-1 | z) = -e^{-\pi ia} \frac{F(a, b, 1 | z)}{F(b, a, 1 | 1-z)}$$

and for $z = \frac{1}{2}$ we even get

$$D(0, 0, 2a - 1 | 1/2) = -e^{-\pi ia}.$$

Sometimes it is useful to use other Pochhammer cycles instead of those used in Y . The corresponding integrals are calculated in Theorem 5.3 of [16] p. 648. For instance for the case $1 = c = a + b$ the $(2, 2)$ -entry of Y is not well-defined. We use the Pochhammer cycle $[\gamma_0, \gamma_1]$ and for the second row of Y we obtain

$$(17) \quad \begin{aligned} & \begin{pmatrix} \int_{[\gamma_0, \gamma_1]} \varphi(x, z) \frac{dx}{x} \\ \int_{[\gamma_0, \gamma_1]} \varphi(x, z) \frac{dx}{1-x} \end{pmatrix} \\ &= (1 - e^{2\pi ia})(1 - e^{2\pi ib})B(a, -b + 1)z^{c-a-1} \begin{pmatrix} F(a - c + 1, a, a - b + 1 | \frac{1}{z}) \\ \frac{a}{a-b} F(a - c + 1, a + 1, a - b + 1 | \frac{1}{z}) \end{pmatrix}. \end{aligned}$$

Note that if $(Y_1, Y_2)^t$ is a column of Y or the above matrix then

$$(18) \quad Y_2 = -b^{-1}(zY_1' + (1 - c)Y_1).$$

3.2 A family of curves associated to the Gaussian System

Let

$$\mu_i = a_i + b_i, \quad a_i \in \mathbb{Z}, \quad 0 \leq b_i < 1$$

and k be the least common denominator of b_0, b_1, b . In this section we assume that μ_i 's are rational non-integer numbers. Define $X(k, z)$ as the smooth curve obtained by the desingularization of

$$y^k = x^{kb_0}(1 - x)^{kb_1}(z - x)^{kb}.$$

Define the differential forms

$$\eta_1 := x^{-a_0}(1 - x)^{-a_1}(z - x)^{-a} \frac{dx}{y}, \quad \eta_2 := x^{-a_0}(1 - x)^{-a_1}(z - x)^{-a} \frac{x dx}{(x - 1)y}.$$

The Pochhammer cycles $[\gamma_i, \gamma_z]$, $i = 0, 1$ lift to cycles $\delta_i \in H_1(X(k, z), \mathbb{Z})$ and

$$Y = \begin{pmatrix} \int_{\delta_1} \eta_1 & \int_{\delta_2} \eta_1 \\ \int_{\delta_1} \eta_2 & \int_{\delta_2} \eta_2 \end{pmatrix}.$$

Proposition 3. ([19] Satz 1,2) *We have the following isogeny for the Jacobian $J_{X(k, z)}$*

$$J_{X(k, z)} \sim T(k, z) \oplus \sum_{d|k} J_{X(d, z)},$$

where $T(k, z)$ is an abelian variety of dimension $\varphi(k)$. Moreover

$$\mathbb{Q}(\zeta_k) \subset \text{End}_0(T) := \mathbb{Q} \otimes \text{End}(T)$$

and $H_1(T(k, t), \mathbb{Q})$ as a $\mathbb{Q}(\zeta_k)$ -vector space is of dimension two.

4 A family of cubic four dimensional Hypersurfaces

The calculation of Gauss-Manin connection in higher dimensions does not seem to be done by hand easily. In [13, 14] the first author has developed algorithms which calculate the Gauss Manin connection and hence Picard-Fuchs equations for families of hypersurfaces in weighted projective spaces. They are implemented in SINGULAR (see [7]) and the corresponding library is `foliation.lib`. In this and the next sections we have used this library for our calculations.

