Hossein Movasati

An Advanced Course in Hodge Theory

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Introduction

The author's search for a counterexample to the Hodge conjecture took more than half of the book [Mov17b]. As Grothendieck's method to solve a mathematical problem is to generalize it so much that a solution comes by itself, I belive that finding a counterexample to the Hodge conjecture must take a completely different way. Instead of generalizations, one has to study so many particular examples such that it comes to mind automatically. Even if the Hodge conjecture is true, the belief that it is false has made the Hodge theory more accessible to those who loves computational mathematics. The history of the Hodge conjecture from a computational point of view is a sad one. Up to the time of writing the present text, there was no explicit generators of the Neron-Severi group of the Fermat variety of degree 12. This job is being done by N. Aoki, see [Aok15]. I belive that once a counterexample is found it would be like a Colombus' egg and there will be explosion of other counterexamples. Once you are in the oceon without compass, all directions might lead you to a land. In [Mov17b] the author thought himself few directions.

The emphasis of the first book [Mov17b] was mainly on hypersurfaces and the study of the Hodge locus through the Fermat variety. This book intends to tell us the Hodge theory of smooth projective varieties and their properties inside families. This is the study of Cech cohomology, hypercohomology, Gauss-Manin connection, infinitesimal variation of Hodge structures, Hodge loci etc. A synopsis of each chapter is explained below.

In Chapter 3 we present a minimum amount of material so that the reader get familiar with Cech cohomologies and hypercohomologies. We always need to represent elements of cohomologies with concrete data and we do this using an acyclic covering. In this chapter we also discuss quasi-isomorphisms and filtration, in a way adopted for computations. This chapter is presented for general sheaves, however, we only apply its content to the sheaf of differential forms.

Chapter 4 is fully dedicated to algebraic de Rham cohomology and the fact that it is isomorphic to the classical de Rham cohomology. The main ingredient here is the Atiyah-Hodge theorem on the de Rham cohomology of affine varieties. The objective in this chapter is to collect all necessary material for computing the integration of elements of algebraic de Rham cohomologies over algebraic cycles.

Chapters 5 is fully dedicated to the description of Gauss-Manin connection of families of algebraic varieties. In this chapter we describe the general context of arbitrary families of projective varieties, whereas in [Mov17b] focuses on the computation of Gauss-Manin connection for tame polynomials, and in particular families of hypersurfaces. The Griffiths transversality is one of the main theorems. It relates the Gauss-Manin connection to the underlying Hodge filtrations.

We do not give concrete applications of the Gauss-Manin connection in Algebraic Geometry, however, its partial data, namely the infinitesimal variation of Hodge structures (IVHS) has successful applications. This includes the famous Noether-Lefschetz theorem which says that a generic surface in the projective space of dimension three has Picard rank one. Chapter 6 is dedicated to this topic.

Most of the machineries in this book are introduced with one mission in mind. We would like to study deformations of Hodge cycles in the moduli space of the underlying variety. This results in the study of the so-called Hodge locus. There are many open questions in this context, even in the case where the Hodge conjecture is well-known.

Some words of philosophy. The truth in mathematics is a state of satisfaction but not vice versa. The classical way of doing mathematics is to prove, and hence to feel, the truth and then to enjoy the consequent satisfaction. However, with the rapid development of mathematics it seems to be very difficult to transfer to an student all the details leading to a truth and then satisfaction. Manytimes we need to use an object and in order to construct it explicitly we spend a lot of time so that the student lose all his/her interest on the subject. In this situation, I think, it is crucial to invest on inducing the state of satisfaction in new learners rather than using the classical methodology of doing mathematics which is defining and proving every thing precisely. In the present text I will try to follow this method in order to introduce one of the main conjectures in Algebraic Geometry, namely the Hodge conjecture.

Chapter 2
Cech cohomology

Il faut faisceautiser. (The motto of french revolution in algebraic and complex geometry, see [Rem95], page 6).

2.1 Introduction

In Chapter 4 [Mov17b] we discussed the axiomatic approach to singular homology and cohomology. These are the first examples of cohomology theories constructed in the first half of 20th century. Almost in the same time, other cohomolgy theories, such as De Rham and Cech cohomologies, and their properties were under construction and intensive investigation.

In this chapter we aim to introduce sheaf cohomologies and its explicit construction using Cech cohomologies. Similar to the case of singular homology and cohomology, the categorical approach to sheaf cohomology and the way that it is used in mathematics, shows that in most of occasions we only need to know a bunch of properties of the sheaf cohomology and not its concrete construction. However, in some other occasions, mainly when we want to formulate some obstructions, we obtain elements in some sheaf cohomologies and so just axioms of Cech cohomology would not work. Therefore, we introduce some properties of sheaf chomology which can be taken as axioms and we also explain its explicit construction using Cech cohomology. We assume that the reader is familiar with sheaves of abelian groups on topological spaces. The reader who is interested in a more elaborated version of this section may consult other books like [BT82], Section 10, [Voi02] Section 4.3.

2.2 Cech cohomology

A sheaf $\mathscr{S}$ of abelian groups on a topological space X is a collection of abelian groups

$$\mathscr{S}(U),\ U \subset X \text{ open}$$

Complete this.

with restriction maps which satisfy certain properties. In particular, $\mathscr{S}(X)$ is called the set of global sections of $\mathscr{S}$. Some other equivalent notations for this are

$$\mathscr{S}(X) = \Gamma(X, \mathscr{S}) = H^0(X, \mathscr{S})$$

It is not difficult to see that for an exact sequence of sheaves of abelian groups

$$0 \to \mathscr{S}_1 \to \mathscr{S}_2 \to \mathscr{S}_3 \to 0. \tag{2.1}$$

we have

$$0 \to \mathscr{S}_1(X) \to \mathscr{S}_2(X) \to \mathscr{S}_3(X)$$

and the last map is not necessarily surjective. In this section we want to construct abelian groups $H^i(X, \mathscr{S}),\ i = 0, 1, 2 \ldots,\ H^0(X, \mathscr{S}) = \mathscr{S}(X)$ such that we have the long exact sequence

$$0 \to H^0(X, \mathscr{S}_1) \to H^0(X, S_2) \to H^0(X, \mathscr{S}_3) \to H^1(X, \mathscr{S}_1) \to H^1(X, S_2) \to H^1(X, \mathscr{S}_3) \to$$

$$H^2(X, \mathscr{S}_1) \to \cdots$$

that is in each step the image and kernel of two consecutive maps are equal.

Let us explain the basic idea behind $H^1(X, \mathscr{S}_1)$. The elements of $H^1(X, \mathscr{S}_1)$ are considered as obstructions to the surjectivity of $H^0(X, \mathscr{S}_2) \to H^0(X, \mathscr{S}_3)$. This map is not surjective, however, we can look at an element $f \in \mathscr{S}_3(X)$ locally and use the surjectivity of $\mathscr{S}_2 \to \mathscr{S}_3$. We fix a covering $\mathscr{U} = \{U_i,\ i \in I\}$ of X such that the exact sequences corresponding to global section of (2.1) over U_i hold, that is,

$$0 \to \mathscr{S}_1(U_i) \to \mathscr{S}_2(U_i) \to \mathscr{S}_3(U_i) \to 0.$$

This covering is taken so that we have exactness at $\mathscr{S}_2(U_i)$ and for the exactness at $\mathscr{S}_3(U_i)$ we need to assume that the set of small open sets giving exactness at $\mathscr{S}_2$ and $\mathscr{S}_3$ in (2.1) is not emprty. This is the case for all examples of the short exact sequence (2.1) in this text (one may also justify this by assuming that $H^1(U_i, \mathscr{S}_1) = 0$).

We take $f_i \in \mathscr{S}_2(U_i),\ \ i \in I$ such that f_i is mapped to f under $\mathscr{S}_2(U_i) \to \mathscr{S}_3(U_i)$. This means that the elements $f_j - f_i$, which are defined in the intersections $U_i \cap U_j$'s, are mapped to zero and so there are elements $f_{ij} \in \mathscr{S}_1(U_i \cap U_j)$ such that f_{ij} is mapped to $f_j - f_i$. It is easy to check that different choices of f_i's lead us to elements

$$f_{ij} + \tilde{f}_j - \tilde{f}_i, \tag{2.2}$$

where $\tilde{f}_i \in \mathscr{S}_1(U_i)$. This lead us to define $H^1(\mathscr{U},\mathscr{S}_1)$ to be the set of $(f_{ij},\ \ i,j \in I)$ modulo those of the form (2.2).

2.3 Covering and direct limit

Let X be a topological space, $\mathscr{S}$ a sheaf of abelian groups on X and $\mathscr{U} = \{U_i,\ i \in I\}$ a covering of X by open sets. In this paragraph we want to define the Cech cohomology of the covering $\mathscr{U}$ with coefficients in the sheaf $\mathscr{S}$. Let $\mathscr{U}^p$ denotes the set of $(p+1)$-tuples $\sigma = (U_{i_0},\dots,U_{i_p}),\ i_0,\dots,i_p \in I$ and for $\sigma \in \mathscr{U}^p$ define

$$|\sigma| = \cap_{j=0}^{p} U_{i_j}.$$

A p-cochain $f = (f_\sigma)_{\sigma\in\mathscr{U}^p}$ is an element in

$$C^p(\mathscr{U},\mathscr{S}) := \prod_{\sigma\in\mathscr{U}^p} H^0(|\sigma|,\mathscr{S})$$

Definition 2.3.1 Let π be the permutation group of the set $\{0,1,2,\dots,p\}$. It acts on $\mathscr{U}^p$ in a canonical way and we say that $f \in C^p(\mathscr{U},\mathscr{S})$ is skew-symmetric if $f_{\pi\sigma} = \operatorname{sign}(\pi) f_\sigma$ for all $\sigma \in \mathscr{U}^p$. The set of skew-symmetric cochains form an abelian subgroup $C_s(\mathscr{U},\mathscr{S})$.

For $\sigma \in \mathscr{U}^p$ and $j = 0,1,\dots,p$ denote by σ_j the element in $\mathscr{U}^{p-1}$ obtained by removing the j-th entry of σ. We have $|\sigma| \subset |\sigma_j|$ and so the restriction maps from $H^0(|\sigma_j|,\mathscr{S})$ to $H^0(|\sigma|,\mathscr{S})$ is well-defined. We define the boundary mapping

$$\delta : C_s^p(\mathscr{U},\mathscr{S}) \to C_s^{p+1}(\mathscr{U},\mathscr{S}),\ (\delta f)_\sigma = \sum_{j=0}^{p+1} (-1)^j f_{\sigma_j}\ |_{|\sigma|}$$

We have to check that

Proposition 2.3.1 *The above map is well-defined, that is, if $f \in C^p(\mathscr{U},\mathscr{S})$ is skew-symmetric then δf is also skew-symmetric.*

Proof. For simplicity, and without loss of generality we can assume that π is permutation of 0 and 1.

From now on we identify σ with $i_0 i_1 \cdots i_p$ and write a p-cochain as $f = (f_{i_0 i_1 \cdots i_p},\ i_j \in I)$. For simplicity we also write

$$(\delta f)_{i_0 i_1 \cdots i_{p+1}} := \sum_{j=0}^{p+1} (-1)^j f_{i_0 i_1 \cdots i_{j-1} \hat{i}_j i_{j+1} \cdots i_{p+1}}$$

where $\hat{i}_j$ means that i_j is removed.

Proposition 2.3.2 *We have*

$$\delta \circ \delta = 0.$$

Proof. Let $f \in C_s^p(\mathscr{U}, \mathscr{S})$. We have

$$(\delta^2 f)_{i_0 i_1 \cdot i_{p+2}} = \sum_{j=0}^{p+2} (-1)^j (\delta f)_{i_0 i_1 \cdots \hat{i}_j \cdot i_{p+2}}$$
$$= \sum_{j=0}^{p+2} \sum_{k=1,\ k \neq j}^{p+2} (-1)^j (-1)^{\tilde{k}} f_{i_0 i_1 \cdots \hat{i}_k \cdots i_{p+2}} = 0$$

where $\tilde{k} = k$ if $k < j$ and $\tilde{k} = k - 1$ if $k > j$. See [BT82] Proposition 8.3.

Now

$$0 \to C_s^0(\mathscr{U}, \mathscr{S}) \xrightarrow{\delta} C_s^1(\mathscr{U}, S) \xrightarrow{\delta} C_s^2(\mathscr{U}, \mathscr{S}) \xrightarrow{\delta} C_s^3(\mathscr{U}, \mathscr{S}) \xrightarrow{\delta} \cdots$$

can be viewed as cochain complexes, i.e. the image of a map in the complex is inside the kernel of the next map.

Definition 2.3.2 The Cech cohomology of the covering $\mathscr{U}$ with coefficients in the sheaf $\mathscr{S}$ is the cohomology groups

$$H^p(\mathscr{U}, \mathscr{S}) := \frac{\text{Kernel}(C_s^p(\mathscr{U}, S) \xrightarrow{\delta} C_s^{p+1}(\mathscr{U}, \mathscr{S}))}{\text{Image}(C_s^{p-1}(\mathscr{U}, S) \xrightarrow{\delta} C_s^p(\mathscr{U}, \mathscr{S}))}.$$

The above definition depends on the covering and we wish to obtain cohomologies $H^p(X, \mathscr{S})$ which depends only on X and $\mathscr{S}$. We recall that the set of all coverings $\mathscr{U}$ of X is directed:

Definition 2.3.3 For two coverings $\mathscr{U}_i = \{U_{i,j},\ j \in I_i\}$, $i = 1, 2$ we write $\mathscr{U}_1 \leq \mathscr{U}_2$ and say that $\mathscr{U}_1$ is a refinement of $\mathscr{U}_2$, if there is a map from $\phi : I_1 \to I_2$ such that $U_{1,i} \subset U_{2,\phi(i)}$ for all $i \in I_1$.

For two covering $\mathscr{U}_1$ and $\mathscr{U}_2$ there is another covering $\mathscr{U}_3$ such that $\mathscr{U}_3 \leq \mathscr{U}_1$ and $\mathscr{U}_3 \leq \mathscr{U}_2$. It is not difficult to show that for $\mathscr{U}_1 \leq \mathscr{U}_2$ we have a well-defined map

$$H^p(\mathscr{U}_2, \mathscr{S}) \to H^p(\mathscr{U}_1, \mathscr{S})$$

which is obtained by restriction from $U_{2,\phi(i)}$ to $U_{1,i}$. For details see [BT82], Lemma 10.4.1 and Lemma 10.4.2.

Definition 2.3.4 The Cech cohomology of X with coefficients in $\mathscr{S}$ is defined in the following way:

$$H^p(X, \mathscr{S}) := \text{dir}\ \lim_{\mathscr{U}} H^p(\mathscr{U}, \mathscr{S}).$$

We may view $H^p(X, \mathscr{S})$ as the union of all $H^p(\mathscr{U}, \mathscr{S})$ for all coverings $\mathscr{U}$, quotiented by the following equivalence relation. Two elements $\alpha \in H^p(\mathscr{U}_1, \mathscr{S})$ and $\beta \in H^p(\mathscr{U}_2, \mathscr{S})$ are equivalent if there is a covering $\mathscr{U}_3 \leq \mathscr{U}_1$ and $\mathscr{U}_3 \leq \mathscr{U}_2$ such that α and β are mapped to the same element in $H^p(\mathscr{U}_3, \mathscr{S})$.

2.4 Acyclic sheaves

Definition 2.4.1 A sheaf $\mathscr{S}$ of abelian groups on a topological space X is called acyclic if

$$H^k(X,\mathscr{S})=0,\ k=1,2,\ldots$$

The main examples of acyclic sheaves that we have in mind are the following:

Proposition 2.4.1 *Let M be a C^∞ manifold. The sheaves $\Omega^i_{M^\infty}$ of C^∞ differential i-forms on M are acyclic.*

Proof. The proof is based on the partition of unity and is left to the reader.

Definition 2.4.2 A sheaf $\mathscr{S}$ is said to be flasque or fine if for every pair of open sets $V\subset U$, the restriction map $\mathscr{S}(U)\to\mathscr{S}(V)$ is surjective.

Proposition 2.4.2 *Flasque sheaves are acyclic.*

See [Voi02], p.103, Proposition 4.34.

2.5 How to compute Cech cohomologies

Definition 2.5.1 The covering $\mathscr{U}$ is called acyclic with respect to $\mathscr{S}$ if $\mathscr{U}$ is locally finite, i.e. each point of X has an open neighborhood which intersects a finite number of open sets in $\mathscr{U}$, and $H^p(U_{i_1}\cap\cdots\cap U_{i_k},\mathscr{S})=0$ for all $U_{i_1},\ldots,U_{i_k}\in\mathscr{U}$ and $p\geq 1$.

Theorem 2.5.1 *(Leray lemma) Let $\mathscr{U}$ be an acyclic covering of a variety X. There is a natural isomorphism*

$$H^\mu(\mathscr{U},\mathscr{S})\cong H^\mu(X,\mathscr{S}).$$

See [Voi02] Theorem 4.41 and Theorem 4.44. A full proof can be found in the book of Godement 1958. See also [GH94] page 40. For a sheaf of abelian groups $\mathscr{S}$ over a topological space X, we will mainly use $H^1(X,\mathscr{S})$. Recall that for an acyclic covering $\mathscr{U}$ of X an element of $H^1(X,\mathscr{S})$ is represented by

$$f_{ij}\in\mathscr{S}(U_i\cap U_j),\ i,j\in I$$

$$f_{ij}+f_{jk}+f_{ki}=0,\ f_{ij}=-f_{ji},\ i,j,k\in I$$

It is zero in $H^1(X,\mathscr{S})$ if and only if there are $f_i\in\mathscr{S}(U_i),\ i\in I$ such that $f_{ij}=f_j-f_i$.

Remark 2.5.1 For sheaves of abelian groups $\mathscr{S}_i,\ i=1,2,\ldots,k$ over a variety X we have:

$$H^p(X,\oplus_i\mathscr{S}_i)=\oplus_i H^p(X,\mathscr{S}_i),\ p=0,1,\ldots.$$

2.6 Resolution of sheaves

A complex of abelian sheaves is the following data:

$$\mathscr{S}^\bullet : \mathscr{S}^0 \stackrel{d_0}{\to} \mathscr{S}^1 \stackrel{d_1}{\to} \cdots \stackrel{d_{k-1}}{\to} \mathscr{S}^k \stackrel{d_k}{\to} \cdots$$

where $\mathscr{S}^k$'s are sheaves of abelian groups and $\mathscr{S}^k \to \mathscr{S}^{k+1}$ are morphisms of sheaves of abelian groups such that the composition of two consecutive morphism is zero, i.e

$$d_{k-1} \circ d_k = 0,\ k = 1, 2, \ldots.$$

A complex $\mathscr{S}^k, k \in \mathbb{N}_0$ is called a resolution of $\mathscr{S}$ if

$$\mathrm{Im}(d^k) = ker(d^{k+1}),\ k = 0, 1, 2, \ldots$$

and there exists an injective morphism $i : \mathscr{S} \to \mathscr{S}^0$ such that $\mathrm{Im}(i) = \ker(d^0)$. We write this simply in the form

$$\mathscr{S} \to \mathscr{S}^\bullet$$

For simplicity we will write $d = d_k$; being clear in the text the domain of the map d.

Definition 2.6.1 A resolution $\mathscr{S} \to \mathscr{S}^\bullet$ is called acyclic if all $\mathscr{S}^k$ are acyclic.