4.1 Some Picard-Fuchs equations

Recall the notations introduced in §2.4 and let

$$f = x_1^3 + x_2^3 + \dots + x_5^3 - x_1 - x_2.$$

The singular values of f are the roots of $27t^3 - 16t$. Let

$$\omega_i = \frac{x_i dx}{df}, \quad \omega_{i,j} = \frac{x_i x_j dx}{df}, \quad \omega_0 = \frac{dx}{df}$$

and $\omega = (\omega_{12}, \omega_2, \omega_1, \omega_0)^t$. Then

$$\nabla(\omega) = \frac{1}{27t^3 - 16t} \begin{pmatrix} 36t^2 - 32/3 & -6t & -6t & 16/9 \\ -24t & 27t^2 - 8 & 8 & -4t \\ -24t & 8 & 27t^2 - 8 & -4t \\ 32 & -18t & -18t & 18t^2 - 16/3 \end{pmatrix} \omega.$$

This gives us a Fuchsian system of order 4. One gets that $\nabla\omega_0$ satisfies

$$(19) \quad (27t^3 - 16t)y'' + (81t^2 - 16)y' + 15ty = 0.$$

According to Theorem 2 the differential form $\nabla(\omega_0)$ is a basis of $H^{3,1}$ and so $\delta \in H_4(M_t, \mathbb{Q})$ is Hodge if and only if

$$\int_{\delta} \nabla\omega_0 = 0.$$

Remark 2. The form $\nabla\omega_{12}$ satisfies

$$(27t^3 - 16t)y'' + (81t^2 - 16)y' - 21ty = 0.$$

Both $\omega_i, i = 1, 2$ satisfy

$$(27t^3 - 16t)y''' + 54t^2y'' - 3ty' + 3y = 0.$$

The 4-forms $\omega_i, i = 3, 4, 5$ satisfy

$$9y + (-9t)y' + (162t^2 - 32)y'' + (81t^3 - 48t)y''' = 0$$

and $\omega_{i,j}, i = 1, 2, j = 3, 4, 5$

$$0y + (-45t)y' + (621t^2 + 32)y'' + (1134t^3 - 192t)y''' + (243t^4 - 144t^2)y'''' = 0$$

and $\omega_{ij}, i, j = 3, 4, 5$

$$0y + (-9t)y' + (81t^2 - 16)y'' + (81t^3 - 48t)y''' = 0.$$

Remark 3. Each variety M_t has the following automorphisms

$$(20) \quad (x_1, x_2, x_3, x_4, x_5) \rightarrow (x_1, x_2, \zeta_3^{i_3} x_3, \zeta_3^{i_4} x_4, \zeta_3^{i_5} x_5), \zeta_3^{i_j} \in G_3.$$

The family $\{M_t\}_{t \in \mathbb{P}^1}$ itself has more automorphisms

$$(x_1, x_2, x_3, x_4, x_5) \rightarrow (-x_1, -x_2, \zeta_6^{i_3} x_3, \zeta_6^{i_4} x_4, x_5 \zeta_3^{i_5}), \zeta_6^{i_j} \in G_6, i_j \text{ odd}.$$

Since $\nabla(\omega_0)$ is an eigen vector with eigen values in $\mathbb{Q}(\zeta_6) = \mathbb{Q}(\zeta_3)$ for the above automorphisms, the period set $\mathcal{P}(M_t, \nabla(\omega_0))$, $t \in \mathbb{C} \setminus C$ is a $\mathbb{Q}(\zeta_6)$ -vector space under the usual multiplication of numbers.

4.2 Associated Gauss system

The differential equation (19) can be written as a system

$$Y' = \begin{pmatrix} 0 & 1 \\ -\frac{15}{27t^2-16} & -\frac{81t^2-16}{27t^3-16t} \end{pmatrix} Y,$$

where $Y = (y, y')^{tr}$. Via the gauge transformation

$$Y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -3t \end{pmatrix} Y$$

we obtain the Fuchsian system

$$Y' = \frac{1}{t} \begin{pmatrix} 0 & -1/3 \\ 0 & 0 \end{pmatrix} + \frac{54t}{27t^2-16} \begin{pmatrix} 0 & 0 \\ 5/6 & -1 \end{pmatrix} Y.$$

This Fuchsian system is a rational pull back via $z = \frac{27}{16}t^2$ of the hypergeometric differential equation:

$$Y' = \frac{1}{z} \begin{pmatrix} 0 & -1/6 \\ 0 & 0 \end{pmatrix} + \frac{1}{z-1} \begin{pmatrix} 0 & 0 \\ 5/6 & -1 \end{pmatrix} Y$$

with

$$(21) \quad \mu_0 = \mu_1 = \frac{1}{6}, \mu = \frac{5}{6}, c = 1, a = \frac{5}{6}, b = \frac{1}{6}.$$

Further,

$$Y \begin{pmatrix} 1 & 0 \\ 0 & (1-\zeta_6)^{-1} \end{pmatrix}$$

has monodromy

$$A_0 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

at 0, respectively 1, and at ∞ it is an element of order 6. Thus the generated group is $SL(2, \mathbb{Z})$.

Now let us discuss the one dimensional geometric model of (19). In this example

$$X(6, z) : y^6 = x(1-x)(z-x)^5$$

which is of genus 5.

$$\eta_1 = \frac{dx}{y}, \quad \eta_2 = \frac{xdx}{(1-x)y}$$

$$X(2, z) : y^2 = x(1-x)(z-x)^5, \quad X(3, z) : y^3 = x(1-x)(z-x)^5.$$

With transformations $(x, y) \mapsto (x, \frac{y}{(z-x)^i})$, $i = 2, 1$ we have:

$$X(2, z) : y^2 = x(1-x)(z-x), \quad X(3, z) : y^3 = x(1-x)(z-x)^2.$$

The genus of $X(2, z)$ (resp. $X(3, z)$) is 1 (resp. 2). We make the transformation $x := z - x$. Then

$$X(6, z) : y^6 = (z-x)(1-z+x)x^5.$$

For $z = \frac{1}{2}$ we have the following new automorphism of $X(6, z)$

$$\sigma(x, y) = (-x, \zeta_{12}y).$$

4.3 Decomposition of $T(6, z)$ into elliptic curves

Lemma 1. *We have*

1. $X(6, z)$ is hyperelliptic of genus 5, where

$$X(6, z) = \mathbb{C}(x_1, y_1), \quad y_1^2 = x_1^{12} + (2-4z)x_1^6 + 1, \quad y_1 = \frac{x^2 - z^2 + z}{x}, \quad x_1 = \frac{y}{x}.$$

- 2.

$$H^0(\mathbb{C}(x_1, y_1), \Omega) = \langle x_1^i \frac{dx_1}{y_1} \mid i = 0, \dots, 4 \rangle \ni \frac{dx}{y}.$$

3. If $z \neq 1/2$ then $\text{Aut}(X(6, z))/\langle \epsilon \rangle = D_6 = \langle \sigma, \tau \rangle$, where ϵ denotes the hyperelliptic involution

$$\epsilon(x_1, y_1) = (x_1, -y_1), \quad \sigma(x_1, y_1) = (\zeta_6 x_1, y_1), \quad \tau(x_1, y_1) = (x_1^{-1}, \frac{y_1}{x_1^6})$$

resp.

$$\begin{aligned} \epsilon(x, y) &= \left(-\frac{z-z^2}{x}, -\frac{(z-z^2)y}{x^2} \right), \quad \sigma(x, y) = (x, \zeta_6 y), \\ \tau(x, y) &= \left(\frac{(x-z)(1-z)}{1-z+x}, \frac{x(x-z)(1-z)}{y(1-z+x)} \right). \end{aligned}$$