Theorem 2.6.1 *Let $\mathscr{S}$ be a sheaf of abelian groups on a topological space X and $\mathscr{S} \to \mathscr{S}^\bullet$ be an acyclic resolution of $\mathscr{S}$ then*

$$H^k(X, \mathscr{S}) \cong H^k(\Gamma(X, \mathscr{S}^\bullet), d),\ k = 0, 1, 2, \ldots. \tag{2.3}$$

where

$$\Gamma(X, \mathscr{S}^\bullet) : \Gamma(\mathscr{S}^0) \stackrel{d_0}{\to} \Gamma(\mathscr{S}^1) \stackrel{d_1}{\to} \cdots \stackrel{d_{k-1}}{\to} \Gamma(\mathscr{S}^k) \stackrel{d_k}{\to} \cdots$$

and

$$H^k(\Gamma(X, \mathscr{S}^\bullet), d) := \frac{\ker(d_k)}{\mathrm{Im}(d_{k-1})}.$$

Proof. Let $\mathscr{U} = \{U_i\}_{i \in I}$ be an acyclic covering of X and let

$$\mathscr{S}^i_j := C^j_s(\mathscr{U}, \mathscr{S}^i),\ \ \mathscr{S}_j := C^j_s(\mathscr{U}, \mathscr{S}),\ \ \Gamma(\mathscr{S}^i) := \Gamma(X, \mathscr{S}^i)$$

Consider the double complex

$$
\begin{array}{ccccccccccccc}
 & & & & \uparrow & & \uparrow & & \uparrow & & & \uparrow & \\
0 \to & \mathscr{S}_n & \to & & \mathscr{S}_n^0 & \to & \mathscr{S}_n^1 & \to & \mathscr{S}_n^2 & \to \cdots \to & & \mathscr{S}_n^n & \to \\
 & \uparrow & & & \uparrow & & \uparrow & & \uparrow & & & \uparrow & \\
0 \to & \mathscr{S}_{n-1} & \to & & \mathscr{S}_{n-1}^0 & \to & \mathscr{S}_{n-1}^1 & \to & \mathscr{S}_{n-1}^2 & \to \cdots \to & & \mathscr{S}_{n-1}^n & \to \\
 & \uparrow & & & \uparrow & & \uparrow & & \uparrow & & & \uparrow & \\
 & \vdots & & & \vdots & & \vdots & & \vdots & & & \vdots & \\
 & \uparrow & & & \uparrow & & \uparrow & & \uparrow & & & \uparrow & \\
0 \to & \mathscr{S}_2 & \to & & \mathscr{S}_2^0 & \to & \mathscr{S}_2^1 & \to & \mathscr{S}_2^2 & \to \cdots \to & & \mathscr{S}_2^n & \to \\
 & \uparrow & & & \uparrow & & \uparrow & & \uparrow & & & \uparrow & \\
0 \to & \mathscr{S}_1 & \to & & \mathscr{S}_1^0 & \to & \mathscr{S}_1^1 & \to & \mathscr{S}_1^2 & \to \cdots \to & & \mathscr{S}_1^n & \to \\
 & \uparrow & & & \uparrow & & \uparrow & & \uparrow & & & \uparrow & \\
0 \to & \mathscr{S}_0 & \to & & \mathscr{S}_0^0 & \to & \mathscr{S}_0^1 & \to & \mathscr{S}_0^2 & \to \cdots \to & & \mathscr{S}_0^n & \to \\
 & & & & \uparrow & & \uparrow & & \uparrow & & & \uparrow & \\
 & & & & \Gamma(\mathscr{S}^0) & \to & \Gamma(\mathscr{S}^1) & \to & \Gamma(\mathscr{S}^2) & \to \cdots \to & & \Gamma(\mathscr{S}^n) & \to \\
 & & & & \uparrow & & \uparrow & & \uparrow & & & \uparrow & \\
 & & & & 0 & & 0 & & 0 & & & 0 &
\end{array}
\tag{2.4}
$$

The up arrows are δ and the left arrow at $\mathscr{S}_q^p$ is $(-1)^q d$, that is, we have multiplied the map d with $(-1)^q$, where q denotes the index related to Cech cohomology. Let us define the map $A : H^k(\Gamma(X, \mathscr{S}^\bullet), d) \to H^k(X, \mathscr{S})$. It sends a d-closed global section ω of $\mathscr{S}^n$ to a δ-closed cocycle $\alpha \in C_s^n(\mathscr{U}, \mathscr{S})$ and the receipie is sketched here:

$$
\begin{array}{cccccccc}
 & 0 & & & & & & \\
 & \uparrow & & & & & & \\
\alpha \to & \omega^0 & \to & 0 & & & & \\
 & \uparrow & & \uparrow & & & & \\
 & \eta^0 & \to & \omega^1 & \to\ 0 & & & \\
 & & & \uparrow & & & & \\
 & & & \eta^1 & \ddots & \ddots & & \\
 & & & & \ddots & \omega^{n-1} & \to & 0 \\
 & & & & & \uparrow & & \uparrow \\
 & & & & & \eta^{n-1} & \to & \omega^n \to 0 \\
 & & & & & & & \uparrow \\
 & & & & & & & \omega
\end{array}
\tag{2.5}
$$

An arrow $a \to b$ means that a is mapped to b under the corresponding map in (5.2). ω^n is the restriction of ω to opens sets U_i's etc. The same diagram 2.5 can be used to explain the map $B : H^k(X, \mathscr{S}) \to H^k(\Gamma(X, \mathscr{S}^\bullet), d)$. In this case we start from α and we reach β. We have to check that

Exercise 1 Show that

1. A and B are well-defined.
2. $A \circ B$ and $B \circ A$ are identity maps.

If we do not care about using d or $(-1)^q d$ we will still get isomorphisms A and B, however, they are defined up to multiplication by -1. The mines sign in $(-1)^q d$ is inserted so that $D := \delta + (-1)^q d$ becomes a differential operator, that is, $D \circ D = 0$. For further details, see [**?**] Chapter 2. Another way to justify $(-1)^q$ is to see it in the double complex of differential (p,q)-forms in a complex manifold. □

Let us come back to the sheaf of differential forms. Let M be a C^∞ manifold. The de Rham cohomology of M is defined to be

$$H^i_{\mathrm{dR}}(M) = H^n(\Gamma(M, \Omega^i_{M^\infty}), d) := \frac{\text{global closed } i\text{-forms on } M}{\text{global exact } i\text{-forms on } M}.$$

Theorem 2.6.2 *(Poincaré Lemma) If M is a unit ball then*

$$H^i_{\mathrm{dR}}(M) = \begin{cases} \mathbb{R} & \text{if } i = 0 \\ 0 & \text{if } i = 0 \end{cases}$$

The Poincaré lemma and Proposition 2.4.1 imply that

$$\mathbb{R} \to \Omega^\bullet_M$$

is the resolution of the constant sheaf $\mathbb{R}$ on the C^∞ manifold M. By Proposition 2.6.1 we conclude that

$$H^i(M, \mathbb{R}) \cong H^i_{\mathrm{dR}}(M), \ i = 0, 1, 2, \ldots$$

where $H^i(M, \mathbb{R})$ is the Cech cohomology of the constant sheaf $\mathbb{R}$ on M.

2.7 Cech cohomology and Eilenberg-Steenrod axioms

Let G be an abelian group and M be a polyhedra. We can consider G as the sheaf of constants on M and hence we have the Cech cohomologies $H^k(X, G), \ k = 0, 1, 2, \ldots$. This notation is already used in Chapter 4 of [Mov17b] to denote a cohomology theory with coefficients group G which satisfies the Eilenberg-Steenrod axioms. The following theorem justifies the usage of the same notation.

Theorem 2.7.1 *In the category of polyhedra the Cech cohomology of the sheaf of constants in G satisfies the Eilenberg-Steenrod axioms.*

Therefore, by uniqueness theorem the Cech cohomology of the sheaf of constants in G is isomorphic to the singular cohomology with coefficients in G. We present this isomorphism in the case $G = \mathbb{R}$ or $\mathbb{C}$.

Recall the definition of integration

$$H^{\mathrm{sing}}_i(M, \mathbb{Z}) \times H^i_{\mathrm{dR}}(M) \to \mathbb{R}, \ (\delta, \omega) \mapsto \int_\delta \omega$$

This gives us

$$H^i_{\mathrm{dR}}(M) \to H_i^{\mathrm{sing}}\check{}(M,\mathbb{R}) \cong H^i_{\mathrm{sing}}(M,\mathbb{R})$$

where $\check{}$ means dual of vector space.

Theorem 2.7.2 *The integration map gives us an isomorphism*

$$H^i_{\mathrm{dR}}(M) \cong H^i_{\mathrm{sing}}(M,\mathbb{R})$$

Under this isomorphism the cup product corresponds to

$$H^i_{\mathrm{dR}}(M) \times H^j_{\mathrm{dR}}(M) \to H^{i+j}_{\mathrm{dR}}(M),\ (\omega_1,\omega_2) \mapsto \omega_1 \wedge \omega_2,\ i,j = 0,1,2,\ldots$$

where $\wedge$ is the wedge product of differential forms.

If M is an oriented manifold of dimension n then we have the following bilinear map

$$H^i_{\mathrm{dR}}(M) \times H^{n-i}_{\mathrm{dR}}(M) \to \mathbb{R},\ (\omega_1,\omega_2) \mapsto \int_M \omega_1 \wedge \omega_2,\ i = 0,1,2,\ldots$$

2.8 Dolbeault cohomology

Let M be a complex manifold and $\Omega^{p,q}_{M^\infty}$ be the sheaf of C^∞ (p,q)-forms on M. We have the complex

$$\Omega^{p,0}_{M^\infty} \xrightarrow{\bar\partial} \Omega^{p,1}_{M^\infty} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \Omega^{p,q}_{M^\infty} \xrightarrow{\bar\partial} \cdots$$

and the Dolbeault cohomology of M is defined to be

$$H^{p,q}_{\bar\partial}(M) := H^q(\Gamma(M,\Omega^{p,\bullet}_{M^\infty}),\bar\partial) = \frac{\text{global } \bar\partial\text{-closed } (p,q)\text{-forms on } M}{\text{global } \bar\partial\text{-exact } (p,q)\text{-forms on } M}$$

Theorem 2.8.1 (Dolbeault Lemma) *If M is a unit disk or a product of one dimensional disks then $H^{p,q}_{\bar\partial}(M) = 0$*

Let Ω^p be the sheaf of holomorphic p-forms on M. In a similar way as in Proposition 2.4.1 one can prove that $\Omega^{p,q}_{M^\infty}$'s are fine sheaves and so we have the resolution of Ω^p:

$$\Omega^p \to \Omega^{p,\bullet}_{M^\infty}.$$

By Proposition 2.6.1 we conclude that:

Theorem 2.8.2 (Dolbeault theorem) *For M a complex manifold*

$$H^q(M,\Omega^p) \cong H^{p,q}_{\bar\partial}(M)$$

We give an example of a domain D in $\mathbb{C}^n$ such that $H^{0,1}_{\bar\partial}(D) \neq 0$. See [**?**], the end of Chapter E.

2.9 Cohomology of manifolds

The first natural sheaves are constant sheaves. For an abelian group G, the sheaf of constants on X with coeficients in G is a sheaf such that for any open set it associates G and the restriction maps are the identity. We also denote by G the corresponding sheaf. Our main examples are $(k,+)$, $k = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$. For a smooth manifold X, the cohomologies $H^i(X,G)$ are isomorphic in a natural way to singular cohomologies and de Rham cohomologies, see respectively [Voi02] Theorem 4.47 and [BT82] Proposition 10.6. We will need the following topological statements.

Proposition 2.9.1 *Let X be a topological space which is contractible to a point. Then $H^p(X,G) = 0$ for all $p > 0$.*

This statement follows from another statement which says that two homotopic maps induce the same map in cohomologies.

Proposition 2.9.2 *Let X be a manifold of dimension n. Then X has a covering $\mathscr{U} = \{U_i,\ i \in I\}$ such that*

1. *all U_i's and their intersections are contractible to points.*
2. *The intersection of any $n+2$ open sets U_i is empty.*

Using both propositions we get an acyclic covering of a manifold and we prove.

Proposition 2.9.3 *Let M be an orientable manifold of dimension m.*

1. *We have $H^i(M,\mathbb{Z}) = 0$ for $i > m$.*
2. *If M is not compact then the top cohomology $H^m(M,\mathbb{Z})$ is zero.*
3. *If M is compact then we have a canonical isomorphism $H^m(M,\mathbb{Z}) \cong \mathbb{Z}$ given by the orientation of M.*

If M is a complex manifold of dimension n, then it has a canonical orientation given by the orientation of $\mathbb{C}$ and so we can apply the above proposition in this case. Note that M is of real dimension $m = 2n$.

2.10 Short exact sequences

In this section we return back to one of the main motivations of the sheaf cohomology, namely, for an exact sequence of sheaves of abelian groups

$$0 \to \mathscr{S}_1 \to \mathscr{S}_2 \to \mathscr{S}_3 \to 0.$$

we have a long exact sequence

$$\cdots \to H^i(X,\mathscr{S}_1) \to H^i(X,\mathscr{S}_2) \to H^i(X,\mathscr{S}_3) \to H^{i+1}(X,\mathscr{S}_1) \to \cdots \tag{2.6}$$

that is in each step the image and kernel of two consecutive maps are equal. All the maps in the above sequence are canonical except those from i-dimensional cohomology to $(i+1)$-dimensional cohomology. In this section we explain how this map is constructed in Cech cohomology. Let $\mathscr{S}_j^i := C_s^j(\mathscr{U}, \mathscr{S}_i)$. The idea is to use:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & \mathscr{S}_{n-1}^1 & \to & \mathscr{S}_n^1 & \to & \mathscr{S}_{n+1}^1 & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & \mathscr{S}_{n-1}^2 & \to & \mathscr{S}_n^2 & \to & \mathscr{S}_{n+1}^2 & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & \mathscr{S}_{n-1}^3 & \to & \mathscr{S}_n^3 & \to & \mathscr{S}_{n+1}^3 & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{2.7}$$

Let us take a covering $U_i,\ i \in I$ and $f = \{f_j,\ j \in I^{i+1}\} \in H^i(X, \mathscr{S}_3)$. By taking the covering smaller, if necessary, we can assume that f is in the image of the map $\mathscr{S}_2 \to \mathscr{S}_3$, that is, there is $g = \{g_j,\ j \in I^{i+1}\}$ such that each g_j is mapped to f_j under $\mathscr{S}_2 \to \mathscr{S}_3$. Now we have $\delta f = 0$ and so δg is mapped to zero under $\mathscr{S}_2 \to \mathscr{S}_3$. We conclude that there is $h = \{h_j,\ j \in I^{i+2}\}$ such that h is mapped to δg under $\mathscr{S}_1 \to \mathscr{S}_2$. We have the map

$$H^i(X, \mathscr{S}_3) \to H^{i+1}(X, \mathscr{S}_1), \quad f \mapsto h$$

which is well-defined and gives us the long exact sequence (2.6).

Chapter 3
Hypercohomology

My mathematics work is proceeding beyond my wildest hopes, and I am even a bit worried - if it's only in prison that I work so well, will I have to arrange to spend two or three months locked up every year? (André Weil writes from Rouen prison, [OR16]).

3.1 Introduction

After a fairly complete understanding of singular homology and de Rham cohomology of manifolds and the invention of Cech cohomology, a new wave of abstraction in mathematics started. H. Cartan and S. Eilenberg in their foundational book [CE56] called *Homological Algebra* took many ideas from topology and replaced it with categories and functors. A. Grothendieck in [Gro57] took this into a new level of abstraction and the by-product of his effort was the creation of many cohomology theories, such as Étale and algebraic de Rham cohomology. Étale cohomology were mainly created in order to solve Weil conjectures, however, the relevant one to integrals is the algebraic de Rham cohomology. Its main ingredient is the concept of hypercohomology of complexes of sheaves which soon after its creation was replaced with derived functors and derived categories. This has made it an abstract concept far beyond concrete computations and the situation is so that the introduction of C. A. Weibel's book [Wei94] starts with: "Homological algebra is a tool used to prove nonconstructive existence theorems in algebra (and in algebraic topology)." In this chapter we aim to introduce hypercohomology without going into categorical approach, and the main reason for this is that we would like to emphasize that its elements can be computed and in the case of complex of differential forms, they can be integrated over topological cycles. For further information the reader is referred to [Voi03, Bry08]

3.2 Hypercohomology of complexes

In this section we assume that the reader is familiar with basic concepts of Cech cohomology, the preliminary material on this can be found in [Mov17a]. Our notations are mainly taken from this reference.

Let us be given a complex of sheaves of abelian groups on a topological space X:

$$\mathscr{S}^\bullet : \quad \mathscr{S}^0 \stackrel{d}{\to} \mathscr{S}^1 \stackrel{d}{\to} \mathscr{S}^2 \stackrel{d}{\to} \cdots \stackrel{d}{\to} \mathscr{S}^n \stackrel{d}{\to} \cdots, \quad d\circ d = 0. \tag{3.1}$$

We would like to associate to $\mathscr{S}^\bullet$ a cohomology which encodes all the Cech cohomologies of individual $\mathscr{S}^i$ together with the differential operators d. We first explain this cohomology using a covering $\mathscr{U} = \{U_i\}_{i\in I}$ of X.

Before proceeding further, let us mention our main example in this chapter. We take a smooth projective variety $X \subset \mathbb{P}^N$ of dimension n over an algebraically closed field k. We take the complex $\Omega_X^\bullet$ of differential forms on X.

Proposition 3.2.1 *There is a covering of X with a $n+1$ affine Zariski open subsets.*

Proof. The covering is going to be

$$U_i := \{g_i \neq 0\}, \ \ i = 0,1,2,\ldots,n,$$

where g_i's are linear homogeneous polynomials in $x_0, x_1, \cdots, x_N$ and we have assumed that the projective space $\mathbb{P}^{N-n-1} \subset \mathbb{P}^N$ given by $g_0 = g_1 = \cdots = g_{n+1} = 0$ does not intersect X. This happens for a generic $\mathbb{P}^{N-n-1}$. For instance for a generic linear $\mathbb{P}^{N-n}$ intersects X in $\deg(X)$ distinct points. This is the definition of the degree of a projective variety. Now we can take $\mathbb{P}^{N-n-1} \subset \mathbb{P}^{N-n}$ such that it does not cross the mentioned $\deg(X)$ points. □

For a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ given by the homogeneous polynomial $f(x_0, x_1, \cdots, x_{n+1})$ of degree d, we have also another useful covering given by

$$U_i := \left\{ \frac{\partial f}{\partial x_i} \neq 0 \right\}, \ \ i = 0,1,2,\ldots,n+1$$

which is called the Jacobian covering of X. Note that for this covering we use the fact that X is smooth. It has $n+2$ open sets. For a fixed $k = 0,1,2,\ldots,n+1$ the open sets $U_0, U_1, \cdots, U_{k-1}, U_{k+1}, \cdots, U_{n+1}$ covers X if X is smooth and $x_k = 0$ intersects X transversely.

Consider the double complex

$$\begin{array}{ccccccccc}
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
\mathscr{S}_n^0 & \to & \mathscr{S}_n^1 & \to & \mathscr{S}_n^2 & \to & \cdots & \to & \mathscr{S}_n^n & \to \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
\mathscr{S}_{n-1}^0 & \to & \mathscr{S}_{n-1}^1 & \to & \mathscr{S}_{n-1}^2 & \to & \cdots & \to & \mathscr{S}_{n-1}^n & \to \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
\mathscr{S}_2^0 & \to & \mathscr{S}_2^1 & \to & \mathscr{S}_2^2 & \to & \cdots & \to & \mathscr{S}_2^n & \to \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
\mathscr{S}_1^0 & \to & \mathscr{S}_1^1 & \to & \mathscr{S}_1^2 & \to & \cdots & \to & \mathscr{S}_1^n & \to \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
\mathscr{S}_0^0 & \to & \mathscr{S}_0^1 & \to & \mathscr{S}_0^2 & \to & \cdots & \to & \mathscr{S}_0^n & \to
\end{array} \tag{3.2}$$

where

$$\mathscr{S}_j^i := C_s^j(\mathscr{U}, \mathscr{S}^i)$$

The horizontal arrows are usual differential operator d of $\mathscr{S}^i$'s and vertical arrows are differential operators δ in the sense of Cech cohomology. The m-th piece of the total chain of (5.2) is

$$\mathscr{L}^m := \oplus_{i=0}^m \mathscr{S}_{m-i}^i$$

with the the differential operator d' which is defined on $\mathscr{S}_j^i$ by:

$$d' = \delta + (-1)^j d \tag{3.3}$$

Our convention for d' is compatible with the one used in [BT82] page 90 and [Bry08] page 14. It is easy to see that $d' \circ d' = 0$, see Exercise 3.1 in this chapter. This also justifies the appearance of the sign $(-1)^j$ in the definition of d'. We define the hypercohomology of the complex $\mathscr{S}^\bullet$ relative to the covering $\mathscr{U}$

$$\mathbb{H}^m(\mathscr{U}, \mathscr{S}^\bullet) = H^m(\mathscr{L}^\bullet, d') := \frac{\ker(\mathscr{L}^m \to \mathscr{L}^{m+1})}{\mathrm{Im}(\mathscr{L}^{m-1} \to \mathscr{L}^m)}$$

Definition 3.2.1 *The hypercohomology $\mathbb{H}^m(X, \mathscr{S}^\bullet)$ is defined to be the direct limit of the total cohomology of the double complex (5.2), i.e.*

$$\mathbb{H}^m(X, \mathscr{S}^\bullet) \cong \mathrm{dirlim}_{\mathscr{U}} \mathbb{H}^m(\mathscr{U}, \mathscr{S}^\bullet).$$

see [Mov17a] for the definition of direct limit.

From a computational point of view this definition is completely useless. We have to search for coverings $\mathscr{U}$ such that $\mathbb{H}^m(\mathscr{U}, \mathscr{S}^\bullet)$ becomes the hypercohomology itself.

Proposition 3.2.2 *If the covering $\mathscr{U}$ is acyclic with respect to all abelian sheaves $\mathscr{S}^i$'s, that is, $\mathscr{U}$ is locally finite and*

$$H^k(U_{i_1} \cap U_{i_2} \cap \cdots U_{i_r}, \mathscr{S}^i) = 0, \ k, r = 1, 2, \ldots, \ i = 0, 1, 2, \ldots. \tag{3.4}$$

then

$$\mathbb{H}^m(X, \mathscr{S}^\bullet) \cong \mathbb{H}^m(\mathscr{U}, \mathscr{S}^\bullet). \tag{3.5}$$

Proof. By definition of the direct limit, we have already a map

$$\mathbb{H}^m(\mathscr{U}, \mathscr{S}^\bullet) \to \mathbb{H}^m(X, \mathscr{S}^\bullet)$$

which sends an α to its euivalence class. We have to show that it is surjective and injective, see Exercise 3.2.

The reader who is mainly interested on computational aspects of hypercohomology may take (3.5) as the definition of hypercohomology. In this way we can describe its elements explicitly. An element of $\mathbb{H}^m(X, \mathscr{S}^\bullet)$ is represented by

$$\omega = \omega^0 + \omega^1 + \cdots + \omega^m, \quad \omega^j \in C_s^{m-j}(\mathscr{U}, \mathscr{S}^j)$$

Each ω^j itself is the following data:

$$\omega^j_{i_0 i_1 \cdots i_{m-j}} \in \mathscr{S}^j(U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_{m-j}})$$

for all $i_0, i_1, \cdots, i_{m-j} \in I$. Such an ω is d'-closed, that is, $d'(\omega) = 0$, if and only if, the following equalities hold

$$\begin{aligned} 0 &= \delta\omega^0 \\ (-1)^{m-1} d\omega^0 &= \delta(\omega^1) \\ (-1)^{m-2} d\omega^1 &= \delta(\omega^2) \\ &\vdots \\ d\omega^{m-1} &= \delta(\omega^m) \\ d\omega^m &= 0 \end{aligned} \tag{3.6}$$

Such an ω is d'-exact, or equivelently it is zero in $\mathbb{H}^m(X, \mathscr{S}^\bullet)$, if and only if there is $\eta = \sum_{j=0}^{m-1} \eta^j, \quad \eta^j \in C_s^{m-1-j}(\mathscr{U}, \mathscr{S}^j)$ such that

$$\begin{aligned} \omega^0 &= \delta(\eta^0) \\ \omega^1 &= (-1)^{m-1} d(\eta^0) + \delta(\eta^1) \\ &\vdots \\ \omega^{m-1} &= -d(\eta^{m-2}) + \delta(\eta^{m-1}) \\ \omega^m &= d\eta_{m-1} \end{aligned} \tag{3.7}$$

In order to better memorize these equalities the diagram below can be helpful

$$\begin{array}{cccccc} 0 & & & & & \\ \omega^0 & 0 & & & & \\ \eta^0 & \omega^1 & 0 & & & \\ & \eta^1 & \ddots & \ddots & & \\ & & \ddots & \omega^{m-1} & 0 & \\ & & & \eta^{m-1} & \omega^m & 0 \end{array} \tag{3.8}$$

Recall that one must use d for horizontal map and δ for vertical maps. We have

$$\mathbb{H}^0(X, \mathscr{S}^\bullet) \cong \left\{\omega \in \mathscr{S}^0(X) \mid d\omega = 0\right\}$$

and $\mathbb{H}^1(X, \mathscr{S}^\bullet)$ is the set of pairs (ω^0, ω^1), where ω^0 consists of $\omega^0_{i_0 i_1} \in \mathscr{S}^0(U_{i_0} \cap U_{i_1}),\ i_0, i_1 \in I$ and ω^1 consists of $\omega^1_{i_0} \in \mathscr{S}^1(U_{i_0}),\ i_0 \in I$ which satisfy the relation

$$\omega^1_{i_1} - \omega^1_{i_0} = -d\omega^0_{i_0 i_1}.$$

Such an ω is taken modulo those of the form $(f_{i_1} - f_{i_0}, df_{i_0})$.