4. If $z = 1/2$ then $\text{Aut}(X(6, z))/\langle \epsilon \rangle = D_{12} = \langle \sigma, \tau \rangle$, where

$$\sigma(x_1, y_1) = (\zeta_{12} x_1, y_1), \quad \tau(x_1, y_1) = (x_1^{-1}, \frac{y_1}{x_1^6})$$

resp.

$$\sigma(x, y) = (-x, \zeta_{12}y) \quad \tau(x, y) = \left(\frac{(x-z)(1-z)}{1-z+x}, \frac{x(x-z)(1-z)}{y(1-z+x)} \right).$$

5. We have

$$-6x_1^4 \frac{dx_1}{y_1} = \frac{dx}{y} = \eta_1.$$

Proof. 1. Let $K = X(6, z)$, $y^6 = (z-x)(1-z+x)x^5$, where $g(K) = 5$ by the Hurwitz-Zeuthen genus formula. To each of the functions $z-x$, $1-z+x$, $x \in \mathbb{C}(x)$ we associate the principal divisors $(z-x) = \frac{\mathfrak{p}_z}{\mathfrak{n}}$, $(1-z+x) = \frac{\mathfrak{p}_{z-1}}{\mathfrak{n}}$, $(x) = \frac{\mathfrak{p}_0}{\mathfrak{n}}$, where the prime divisor \mathfrak{n} of $\mathbb{C}(x)$ corresponds to the place $p = \infty$ and the prime divisor \mathfrak{p}_0 (resp. \mathfrak{p}_z and \mathfrak{p}_{z-1}) corresponds to the prime polynomial $p = x$ (resp. $p = x-z$ and $p = x-z+1$) (see Chap. III, §2 in [4]). The prime divisor \mathfrak{p}_0 of $\mathbb{C}(x)$ decomposes in K into $\mathfrak{p}_{0,1}^{e_1} \cdots \mathfrak{p}_{0,r}^{e_r}$, where $\sum e_i = \deg(K/\mathbb{C}(x)) = 6$. Comparing the order of the prime divisor $\mathfrak{p}_{0,1}$ in the principal divisors (y) , $(z-x)$, $(1-z+x)$, (x) of K we get

$$\text{ord}_{\mathfrak{p}_{0,1}}(y^6) = 6 \text{ord}_{\mathfrak{p}_{0,1}}(y) = \text{ord}_{\mathfrak{p}_{0,1}}((z-x)(1-z+x)(x)^5) = 5 \text{ord}_{\mathfrak{p}_{0,1}}(x) = 5e_1.$$

Hence $\text{ord}_{\mathfrak{p}_{0,1}}(y) = 5$, $\text{ord}_{\mathfrak{p}_{0,1}}(x) = 6$ and $r = 1$. Thus the prime divisor \mathfrak{p}_0 of $\mathbb{C}(x)$ decomposes into $\mathfrak{p}_{0,1}^6$ in K . (Hence we denote by \mathfrak{p}_0 also the prime divisor $\mathfrak{p}_{0,1}$ of K .)

Proceeding as above we obtain that the principal divisors associated to the functions $y, x, x-z, x-z+1$ in K are

$$(y) = \frac{\mathfrak{p}_0^5 \mathfrak{p}_z \mathfrak{p}_{z-1}}{\mathfrak{n}^7}, \quad (x) = \frac{\mathfrak{p}_0^6}{\mathfrak{n}^6}, \quad (x-z) = \frac{\mathfrak{p}_z^6}{\mathfrak{n}^6}, \quad (x-z+1) = \frac{\mathfrak{p}_{z-1}^6}{\mathfrak{n}^6}.$$