When the covering $\mathscr{U} := \{U_0, U_1, \cdots, U_n\}$ consists of n open sets and $\mathscr{S}^m = 0,\ m > n$ then by definition

$$\mathbb{H}^m(X, \mathscr{S}^\bullet) = 0, \ \ m > 2n$$

Moreover, if we define $\check{U}_i := U_0 \cap U_1 \cap \cdots \cap U_{i-1} \cap U_{i+1} \cap \cdots \cap U_n$ then

$$\mathbb{H}^{2n}(X, \mathscr{S}^\bullet) \cong \frac{\mathscr{S}^n(U_0 \cap U_1 \cap \cdots \cap U_n)}{d\mathscr{S}^{n-1}(U_0 \cap U_1 \cap \cdots \cap U_n) + \mathscr{S}^n(\check{U}_0) + \mathscr{S}^n(\check{U}_1) + \cdots + \mathscr{S}^n(\check{U}_n)}. \tag{3.9}$$

For simplicity, we have not written the restriction maps. Note that the last $n+1$ terms in the denominator of (3.9) form the set $\{\delta\eta^n = \eta^n_0 - \eta^n_1 + \cdots + (-1)^n \eta^n_n \mid \eta^n_i \in \mathscr{S}^n(\check{U}_i)\}$. For $n = 1$ we get

$$\mathbb{H}^2(X, \mathscr{S}^\bullet) \cong \frac{\mathscr{S}^1(U_0 \cap U_1)}{d\mathscr{S}^0(U_0 \cap U_1) + \mathscr{S}^1(U_1) + \mathscr{S}^1(U_0)}.$$

3.3 Acyclic sheaves and hypercohomology

Proposition 3.3.1 *If all the sheaves $\mathscr{S}^i$ are acyclic, that is, $H^j(X, \mathscr{S}^i) = 0,\ j = 1, 2, \ldots$ then*

$$\mathbb{H}^m(X, \mathscr{S}^\bullet) \cong H^m(H^0(X, \mathscr{S}^\bullet), d) := \frac{\ker(H^0(X, \mathscr{S}^m) \to H^0(X, \mathscr{S}^{m+1}))}{\operatorname{Im}(H^0(X, \mathscr{S}^{m-1}) \to H^0(X, \mathscr{S}^m))}.$$

Proof. We Use Proposition 3.2.2 and consider $\mathbb{H}^m(\mathscr{U}, \mathscr{S}^\bullet)$ relative to a acyclic covering $\mathscr{U}$. We have already a map

$$f : H^m(H^0(X, \mathscr{S}^\bullet), d) \to \mathbb{H}^m(\mathscr{U}, \mathscr{S}^\bullet) \tag{3.10}$$

which is obtained by restricting a global section of $\mathscr{S}^m$ to the open sets of the covering $\mathscr{U}$. We have to define its inverse

$$f^{-1} : \mathbb{H}^m(\mathscr{U}, \mathscr{S}^\bullet) \to H^m(H^0(X, \mathscr{S}^\bullet), d) \tag{3.11}$$

Let us take an element $\omega = \sum_{j=0}^m \omega^j \in \mathscr{L}^m$ which is d'-closed. We have written the ingredient equalities derived from this in (3.6). In particular, $\delta\omega^0 = 0$ and by our hypothesis $\omega^0 = \delta\eta^0$ for some $\eta^0 \in \mathscr{S}^0_{m-1}$. The elements ω and $\omega - d'\eta^0$ represesnt the same object in $\mathbb{H}^m(\mathscr{U}, \mathscr{S}^\bullet)$, and so we can assume that $\omega^0 = 0$. This process continues and finally we get an element in $\mathscr{S}^m_0$ which is is equivalent to ω in $\mathbb{H}^m(\mathscr{U}, \mathscr{S}^\bullet)$, and moreover, it is both δ and d-closed. This gives us a global section $f^{-1}(\omega) \in H^0(X, \mathscr{S}^m)$. We have to check that f^{-1} is well-defined, and it is the inverse of f. These details are left to the reader, see Exercise 3.3.

3.4 Quasi-isomorohism and hypercohomology

A morphism between two complexes $\mathscr{S}^\bullet$ and $\check{\mathscr{S}}^\bullet$ is the following commutative diagram:

$$\begin{array}{ccccccccc}
\cdots & \to & \mathscr{S}^{n-1} & \to & \mathscr{S}^n & \to & \mathscr{S}^{n+1} & \to & \cdots \\
\cdots & & \downarrow & & \downarrow & & \downarrow & & \cdots \\
\cdots & \to & \check{\mathscr{S}}^{n-1} & \to & \check{\mathscr{S}}^n & \to & \check{\mathscr{S}}^{n+1} & \to & \cdots
\end{array}$$

It induces a canonical map in the hypercohomologies

$$\mathbb{H}^m(X, \mathscr{S}) \to \mathbb{H}^m(X, \check{\mathscr{S}}).$$

For a morphism $f : \mathscr{S} \to \check{\mathscr{S}}$ of complexes we have a canonical morphism

$$H^m(f) : H^m(\mathscr{S}) \to H^m(\check{\mathscr{S}})$$

where for a complex $\mathscr{S}$ we have defined

$$H^m(\mathscr{S}) := \text{The sheaf constructed from the presheaf } \frac{\ker(d^m)}{\operatorname{Im}(d^{m-1})},\ m \in \mathbb{Z}$$

Definition 3.4.1 A morphism f of complexes is called a quasi-isomorphism if the induced morphisms $H^m(f),\ m \in \mathbb{Z}$ are isomorphisms.

Let X be a smooth variety over the field of complex numbers and let X^∞ be the underlying C^∞ manifold. Let $\Omega_X^\bullet$ be the complex of (algebraic) differential forms in X. We also consider the complex $\Omega_{X^\infty}^\bullet$ of C^∞ differential forms in X for the Zariski topology of X^∞. We will only need to consider Zariski open sets. In Chapter 4 we are going to show that the natural inclusion $\Omega_X^\bullet \to \Omega_{X^\infty}^\bullet$ is a quasi-isomorphism.

Proposition 3.4.1 *If $f : \mathscr{S} \to \check{\mathscr{S}}$ be a quasi-isomorphism then the induce map in hypercohomologies $f_* : \mathbb{H}^m(X, \mathscr{S}^\bullet) \to \mathbb{H}^m(X, \check{\mathscr{S}}^\bullet)$ is an isomorphism.*

Proof. We have to define the inverse map

$$f_*^{-1} : \mathbb{H}^k(X, \check{\mathscr{S}}^\bullet) \to \mathbb{H}^k(X, \mathscr{S}^\bullet) \tag{3.12}$$

We take a covering $\mathscr{U}$ acyclic with respect to all abelian sheaves $\mathscr{S}^i$ and $\check{\mathscr{S}}^i$ and use Proposition (3.2.2). Let $\check{\omega} = \sum_{j=0}^m \check{\omega}^j \in \mathbb{H}^k(X, \check{\mathscr{S}})$. We start looking at $\check{\omega}^m$. We have $d(\check{\omega}^m) = 0$ and so there is

$$\omega^m \in \mathscr{S}_0^m, d\omega^m = 0$$

such that

$$\check{\omega}^m - f_*\omega^m = d\check{\eta}^{m-1}$$

for some $\check{\eta}^{m-1} \in \check{\mathscr{S}}_0^{m-1}$. For this we have used the fact that f is a quasi-isomorphism. We replace $\check{\omega}$ with $\check{\omega} - d'\check{\eta}^{m-1}$ and in this way we can assume that

$$\check{\omega}^m = f_*\omega^m.$$

Now we look at the $(m-1)$-level in which we have

$$d\check{\omega}^{m-1} = \delta\check{\omega}^m = \delta f_*\omega^m = f_*(\delta\omega^m) \tag{3.13}$$

$\delta\omega^m$ is a d-closed element and using the above equality we know that it is mapped to zero under $H^m(f)$. Therefore, using the property that f is a quasi-isomorphism, we conclude that $\delta\omega^m$ is d-exact, that is, there exists $\omega^{m-1} \in \mathscr{S}_1^{m-1}$ such that

$$\delta\omega^m - d\omega^{m-1} = 0. \tag{3.14}$$

Combining (3.13) and (3.14) we get

$$d\left(\check{\omega}^{m-1} - f_*\omega^{m-1}\right) = 0. \tag{3.15}$$

We are now in a similar situation as in the beginning. Since f is quasi-isomorphism we have

$$\check{\omega}^{m-1} - f_*\omega^{m-1} = d\check{\eta}^{m-2}, \text{ for some}, \check{\eta}^{m-2} \in \mathscr{S}_1^{m-2}$$

and by adding $-d'\check{\eta}^{m-2}$ to $\check{\omega}$ we can assume that $\check{\omega}^{m-1} = f_*\omega^{m-1}$. In the $(m-2)$-the level we get $\omega^m \in \mathscr{S}_0^m, \omega^{m-1} \in \mathscr{S}_1^{m-1}, \omega^{m-2} \in \mathscr{S}_2^{m-2}$ with the following identities

$$\check{\omega}^m = f_*\omega^m,\ \check{\omega}^{m-1} = f_*\omega^{m-1},\ -d\omega^{m-1} + \delta\omega^m = 0,$$

and

$$d\left(\check{\omega}^{m-2} - f_*\omega^{m-2}\right) = 0. \tag{3.16}$$

This process stops at $(m+1)$-th step and we get $\omega = \sum_{i=0}^{m} \omega^m$ which is d'-closed and $f_*\omega = \check{\omega}$. We define $f_*^{-1}(\check{\omega})$ to be equal to ω. It is left to the reader to prove that f_*^{-1} is well-defined and it is inverse to f_*, see Exercise 3.4.

Proposition 3.4.2 *Let $\mathscr{S} \to \mathscr{S}^\bullet$ be a resolution of $\mathscr{S}^\bullet$. Then*

$$\mathbb{H}^k(X, \mathscr{S}^\bullet) \cong H^k(X, \mathscr{S})$$

Proof. Our hypothesis is the same as to say that that the complex $\cdots \to 0 \to \mathscr{S} \to 0 \to \cdots$ with $\mathscr{S}$ in the 0-th place, is quasi-isomorphic to the complex $\mathscr{S}^\bullet$ and so by Proposition 3.4.1 the proof is done.

3.5 A description of an isomorphism

Let us now be given a quasi-isomorphism of complexes $\mathscr{S}^\bullet \to \check{\mathscr{S}}^\bullet$ and assume that all the abelian sheaves $\check{\mathscr{S}}^m$ are acyclic. We use Proposition 3.3.1 and Proposition 3.4.1 and we get an isomorphism

$$\mathbb{H}^m(X, \mathscr{S}) \cong H^m(\check{\mathscr{S}}^\bullet(X), d).$$

For later applications we need to describe the two maps

$$\begin{aligned} A:\ & \mathbb{H}^m(X, \mathscr{S}) \to H^m(\check{\mathscr{S}}^\bullet(X), d) \\ A^{-1}:\ & H^m(\check{\mathscr{S}}^\bullet(X), d) \to \mathbb{H}^m(X, \mathscr{S}). \end{aligned}$$

such that both $A \circ A^{-1}$ and $A^{-1} \circ A$ are identity maps. We take an acyclic covering $\mathscr{U} = \{U_i\}_{i\in I}$ with respect to all sheaves $\mathscr{S}^i$ and $\check{\mathscr{S}}^i$. The map A is obtained by the composition of the maps

$$\mathbb{H}^m(X, \mathscr{S}^\bullet) \to \mathbb{H}^m(X, \check{\mathscr{S}}^\bullet) \to H^m(\check{\mathscr{S}}^\bullet(X), d)$$

The first map is simply induced by the quasi-isomorphism. The second map is the map (3.11). Its explicit desciption is given in the proof of Proposition 3.3.1. The map A^{-1} is the composition of

$$H^m(\check{\mathscr{S}}^\bullet(X), d) \to \mathbb{H}^m(X, \check{\mathscr{S}}^\bullet) \to \mathbb{H}^m(X, \mathscr{S}^\bullet)$$

The first map is obtained by restricting a global section of $\check{\mathscr{S}}^m$ to the open sets of the covering $\mathscr{U}$. The second map is (3.12) and its explicit description is given in the proof of Proposition 3.4.1.

3.6 Filtrations

For a complex $\mathscr{S} = \mathscr{S}^\bullet$ (for simplicity we drop $\bullet$) and $k \in \mathbb{Z}$ we define the truncated complexes

$$\mathscr{S}^{\leq k}: \quad \cdots \mathscr{S}^{k-1} \to \mathscr{S}^k \to 0 \to 0 \to \cdots$$

and

$$\mathscr{S}^{\geq k}: \quad \cdots \to 0 \to 0 \to \mathscr{S}^k \to \mathscr{S}^{k+1} \to \cdots$$

We have canonical morphisms of complexes:

$$\mathscr{S}^{\leq k} \to \mathscr{S},\ \mathscr{S}^{\geq k} \to \mathscr{S}$$

Assume that $\mathscr{S}$ is a left-bounded complex, that is,

$$\mathscr{S}: \ 0 \to \mathscr{S}^0 \to \mathscr{S}^1 \to \cdots$$

The morphism $\mathscr{S}^{\geq i} \to \mathscr{S}$ induces a map in hypercohomologies and we define

$$F^i := \mathrm{Im}(\mathbb{H}^m(X, \mathscr{S}^{\geq i}) \to \mathbb{H}^m(X, \mathscr{S}))$$

This gives us the filtration

$$\cdots \subset F^i \subset F^{i-1} \subset \cdots \subset F^1 \subset F^0 := \mathbb{H}^m(X, \mathscr{S}).$$

The differential map $d: \mathscr{S}^i \to \mathscr{S}^{i+1}$ induces the maps

$$d_1: H^j(X, \mathscr{S}^i) \to H^j(X, \mathscr{S}^{i+1}),\ j \in \mathbb{N}_0.$$

If all these maps are zero then we can define the maps

$$d_2: H^j(X, \mathscr{S}^i) \to H^{j-1}(X, \mathscr{S}^{i+2}),\ j \in \mathbb{N}_0$$

which are defined as follows: we take $\omega^i \in H^j(X, \mathscr{S}^i)$ and since d_1 is zero we have $\omega^{i+1} \in \mathscr{S}^{i+1}_{j-1}$ such that $\delta\omega^{i+1} + (-1)^j d\omega^i = 0$. Under d_2 the element ω^i is mapped to $d\omega^{i+1}$. In a similar way, we define

$$d_r: H^j(X, \mathscr{S}^i) \to H^{j-r+1}(X, \mathscr{S}^{i+r}),\ j \in \mathbb{N}_0.$$

and if $d_k,\ k \leq r$ are all zero we define d_{r+1}.

Proposition 3.6.1 *Assume that all d_r's constructed above are zero. Then we have canonical isomorphisms*

$$F^i/F^{i+1} \cong H^{m-i}(X, \mathscr{S}^i)$$

Proof. We have a canonical map $F^i \to H^{m-i}(X, \mathscr{S}^i)$ which sends $\omega^i + \omega^{i+1} + \cdots + \omega^m$ to ω^i. It is well-defined and its kernel is F^{i+1}. Therefore, we have an embedding

$F^i/F^{i+1} \hookrightarrow H^{m-i}(X, \mathscr{S}^i)$. The surjectivity follows from the hypothesis. This has been always our source of inspiration for the definition of d_r's. □

The reader who is familiar with spectral sequences has noticed that the hypothesis in Proposition 3.6.1 is equivalent to say that the spectral sequence associated to the double complex (5.2) degenerates at E_1. For further detail on this see [Voi02], 8.3.2, Theorem 8.21 and Proposition 8.25. In general we have $d_r : E^{p,q} \to E_r^{p+r,q-r+1}$ which are defined in the following way:

1. $E_0^{p,q} = \mathscr{S}_q^p$ and $d_0 = \delta$.
2. $$E_{r+1}^{p,q} := \frac{\ker\left(E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-r+1}\right)}{\operatorname{Im}\left(E_r^{p-r,q+r-1} \xrightarrow{d_r} E_r^{p,q}\right)}$$
   and so $E_1^{p,q} = H^q(X, \mathscr{S}^p)$ and d_1 is induced by d. If d_1 is zero then $E_2^{p,q} = E_1^{p,q}$ and d_2 is defined as we did before Proposition 3.6.1.

3.7 Exercises

3.1. For the operator d' in (3.3) of the double complex (5.2) show that $d' \circ d' = 0$.

3.2. Complete the proof of Proposition 3.2.2.

3.3. Prove that the map (3.11) is well-defined, and it is inverse to (3.10).

3.4. Complete the details of the proof of Proposition 3.4.1.

3.5 (Deligne's cohomology). For the complex $\mathscr{S}^\bullet$ if the the kernel of $d : \mathscr{S}^0 \to \mathscr{S}^1$ is non-trivial then we can take any abelian subgroup B of $\ker(d)$ and form the new complex $B \to \mathscr{S}^\bullet$ and take its hypercohomology. Discuss this.

Chapter 4
Algebraic de Rham cohomology

I do feel however that while we wrote algebraic GEOMETRY they [Weil, Zariski, Grothendieck] make it ALGEBRAIC geometry with all that it implies. Solomon Lefschetz in [Lef68].

4.1 Introduction

The main objective in this chapter is to define the algebraic de Rham cohomology $H^m_{\rm dR}(X)$ of a smooth algebraic veriety X defined over a field k of characteristic zero. When $\mathsf{k} = \mathbb{C}$ we have the underlying $C^inftу$ manifold X^∞ and we aim construct explicitly the isomorphism between the algebraic $H^m_{\rm dR}(X)$ and the classical $H^m_{\rm dR}(X^\infty)$ de Rham cohomologies. In this way, we are able to write down explicit formulas for cup products, Gauss-Manin connection etc, in algebraic de Rham cohomology. A. Grothendieck is the main responsible for the definition of algebraic de Rham cohomology, however, it must be noted that he was largely inspired by the work of Atiyah and Hodge in [HA55]. His paper [Gro66] was originally written as a letter to Atiyah and Hodge and it would be fair to call this Atiyah-Hodge-Grothendieck algebraic de Rham cohomology. However, in the literature, and mainly in Algebraic Geometry we find the name Grothendieck's algebraic de Rham cohomology. Algebraic de Rham cohomology for singular schemes has been studied by several authors during the seventies, see for example Hartshorne's work [Har75] and the references therein.

4.2 Atiyah-Hodge theorem

Algebraic de Rham cohomology was introduced after many efforts in order to understand the de Rham cohomology of affine varieties. This is in some sense natural becuase the integration domain of integrals are usully supported in affine varieties. In this section we are going to state a theorem due to Atiyah and Hodge which was the starting point for Grothendieck in order to define algebraic de Rham cohomology.

Theorem 4.2.1 (Atiyah-Hodge Theorem, [HA55]) *Let X be an affine smooth variety over the field $\mathbb{C}$ of complex numbers. Then the canonical map*

$$H^q(\Gamma(\Omega^\bullet_{X/\mathbb{C}}), d) \to H^q_{\mathrm{dR}}(X^\infty)$$

is an isomorphism of $\mathbb{C}$-vector spaces.

Proof. See [Nar68], p. 86.

In the next section we will need the following theorem.

Theorem 4.2.2 *(Serre vanishing theorem) Let X be an affine variety over the field $\mathbb{C}$ of complex numbers. Then*

$$H^i(X, \Omega^j_X) = 0,\ i = 1,2,\ldots, j = 0,1,2,\ldots.$$

Proof. The Theorem is valid in general for coherent sheaves.

4.3 Algebraic De Rham cohomology

Following the historical path which resulted in the definition of algebraic de Rham cohomology, we are now ready to define it for arbitrary algebraic varieties.

Definition 4.3.1 Let X be a smooth variety over a field k. We consider the complex $(\Omega^\bullet_{X/k}, d)$ of regular differential forms on X. The (algebraic) de Rham cohomology of X is defined to be the hypercohomology

$$H^q_{\mathrm{dR}}(X/k) := \mathbb{H}^q(X, \Omega^\bullet_{X/k}),\ q = 0,1,2,\ldots.$$

Let X be an smooth affine variety over $\mathbb{C}$. We have

$$H^i_{\mathrm{dR}}(X/\mathbb{C}) \cong H^i(\Gamma(X, \Omega^\bullet_X), d) \cong H^i_{\mathrm{dR}}(X^\infty)$$

The first isomorphism follows from Theorem 4.2.2 and Proposition 3.3.1. The second isomorphism is the statement of Theorem 4.2.1.