Hence, the degree of the denominator of

$$\left(\frac{y}{x}\right) = \frac{\mathfrak{p}_z \mathfrak{p}_{z-1}}{\mathfrak{n} \mathfrak{p}_0}$$

is two. Thus K has to be hyperelliptic, where $K = \mathbb{C}(x_1, y_1)$, $x_1 = \frac{y}{x}$, $y_1^2 = f(x_1)$, $f(x_1) \in \mathbb{C}[x_1]$, $\deg(f) = 12$. Using that

$$x_1^6 = \frac{(z-x)(x-1+z)}{x}$$

and that there exists a $\sigma \in \text{Aut}(K/\mathbb{C})$ with $\sigma(x_1) = \zeta_6 x_1$ we have to solve the following equation in order to determine y_1 :

$$x_1^{12} + ax_1^6 + 1 = y_1^2, \quad a \in \mathbb{C}.$$

Hence, we can assume that $y_1 = \frac{x^2 + a_1 x + a_2}{x}$, $a_1, a_2 \in \mathbb{C}$. This gives after some computations

$$a = 2 - 4z, \quad a_1 = 0, \quad a_2 = -z^2 + z.$$

2. It is well known that

$$H^0(\mathbb{C}(x_1, y_1), \Omega) = \langle x_1^i \frac{dx_1}{y_1} \mid i = 0, \dots, 4 \rangle.$$

Since

$$\left(\frac{dx}{y}\right) = \frac{\mathfrak{d}_{K/\mathbb{C}(x)}}{\mathfrak{n}_x^2} \frac{1}{(y)} = \frac{\mathfrak{p}_0^5 \mathfrak{p}_z^5 \mathfrak{p}_{z-1}^5 \mathfrak{n}^5}{\mathfrak{n}^{12}} \frac{\mathfrak{n}^7}{\mathfrak{p}_0^5 \mathfrak{p}_z \mathfrak{p}_{z-1}} = \mathfrak{p}_z^4 \mathfrak{p}_{z-1}^4,$$

where $\mathfrak{d}_{K/\mathbb{C}(x)}$ denotes the pseudo different and \mathfrak{n}_x the divisor in K of the denominator divisor of $x \in \mathbb{C}(x)$, (See Chap. III, §2, §3 in [4].) is an positive divisor the claim follows.

Items 3, 4 and 5 are proved by easy computations. \square

We will show that the vector space of holomorphic differentials can be written as a direct sum $\oplus U_i$, where $U_i = H^0(E_i, \Omega)$, where E_i is an elliptic subfield of K . Thus we will study some subfields fixed by automorphisms of $\text{Aut}(X(6, z))$:

Lemma 2. *Let $\tau, \sigma \in \text{Aut}(K)$ be as in Lemma 1.*

$$H^0(X(6, z), \Omega) = \oplus_{i=1}^5 H^0(E_i, \Omega), \text{ where}$$

1. E_1 is contained in the hyperelliptic function field $K^{\langle \sigma \rangle}$ of genus 2 and

$$E_1 : \tilde{y}^2 = \tilde{x}^3 - 3\tilde{x} + 2 - 4z, \quad \tilde{x} = (x_1 + x_1^{-1})^2 - 2, \quad \tilde{y} = y_1 x_1^{-3},$$

where

$$j(E_1) = \frac{-2^4 3^3}{z(z-1)}, \quad \frac{d\tilde{x}}{\tilde{y}} = 2(x_1^4 - 1) \frac{dx_1}{y_1}.$$

2. E_2 is contained in the hyperelliptic function field $K^{\langle \tau\sigma \rangle}$ of genus 2 and

$$E_2 : \tilde{y}^2 = \tilde{x}^3 - 3\zeta_6^2 \tilde{x} + 2 - 4z, \quad \tilde{x} = (x_1 + \zeta_6 x_1^{-1})^2 - 2\zeta_6, \quad \tilde{y} = y_1 x_1^{-3},$$

where

$$j(E_2) = \frac{-2^4 3^3}{z(z-1)}, \quad \frac{d\tilde{x}}{\tilde{y}} = 2(x_1^4 - \zeta_6^2) \frac{dx_1}{y_1}.$$