For an arbitrary algebraic variety X over complex numbers, the Atiyah-Hodge theorem implies that the $\Omega^\bullet_{X/\mathbb{C}} \to \Omega^\bullet_{X^\infty}$ is a quasi-isomorphism. In this way we can use Proposition 3.3.1 and Proposition 3.4.1 and we get an isomorphism

$$H^m_{\mathrm{dR}}(X/k) \cong H^m_{\mathrm{dR}}(X^\infty).$$

In §3.5 we have described this isomorphism and its inverse explicitly. Since this will be an important tool for later applications, we are going to explain again explicit construction of the maps

$$A: \ H^m_{\mathrm{dR}}(X^\infty) \to H^m_{\mathrm{dR}}(X/\mathbb{C}) \tag{4.1}$$
$$A^{-1}: \ H^m_{\mathrm{dR}}(X/\mathbb{C}) \to H^m_{\mathrm{dR}}(X^\infty) \tag{4.2}$$

We first describe the map (4.1). Let us take an element $H^m_{\mathrm{dR}}(X^\infty)$. This is represented by a closed differential m-form $\check{\omega}$. In the following differential forms with ˇ are C^∞-forms and those without ˇ are algebraic differential forms. We restrict $\check{\omega}$ to each open set U_i, say $\check{\omega}^m_i = \check{\omega}|_{U_i}$. Using Atiyah-Hodge theorem we find algebraic differential m-forms ω^m_i and C^∞ $(m-1)$-forms $\check{\eta}^{m-1}_i$ in U_i such that

$$\check{\omega}^m_i = \omega^m_i - d\check{\eta}^{m-1}_i, \quad d\omega^m = 0$$

In the 0-th level we have:

$$\omega^m_j - \omega^m_i = d(\check{\eta}^{m-1}_j - \check{\eta}^{m-1}_i) \text{ in } U_i \cap U_j$$

Again we use Atiyah-Hodge theorem. The right hand side of the above equality represents the zero element in the $(m-1)$-th de Rham cohomology of $U_i \cap U_j$. Therefore, there are algebraic differential $m-1$-forms ω^{m-1}_{ij} such that

$$\omega^m_j - \omega^m_i - d\omega^{m-1}_{ij} = 0$$

and so

$$d(\check{\eta}^{m-1}_j - \check{\eta}^{m-1}_i - \omega^{m-1}_{ij}) = 0.$$

In intersections $U_i \cap U_j$ we can add closed algebraic differential forms to ω^{m-1}_{ij} and assume that

$$\check{\eta}^{m-1}_j - \check{\eta}^{m-1}_i - \omega^{m-1}_{ij} = d\check{\eta}^{m-2}_{ij}.$$

The process repeats in the 1-the level. Summing these equalities with a proper mines sign in the intersection of three open sets $U_i \cap U_j \cap U_k$ (taking δ) we get

$$\omega^{m-1}_{jk} - \omega^{m-1}_{ik} + \omega^{m-1}_{ij} = -d(\eta^{m-1}_{jk} - \eta^{m-1}_{ik} + \eta^{m-1}_{ij})$$

Again we have an algebraic $(m-1)$-form which is exact using C^∞ differential forms and so it must be exact using algebraic differential forms etc. This process at the k-the level gives us

$$\begin{aligned}
\delta\omega^{m-k} + (-1)^{k+1} d(\delta\check{\eta}^{m-k-1}) &= 0 \\
\delta\omega^{m-k} + (-1)^{k+1} d\omega^{m-k-1} &= 0 \\
d\left(\delta\check{\eta}^{m-k-1} - \omega^{m-k-1}\right) &= 0 \\
\delta\check{\eta}^{m-k-1} - \omega^{m-k-1} - (-1)^{k+2} d\check{\eta}^{m-k-2} &= 0
\end{aligned}$$

At the $(m-1)$-the level we get the element $\omega = \omega^0 + \omega^1 + \cdots + \omega^m$ of $H^m_{\mathrm{dR}}(X/\mathbb{C})$. Note that the last equality for $k = m-1$ is just $\delta\check{\eta}^0 - \omega^0 = 0$.

Let us now describe the map (4.2). We take am element $H^m_{\mathrm{dR}}(X/\mathbb{C})$. It is represented by $\omega = \sum_{j=0}^m \omega^j$ such that $d'\omega = 0$. In particular, $\delta\omega^0 = 0$. Since the Cech cohomology of the sheaf of C^∞ differential forms is zero (except in dimension zero), we have

$$\omega^0 = \delta\check{\eta}^0$$

for some C^∞ differential forms $\check{\eta}^0$. We replace ω and $\omega - \delta\eta^0$ and we so we can assume that $\omega^0 = 0$. This process continues with ω^1 with $\delta\omega^0 = 0$. In the final step we get a closed C^∞ m-form in X^∞. The complete description for $m = 0, 1, 2$ is left to the reader, see Exercise 4.1.

4.4 Hodge filtration

For the complex of differential forms $\Omega^\bullet_{X/k}$ we have the complex of truncated differential forms:

$$\Omega^{\bullet \geq i}_{X/k}: \quad \cdots \to 0 \to 0 \to \Omega^i_{X/k} \to \Omega^{i+1}_{X/k} \to \cdots$$

and a natural map

$$\Omega^{\bullet \geq i}_{X/k} \to \Omega^\bullet_{X/k}$$

We define the Hodge filtration

$$0 = F^{m+1} \subset F^m \subset \cdots \subset F^1 \subset F^0 = H^m_{\mathrm{dR}}(X/k)$$

as follows

$$F^q = F^q H^m_{\mathrm{dR}}(X/k) = \mathrm{Im}\left(\mathbb{H}^m(X, \Omega^{\bullet \geq i}_{X/k}) \to \mathbb{H}^m(X, \Omega^\bullet_{X/k})\right)$$

Theorem 4.4.1 *Let* k *be an algebraically closed field of characteristic zero and X be a smooth projective variety over* k*. We have*

$$F^q/F^{q+1} \cong H^{m-q}(X, \Omega^q)$$

By Lefschetz principle we can assume that X is defined over complex numbers. We denote by X^{an} the underlying complex manifold of X. Let $\check{\Omega}^i_{X^{\mathrm{an}}}$ be the sheaf of closed holomorphic differential form on X^{an}.

Theorem 4.4.2 *(Dolbeault) Let X be a smooth projective variety over* $\mathbb{C}$*. The maps*

$$H^j(X^{\mathrm{an}}, \Omega^i_{X^{\mathrm{an}}}) \to H^j(X^{\mathrm{an}}, \check{\Omega}^{i+1}_{X^{\mathrm{an}}}),\ i, j \in \mathbb{N}_0,$$

induced by the differential map $d : \Omega^i_{X^{\mathrm{an}}} \to \check{\Omega}^{i+1}_{X^{\mathrm{an}}}$ *are all zero.*

This theorem is taken from Griffiths' article [Gri69a], II. He uses this property in order to prove that the Hodge filtration varies holomorphically. He associate this proposition to Dolbeault, and in the appendix of [Gri69b], he gives a proof using Laplacian operators. The situation is quit similar to the case of Hard Lefschetz theorem, where the only available proof is done using harmonic forms. Theorem 4.4.2 is equivalent to the fact the Frölicher spectral sequence of X^{an} degenerates at E_1, see for instance Theorem 8.28 in Voisin's book [Voi02]. In this book the mentioned fact is basically deriven from the Hodge decomposition.

Proof (Proof of Theorem 4.4.1). We need to check that the hypothesis of Proposition 3.6.1 are satisfied in our case. By GAGA principle the natural morphisms

$$H^j(X,\Omega^i_X) \to H^j(X^{\mathrm{an}},\Omega^i_{X^{\mathrm{an}}})$$

are isomorphisms of $\mathbb{C}$-vector spaces and under this identifications the algebraic d_r coincides with the holomorphic d_r. Therefore, it is enough to check that the holomorphic d_r satisfies the hypothesis of Proposition 3.6.1. In the holomorphic context, by Theorem 4.4.2 we have $H^j(X^{\mathrm{an}},\check{\Omega}^i_{X^{\mathrm{an}}}) \cong H^j(X^{\mathrm{an}},\Omega^i_{X^{\mathrm{an}}})$. For this we write the long exact sequence of the short exact sequence:

$$0 \to \check{\Omega}^i_{X^{\mathrm{an}}} \to \Omega^i_{X^{\mathrm{an}}} \to \check{\Omega}^{i+1}_{X^{\mathrm{an}}} \to 0.$$

In the definition if d_1 we can choose a representative of $\omega^i \in H^j(X^{\mathrm{an}},\Omega^i_{X^{\mathrm{an}}})$ such that $d\omega^i = 0$, and so, by definition all d_r's are zero. □

According to [Voi02] page 207, the algebraic proof of degeneracy at E_1 of the complex of algebraic differential forms (and hence an algebraic proof of Theorem 4.4.1) is done by Deligne and Illusie in 1987 and Illusie in 1996.

4.5 Cup product

In the usual de Rham cohomology we have the cup/wedge product

$$H^m_{\mathrm{dR}}(X^\infty) \times H^n_{\mathrm{dR}}(X^\infty) \to H^{n+m}_{\mathrm{dR}}(X^\infty),\ \omega,\alpha \mapsto \omega\wedge\alpha$$

and it is natural to ask for the corresponding bilinear map in algebraic de Rham cohomology. For partial result in this direction see [CG80].

Theorem 4.5.1 *The cup product of $\omega \in H^m_{\mathrm{dR}}(X/\mathsf{k})$ and $\alpha \in H^n_{\mathrm{dR}}(X/\mathsf{k})$ is given by $\gamma = \omega \cup \alpha$, where*

$$\begin{aligned}
\gamma^{n+m}_{i_0} &= \omega^m_{i_0} \wedge \alpha^n_{i_0} \\
\gamma^{n+m-1}_{i_0 i_1} &= (-1)^m \omega^m_{i_0} \wedge \alpha^{n-1}_{i_0 i_1} + \omega^{m-1}_{i_0 i_1} \wedge \alpha^n_{i_1} \\
\vdots \quad & \quad \vdots \\
\gamma^{n+m-j}_{i_0 i_1 \cdots i_j} &= \sum_{r=0}^{j} (-1)^{m(j-r)+r(j-1)} \omega^{m-r}_{i_0 \cdots i_r} \wedge \alpha^{n-j+r}_{i_r \cdots i_j} \\
\gamma^0 &= \omega^0_{i_0 \cdots i_m} \wedge \alpha^0_{i_m \cdots i_{n+m}}
\end{aligned}$$

Proof. We have to prove that under the canonical isomorphism between the classical and algebraic de Rham cohomology, the wedge product is transformed in the product given in the theorem. Let us take $\check{\omega} \in H^m_{\mathrm{dR}}(X^\infty)$ and $\alpha \in H^n_{\mathrm{dR}}(X^\infty)$ and construct the algebraic counterpart of $\check{\omega}, \check{\alpha}, \check{\omega} \cup \check{\alpha}$. For this we will use the explicit construction of (4.1). Followng this, it is possible to compute the first two lines in Theorem 4.5.1. Once the general formula is guessed the proof is as bellow: First we check that $d'\gamma = 0$.

$$
\begin{aligned}
(\delta\gamma^{n+n-j})_{i_0\cdots i_{j+1}} &= \sum_{k=0}^{j+1}(-1)^k\gamma^{m+n-j}_{i_0\cdots\check{i}_k\cdots i_{j+1}} \\
&= \sum_{k=0}^{j+1}\sum_{r=0}^{k-1}(-1)^{k+m(j-r)+r(j-1)}\omega^{m-r}_{i_0\cdots i_r}\wedge\alpha^{n-j+r}_{i_r\cdots\check{i}_k\cdots i_{j+1}} \\
&+ \sum_{k=0}^{j+1}\sum_{r=k+1}^{j+1}(-1)^{k+m(j-r+1)+(r-1)(j-1)}\omega^{m-r+1}_{i_0\cdots\check{i}_k\cdots i_r}\wedge\alpha^{n-j+r}_{i_r\cdots i_{j+1}} \\
&= \sum_{r=0}^{j}\sum_{k=r+1}^{j+1}(-1)^{k+m(j-r)+r(j-1)}\omega^{m-r}_{i_0\cdots i_r}\wedge\alpha^{n-j+r}_{i_r\cdots\check{i}_k\cdots i_{j+1}} \\
&+ \sum_{r=1}^{j+1}\sum_{k=0}^{r-1}(-1)^{k+m(j-r+1)+(r-1)(j-1)}\omega^{m-r+1}_{i_0\cdots\check{i}_k\cdots i_r}\wedge\alpha^{n-j+r}_{i_r\cdots i_{j+1}} \\
&= \sum_{r=0}^{j}(-1)^{r+m(j-r)+r(j-1)}\omega^{m-r}_{i_0\cdots i_r}\wedge\left[(\delta\alpha^{n-j+r})_{i_r\cdots i_{j+1}}-\alpha^{n-j+r}_{i_{r+1}\cdots i_{j+1}}\right] \\
&+ \sum_{r=1}^{j+1}(-1)^{m(j-r+1)+(r-1)(j-1)}\left[(\delta\omega^{m-r+1})_{i_0\cdots i_r}-(-1)^r\omega^{m-r+1}_{i_0\cdots i_{r-1}}\right]\wedge\alpha^{n-j+r}_{i_r\cdots i_{j+1}} \\
&= \sum_{r=0}^{j}(-1)^{j+m(j-r)+r(j-1)}\omega^{m-r}_{i_0\cdots i_r}\wedge d\alpha^{n-j+r-1}_{i_r\cdots i_{j+1}} \\
&+ \sum_{r=1}^{j+1}(-1)^{r-1+m(j-r+1)+(r-1)(j-1)}(d\omega^{m-r})_{i_0\cdots i_r}\wedge\alpha^{n-j+r}_{i_r\cdots i_{j+1}} \\
&= (-1)^j d\left(\sum_{r=0}^{j+1}(-1)^{m(j+1-r)+rj}\omega^{m-r}_{i_0\cdots i_r}\wedge\alpha^{n-j-1+r}_{i_r\cdots i_{j+1}}\right) \\
&\quad (-1)^j d\gamma^{m+n-j-1}_{i_0\cdots i_{j+1}}
\end{aligned}
$$

Now we have to prove that $\check{\alpha}\wedge$ and γ induce the same element in $\mathbb{H}^{n+m}(X^\infty,\Omega^\bullet_{X^\infty})$.

For the description of cup product in Cech cohomology $H^m(X,\mathbb{Z})$ see [Bry08]. For simple applications of cup product in the case of elliptic curves see [Mov12].

4.6 Cup product for hypersurfaces

The discussion of the algebraic cup product for hypersurfaces is partially done in Carlson and Griffiths' article [CG80]. Let $X\subset\mathbb{P}^{n+1}$ be a smooth hypersurface of degree d given by $f=0$. Recall that for a monomial $x^i=x_0^{i_0}x_1^{i_1}\cdots x_{n+1}^{i_{n+1}}$ of degree $(k+1)d-n-2$

$$\omega_i=\text{Residue}\left(\frac{x^i\cdot\Omega}{f^{k+1}}\right)\in H^n_{\text{dR}}(X).$$

where $\Omega := \sum_{i=0}^{n+1}(-1)^i x_i \widehat{dx_i}$. We say that ω_i has adjoint level k. We consider the Jacobian covering $\mathscr{J}_X$ of $\mathbb{P}^{n+1}$:

$$\mathscr{J}_X := \{U_j,\ \ j = 0,1,2,\cdots,n+1\},\ \ U_j := \left\{\frac{\partial f}{\partial x_j} \neq 0\right\}.$$

Since X is smooth, this is a covering of $\mathbb{P}^{n+1}$ and hence X itself. For a vector field Z in $\mathbb{C}^{n+2}$, let ι_Z denote the contraction of differential forms along Z and for a multi-index $j = (j_0, ..., j_l)$ with $|j| := l$ let

$$\Omega_j := \iota_{\frac{\partial}{\partial x_0}} \circ \iota_{\frac{\partial}{\partial x_1}} \circ \cdots \circ \iota_{\frac{\partial}{\partial x_l}} \Omega \tag{4.3}$$

$$f_j := \frac{\partial f}{\partial x_{j_0}} \cdot \frac{\partial f}{\partial x_{j_1}} \cdots \frac{\partial f}{\partial x_{j_l}}. \tag{4.4}$$

Theorem 4.6.1 (Carlson-Griffiths, [CG80], page 7) *Let ω_i be a differential form of adjoint level k. Then, in $F^k/F^{k+1} \cong H^k(X, \Omega_X^{n-k})$, is represented by the cocycle*

$$(\omega_i)^{n-k,k} = \frac{(-1)^{n+\binom{k+1}{2}}}{k!}\left\{\frac{x^i \Omega_j}{f_j}\right\}_{|j|=k} \tag{4.5}$$

with respect to the Jacobian covering.

The above theorem gives us only a folrmula for $(\omega_i)^{n-k,k}$ and we need the full expression of ω_i in the algebraic de Rham cohomology of X. Once this is done we need a formula for the cup product of ω_i's. A partiall result is also given by Carlson and Griffiths. For ω_i and ω_j of adjoint level k and $n-k$ we have

$$\omega_i \cup \omega_j = \frac{x^{i+j}\Omega}{f_0 f_1 \cdots f_{n+1}}. \tag{4.6}$$

4.7 The cohomological class of an algebraic cycle

Let $Z \subset X$ be an algebraic cycle of codimension m. In this section we construct

$$[Z] \in H^m_{\mathrm{dR}}(X). \tag{4.7}$$

4.8 Polarization

Let $\mathbb{P}^n$ be a projective space of dimension n with the projective coordinate system $[x_1 : x_2 : \cdots : x_{n+1}]$ and $f_{ij} := \frac{x_j}{x_i}$. The rational function f_{ij} has a zero (resp. pole) of order 1 at $x_j = 0$ (resp. $x_i = 0$).

Definition 4.8.1 We call

$$\theta \in H^2_{\mathrm{dR}}(X), \ \ \theta_{ij} := \frac{d(\frac{z_j}{z_i})}{\frac{z_j}{z_i}}.$$

the polarization.

Proposition 4.8.1 *The polarization in the usual de Rham cohomology is given by*

$$\bar{\partial}\partial \log(\sum_{i=1}^{n+1} |x_i|^2).$$

and it for any linear $\mathbb{P}^1 \subset \mathbb{P}^n$ we have

$$\int_{\mathbb{P}^1} \theta = 2\pi i.$$

Proof. We have $\delta\theta = 0$ and $d\theta = 0$. Therefore. $d'\theta = 0$ and hence it induces an element in the algebraic de Rham cohomology $H^2_{\mathrm{dR}}(X)$. We define:

$$p_i : \mathbb{P}^n \backslash \{x_i = 0\} \to \mathbb{R}^+, \ \ p_i(x) := \sum_{j=1}^{n+1} \left| \frac{x_j}{x_i} \right|^2$$

We have $p_j = \left|\frac{x_j}{x_i}\right|^2 p_i$ and so $\omega_j - \omega_i = \frac{df_{ij}}{f_{ij}}$, where $\omega_i := \frac{\partial p_i}{p_i} = \partial \log p_i$. The $(1,1)$-forms $\bar{\partial}\partial \log p_i$'s in the intersection of U_i's coincide and gives us the element in the usual de Rham cohomology corresponding to θ. For the second statement see the section of Chern classes in my lecture notes [Mov17a]. □

Using Theorem 4.5.1 we know that $\theta^m \in H^{2m}_{\mathrm{dR}}(X)$ has only the middle piece:

$$(\theta^m)_{i_0 i_1 \cdots i_m} = \frac{df_{i_0 i_1}}{f_{i_0 i_1}} \wedge \frac{df_{i_1 i_2}}{f_{i_1 i_2}} \wedge \cdots \wedge \frac{df_{i_{m-1} i_m}}{f_{i_{m-1} i_m}}. \tag{4.8}$$

where $f_{ij} := \frac{x_j}{x_i}$.

4.9 Top cohomology

In this section we describe the isomorphism

$$\mathrm{Tr} : H^{2n}_{\mathrm{dR}}(X) \cong 2\mathsf{k}$$

over the field k. In the complex context $\mathsf{k} = \mathbb{C}$, it is given by

$$\omega \to \frac{1}{(2\pi i)^n} \int_X \omega.$$

Proposition 4.9.1 *The top cohomology $H^{2n}_{\rm dR}(X)$ is of dimension 1 and it is generated by the restriction of $\theta^n \in H^{2n}_{\rm dR}(\mathbb{P}^N)$.*

Let $\mathbb{P}^{N-n-1} \subset \mathbb{P}^N$ be a projective space such that $\mathbb{P}^{N-n-1} \cap X = \emptyset$. We assume that it is given by linear equations $g_0 = g_1 = \cdots = g_n = 0$ and consider the corresponding covering. In this covering a generator of $H^{2n}_{\rm dR}(X)$ is given by

$$\frac{df_{i_0i_1}}{f_{i_0i_1}} \wedge \frac{df_{i_1i_2}}{f_{i_1i_2}} \wedge \cdots \wedge \frac{df_{i_{m-1}i_m}}{f_{i_{m-1}i_m}} \tag{4.9}$$

where $f_{ij} := \frac{g_j}{g_i}$.

4.10 Poincaré duality

In this section we prove:

Theorem 4.10.1

4.11 Periods of algebraic cycles

In this section we take a smooth projective variety $X \subset \mathbb{P}^N$ over a field k of characteristic 0 and assume that k is small enough so that we have an embedding $\mathsf{k} \hookrightarrow \mathbb{C}$. Since any projective variety X over k uses a finite number of coefficients in k, the assumption on the embedding of k in $\mathbb{C}$ is not problematic. We start this section with the following remarkable property of algebraic cycles.

Proposition 4.11.1 *(Deligne, [DMOS82] Proposition 1.5) Let X be a projective smooth variety over an algebraically closed field $\mathsf{k} \subset \mathbb{C}$ and let Z be an irreducible subvariety in X of dimension $\frac{m}{2}$. For any element of the algebraic de Rham cohomology $H^m_{\rm dR}(X/\mathsf{k})$, we have*

$$\frac{1}{(2\pi i)^{\frac{m}{2}}} \int_{[Z]} \omega \quad \in \quad \mathsf{k} \tag{4.10}$$

where $[Z] \subset H_n(X^{\rm an}, \mathbb{Z})$ is the homololgy class of Z.