3.

$$E_3 = K^{\langle \sigma^2 \rangle} = X(2, z) : \tilde{y}^2 = \tilde{x}^4 + (2 - 4z)\tilde{x}^2 + 1, \quad \tilde{x} = x_1^3, \quad \tilde{y} = y_1,$$

where

$$j(E_3) = 2^8 \frac{(z^2 - z + 1)^3}{z^2(z-1)^2}$$

and

$$\frac{d\tilde{x}}{\tilde{y}} = 3x_1^2 \frac{dx_1}{y_1}.$$

4. E_4 and E_5 are two isomorphic subfields of the hyperelliptic subfield $K^{\langle \sigma^3 \rangle} = X(3, z)$ of genus 2.

$$E_4 : \tilde{y}^2 = \tilde{x}^3 - 6\tilde{x}^2 + 9\tilde{x} - 4z, \quad \tilde{x} = x_1^4, \quad \tilde{y} = y_1,$$

where

$$j(E_4) = \frac{-2^4 3^3}{z(z-1)}, \quad \frac{d\tilde{x}}{\tilde{y}} = 2x_1 \frac{dx_1}{y_1}$$

$$E_5 : \tilde{y}^2 = \tilde{x}^3 - 6\tilde{x}^2 + 9\tilde{x} - 4z, \quad \tilde{x} = x_1^2 + x_1^{-2}, \quad \tilde{y} = y_1 x_1^{-6}$$

where

$$j(E_5) = \frac{-2^4 3^3}{z(z-1)}, \quad \frac{d\tilde{x}}{\tilde{y}} = 2x_1^3 \frac{dx_1}{y_1}.$$

Proof. The claim follows from Lemma 1. □

Corollary 2. *Let $E_i, i = 1, \dots, 5$ the elliptic subfields of $X(6, z)$ from Lemma 2. Then*

$$H^0(X(6, z), \Omega) = \oplus_{i=1}^5 H^0(E_i, \Omega).$$

Proof. By Lemma 2 we know that the elliptic differentials in E_i , $i = 1, \dots, 5$ generate $H^0(X(6, z), \Omega)$. \square

Corollary 3. *We have the isogeny*

$$T(6, z) \sim E^2,$$

where

$$E : y^2 = x^3 - 3x + 2 - 4z.$$

Remark 4. By a result of Ekedahl and Serre [6] it follows already from the existence of the hyperelliptic subfield $X(3, z)$ of genus 2 that the Jacobian of $X(6, z)$ is isogenous to a direct sum of elliptic curves. Using the structure of the automorphism group it follows that E_1 is isogenous to E_2 by a result of Lange and Recillas [12].

4.4 Proof of Theorem 1

Recall the definition (9). Both the period sets $\mathcal{P}(M_t, \nabla(\omega_0))$ and $\mathcal{P}(E_t, \frac{dx}{y})$ satisfy the Gauss hypergeometric equation (1) with parameters a, b, c given in (21). This furnishes us with the hypothesis of the Conjecture in §2.3.

Lemma 3. *The space $\mathcal{P}(M_t, \nabla(\omega_0))$ is spanned over \mathbb{Q} by $\Gamma(\frac{1}{3})^3 \cdot \mathbb{Q}(\zeta_3) \cdot \mathcal{P}(E_t, \frac{dx}{y})$.*