Proof. The cohomology ring of the projective variety $\mathbb{P}^N$ is generated by the polarization $\theta \in H^2_{\rm dR}(\mathbb{P}^N)$ which satisfies $\int_{[\mathbb{P}^1]} \theta = 2\pi i$. Therefore, for a smooth projective variety $\tilde{Z} \subset \mathbb{P}^N$ of dimension $\frac{m}{2}$ defined over k we have

$$\int_{[\tilde{Z}]} \theta^{\frac{m}{2}} = \deg(\tilde{Z}) \cdot (2\pi i)^{\frac{m}{2}} \tag{4.11}$$

and the fact that $\theta^{\frac{m}{2}}$ restricted to $\tilde{Z}$ generates the k-vector space $H^m_{\rm dR}(\tilde{Z})$. Let $Z \subset X$ be as in the proposition. We have a desingularization morphism $\tilde{Z} \to Z$ which composed with the inclusion gives us a map $f: \tilde{Z} \to X$ which is still defined over k. We have $f^*\omega \in H^m_{\rm dR}(\tilde{Z})$ and so there is a $a \in$ k such that $f^*\omega = a \cdot \theta^{\frac{m}{2}}$. This together with (4.11) implies the desired statement. □

4.12 Proof of the Hodge decomposition

In this section we present a new proof of the Hodge decomposition which does not use harmonic forms, instead it uses the algebraic de Rham cohomology introduced in the previous sections.

Let X be an algebraic variety over $\mathbb{C}$ and $\mathscr{O}_X$ be the sheaf of regular functions on X. We consider the sheaf $\mathscr{O}_{X^{\rm ah}}$ on X such that is sections over an open set $U \subset X$ is given by:

$$\mathscr{O}_{X^{\rm ah}}(U) := \text{The } \mathbb{C} \text{ algebra generated by the elements of , } \mathscr{O}_X(U), \quad \overline{\mathscr{O}_X(U)}.$$

We also call $\mathscr{O}_{X^{\rm ah}}$ the sheaf of holomorphic and anti-holomorphic functions. From this we can construct the complex of differential (p,q)-forms $\Omega^{p,q}_{X^{\rm ah}}$.

Conjecture 4.12.1 *The sheaves $\Omega^{p,q}_{X^{\rm ah}}$ are acyclic, that is,*

$$H^m(X, \Omega^{p,q}_{X^{\rm ah}}) = 0, \quad \forall m \geq 1.$$

I do not have any evidence why this conjecture must be true. If so, we do need the big class of C^∞-functions. Just polynomials and their complex conjugate would be enough for the proofs of many classical theorems in Hodge theory, mainly proved by functional or harmonic analysis. If the above conjecture is false it means that we really need the bump function used in the defintion of partition of unity, and maybe even more. However, the desire of defining a small of C^∞ functions which would do the job of Hodge decomposition will be still there.

4.13 Positivity of the the polarization

I think is is possible to give a proof of the following theorem, only using algebraic differential forms and their complex conjugates, without using Harmonic forms.

Proposition 4.13.1 *The Hodge filtration with respect to the polarization satisfies the following properties:*

$$\langle F^p, F^q \rangle = 0, \ \forall p,q, \ p+q > m, \tag{4.12}$$

$$(-1)^{\frac{m(m-1)}{2}+p}(\sqrt{-1})^{-m}\langle \omega, \overline{\omega} \rangle > 0, \ \forall \omega \in F^p_0 \cap \overline{F^{m-p}_0}, \ \ \omega \neq 0. \tag{4.13}$$

Final remarks

In this section let $X/\mathbb{C}$ be a smooth algebraic variety over the field of complex numbers, and let X^{an} be the underlying complex manifold of X. Let also $\Omega^\bullet_{X/\mathbb{C}}, \Omega^\bullet_{X^{an}}$ and $\Omega^\bullet_{X^\infty}$ be the complex of the sheaves of algebraic, holomorphic and C^∞ differential forms on X, respectively C^∞. The first complex is automatically defined over Zariski topology of $X/\mathbb{C}$, whereas the two others are defined over the usual topology of X^{an} and X^∞. By Poincaré lemma both $\Omega^\bullet_{X^{an}}$ and $\Omega^\bullet_{X^\infty}$ are the resoultions of the constant sheaf $\mathbb{C}$ and hence

$$\mathbb{H}^m(X^{an},\ \Omega^\bullet_{X^{an}}) \cong H^m(X^{an},\mathbb{C}),\ m = 0,1,2,\dots.$$

$$H^m_{\mathrm{dR}}(X^\infty) \cong \mathbb{H}^m(X^{an},\ \Omega^\bullet_{X^\infty}) \cong H^m(X^{an},\mathbb{C}),\ m = 0,1,2,\dots.$$

From another side the inclusion $\Omega^\bullet_{X/\mathbb{C}} \to \Omega^\bullet_{X^{an}}$ is a quasi-isomorphism and so

$$H^m_{\mathrm{dR}}(X/\mathbb{C}) \cong H^q_{\mathrm{dR}}(X^{an}),\ m = 0,1,2,\dots$$

Note that X is an affine variety X^{an} is Stein by Cartan's B theorem

$$H^i(X^{an},\Omega^j_{X^{an}}) = 0,\ i = 1,2,\dots, j = 0,1,2,\dots.$$

4.14 Hodge filtration of affine varieties

Let X be an affine variety. It follows from Theorem 4.2.2 and Theorem 4.4.1 that the Hodge filtration of X defined in (4.4) is trivial, that is, $F^m = F^{m-1} = \cdot = F^1 = F^0$. Therefore, the truncated complexes $\Omega^{\bullet \geq i}_X$ give us the correct Hodge filtration for projective varieties and beyond this we may be still in trouble to define the correct Hodge filtrations. Here, one has to calrify what correct means. Deligne in [Del71] defines the mixed Hodge structure of affine varieties and in particular its basic ingredient, namely the Hodge filtration. In this section we give an exposition of this topic. Even if one is interested in projective varieties, computations are usually done in affine varieties, and this topic is indespensible for further study of projective varieties. The reader may also consult Voisin's books [Voi02] Section 8.2.3 and [Voi03] Section 6.1.

Definition 4.14.1 Let X be a projective variety and $D = \sum_{i=1}^s D_i$ be a divisor. We say that Y is a normal crossing divisor of ....

Definition 4.14.2 The sheaf of logarithmic differential forms.

...

4.15 Hard Lefschez theorem

4.16 Exercises

4.1. Describe as explicit as possible the maps (4.1) and (4.2).

4.2. One can further describe the map (4.2) explicitly, provided that we specify the partition of unity used in our algebraic contect. For this we may need to add exponent of algebraric functions into the sheaf $\mathscr{O}_X$ of regular functions in a variety X. From another side, it is quit possible to redefine the C^∞ de Rham cohomology $H^m_{\mathrm{dR}}(X^{an})$, using a minimum data of C^∞ functions and not all of them. For instance, we may add the complex conjugate of algebraric functions into the sheaf $\mathscr{O}_X$. Develope these ideas as much as you can.

4.3. Let $\mathscr{O}_X^{ah}$ be the the $\mathbb{C}$-algebra generated by holomorphic and anti-holomorphic functions. From this we construct Ω_X^a. What are the Cech cohomologies of these sheafs?

Chapter 5
Gauss-Manin connection: general theory

In 1958 Yu. Manin solved the Mordell conjecture over function fields and A. Grothendieck after reading his article invented the name Gauss-Manin connection. I did not find any simple exposition of this subject, the one which could be understandable by Gauss's mathematics. I hope that the following explains the presence of the name of Gauss on this notion. Our story again goes back to integrals. Many times an integral depend on some parameter and so the resulting integration is a function in that parameter. For instance take the elliptic integral

$$\int_\delta \frac{Q(x)dx}{\sqrt{P(x)}} \tag{5.1}$$

and assume that P and Q depends on the parameter t and the interval δ does not depend on t. In any course in calculus we learn that the integration and derivation with respect to t commute:

$$\frac{\partial}{\partial t}\int_\delta \frac{Q(x)}{\sqrt{P(x)}}dx = \int_\delta \frac{\partial}{\partial t}\left(\frac{Q(x)}{\sqrt{P(x)}}\right)dx.$$

As before we know that the right hand side of the above equality can be written as a linear combination of two integrals $\int_\delta \frac{dx}{\sqrt{P}}$ and $\int_\delta \frac{xdx}{\sqrt{P(x)}}$. This is the historical origin of the notion of Gauss-Manin connection, that is, derivation of integrals with respect to parameters and simplifying the result in terms of integrals which cannot be simplified more.

5.1 De Rham cohomology of projective varieties over a ring

Recall our convention about the ring R in §**??** and let $\mathsf{T} := \mathrm{Spec}(\mathsf{R})$ be the corresponding variety over k.

Remark 5.1.1 The usage of T is just for the sake of compatibility of our notations with those in the literature.

Let X be a projective, smooth, reduced variety over k and let $\mathsf{X}\to\mathsf{T}$ be a morphism of varieties over k. We also say that X is a variety over the ring R.

We have the complex of sheaves of differential forms $(\Omega^\bullet_{\mathsf{X}/\mathsf{T}},d)$ and we define the i-th relative de Rham cohomology of X as the i-th hypercohomology of the complex $(\Omega^\bullet_{\mathsf{X}/\mathsf{T}},d)$, that is

$$H^i_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})=\mathbb{H}^i(\Omega^\bullet_{\mathsf{X}/\mathsf{T}},d).$$

We explain how the elements of the hypercohomology look like and how to calculate it.

Let $\mathscr{U}=\{U_i\}_{i\in I}$ be any open covering of X by affine subsets, where I is a totally ordered finite set. We have the following double complex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
(\Omega_{\mathsf{X}/\mathsf{T}})^0_2 & \to & (\Omega_{\mathsf{X}/\mathsf{T}})^1_2 & \to & (\Omega_{\mathsf{X}/\mathsf{T}})^2_2 & \to & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
(\Omega_{\mathsf{X}/\mathsf{T}})^0_1 & \to & (\Omega_{\mathsf{X}/\mathsf{T}})^1_1 & \to & (\Omega_{\mathsf{X}/\mathsf{T}})^2_1 & \to & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
(\Omega_{\mathsf{X}/\mathsf{T}})^0_0 & \to & (\Omega_{\mathsf{X}/\mathsf{T}})^1_0 & \to & (\Omega_{\mathsf{X}/\mathsf{T}})^2_0 & \to & \cdots
\end{array} \tag{5.2}$$

Here $(\Omega_{\mathsf{X}/\mathsf{T}})^i_j$ is the product over $I_1\subset I$, $\#I_1=j+1$ of the set of global sections ω_σ of $\Omega^i_{\mathsf{X}/\mathsf{T}}$ in the open set $\sigma=\cap_{i\in I_1}U_i$. The horizontal arrows are usual differential operator d of $(\Omega_{\mathsf{X}/\mathsf{T}})^i_{X/T}$'s and vertical arrows are differential operators δ in the sense of Cech cohomology, that is,

$$\delta:(\Omega_{\mathsf{X}/\mathsf{T}})^i_j\to(\Omega_{\mathsf{X}/\mathsf{T}})^i_{j+1},\ \ \{\omega_\sigma\}_\sigma\mapsto\{\sum_{k=0}^{j+1}(-1)^k\omega_{\tilde\sigma_k}\mid_{\tilde\sigma}\}_{\tilde\sigma}. \tag{5.3}$$

Here $\tilde\sigma_k$ is obtained from $\tilde\sigma$, neglecting the k-th open set in the definition of $\tilde\sigma$. The k-th piece of the total chain of (5.2) is

$$\mathscr{L}^k:=\oplus_{i=0}^k(\Omega_{\mathsf{X}/\mathsf{T}})^i_{k-i}$$

with the differential operator

$$d'=d+(-1)^k\delta:\mathscr{L}^k\to\mathscr{L}^{k+1}. \tag{5.4}$$

The relative de Rham cohomology $H^k_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$ is the total cohomology of the double complex (5.2), that is,

$$H^k_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}) := \mathbb{H}^k(M, \Omega^\bullet_{\mathsf{X}/\mathsf{T}}) := \frac{\ker(\mathscr{L}^k \stackrel{d}{\to} \mathscr{L}^{k+1})}{\mathrm{Im}(\mathscr{L}^{k-1} \stackrel{d}{\to} \mathscr{L}^k)}.$$

5.2 Gauss-Manin connection

What we do in this section in the framework of Algebraic Geometry is as follows: Let X be a smooth reduced variety over R. We construct a connection

$$\nabla : H^i_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}) \to \Omega^1_{\mathsf{T}} \otimes_{\mathsf{R}} H^i_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$$

where Ω^1_{T} is by definition $\Omega^1_{\mathsf{R}/\mathsf{k}}$, that is, the R-module of differential attached to R. By definition of a connection, ∇ is k-linear and satisfies the Leibniz rule

$$\nabla(r\omega) = dr \otimes \omega + r\nabla\omega.$$

A vector field v in T is an R-linear map $\Omega^1_{\mathsf{T}} \to \mathsf{R}$. We define

$$\nabla_v : H^i_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}) \to H^i_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$$

to be ∇ composed with

$$v \otimes \mathrm{Id} : \Omega^1_{\mathsf{T}} \otimes_{\mathsf{R}} H^i_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}) \to \mathsf{R} \otimes_{\mathsf{R}} H^i_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}) = H^i_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}).$$

If R is a polynomial ring $\mathbb{Q}[t_1, t_2, \ldots]$ then we have vector fields $\frac{\partial}{\partial t_i}$ which are defined by the rule

$$\frac{\partial}{\partial t_i}(dt_j) = 1 \text{ if } i = j \text{ and } = 0 \text{ if } i \neq j.$$

In this case we simply write $\frac{\partial}{\partial t_i}$ instead of $\nabla_{\frac{\partial}{\partial t_i}}$.

Sometimes it is useful to choose a basis $\omega_1, \omega_2, \ldots, \omega_h$ of the R-modular $H^i(\mathsf{X}/\mathsf{T})$ and write the Gauss-Manin connection in this basis:

$$\nabla \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_h \end{pmatrix} = A \otimes \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_h \end{pmatrix} \tag{5.5}$$

where A is a $h \times h$ matrix with entries in Ω^1_{T}.

5.3 Construction

Let X be a variety over the ring R and $\mathsf{T} := \mathrm{Spec}(\mathsf{R})$ Let us take a covering $\mathscr{U} = \{U_i\}_{i\in I}$ of X by affine open sets. In this section we need to distinguish between the differential operator relative to R

$$d_{\mathsf{R}} : \Omega^k_{\mathsf{X}/\mathsf{T}} \to \Omega^{k+1}_{\mathsf{X}/\mathsf{T}}$$

and the differential operator d_{k} relative to k:

$$d_{\mathsf{k}} : \Omega^k_{\mathsf{X}} \to \Omega^{k+1}_{\mathsf{X}}$$

We also need to consider the double complex similar to (5.2) relative to k: (5.4).

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \\ \uparrow & & \uparrow & & \uparrow & & \\ (\Omega_{\mathsf{X}})^0_2 & \to & (\Omega_{\mathsf{X}})^1_2 & \to & (\Omega_{\mathsf{X}})^2_2 & \to & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ (\Omega_{\mathsf{X}})^0_1 & \to & (\Omega_{\mathsf{X}})^1_1 & \to & (\Omega_{\mathsf{X}})^2_1 & \to & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ (\Omega_{\mathsf{X}})^0_0 & \to & (\Omega_{\mathsf{X}})^1_0 & \to & (\Omega_{\mathsf{X}})^2_0 & \to & \cdots \end{array} \tag{5.6}$$

We have a projection map from the double complex (5.6) to (5.2).

Let $\omega \in H^k_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$. By our definition ω is represented by $\oplus_{i=0}^k \omega_i,\ \omega_i \in (\Omega_{\mathsf{X}/\mathsf{T}})^i_{k-i}$ and ω_i is a collection of i-forms $\{\omega_{i,\sigma}\}_\sigma$. We have By definition we have $d'_{\mathsf{R}}\omega = 0$. Since the differential map d_{R} used in the definition of d'_{R} is relative to R, that is, by definition $dr = 0, r \in \mathsf{R}$. Now, let us take any element $\check{\omega}$ in the double complex (5.6) which is mapped into ω under the canonical projection. We apply d_{k} on $\check{\omega}$ and the result is not necessarily zero. However, by our choise of ω we have $d'_{\mathsf{R}}\omega = 0$, and so, $d'\check{\omega}$ maps to zero in the double complex (5.2). This is equivalent to say that

$$d'_{\mathsf{k}}\check{\omega} = \check{\eta} = \oplus_{i=0}^{k+1}\check{\eta}_i\ \check{\eta}_i \in \Omega^1_{\mathsf{T}} \wedge (\Omega_{\mathsf{X}})^i_{k+1-i}$$

We map $\check{\eta}$ into the wedge product of ω^1_{T} with the double complex (5.2) and we get an element

$$\check{\eta} \in \Omega^1_{\mathsf{T}} \otimes_{\mathsf{R}} (\Omega_{\mathsf{X}/\mathsf{T}})^i_{k+1-i}.$$

In this process we have replaced the notation $\wedge$ with the tensor product $\otimes_{\mathsf{R}}$. Since $d'_k \circ d'_{\mathsf{k}}\check{\omega} = 0$

$$(\mathrm{id} \otimes_{\mathsf{R}} d'_{\mathsf{R}})(\check{\eta}) = 0.$$

This gives us an element in $\Omega^1_T \otimes_{\mathsf{R}} H^k_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$ which is by definition $\nabla\omega$.

5.4 Griffiths transversality

Let X/T be as before. The relative algebraic de Rham cohomology $H^m(\mathsf{X}/\mathsf{T})$ carries a natural filtration which is called then Hodge filtration:

$$0 = F^{m+1} \subset F^m \subset \cdots F^1 \subset F^0 = H^m(\mathsf{X}/\mathsf{T})$$

Its ingredients are defined by

$$F^i = F^i H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}) = \mathrm{Im}\left(\mathbb{H}^m(X, \Omega^{\bullet \geq i}_{\mathsf{X}/\mathsf{T}}) \to \mathbb{H}^m(\mathsf{X}, \Omega^{\bullet}_{\mathsf{X}/\mathsf{T}})\right)$$

Theorem 5.4.1 (Griffiths transversality) *The Gauss-Manin connection maps F^i to $\Omega^1_{\mathsf{T}} \otimes_{\mathsf{R}} F^{i-1}$, that is,*

$$\nabla(F^i) \subset \Omega^1_{\mathsf{T}} \otimes_{\mathsf{R}} F^{i-1}, \ \ i = 1, 2, \ldots, m.$$

Proof. This follows from the definition of the Gauss-Manin connection.

5.5 Geometric Gauss-Manin connection

Let us now assume that $\mathsf{k} = \mathbb{C}$. The main motivation, which is also the historical one, for defining the Gauss-Manin connection is the following: For any $\omega \in H^i_{\mathrm{dR}}(X)$ and a continuous family of cycles $\delta_t \in H_i(X_t, \mathbb{Z})$ we have

$$d\left(\int_{\delta_t} \omega\right) = \int_{\delta_t} \nabla \omega. \tag{5.7}$$

Here, by definition

$$\int_{\delta_t} \alpha \otimes \beta = \alpha \int_{\delta_t} \beta,$$

where $\beta \in H^i_{\mathrm{dR}}(X)$ and $\alpha \in \Omega^1_T$. Integrating both side of the equality (5.5) over a a basis $\delta_1, \delta_2, \ldots, \delta_h \in H_i(X_t, \mathbb{Q})$ we conclude that

$$d([\int_{\delta_j} \omega_i]) = [\int_{\delta_j} \omega_i] \cdot A. \tag{5.8}$$

5.6 Algebraic vs. Analytic Gauss-Manin connection

5.7 A consequence of global invariant cycle theorem

We derive a consequence of gloabl invariant cycle theorem. For the references on this topic see [Mov17b] §6.11.

Let $Y, X, \pi : U \to B$ as before, but defined over a field k of characteristic zero. Let

Theorem 5.7.1 *Let δ be a global section of the relative algebraic de Rham cohomology $H^m_{\rm dR}(U/B)$. If $\nabla\delta = 0$ then its evaluation δ_0 at $0 \in B$ is in the image of the map*

$$H^{\mathsf{m}}_{\rm dR}(X) \to H^{\mathsf{m}}_{\rm dR}(Y).$$

induced by inclusion $Y \hookrightarrow X$. Conversely, any δ_0 in the image of the above map comes from a global section δ with $\nabla\delta = 0$.

Proof. The proof is done for $\mathsf{k} = \mathbb{C}$. By our hypothesis δ is a glonal section of $R^{\mathsf{m}}\pi^*\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{C}$ and so it is invariant under the monodromy. Therefore,

$$\delta_0 \in H^m(Y, \mathbb{Q})^\rho \otimes_{\mathbb{Q}} \mathbb{C}.$$

1. Show that the above definition does not depend on the choice of covering, that is, if $\mathscr{U}_1$ and $\mathscr{U}_2$ are two open covering of X then the corresponding hypercohomologies are isomorphic in a canonical way.
2. For which varieties X, we have $H^0_{\rm dR}(X) = \mathsf{R}$.

Chapter 6
Infinitesimal variation of Hodge structures

In its early phase (Abel, Riemann, Weierstrass), algebraic geometry was just a chapter in analytic function theory [...] A new current appeared however (1870) under the powerfull influence of Max Noether who really put "geometry" and more "birational geometry" into algebraic geometry [...] The next step in the same direction was taken by Castelnuovo (1892) and Enriques (1893). They applied analogous methods to the creation of an entirely new theory of algebraic surfaces. Their basic instrument was the study of linear systems of curves on a surface. Many new birationally invariant properties were discovered and an entirely new and beautiful chapter of geometry was opened. In 1902 the Castelnuovo-Enriques team was enriched by the brilliant personality of Severi. More than his associates he was interested in the contacts with the analytic theory developed since 1882 by Émile Picard. The most important contribution of Severi, his theory of the base was in fact obtained by utilizing the Picard number p. The theory of the great Italian geometers was essentially, like Noether's, of algebraic nature. Curiously enough this holds in good part regarding the work of Picard. This was natural since in his time Poincaré's creation of algebraic topology was in its infancy. Indeed when I arrived on the scene (1915) it was hardly further along. [...] I cannot refrain, however, from mention of [....] the systematic algebraic attack on algebraic geometry by Oscar Zariski and his school, and beyond that of André Weil and Grothendieck.

Gauss-Manin connection carries many information of the underlying family of algebraic varieties, however, in general it is hard to compute it and verify some of its properties. For this reason it is sometimes convenient to break it into smaller peices and study these by their own. Infinitesimal variation of Hodge structures, IVHS for short, is one of these pieces of the Gauss-Manin connection, and we explain it in this chapter. It was originated by the articles of Griffiths around sixties [Gri68a, Gri68b, Gri69a], and was introduced by him and his collaborators in the subsequence articles [CG80, CGGH83]. See for instance page 183 of the last article for some historical comments on this.

6.1 Starting from Gauss-Manin connection

Recall that for a family of projective varieties $\mathsf{X} \to \mathsf{T}$ we have the Gauss-Manin connection

$$\nabla : H^m(\mathsf{X}/\mathsf{T}) \to \Omega^1_{\mathsf{T}} \otimes_{\mathscr{O}_{\mathsf{T}}} H^m(\mathsf{X}/\mathsf{T})$$

We have also the Hodge filtration

$$0 = F^{m+1} \subset F^m \subset \cdots F^1 \subset F^0 = H^m(\mathsf{X}/\mathsf{T})$$

and the Gauss-Manin connection satisfies the so called Griffiths transversality:

$$\nabla(F^i) \subset \Omega^1_{\mathsf{T}} \otimes_{\mathscr{O}_{\mathsf{T}}} F^{i-1}, \;\; i = 1, 2, \ldots, m.$$

Therefore, the Gauss-Manin connection induces well-defined maps

$$\nabla_i : \frac{F^i}{F^{i+1}} \to \Omega^1_{\mathsf{T}} \otimes_{\mathscr{O}_{\mathsf{T}}} \frac{F^{i-1}}{F^i} \tag{6.1}$$

In Theorem 4.4.1 we have learned that

$$\frac{F^i}{F^{i+1}} \cong H^{m-i}(\mathsf{X},\ \Omega^i_{\mathsf{X}/\mathsf{T}}).$$

Therefore, we get the following R-linear map

$$\nabla_i : H^{m-i}(\mathsf{X},\ \Omega^i_{\mathsf{X}/\mathsf{T}}) \to \Omega^1_{\mathsf{T}} \otimes_{\mathsf{R}} H^{m-i+1}(\mathsf{X},\ \Omega^{i-1}_{\mathsf{X}/\mathsf{T}}) \tag{6.2}$$

After analysing the definition of the Gauss-Manin connection we get the following description of ∇_i. Let $\omega \in H^{m-i}(\mathsf{X},\ \Omega^i_{\mathsf{X}/\mathsf{T}})$. It is given by a cocycle in

$$(\Omega_{\mathsf{X}/\mathsf{T}})^i_{m-i} = \frac{(\Omega_{\mathsf{X}})^i_{m-i}}{\Omega^1_{\mathsf{T}} \wedge (\Omega_{\mathsf{X}})^{i-1}_{m-i}}.$$

Let $\check{\omega} \in (\Omega_{\mathsf{X}})^i_{m-i}$ such that it maps to ω under the canonical projection. We have

$$\delta(\check{\omega}) \in \Omega^1_{\mathsf{T}} \wedge (\Omega_{\mathsf{X}})^{i-1}_{m-i}$$

Projecting this in $\Omega^1_{\mathsf{T}} \otimes_{\mathsf{R}} (\Omega_{\mathsf{X}/\mathsf{T}})^{i-1}_{m-i}$ we get $\nabla_i(\omega)$.

Sometimes it is usefull to write (6.2) in the following way:

$$\nabla_i : \Theta_{\mathsf{T}} \to \hom\left(H^{m-i}(\mathsf{X},\ \Omega^i_{\mathsf{X}/\mathsf{T}}), H^{m-i+1}(\mathsf{X},\ \Omega^{i-1}_{\mathsf{X}/\mathsf{T}})\right) \tag{6.3}$$

6.2 Algebraic polarization

A smooth projective variety comes with an embedding $X \subset \mathbb{P}^N$, and this gives us the polarization $\theta \in H^2(X,\mathbb{Z}) \cap H^{11}$ which is Poincaré dual $[Y] \subset H_{2n-2}(X,\mathbb{Z})$, where Y is a smooth hyperplance section of X. We would like to introduce this in the framework of algebraic de Rham cohomology.

Recall that θ is the Chern class of the line bundle $\mathscr{O}(1)\mid_X$:

$$\theta := \mathsf{c}(\mathscr{O}(1)).$$

We make the following composition

$$H^1(X,\mathscr{O}_X^*) \to H^2(X,\mathbb{Z}) \to H^2_{\mathrm{dR}}(X)$$

where the first one is the Chern class map and define the resulting map

$$\mathsf{c} : H^1(X,\mathscr{O}_X^*) \to H^2_{\mathrm{dR}}(X) \tag{6.4}$$

in the algebraic context. Its image is in the F^1 piece of the de Rham cohomology. Therefore, we will have also the map

$$\mathsf{c} : H^1(X,\mathscr{O}_X^*) \to H^1(X,\Omega_X^1) \tag{6.5}$$

that we will denote it by the same letter c. We will call c the algebraic Chern class map.

Proposition 6.2.1 *The algebraic Chern class of a line bundle in $H^1(X,\mathscr{O}_X^*)$ given by a cocycle $\{f_{ij}\}_{i,j\in I}$ is given by $\omega = \omega_0^2 + \omega_1^1 + \omega_2^0$, where ω_0^2 and ω_2^0 are zero and ω_1^1 is given by $\{\frac{df_{ij}}{f_{ij}}\}_{i,j\in I}$.*

Proof.

6.3 Kodaira-Spencer map I

Now we define the Kodaira-Spencer map

$$\mathsf{K} : H^0(\mathsf{T},\Theta_{\mathsf{T}}) \to H^1(\mathsf{X},\Theta_{\mathsf{X}/\mathsf{T}}) \tag{6.6}$$

For a global vector field v in T we define $\mathsf{K}(v)$ in the following way. We choose an acyclic covering $\mathscr{U} = \{U_i\}_{i\in I}$ and vector fields v_i in U_i such that v_i is mapped to v under $\mathsf{X} \to \mathsf{T}$. The vector fields $v_i - v_j$ are tangent to the fibers of $\mathsf{X} \to \mathsf{T}$ and they give us $\mathsf{K}(v)$.

The Kodaira-Spencer map K is related to the connecting homomorphism of the long exact sequence attached to the short exact sequence:

$$0 \to \Theta_{\mathsf{X}/\mathsf{T}} \to \check{\Theta}_{\mathsf{X}} \to \pi^* \Theta_{\mathsf{T}} \to 0 \tag{6.7}$$

where $\check{\Theta}_{\mathsf{X}} \subset \Theta_{\mathsf{X}}$ is by definition the sheaf of vector fields in X which are mapped to vector fields in T, and $\pi^*\Theta_{\mathsf{T}}$ is a sheaf in X such that its section in a Zariski open set X are just the elements of Θ_{T}.

Therefore, we have

$$H^0(\mathsf{X}, \pi^*\Theta_{\mathsf{T}}) = H^0(\mathsf{T}, \Theta_{\mathsf{T}}), \;\; H^1(\mathsf{X}, \pi^*\Theta_{\mathsf{T}}) = H^1(\mathsf{T}, \Theta_{\mathsf{T}}) \tag{6.8}$$

The long exact sequence of (6.7) turns out to be

$$\cdots \to H^0(\mathsf{X}, \check{\Theta}_{\mathsf{X}}) \to H^0(\mathsf{T}, \Theta_{\mathsf{T}}) \xrightarrow{\mathsf{K}} H^1(\mathsf{X}, \Theta_{\mathsf{X}/\mathsf{T}}) \to H^1(\mathsf{X}, \check{\Theta}_{\mathsf{X}}) \to H^1(\mathsf{T}, \Theta_{\mathsf{T}}) \to \cdots \tag{6.9}$$

We conclude that

Proposition 6.3.1 *Assume that $H^1(\mathsf{T}, \Theta_{\mathsf{T}}) = 0$. We have $H^1(\mathsf{X}, \check{\Theta}_{\mathsf{X}}) = 0$ if and only if the Kodaira-Spencer map is surjective.*

For the main purpose of the present book, it would be essential to compute $H^i(\mathsf{X}, \check{\Theta}_{\mathsf{X}}), \; i = 0, 1$.

Let us now consider the parameter space T of smooth hypersurfaces $X \subset \mathbb{P}^{m+1}$. In this case T is the affine space $\mathbb{A}_k^a$ mines a discriminant locus $\{\Delta = 0\}$. It has the coordinate system $(t_\alpha, \alpha \in \check{I})$, where

$$\check{I} := \left\{(\alpha_0, \alpha_1, \ldots, \alpha_{m+1}) \mid 0 \le \alpha_e \le d, \quad \sum \alpha_e = d\right\}$$

The polynomial expressions of Δ in terms of the variables t_α is in general huge. The variety $\mathsf{X} \subset \mathbb{P}^{m+1} \times \mathsf{T}$ is given by

$$\mathsf{X}: \;\; g = 0.$$

$$\text{where } g := \sum_{\alpha \in \check{I}} t_\alpha x^\alpha = 0$$

and $\mathsf{X} \to \mathsf{T}$ is the projection on T. Let $\frac{\partial}{\partial t_\alpha}, \;\; \alpha \in \check{I}$ be canonical vector fields in T.

Since the fibers of $\mathsf{X} \to \mathsf{T}$ are smooth, we can take the Jacobian covering $\mathscr{U} = \{U_i\}_{j=0,1,2,\ldots,m+1}$ of X, where

$$U_j : \frac{\partial g}{\partial x_j} \neq 0.$$

In the affine open set U_i a pull-back of $\frac{\partial}{\partial t_\alpha}$ is given by

$$\frac{\partial}{\partial t_\alpha} - \frac{\frac{\partial g}{\partial t_\alpha}}{\frac{\partial g}{\partial x_j}} \frac{\partial}{\partial x_j}$$

Therefore, the Kodaira-Spencer map is given by

$$\mathsf{K}(\frac{\partial}{\partial t_\alpha}) := \left\{ \frac{\partial g}{\partial t} \left((\frac{\partial g}{\partial x_j})^{-1} \frac{\partial}{\partial x_j} - (\frac{\partial g}{\partial x_i})^{-1} \frac{\partial}{\partial x_i} \right) \right\}_{i,j=0,1,\ldots,m+1}$$

Proposition 6.3.2 *For the universal family of hypersurfaces* $\mathsf{X} \to \mathsf{T}$, *we have* $H^1(\mathsf{X}, \Theta_{\mathsf{X}}) = 0$ *except for hypersurface of dimension two and degree four. In this exceptional case it is a one dimensional* $\mathbb{Q}$*-vector spaces.*

This proposition will be proved in the next section.

6.4 Kodaira-Spencer map II

Most of the time we take a point $0 \in \mathsf{T}$, set $X := \mathsf{X}_0$ and specialize (6.6) at a point $0 \in \mathsf{T}$:

$$\mathsf{K} : \mathbf{T}_{\mathsf{T},0} \to H^1(X, \Theta_X) \tag{6.10}$$

In the literature we mainly find this map. The next discussion is take from Voisin's book [Voi03] Lemma 6.15. We start with

$$\begin{array}{ccccccccc} 0 \to & \mathbf{T}_X & \to & \mathbf{T}_{\mathsf{X}}|_X & \to & \mathbf{N}_{X\subset\mathsf{X}} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathbf{T}_X & \to & \mathbf{T}_{\mathbb{P}^{n+1}}|_X & \to & \mathbf{N}_{X\subset\mathbb{P}^{n+1}} & \to 0 \end{array} \tag{6.11}$$

Here, $\mathbf{N}_{A\subset B}$ is the normal bundle of A inside B, the first down arrow map is the identity, the second is the derivation of the projection $\mathsf{X} \to \mathbb{P}^{n+1}$ and the third is the map induced in the quotient. We note that $\mathbf{N}_{X\subset\mathsf{X}}$ is the trivial bundle in X. In fact, a trivialization of this bundle is given by the restriction of the derivation of the map $\mathsf{X} \to \mathsf{T}$ to the points of X. Therefore, we have canonical identifications

$$H^0(X, \mathbf{N}_{X\subset\mathsf{X}}) = \mathbf{T}_{\mathsf{T},0},$$

$$H^1(X, \mathbf{N}_{X\subset\mathsf{X}}) = \mathbf{T}_{\mathsf{T},0} \otimes_{\mathsf{k}} H^1(X, \mathscr{O}_X).$$

Now we write the long exact sequence of (6.11) and we use the above data

$$\begin{array}{ccccccccc} H^0(X, \mathbf{T}_{\mathsf{X}}|_X) & \to & \mathbf{T}_{\mathsf{T},0} & \xrightarrow{\mathsf{K}} & H^1(X, \mathbf{T}_X) & \to & H^1(X, \mathbf{T}_{\mathsf{X}}|_X) & \to & \mathbf{T}_{\mathsf{T},0} \otimes_{\mathsf{k}} H^1(X, \mathscr{O}_X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^0(X, \mathbf{T}_{\mathbb{P}^{n+1}}|_X) & \xrightarrow{b} & H^0(X, \mathbf{N}_{X\subset\mathbb{P}^{n+1}}) & \xrightarrow{c} & H^1(X, \mathbf{T}_X) & \to & H^1(X, \mathbf{T}_{\mathbb{P}^{n+1}}|_X) & \to & H^1(X, \mathbf{N}_{X\subset\mathbb{P}^{n+1}}) \\ \uparrow a & & & & & & & & \\ H^0(X, \mathbf{T}_{\mathbb{P}^{n+1}}) & & & & & & & & \end{array} \tag{6.12}$$

The last up arrow map is just the restriction map that we will need later.

Let us now focus on the case of hypersurfaces X in $\mathbb{P}^{n+1}$ given by the homogeneous polynomial $g \in \mathsf{k}[x]$ of degree d. In this case we have many canonical identifications and vanishings so that (6.13) becomes

$$\begin{array}{ccccccccc} H^0(X,\mathbf{T}_{\mathsf{X}}|_X) & \to & \mathsf{k}[x]_d & \stackrel{\mathsf{K}}{\to} & H^1(X,\mathbf{T}_X) & \to & H^1(X,\mathbf{T}_{\mathsf{X}}|_X) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^0(X,\mathbf{T}_{\mathbb{P}^{n+1}}|_X) & \stackrel{b}{\to} & \mathsf{k}[x]_d & \stackrel{c}{\to} & H^1(X,\mathbf{T}_X) & \to & H^1(X,\mathbf{T}_{\mathbb{P}^{n+1}}|_X) & \to & 0 \\ \uparrow a & & & & & & & & \\ \langle x_i\frac{\partial}{\partial x_j},\ i,j=1,2,\ldots,n+1\rangle & & & & & & & & \end{array} \tag{6.13}$$

Let us explain the details of this. The identification

$$\mathbf{T}_{\mathsf{T},0}=\mathsf{k}[x]_d$$

is done in the following way. For $p\in\mathsf{k}[x]_d$ we first consider the curve $g+tp,\ t\in\mathbb{A}_{\mathsf{k}}$ whose derivation at $t=0$ gives the corresponding vector in $\mathbf{T}_{\mathsf{T},0}$. We have also

$$H^0(X,\mathbf{N}_{X\subset\mathbb{P}^{n+1}})\cong H^0(\mathbb{P}^{n+1},\mathscr{O}_{\mathbb{P}^{n+1}}(d))\cong\mathsf{k}[x]_d$$

The map $\mathbf{T}_{\mathsf{T},0}\to H^0(X,\mathbf{N}_{X\subset\mathbb{P}^{n+1}})$ turns out to be the identity map. The k-vector space $H^0(X,\mathbf{T}_{\mathbb{P}^{n+1}})$ is generated by $x_j\frac{\partial}{\partial x_i}$ and the composition $b\circ a$ after these identifications is

$$x_j\frac{\partial}{\partial x_i}\to x_j\frac{\partial g}{\partial x_i}$$

We will need the following:

Theorem 6.4.1 (Bott, [Bot57]) *For a smooth hypersurface X in $\mathbb{P}^{n+1}$ we have*

$$H^1(\mathbb{P}^{n+1},\mathbf{T}_{\mathbb{P}^{n+1}}(X))=0,$$

where $\mathbf{T}_{\mathbb{P}^{n+1}}(X)$ is the sheaf of vector fields in $\mathbb{P}^{n+1}$ vanishing along X. Further, if $n\neq 2$ and the degree of X is not 4 then

$$H^1(X,\mathbf{T}_{\mathbb{P}^{n+1}}|_X)=0$$

The first part of Bott's theorem implies that the maps a and c are surjective. Finally we get

Theorem 6.4.2 *For a fixed hypersurface $X\subset\mathbb{P}^{n+1}$ given by the homogeneous polynomial g we have the Kodaira-Spencer map*

$$(\mathbb{C}[x]/\mathrm{jacob}(g))_d\stackrel{\mathsf{K}}{\to}H^1(X,\Theta_X)$$

given by

$$\mathsf{K}(x^\alpha):=\left\{x^\alpha\left((\frac{\partial g}{\partial x_j})^{-1}\frac{\partial}{\partial x_j}-(\frac{\partial g}{\partial x_i})^{-1}\frac{\partial}{\partial x_i}\right)\right\}_{i,j=0,1,\ldots,m+1}$$

for $x^\alpha\in(\mathbb{C}[x]/\mathrm{jacob}(g))_d$. For $(n,d)\neq(2,4)$ it is an isomorphism and for $(n,d)=(2,4)$ it is an injection whose image is of codimension 1 in $H^1(X,\Theta_X)$.

6.5 A theorem of Griffiths

Under the assumption that $H^1(\mathsf{X},\Theta_{\mathsf{X}})=0$, we are going to define a canonical map

$$\bar{\nabla}_i : H^1(\mathsf{X},\Theta_{\mathsf{X}/\mathsf{T}}) \to \hom\left(H^{m-i}(\mathsf{X},\ \Omega^i_{\mathsf{X}/\mathsf{T}}), H^{m-i+1}(\mathsf{X},\ \Omega^{i-1}_{\mathsf{X}/\mathsf{T}})\right) \tag{6.14}$$

Let $v=\{v_{ij}\}\in H^1(\mathsf{X},\Theta_{\mathsf{X}/\mathsf{T}})$ and $\omega\in H^{m-i}(\mathsf{X},\ \Omega^i_{\mathsf{X}/\mathsf{T}})$. We want to define $\bar{\nabla}_i(v)(\omega)\in H^{m-i+1}(\mathsf{X},\ \Omega^{i-1}_{\mathsf{X}/\mathsf{T}})$. We take an acyclic covering $\mathscr{U} := \{U_i\}_{i\in I}$ and a cocycle $\check{\omega}\in(\Omega_{\mathsf{X}})^i_{m-i}$ which maps to ω under the canonical projection $\Omega^i_{\mathsf{X}}\to\Omega^i_{\mathsf{X}/\mathsf{T}}$. The ingredients of $\delta\check{\omega}$ are sections of the sheaf $\Omega^1_{\mathsf{T}}\wedge\Omega^{i-1}_{\mathsf{X}}$.

We have $H^1(\mathsf{X},\Theta_{\mathsf{X}})=0$ and $\Theta_{\mathsf{X}/\mathsf{T}}\subset\Theta_{\mathsf{X}}$. Therefore, we have vector fields v_i in U_i such that $v_{ij}=v_j-v_i$. In any intersections $U_0\cap U_1\cap\cdots\cap U_{i_m}\cap U_{m-i+1}$ we have

$$0 = i_{v_{ij}}(\delta\check{\omega}) = i_{v_j}(\delta\check{\omega}) - i_{v_i}(\delta\check{\omega})$$

The first equality is in $\Omega^{i-1}_{\mathsf{XT}}$. Therefore, $\{i_{v_j}(\delta\check{\omega})\}$ does not depend on the choice of j. This gives us the desired element in $H^{m-i+1}(\mathsf{X},\ \Omega^{i-1}_{\mathsf{X}/\mathsf{T}})$.

Theorem 6.5.1 (Griffiths) *The map* (6.3) *factors through the Kodaira-Spencer map* (6.6) *and one gets the map* $\bar{\nabla}_i$, *that is,*

$$\nabla_i = \bar{\nabla}_i \circ \mathsf{K}$$

Proof. This follows from the definition of ∇_i and $\bar{\nabla}_i$.

In many situtation we want to use

$$\bar{\nabla}_i : H^1(X,\Theta_X) \to \hom\left(H^{m-i}(X,\ \Omega^i_X), H^{m-i+1}(X,\ \Omega^{i-1}_X)\right) \tag{6.15}$$

which is (6.14) over a fixed fiber $X=\mathsf{X}_t$ of $\mathsf{X}\to\mathsf{T}$. Since we have used $H^1(\mathsf{X},\Theta_{\mathsf{X}})=0$, we cannot simply take T as a one point variety and proceed the definition of $\bar{\nabla}_i$ as before.

6.6 IVHS for hypersurfaces

Let us consider a hyper surface $\mathsf{X}\subset\mathbb{P}^{m+1}$ given by the homogeneous poynomial g. Griffiths theorem gives a basis of the algebraic de Rham cohomology $H^m_{\mathrm{dR}}(\mathbb{P}^{m+1}-X)$. Recall that we have a long exact sequence

$$\cdots\to H^{n+1}_{\mathrm{dR}}(\mathbb{P}^{n+1})\to H^{n+1}_{\mathrm{dR}}(\mathbb{P}^{n+1}-X)\to H^n_{\mathrm{dR}}(X)\to H^{n+2}(\mathbb{P}^{n+1})\to\cdots$$

For n odd we have $H^n_{\mathrm{dR}}(X)_0 = H^n_{\mathrm{dR}}(X)$ and for n even $H^n_{\mathrm{dR}}(X) = H^n_{\mathrm{dR}}(X)_0\oplus\theta^{\frac{n}{2}}$, where $\theta\in H^2_{\mathrm{dR}}(X)$ is the polarization of X. Under the residue map $H^{n+1}_{\mathrm{dR}}(\mathbb{P}^{n+1}-$

$X) \to H^n_{\rm dR}(X)_0$ is an isomorphism. Since $\theta^{\frac{n}{2}} \in F^{\frac{n}{2}} H^n_{\rm dR}(X)$, we get the following identifications

$$H^k(X, \Omega_X^{n-k})_0 \cong (\mathbb{C}[X]/J)_{(k+1)d-n-2}, \ \ k = 0, 1, \ldots, m. \tag{6.16}$$

For n even and $k = \frac{n}{2}$, we have $H^k(X, \Omega_X^{n-k}) = H^k(X, \Omega_X^{n-k})_0 + \theta^{\frac{n}{2}}$. For all other n and k, $H^k(X, \Omega_X^{n-k})_0 = H^k(X, \Omega_X^{n-k})$.