Proof. Let $f_1 = x_1^3 - x_2^3 - x_1 - x_3$ and $f_2 = t - (x_3^2 + x_4^3 + x_5^3)$ and fix a point $b \in \mathbb{C}$ which is not a critical value neither for f_1 nor f_2 . Let also $\gamma : [0, 1] \rightarrow \mathbb{C}$, $\gamma(\frac{1}{2}) = b$ be a path in \mathbb{C} which connects the critical point $t \in \mathbb{C}$ of f_2 to a critical point c of f_1 such that δ_{1b} (resp. δ_{2b}) vanishes along γ (resp. γ^{-1}) when s goes to 0 (resp. 1). One can show that the union

$$\delta_t = \cup_{s \in [0, 1]} \delta_{1\gamma(s)} \times \delta_{2\gamma(s)}$$

defines a well-defined cycle in $H_4(M_t, \mathbb{Z})$. In fact it is the join of topological spaces δ_{1b} and δ_{2b} and

$$I(t) := \int_{\delta_t} \omega_0 = \int_0^1 I_1(s) I_2(s) ds,$$

$$I_1(s) = \int_{\delta_{1\gamma(s)}} \frac{dx_1 \wedge dx_2}{df_1}, \quad I_2(s) = \int_{\delta_{2\gamma(s)}} \frac{dx_3 \wedge dx_4 \wedge dx_5}{df_2}.$$

(see [2] p. 53). A topological argument similar to the one in [2] Theorem 2.1 or [11] §5 implies that the cycles δ_t generate $H_4(M_t, \mathbb{Q})$. A simple integration shows that $I_2(s)$ is constant and the space of such integrals is equal to $\Gamma(\frac{1}{3})^3 \mathbb{Q}(\zeta_3)$ (see [5] Lemma 7.12). This means that up to $\Gamma(\frac{1}{3})^3 \mathbb{Q}(\zeta_3)$ the integral $I(t)$ reduces to $\int_{\Delta} dx_1 \wedge dx_2$, where $\Delta := \cup_{s \in [0, 1]} \delta_{1\gamma(s)} \in H_2(\mathbb{C}^2, f_1^{-1}(t), \mathbb{Z})$. By Stokes theorem this integral is

$$\int_{\delta_{1t}} x_1 dx_2.$$

Since $\frac{\partial I}{\partial t} = \int_{\delta_t} \nabla(\omega_0)$, we conclude that

$$\int_{\delta_t} \nabla(\omega_0) \in \Gamma(\frac{1}{3})^3 \mathbb{Q}(\zeta_3) \mathcal{P}(\{f_1 = t\}, \frac{dx_1 \wedge dx_2}{df_1}).$$

Now the elliptic curve $\{f_1 = t\}$ is isomorphic to the one in (4) by the mapping

$$(x_1, x_2) \rightarrow \left(\frac{3t}{x_1 + x_2} + 1, 9t \left(\frac{x_2}{x_1 + x_2} - \frac{1}{2} \right) \right)$$

and under this mapping $\frac{dx_1 \wedge dx_2}{df_1}$ goes to $-\frac{dx}{2y}$ □

Now let us prove Theorem 1. The proof of the first and second part is as follows: The map $\delta \in H_4(M_t, \mathbb{Q}(\zeta_3)) \rightarrow \int_\delta \nabla(\omega_0) \in \mathcal{P}(M_t, \nabla(\omega_0))$ is surjective. Now Lemma 3, Corollary 1 and Corollary 2 imply that $\mathcal{P}(M_t, \nabla(\omega_0))$ is of dimension one if and only if E_t is elliptic.

By Proposition 1 if $\delta \in H_4(M_t, \mathbb{Q})$ is a Hodge cycle then

$$(22) \quad \frac{\partial}{\partial t} \left(\int_\delta \nabla(\omega_0) \right) = \int_\delta \nabla^2(\omega_0) \in \pi^2 \bar{\mathbb{Q}}.$$