Assume that the polynomial g depends on a parameter t. According to the definition of the Gauss-Manin connection, we have

$$\nabla_{\frac{\partial}{\partial t}} \left(\frac{P\eta_\alpha}{g^k} \right) = \left(-k \frac{\frac{\partial g}{\partial t} P\eta_\alpha}{g^{k+1}} \right) \otimes dt \tag{6.17}$$

We have

$$\left(\sum_{i=1}^{n+1} A_i \frac{\partial g}{\partial x_i} \right) \frac{\eta_\alpha}{g^{k+1}} = \frac{1}{k} \left(\sum_{i=1}^{n+1} \frac{\partial A_i}{\partial x_i} \right) \frac{\eta_\alpha}{q^k} + \text{exact terms.} \tag{6.18}$$

We conclude that

Proposition 6.6.1 *The infinitesimal variation of Hodge structures*

$$H^1(X, \Theta_X) \times H^{m-k}(X, \Omega_X^k) \to H^{m-k+1}(X, \Omega_X^{k-1})$$

for a hypersurface $X : g = 0$ *of degree* d *and dimension* n *is given by the multiplication of polynomials*

$$(\mathbb{C}[X]/J)_d \times (\mathbb{C}[X]/J)_{(k+1)d-\mathsf{m}-2} \to (\mathbb{C}[X]/J)_{(k+2)d-\mathsf{m}-2}, \ \ (F, G) \mapsto FG, \tag{6.19}$$

provided that $(n, d) \neq (2, 4)$. *In this exceptional case, the same statement is true if we replace* $H_1(X, \Theta_X)$ *with the image of the Kodaira-Spencer map.*

6.7 Griffiths-Dwork method

We can use Theorem **??** and find a basis of $H^m_{\rm dR}(\mathbb{P}^{m+1} - X)$. Applying the Gauss-Manin connection to the elements of this basis, we can write the right hand side of (6.17) in terms of the basis. For this we may use (6.18) in order to reduce the pole order. This is mainly known as Griffiths-Dwork method.

6.8 Noether-Lefschetz theorem

Theorem 6.8.1 (Noether-Lefschetz theorem) *For* $d \geq 4$ *a generic surface* $X \subset \mathbb{P}^3$ *of degree has Picard group* $\mathrm{Pic}(X) \cong \mathbb{Z}\mathscr{O}(1)$, *that is, every curve* $C \subset X$ *is a complete intersction of* X *with another surface.*

Let $\mathbb{P}^N$ be the projectivization of the parameter space of surfaces in $\mathbb{P}^3$. Here by a generic surface we mean that there is a countable union V of proper subvarieties of $\mathbb{P}^N$ such that X is in $\mathbb{P}^N - V$.

For the proof of Noether-Lefschez theorem, we need varieties which parameterize all curves in $\mathbb{P}^3$. This is done using Hilbert schemes. Let us consider the set $\mathsf{H}_{g,n}$ of all curves of genus g and degree d in $\mathbb{P}^3$. We know that $\mathsf{H}_{g,n}$ is a projective variety, see for instance [ACG11]. It maynot be irreducible. We call $\mathsf{H}_{g,n}$ the Hilbert scheme of degree n and genus g curves in $\mathbb{P}^3$. Let also T be the space of surfaces of degree d surfaces in $\mathbb{P}^3$. we have the following incidence variety

$$\Sigma_{g,n} := \{(C,X) \in \mathsf{H}_{g,n} \times \mathsf{T} \mid C \subset X\}$$

which is again a projective variety. For particular classes of (g,n), we have an irreducible component $\tilde{\Sigma}_{g,n}$ which parametrizes the pairs (C,X), where C is a complete intersection of X with another surfaces. The closure $\check{\Sigma}_{g,n}$ of $\Sigma_{g,n} - \tilde{\Sigma}_{g,n}$ is still a projective variety and the projection $\check{\Sigma}_{g,n} \to \mathsf{T}$ is a proper map. The image of this map is a finite union of irreducible subvarieties of T. The union of all these for all (g,n) is called the Noether-Lefschetz loci in T.

Proof. We give two proofs. The first one is topological and is due to Lefschetz. The second one uses IVHS for surfaces.

First proof: If the theorem is not true then there is a cycle $\delta \in H_2(X,\mathbb{Z})$ which is invariant under the monodromy. This is not possible except when δ is the homology class of a complete intersection of X with another surface.

Second proof: Let us assume the theorem is not true. This means that there is some g and n and a component Σ of $\check{\Sigma}_{g,n}$ such that the projection $\pi : \Sigma \to \mathsf{T}$ is surjective. For each $t \in \mathsf{T}$ we want to choose a point in $\pi^{-1}(t)$, and hence, a curve $C_t \subset \mathsf{X}_t$ which varies continously with t. We choose a smooth point $t \in \Sigma$ such that the drivative of π at t is surjective. By implicit function theorem, we can choose an analytic subvariety $A \subset \Sigma$ such that π_A is a biholomorphism between A and (Σ,t). This gives us an an analytic family (X_t, C_t) of hypersurfaces X_t and curves $C_t \subset X_t$ for t in an open set of T. For the rest of our argument, we only need the topological class $\delta_t := [C_t] \in H_2(X_t,\mathbb{Z})$ of C_t. We have find a continous family of topological cycles δ_t, t being in an open subset of T, such that, $\int_{\delta_t} \omega = 0$, for all global 2-forms ω in X_t. We get

$$\int_{\delta_t} \nabla_{\frac{\partial}{\partial t}} \omega = 0, \; \forall \; \frac{\partial}{\partial t} \in \Theta_{\mathsf{T}}, \;\; \omega \in H^0(X, \Omega_X).$$

For a fixed $X = X_t$ among this family the map

$$H^1(X,\Theta_X) \times H^0(X,\Omega_X^2) \to H^1(X,\Omega_X^1)_0$$

is surjective. For $d = 4$ we use the image of the Kodaira-Spencer map insetad of the full $H^1(X,\Theta_X)$. Therefore, $H^0(X,\Omega_X^2) + \Theta_{\mathsf{T}} H^0(X,\Omega_X^2)$ and its complex conjugate generate the whole primitive cohomology $H^2_{\mathrm{dR}}(X)_0$. Since we have

$$\int_{\delta_t} H^0(X,\Omega_X^2)+\Theta_{\mathsf{T}}H^0(X,\Omega_X^2)+\overline{H^0(X,\Omega_X^2)+\Theta_{\mathsf{T}}H^0(X,\Omega_X^2)}=0$$

we conclude that δ_t is the Poincaré dual to the polarization θ. This is the same as to say that C_t is homolog to an a curve D_t which is a complete intersection. Since $H^1(X,\mathscr{O}_X)=0$, we conclude that C_t-D_t is a zero divisor of a function on X.

6.9 Algebraic deformations

Let us now consider the following IVHS

$$\bar{\nabla}: H^1(X,\Theta_X)\times H^1(X,\Omega_X^1)\to H^2(X,\Omega_X^0).$$

The polarization θ is originally defined as an element of $H^2(X,\mathbb{Z})$, therefore, if $X\subset \mathbb{P}^N$ depends on a parametr then it is a flat section of the Gauss-Manin connection. Therefore, it is natural to define

$$H^1(X,\Theta_X)_\theta := \left\{\ a\in H^1(X,\Theta_X)\quad \mid\quad \bar{\nabla}(a,\theta)=0,\ \right\} \tag{6.20}$$

In the case of a hypersurface $X\subset\mathbb{P}^{n+1}$, by Lefschetz theorems we know that for $n\geq 3$ we have $H^2(X,\Omega_X^0)=0$. Therefore, in this case we have

$$H^1(X,\Theta_X)_\theta = H^1(X,\Theta_X),$$

For $n=2$, the dimension of $H^2(X,\Omega_X^0)$ is the Hodge number h^{20}, and so, the
*****

A hypersurface $X\subset\mathbb{P}^3$ of degree 4 is called a K3 surface. Using Serre duality we have

$$H^1(X,\Theta_X)\cong H^1(X,\Omega_X^1)$$

Note that Ω_X^1 is dual to Θ_X and Ω_X^2 is the trivial line bundle. We find that the dimension of $H^1(X,\Theta_X)$ is the hodge number h^{11} of X. This is $h^{11}=20$. From another side $\dim(\mathbb{C}[x]/J)_4=19$. We conclude that the complex moduli space of a K3 surface is of dimension 20. Algebraic deformations correspond to a 19 dimensional subspace of this space.

Exercises

1. Prove that the vector space $H^0(\mathbb{P}^n,\Theta_{\mathbb{P}^n})$ of global vector fields in $\mathbb{P}^n$ is generated by $x_i\frac{\partial}{\partial x_j}$, where x_i's are coordinates of $\mathbb{P}^n$. Therefore, it is of dimension n^2.

Chapter 7
Hodge cycles and Gorenstein rings

In this chapter we describe a relation between Gorenstein rings and Hodge cycles. We have used mainly [Dan14] and [MV17] and [Voi03].

7.1 Gorenstein rings

Preliminareis on Gorenstein rings, see this link.

Theorem 7.1.1 (Macaulay's theorem)

7.2 Zariski tangent space of Hodge loci

Let $X_0 \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d and dimension n, and

$$N := (\frac{n}{2}+1)(d-2).$$

Any non-torsion Hodge cycle $\delta_0 \in H_n(X_0,\mathbb{Z})/\mathbb{Z}[Z_\infty]$ defines a ring R whose a-th graded piece for $a \leq N$ is $R_a := \mathbb{C}[x]_a/I_a$, where

$$I_a := \left\{ Q \in \mathbb{C}[x]_a \left| \int_{\delta_0} \frac{QP\Omega}{f^{\frac{n}{2}+1}} = 0, \;\; \forall P \in \mathbb{C}[x]_{N-a} \right. \right\}.$$

By definition $I_m = \mathbb{C}[x]_m$ for all $m \geq N+1$ and so $R_a = 0$. Note that if δ_0 is a rational multiple of Z_∞, then we have $R = \mathbb{C}[x]$ which we discard it. It turns out that R_N is a one dimensional vector space, and so, R is a Gorenstein ring of socle degree N. Let

$$J := \mathrm{jacob}(f).$$

Proposition 7.2.1 *We have*

$$J \subset I$$

and hence we have a canonical surjective map $\mathbb{C}[x]/J \to R$.

Proof. For this we use the formula

$$\frac{d\omega}{f^{i-1}} = (i-1)\frac{df \wedge \omega}{f^{i-1}} + d\left(\frac{\omega}{f^{i-1}}\right)$$

and the fact that δ_0 is Hodge.

we have the following natural isomorphism of one dimensional vector spaces:

$$H : R_N \to \mathbb{C},\ P \mapsto \frac{1}{(2\pi i)^{\frac{n}{2}}} \int_{\delta_0} \frac{P\Omega}{f^{\frac{n}{2}+1}} \tag{7.1}$$

and hence we get:

$$H : R_a \times R_{N-a} \to R_N \cong \mathbb{C}.$$

The following is a consequence of the Hodge conjecture.

Conjecture 7.2.1 *If f is defined over a field $\mathsf{k} \subset \mathbb{C}$ then I, R and H are defined over an algebraic extesion of k.*

The following particular case is proved by P. Deligne.

Theorem 7.2.1 *If f is the Fermat polynomial, and hence defined over $\mathbb{Q}$, then I, R and H are defined over an abelian extension of $\mathbb{Q}$.*

A major problem in our way is that for a generic f there is no no-zero primitive Hodge cycle, and we might be interested to translate this into non-existence of Gorenstein rings of socle degree N for such polynomials. Note that Conjecture 7.2.1 and Theorem 7.2.1 are the only manifestation of the fact that δ_0 has coefficients in $\mathbb{Z}$.

7.3 Hodge locus

Let $\mathsf{T} \subset \mathbb{C}[x]_d$ be the parameter space of smooth hypersurface of degree d and dimension d in $\mathbb{P}^{n+1}$. For $t \in \mathsf{T}$ we have the hypersurface X_t given by $f_t \in \mathbb{C}[x]_d$. We fix $0 \in \mathsf{T}$. Recall the following definition of Hodge locus from [Mov17b]. For a Hodge cycle $\delta \in \mathrm{Hodge}_n(X, \mathbb{Z})_{\mathfrak{0}}$, let $\delta_t \in H_n(X_t, \mathbb{Z})_{\mathfrak{0}}$ be the monodromy of δ to the hypersurface X_t. Let $\mathscr{O}_{\mathsf{T},0}$ be the ring of holomorphic functions in a neighborhood of 0 in T. We have the elements

$$F_{P,i}(t) := \int_{\delta_t} \frac{P\Omega}{f^i} \in \mathscr{O}_{\mathsf{T},0},\ \ P \in \mathbb{C}[x]_{N-(\frac{n}{2}-i+1)d},\ i = 1, 2, \ldots, \frac{n}{2}.$$

Definition 7.3.1 The (analytic) Hodge locus passing through 0 and corresponding to δ is the analytic variety

$$V_\delta := \left\{ t \in (\mathsf{T},0) \;\middle|\; F_{P,i}(t)=0,\ \forall P \in \mathbb{C}[x]_{N-(\frac{n}{2}-i+1)d},\ i=1,2,\ldots,\frac{n}{2} \right\}. \tag{7.2}$$

We consider it as an analytic scheme with

$$\mathscr{O}_{V_\delta} := \mathscr{O}_{\mathsf{T},0} \Big/ \left\langle P \in \mathbb{C}[x]_{N-(\frac{n}{2}-i+1)d},\ i=1,2,\ldots,\frac{n}{2} \right\rangle. \tag{7.3}$$

In the two dimensional case, that is $\dim(X_t)=2$, the Hodge locus is usually called Noether-Lefschetz locus.

Theorem 7.3.1 *Let $V_{\delta_0} \subset (\mathsf{T},0)$ be the Hodge locus passing through 0 and corresponding to δ_0. The Zariski tangent space of V_{δ_0} at 0 is canonically identified with I_d.*

Proof. This follows by our definition of Hodge locias above. Note that

$$\frac{\partial}{\partial t_\alpha}\int_{\delta_t}\frac{P\Omega}{f^i} = \int_{\delta_t}\frac{QP\Omega}{f^{i+1}}, \quad Q := -\frac{\partial f}{\partial t_\alpha}. \tag{7.4}$$

This implies that the linear part of $F_{P,i}$ for $i<\frac{n}{2}$ and only the linear parts of $F_{P,\frac{n}{2}}$ contribute to the tangent space of V_{δ_0} at $0 \in \mathsf{T}$. □

Im [Voi02] terminology, ${}^t\nabla(\delta_0^{\mathsf{pd}}) = I_d$ is the Zariski tangent space of V_{δ_0} at 0.

Proposition 7.3.1 *If $\delta_0 = [Z]$ and Z is given by the ideal $\mathscr{I}_Z$ then*

$$\mathscr{I}_{Z,d} \subset I_d. \tag{7.5}$$

In particular, if the primitive part of the cycles $[Z_1],[Z_2],\ldots,[Z_k]$ form a one dimensional sybspace of $H_n(X,\mathbb{Q})_0$ then

$$\sum_{i=1}^{k}\mathscr{I}_{Z_i,d} \subset I_d. \tag{7.6}$$

Proof. This follows from Carlson-Griffiths Theorem used in [MV17]. □

Definition 7.3.2 *An algebraic cycle Z_1 is called perfect if there are other algebraic cycles $Z_i,\ i=2,\ldots,k$ as in the above definition such that* (7.6) *is an equality.*

Assume that $n \geq 2$ is even and $f \in \mathbb{C}[x]_d$ is of the following format:

$$f = f_1 f_{\frac{n}{2}+2} + f_2 f_{\frac{n}{2}+3} + \cdots + f_{\frac{n}{2}+1} f_{n+2},\ \ f_i \in \mathbb{C}[x]_{d_i},\ \ f_{\frac{n}{2}+1+i} \in \mathbb{C}[x]_{d-d_i}, \tag{7.7}$$

where $1 \leq d_i < d,\ i = 1, 2, \ldots, \frac{n}{2}+1$ is a sequence of natural numbers. Let $X \subset \mathbb{P}^{n+1}$ be the hypersurface given by $f = 0$ and $Z \subset X$ be the algebraic cycle given by

$$Z: \quad f_1 = f_2 = \cdots = f_{\frac{n}{2}+1} = 0.$$

We call Z a complete intersection algebraic cycle in X. In $H_n(X,\mathbb{Z})$ the homology classes of all cycles

$$g_1 = g_2 = \cdots = g_{\frac{n}{2}+1} = 0,\ g_i \in \{f_i, f_{\frac{n}{2}+1+i}\}$$

are equal up to sign and up to $\mathbb{Z}[Z_\infty]$. Let us denote it by δ_0. This with Proposition 7.3.1 imply that Z is perfect and

$$J_d := \langle f_1, f_2, \ldots, f_{n+2} \rangle_d \subset I_d.$$

Now Macaulay's theorem (see [Voi03] Theorem 6.19) implies that the ring $(\mathbb{C}[x]/J)$ is also Groenstein of socle degree N and so $I = J$, and in particular, $I_d = J_d$. Note that J_d is the tangent space of $\mathsf{T}_{\underline{d}}$ at X and this proves the theorem.

In the case of Fermat variety X_n^d and

$$x_{2i-2}^d - x_{2i-1}^d = f_i f_{\frac{n}{2}+1+i}, f_i \in \mathbb{C}[x_{2i-2}, x_{2i-1}]_{d_i}, f_{\frac{n}{2}+1+i} \in \mathbb{C}[x_{2i-2}, x_{2i-1}]_{d-d_i}, i = 1, \ldots, \frac{n}{2}+1$$

the Macaulay's theorem is an easy exercise in commutative algebra.

Chapter 8
Bloch's semi-regularity

In this section we review Bloch's semi-regularity map introduced in [Blo72] and some developments alterwards in [Ran93] and [DK16].

8.1 Normal bundle

Let $Z \subseteq X$ be projective varieties over complex numbers and let

$$\begin{aligned}
&\Theta_X := \text{the sheat vector fields in X}\\
&\Theta_{X,Z} := \text{the subsheaf of } \Theta_X \text{ containing vectors tangent to Z}\\
&N_{Z\subseteq X} := \text{ the normal bundle of X}
\end{aligned}$$

We have the short exact sequence

$$0 \to \Theta_{X,Z} \to \Theta_X \to N_{Z\subseteq X} \to 0 \tag{8.1}$$

This might be taken as the definition of the normal bundle.

Proposition 8.1.1 *We have canonical isomorphism*

$$N_{Z\subseteq X} \cong \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{I}_{Z\subset X} \to \mathscr{O}_X).$$

$$v \mapsto (f \mapsto df(v)).$$

Proof. □

The long exact sequence of 8.1 gives us the map α and γ in

$$H^0(X, N_{Z\subseteq X}) \to H^1(X, \Theta_{X,Z}) \xrightarrow{\gamma} H^1(X, \Theta_X) \xrightarrow{\alpha} H^1(X, N_{Z\subseteq X}) \to \dots \tag{8.2}$$

Let us consider the following IVHS

$$\bar{\nabla} : H^{\frac{n}{2}}(X, \Omega_X^{\frac{n}{2}}) \to \mathrm{Hom}\left(H^1(X, \Theta_X), H^{\frac{n}{2}+1}(X, \Omega_X^{\frac{n}{2}-1})\right)$$

and $[Z]^{\mathsf{pd}} \in H^{\frac{n}{2}}\left(X, \Omega^{\frac{n}{2}}\right)$ be the cohomology class of Z.

Theorem 8.1.1 *The Bloch semi-regularity map is such that the following commutes*

$$\begin{array}{ccc} & H^1(X, N_{Z\subseteq X}) & \\ \alpha \nearrow & & \searrow \beta \\ H^1(X, \Theta_X) & \xrightarrow{\bar{\nabla}([Z]^{\mathsf{pd}})} & H^{\frac{n}{2}+1}\left(X, \Omega^{\frac{n}{2}-1}\right) \end{array} \tag{8.3}$$

For this [Dan2017] Theorem 33 and [BuchweitzFlenner2000] Theorem 4.5,5.5. From this we get:

Definition 8.1.1 We say that the algebraic cycle $Z \subset X$ is semi-regular if the semi-regularity map β is injective. It is called weakly semi-regular if one of the following equivalent conditions hold.

$$\mathrm{Im}(\alpha) \cap \ker(\beta) = \{0\} \quad \Leftrightarrow \quad \ker\left(\bar{\nabla}[Z]^{\mathsf{pd}}\right) = \ker(\alpha)$$

The fact that these are equivalent conditions follows from Theorem 8.1.1

Conjecture 8.1.1 *The pair (X,Z) is weakly semi-regular if and only if it satisfies the alternative Hodge conjecture.*

Let $\mathrm{Hilb}(X)$ and $\mathrm{Hilb}(X,Z)$ be the Hilbert scheme parametrizing deformations of X and the pair (X,Z), respectively. We have the canonical map

$$\kappa : \mathrm{Hilb}(X,Z) \to \mathrm{Hilb}(X)$$

and we denote by V_Z its image. We denote by $0 \in \mathrm{Hilb}(X)$ and $0 \in \mathrm{Hilb}(X,Z)$ the points corresponding to X and (X,Z), respectively. We would like to get some information about the tangent space of V_Z at 0. Since V_Z is given as the image of another variety, we will be able to get some information about the $\mathrm{Im}(D_0\kappa)$ which might be stricktly smaller than $\mathbf{T}_0 V_Z$. Let us now consider the diagram

$$\begin{array}{ccccccc} H^0(X, N_{Z\subseteq X}) & \to & H^1(X, \Theta_{X,Z}) & \overset{\gamma}{\to} & H^1(X, \Theta_X) & \xrightarrow{\alpha} & H^1(X, N_{Z\subseteq X}) \\ & & \uparrow KS_{X,Z} & & \uparrow KS_X & \check{\alpha} \nearrow & \\ & & \mathbf{T}_0\mathrm{Hilb}(X,Z) & \overset{D_0\kappa}{\to} & \mathbf{T}_0\mathrm{Hilb}(X) & & \end{array}$$

Let G be the linear reductive group acting on both $\mathrm{Hilb}(X)$ and $\mathrm{Hilb}(X)$. We identify elements of $\mathrm{Lie}(\mathsf{G})$ with global vector fields in $\mathrm{Hilb}(X)$ and $\mathrm{Hilb}(X,Z)$. In this way the the differnetial $D\kappa$ of κ is the identity map on $\mathrm{Lie}(\mathsf{G})$. In the following proposition we need that the kernel of KS_X is given by $\mathrm{Lie}(\mathsf{G})$. For instance, this is the case for hypersurfaces.