Let $Y_1(t), Y_2(t)$ be a basis of $\mathcal{P}(E_t, \frac{dx}{y})$ and $\frac{Y_1(t_0)}{Y_2(t_0)} = r \in \mathbb{Q}(\zeta_3)$ at some point t_0 . Then according to Lemma 3 we have $\Gamma(\frac{1}{3})^3 Y_1(t) = \int_{\delta_{1,t}} \nabla(\omega_0)$ and $\Gamma(\frac{1}{3})^3 r Y_2(t) = \int_{\delta_{2,t}} \nabla(\omega_0)$ for some $\delta_{1,t}, \delta_{2,t} \in H_4(M_t, \mathbb{Q})$ and all t in a neighborhood of t_0 . This implies that $\delta_{1,t_0} - \delta_{2,t_0}$ is a Hodge cycle. Now we use (22) and we conclude that

$$\Gamma(\frac{1}{3})^3 (Y_1'(t_0) - r Y_2'(t_0)) \sim \pi^2.$$

By Legendre theorem on the determinant of the period matrix of elliptic curves (or the relation (15)), we have

$$Y_1'(t_0) - a Y_2'(t_0) = \frac{(Y_1' Y_2 - Y_1 Y_2')(t_0)}{Y_2(t_0)} \sim \frac{\pi}{Y_2(t_0)}$$

and so $Y_1(t_0), Y_2(t_0) \sim \frac{1}{\pi} \Gamma(\frac{1}{3})^3$. Using the formula (13) the third part of the above theorem is proved.

5 Other examples

In this section we give a list of cubic hypersurfaces such that the corresponding integral $\int_{\delta_t} \nabla(\frac{dx}{f})$ for the definition of Hodge cycles satisfy a Picard-Fuchs equation of order two. For calculations we have used the library `brho.lib` written in SINGULAR (see [14, 13]). Let $\omega_0 = \frac{dx}{df}$.

1. For

$$f = x_1^3 + \cdots + x_5^3 - x_1 x_2$$

$\int_{\delta_t} \omega_0$ satisfies

$$(27t^2 + t)y'' + 6y = 0$$

and so $\int_{\delta_t} \nabla \omega_0$ satisfies

$$(27t^2 + t)y'' + 54ty' - 6y = 0.$$

2. For

$$f = x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 - x_1^2 - x_2^2$$

$\int_{\delta_t} \omega_0$ satisfies

$$(405t + 60)y' + (2187t^2 + 648t + 32)y'' + (729t^3 + 324t^2 + 32t)y''' = 0$$

and $y' = \int_{\delta_t} \nabla \omega_0$. Note that the coefficient of y''' is $t(27t + 4)(27t + 8)$.

3. For

$$f = x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 - x_1^2 - x_2x_1$$

$\int_{\delta_t} \omega_0$ satisfies

$$0y + (4374t^2 - 1296t - 108)y' + (39366t^3 + 3645t^2 - 1188t + 2)y'' + (19683t^4 + 6561t^3 - 621t^2 + 2t)y''' = 0.$$

The coefficient of y'' is $t(729t^2 + 297t - 1)(27t - 2)$ and so there is an apparent singularity at $t = \frac{2}{27}$. Note that the critical values of f are the roots of $(729t^2 + 297t - 1)t$.

4. For

$$f = x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 - x_1^2 - x_1$$

$\int_{\delta_t} \omega_0$ satisfies

$$(27t + 11)y' + (81t^2 + 66t - 15)y'' = 0.$$

The coefficient of y'' is $3(27t - 5)(t + 1)$.

5. For

$$f = x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 - x_1 - x_1x_2$$

the set of critical values is the roots of $S(t) = (19683t^4 + 2187t^3 - 5751t^2 + 541t + 433)$ and $\int_{\delta_t} \omega_0$ satisfies

$$(3188646t^4 + 472392t^3 + 6403536t^2 + 236844t - 314922)y' + (28697814t^5 + 4782969t^4 + 25824096t^3 + 1427382t^2 - 4369464t + 152443)y'' + (14348907t^6 + 2657205t^5 + 2322594t^4 + 794610t^3 - 1524204t^2 + 199207t + 140725)y''' = 0.$$

The coefficient of y''' is

$$(19683t^4 + 2187t^3 - 5751t^2 + 541t + 433)(729t^2 + 54t + 325)$$

and so there are two apparent singularities.

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