Proposition 8.1.2 *If the Kodaira-Spencer map $KS_{X,Z}$ is surjective and the kernel of KS_X is given by* $\mathrm{Lie}(\mathsf{G})$ *then*

$$\mathrm{Im}(D_0\kappa) = \ker(\check{\alpha}).$$

Proof. The inclusion $\mathrm{Im}(D_0\kappa) \subset \ker(\check{\alpha})$ is trivial and does not need any hypothesis. We prove $\ker(\check{\alpha}) \subset \mathrm{Im}(D_0\kappa)$. Let $a_1 \in \ker(\check{\alpha})$ and using the first hypothesis we find

$$\begin{array}{ccccc} \to a_3 & \xrightarrow{\gamma} & a_2 & \xrightarrow{\alpha} & 0 \\ \uparrow & & \uparrow & \nearrow & \\ a_4 & \xrightarrow{D_0\kappa} & a_1, a_5 & & \end{array}$$

In the final step, we find $a_5 = D_0\kappa(a_4)$. The element $a_1 - a_5$ is in the kernel of KS_X and by the second hypothesis $a_1 - a_5 \in \mathrm{Lie}(\mathsf{G})$. The map $D\kappa$ is identity restricted to $\mathrm{Lie}(\mathsf{G}) \subset \mathbf{T}\mathrm{Hilb}(X,Z)$.
□

8.2 Complete intersection algebraic cycles

Consider the short exact sequence

$$0 \to N_{Z\subseteq X} \to N_{Z\subseteq \mathbb{P}^{n+1}} \to N_{X\subseteq \mathbb{P}^{n+1}}|_Z \to 0$$

all sheaves over Z, and the corresponding long exact sequence

$$H^0(Z, N_{Z\subseteq\mathbb{P}^{n+1}}) \xrightarrow{i} H^0(Z, N_{X\subseteq\mathbb{P}^{n+1}}) \to H^1(Z, N_{Z\subseteq X}) \to H^1(Z, N_{Z\subseteq\mathbb{P}^{n+1}})$$

Definition 8.2.1 Let f be a homogeneous polynomial of the form

$$f := \sum_{i=1}^{\frac{n}{2}+1} f_i \, f_{\frac{n}{2}+1+i} = 0.$$

We call $Z: \ f_1 = f_2 = \ldots = f_{\frac{n}{2}+1}$ a complete intersection algebraic cycle.

Proposition 8.2.1 *Let $Z \subseteq X$ be a complete intersection algebraic cycle. We have*

1. $N_{X\subseteq\mathbb{P}^{n+1}} \simeq \mathscr{O}_X(d)$
2. $N_{Z\subseteq\mathbb{P}^{n+1}} \simeq \bigoplus_{i=1}^{\frac{n}{2}+1} \mathscr{O}_Z(d_i)$

The map $N_{Z\subseteq\mathbb{P}^{n+1}} \to N_{X\subseteq\mathbb{P}^{n+1}}|_Z$ is given by:

$$\bigoplus_{i=1}^{\frac{n}{2}+1} \mathscr{O}_Z(d_i) \to \mathscr{O}_Z(d)$$

$$(h_1, h_2, \dots h_{\frac{n}{2}+1}) \to \sum_{i=1}^{\frac{n}{2}+1} h_i \, f_{\frac{n}{2}+1+i}$$

Here, we have used the restriction of $f_s \in H^0(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(s)) \cong \mathbb{C}[x]_s$ for $s = \frac{n}{2} + 2, \cdots, n+2$ to Z. By Proposition 8.2.1 we have

$$H^1(Z, N_{Z \subseteq \mathbb{P}^{n+1}}) = \oplus_{i=1}^{\frac{n}{2}+1} H^1(Z, \mathscr{O}_Z(d_i)), \tag{8.4}$$

Therefore, in order to describe $H^1(Z, N_{Z \subseteq X})$ explicity, we need that the above cohomologies are zer and then we need to describe the map i explicity:

Proposition 8.2.2 *If*

$$H^1(Z, \mathscr{O}_Z(a)) = 0, \tag{8.5}$$
$$H^1(\mathbb{P}^{n+1}, I_Z(a)) = 0, \quad a = d_1, d_2, \dots d_{\frac{n}{2}+1}, \tag{8.6}$$

then

$$H^1(Z, N_{Z \subseteq X}) \simeq \left(\frac{\mathbb{C}[x]}{\langle f_1, f_2, \dots f_{\frac{n}{2}+1}, f_{\frac{n}{2}+2}, \dots, f_{n+2} \rangle} \right)_d \tag{8.7}$$

Proof. From (8.4) and (8.5) we have

$$H^1(Z, N_{Z \subseteq X}) \simeq \frac{H^0(Z, \mathscr{O}_Z(d))}{Im \left(\bigoplus_{i=0}^{\frac{n}{2}+1} H^0(Z, \mathscr{O}_Z(d_i)) \xrightarrow{i} H^0(Z, \mathscr{O}_Z(d)) \right)}$$

Let us take

$$0 \to I_Z \to \mathscr{O}_{\mathbb{P}^{n+1}} \to \mathscr{O}_Z \to 0$$

and tensor it with $\mathscr{O}_{\mathbb{P}^{n+1}}(k)$

$$0 \to I_Z(k) \to \mathscr{O}_{\mathbb{P}^{n+1}}(k) \to \mathscr{O}_Z(k) \to 0 \tag{8.8}$$

and consider the corresponding long exact sequence

$$\dots H^0(\mathbb{P}^{n+1}, I_Z(k)) \to H^0(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(k)) \xrightarrow{\alpha} H^0(Z, \mathscr{O}_Z(k)) \to H^1(\mathbb{P}^{n+1}, I_Z(k)) \to \cdots$$

Therefore, if $H^1(\mathbb{P}^{n+1}, I_Z(k)) = 0$ then the restriction map α is surjective and

$$H^0(Z, \mathscr{O}_Z(k)) \cong \frac{\mathbb{C}[x]_k}{(I_Z)_k}$$

□

Proposition 8.2.3 *If for a complete intersection Z we have have* (8.5) *and* (8.6) *then Z is semi-regular.*

Proof. Let $I := \langle f_1, f_2, \dots f_{\frac{n}{2}+1}, f_{\frac{n}{2}+2}, \dots, f_{n+2}\rangle$. The diagram (8.3) becomes

$$\begin{array}{ccc} & \mathbb{C}[x]_d/I_d & \\ \alpha \nearrow & & \searrow \beta \\ \mathbb{C}[x]_d & \xrightarrow{\bar{\nabla}([Z]^{\mathsf{pd}})} & \mathbb{C}[x]_{(\frac{n}{2}+2)d-n-2} \end{array}$$

Note that $\ker(\alpha) = I_d$ which is the Zariski tangent space of the Hodge locus.

Proposition 8.2.4 *A linear cycle $\mathbb{P}^{\frac{n}{2}}$ inside a smooth hypersurface of dimension n is semi-regular.*

Proof. We need to check (8.5) and (8.6).

8.3 Castelnuevo-Mumford regularity

Definition 8.3.1 *The Castelnuevo-Mumford regularity of a scheme $Z \subset \mathbb{P}^{n+1}$ is the smallest r such that*

$$H^i\left(\mathbb{P}^{n+1}, I_Z(r-i)\right) = 0,\ \forall i \geq 1 \tag{8.9}$$

Theorem 8.3.1 *The Castelnuevo-Mumford regularity of a complete intersection of type $d_1, d_2, \dots, d_s$ is less than or equal $d_1 \cdot d_2 \cdots d_s$*

Proof. We first prove this for a hypersurface Z of degree d_1 for which we have

$$I_Z = \mathscr{O}_{\mathbb{P}^{n+1}}(-Z) \cong \mathscr{O}_{\mathbb{P}^{n+1}}(-d_1).$$

and so $I_Z(r-i) \cong \mathscr{O}_{\mathbb{P}^{n+1}}(r-i-d_1)$. Using [Har77] Theorem 5.1, we know that for all $r \geq d_1$ we have (8.9). Now assume that for a complete intersection Z_{s-1} of type $d_1, d_2, \dots, d_{s-1}$ and all $r \geq d_1 \cdot d_2 \cdots d_{s-1}$ we have (8.9) and consider Z_s.

The following short exact sequence might be useful

$$O \to I_Z \cdot \Theta_X \to \Theta_{X,Z} \to \Theta_Z \to 0$$

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Chapter 9
Garbage

9.1 Integral Hodge Conjecture (work with Roberto and Enzo)

In order to produce `A` run the following commands.

```
LIB "foliation.lib";
int d=8; int n=2;
intvec mlist=d;  for (int i=1;i<=n; i=i+1){mlist=mlist,d;}
ring r=(0,z), (x(1..n+1)),dp;
poly cp=cyclotomic(d);   int degext=deg(cp) div deg(var(1));
cp=subst(cp, x(1),z);
minpoly =number(cp); //--z is the d-th root of unity---
list ll=MixedHodgeFermat(mlist); list J=ll[1][1]+ll[2][1];
for (i=2; i<=(n div 2); i=i+1){J=J+ll[1][i]+ll[2][i];}
list Jexp;  for (i=1; i<=size(J); i=i+1)
          {Jexp=insert(Jexp, leadexp(J[i]),  size(Jexp));}
matrix A=DimHodgeCycles(mlist, Jexp);
(d-1)^(n+1)-rank(A);
```

Lattice of Hodge cycles supported in linear cycles

For our article it is better to compute the following `A`. This lattice is explained [Mov17b], 16.17.

```
LIB "foliation.lib";
int d=8; int n=2;
intvec mlist=d;  for (int i=1;i<=n; i=i+1){mlist=mlist,d;}
ring r=(0,z), (x(1..n+1)),dp;
poly cp=cyclotomic(d);   int degext=deg(cp) div deg(var(1));
cp=subst(cp, x(1),z);
minpoly =number(cp); //--z is the d-th root of unity---
list ll=MixedHodgeFermat(mlist); list J=ll[1][1]+ll[2][1];
```

```
for (int i=2; i<=(n div 2); i=i+1){J=J+ll[1][i]+ll[2][i];}
//adding differnetial forms with zero period over linear cycles.
int nh=n div 2;  list Pn2=LinearCoho(mlist,0);
J=J+RemoveList(ll[1][nh+1], Pn2[1]);

list Jexp;  for (i=1; i<=size(J); i=i+1)
          {Jexp=insert(Jexp, leadexp(J[i]),  size(Jexp));}
matrix A=DimHodgeCycles(mlist, Jexp);
(d-1)^(n+1)-rank(A);
```

In order to produce the intersection matrix we first produce the intersection matrix of vanishing cycles. This is a $\mu \times \mu$ matrix Ψ produced by:

```
poly f; for (i=1; i<=n+1; i=i+1){f=f+var(i)^mlist[i];  }
ideal I=std(jacob(f)); I=kbase(I);
list Il=I[1..size(I)];
matrix Psi=IntersectionMatrix(Il);
```

(run this right after the first code)

Lattices

For a lattice V, $V_p := V/pV$ is a $\mathbb{F}_p$-vector space with the induced $\mathbb{F}_p$-bilinear map $V_p \times V_p \to \mathbb{F}_p$. In a similar we define $\check{V}_p$. It is easy to see that

1. For a lattice V, we have $\det(\check{V}) = \pm 1$ if and only if $\mathrm{rank}\check{V} = \mathrm{rank}(\check{V}_p)$ for all prime p.
2. For a basis v_i of V (resp. V_p), $\mathrm{rank}(\check{V})$ (resp. $\mathrm{rank}(\check{V}_p)$) is the rank of the the matrix $[v_i \cdot v_j]$.

The following algorithm computes $\det(\check{V})$ starting from $A := [v_i \cdot v_j]$.

1. Compute $\mathrm{rank}(A)$ over $\mathbb{Q}$ and call it ρ.
2. Find a $\rho \times \rho$-block B of A such that $\det(B) \neq 0$ and find all the primes $\det(B) = p_1^{n_1} \cdots p_s^{n_s}$.
3. Among p's above collect all p such that $\mathrm{rank}(A)$ over $\mathbb{F}_p$ is strictly less that $\mathrm{rank}(A)$ over $\mathbb{Q}$. Call these the bad primes.
4. Let p be a bad prime and so $\rho_p := \mathrm{rank}(A/\mathbb{F}_p)$ is strictly less than ρ. Find ρ_p rows of $A/\mathbb{F}_p$ linearly independent over $\mathbb{F}_p$. Let $v_1, v_2, \ldots, v_{\rho_p} \in V$ be the corresponding elements. Write any other line (corresponding to v) as a $\mathbb{F}_p$-linear combination of these rows. This gives us an element $v - \sum_{i=1}^{\rho_p} a_i v_i \in V$, with a_i's integers between 0 and $p-1$, such that

$$p \Big| (v - \sum_{i=1}^{\rho_p} a_i v_i) \cdot w, \quad \text{for all } w \in V.$$

Substitute v with $v-\sum_{i=1}^{\rho_p} a_i v_i$ and wirte the bilinear form in this new basis, let us call it $v_1, v_2, \ldots, v_{\rho_p}, w_1, w_2, \ldots, w_{k_p}$. The bilinear form in this new basis has the matrix such that it has $\rho_p \times \rho_p$ block of non-zero determinant and elsewhere its entries are divisable by p.

5. Define a new lattice W replacing w_i's with $\frac{w_i}{p}$. We have

$$\det(\check{W}) = \det(\check{V})/p^{\rho-\rho_p}$$

6. Repeat the algorithm for W.

polarization

Let $\theta \in V$ be an element with $d := \theta \cdot \theta$ and call it the ploarization. We consider the lattice $\theta^\perp$ of elements orthogonal to θ and call it the primitive part of V (with respect to the polarization). We will need the following proposition in the discussion of comparing the lattice of Hodge cycles and primitive Hodge cycles.

Proposition 9.1.1 *If there is an element $w \in V$ such that $w \cdot \theta = 1$ then*

$$\det(\theta^\perp) = d \cdot \det(V).$$

Proof. The property $w \cdot \theta = 1$ implies that $V = \mathbb{Z}w \oplus \theta^\perp$: every $v \in V$ can be written uniquely as $v := (v \cdot \theta)w + \tilde{v}$ for some unique $\tilde{u} \in V$. In particular, $\theta = dw + \tilde{\theta}$. We take an arbitrary basis $v_1, v_2, \ldots, v_\mu$ of $\theta^\perp$ and write $\theta, v_1, v_2, \ldots, v_\mu$ in terms of $w, v_1, v_2, \ldots, v_\mu$ and conclude that

$$d \cdot \det(\theta^\perp) = \det(\mathbb{Z}\theta + \theta^\perp) = d^2 \det(V)$$

which implies the desired statement.

Proposition 9.1.2 *Let V be a lattice and $W \subset V$ be its sublattice of the same rank μ. We have*

$$\frac{V}{W} \cong \mathbb{Z}/a_1\mathbb{Z} \times \mathbb{Z}/a_1\mathbb{Z} \times \cdots \times \mathbb{Z}/a_\mu\mathbb{Z}$$

where $a_1, a_2, \cdots, a_\mu$ are the elementary divisors of the Smith normal form of the base change matrix $\mathtt{A}$, that is, if the entries of $[v_i]_{\mu\times 1}$ and $[w_i]_{\mu\times 1}$ form a basis of V and W respectively, then $[w_i] = \mathtt{A}[v_i]$. In particular,

$$\det(W) = (a_1 \cdot a_2 \cdots a_\mu)^2 \det(V)$$

Proof. We just make the change of basis $\mathtt{U}[w_i]$ and $\mathtt{T}^{-1}[v_i]$ and get the result.

Writing linear cycles in terms of vanishing cycles

In this section we explain how to write linear cycles in terms of vanishing cycles. The main ingredients are the integration of ω_β over both cycles. Over linear cycles it is done in [MV17]. Over vanishing cycles it is done in [Mov17b], Chapter 16, formula (16.25) and we reproduce it here:

$$(2\pi i)^{-\frac{n}{2}}\int_{\delta_{\beta'}}\mathrm{Resi}\left(\frac{x^\beta dx}{(g-1)^{\frac{n}{2}+1}}\right)=\frac{(-1)^n}{\frac{n}{2}!d^{n+1}}\frac{\prod_{i=1}^{n+1}\left(\zeta_d^{(\beta_i'+1)(\beta_i+1)}-\zeta_d^{\beta_i'(\beta_i+1)}\right)}{\prod_{j\in A}\left(\zeta_{2d}^{\beta_j+1}+\zeta_{2d}^{\beta_{\sigma(j)}+1}\right)} \tag{9.1}$$

for $\beta\in I^{\frac{n}{2},\frac{n}{2}}_{\mathbb{P}^{\frac{n}{2}}}$, where

$$I^{\frac{n}{2},\frac{n}{2}}_{\mathbb{P}^{\frac{n}{2}}}:=\left\{\beta\in I\left|\frac{\beta_i+1}{d}+\frac{\beta_{\sigma(i)}+1}{d}=1,\ \ i=0,1,2,\dots,n+1,\ \text{ for some } \sigma\right.\right\}, \tag{9.2}$$

and σ is a permutation of $0,1,\dots,n+1$ without fixed point and with σ^2 being identity. Here, $\frac{\beta_0+1}{m_0}:=\frac{n}{2}+1-A_\beta$ arises from the projectivization of the affine Fermat variety.

Three lines forming the capital letter E.

9.2 Harris-Voisin conjecture

Let us consider two Hodge cycles $\delta_k\in \mathrm{Hodge}_n(X_n^d,\mathbb{Z})_{\check{0}},\ \ k=1,2$ and the corresponding matrices $[\mathsf{p}_{i+j}(\delta_k)],\ \ k=1,2$. Let $V_{\delta_1},V_{\delta_2},V_{\delta_1+\delta_2}\subset(\mathsf{T},0)$ be the Hodge loci passing through 0 and corresponding to the Hodge cycles δ_1,δ_2 and $\delta_1+\delta_2$, respectively. We are interested in cases for which $V_{\delta_1+\delta_2}$ is not contained in none of V_{δ_1} and V_{δ_2}. This phenomena produces more components of the Hodge locus using the known ones. In general, we have

$$\ker([\mathsf{p}_{i+j}(\delta_1)])\cap\ker[\mathsf{p}_{i+j}(\delta_2)]\subset\ker[\mathsf{p}_{i+j}(\delta_1+\delta_2)] \tag{9.3}$$

and since V_{δ_k}'s might have complicated singularities we can only get partial information, looking at the Zariski tangent spaces of $V_{\delta_k},\ \ k=1,2$.

Proposition 9.2.1 *Following the notation as above, assume that*

1. V_{δ_k}, $k=1,2$ *are smooth varieties at* 0*. This is satisfied when both* $[\mathsf{p}_{i+j}(\delta_k)]),\ \ k=1,2$ *have maximal rank.*
2. *They intersect each other transversely at* 0*, that is, the codimension of the left hand side of* (9.3) *is equal to the sum of codimensions of* $\ker([\mathsf{p}_{i+j}(\delta_1)])]$ *and* $\ker([\mathsf{p}_{i+j}(\delta_2)])$.
3. $\ker[\mathsf{p}_{i+j}(\delta_1+\delta_2)]$ *is contained in none of* $\ker([\mathsf{p}_{i+j}(\delta_k)])],\ \ k=1,2.$

Then $V_{\delta_1+\delta_2}$ is not contained in none of V_{δ_1} and V_{δ_2}.

Proof. The first and second hypothesis imply that $V_{\delta_1} \cap V_{\delta_2}$ is smooth at 0 and its tangent space is $\ker([\mathsf{p}_{i+j}(\delta_1)]) \cap \ker[\mathsf{p}_{i+j}(\delta_2)]$. This together with the third hypothesis implies that $V_{\delta_1+\delta_2}$ is not contained in none of V_{δ_1} and V_{δ_2}. □

Some new directions for this article.

1. Our arguments work also for degree d cycles $\mathbb{P}^{\frac{n}{2}} \to X$. We have to develope this case too.
2. We might determine explicit linear combinations of $\mathbb{P}^{\frac{n}{2}}$ which are general, see my book.

One can remove the hypothesis that X_t is in a Zariski open neighborphood of 0 by computing periods. This is being done by [Vil18]. Definition of Hodge Locus

We want to find the first d such that the number of special algebraic cycles $\mathbb{Z}$'s is infinite. After proving that the corrsponding NL components are distinct, we get an explicit counterexample for Harris' conjecture.

Proposition 9.2.2 *If for two algebraic cycles Z_1 and Z_2 in the fermat variety, there is no inclusion between $\ker[\mathsf{p}_{i+j}(Z_1))]$ and $\ker[\mathsf{p}_{i+j}(Z_2))]$, then there is no inclusion beween NL_{Z_1} and NL_{Z_2} as analytic schemes. Morover, if*

$$\operatorname{codim}(\ker[\mathsf{p}_{i+j}(Z_1))] \cap \ker[\mathsf{p}_{i+j}(Z_2))]]) > \binom{d-1}{3} \tag{9.4}$$

then there is no inclusion beween NL_{Z_1} and NL_{Z_2} as analytic varieties.

At best we may hope that $\ker[\mathsf{p}_{i+j}(Z_1))] \cap \ker[\mathsf{p}_{i+j}(Z_2))] = \{0\}$. A necessary condition so that this happens is:

$$2\binom{d-1}{3} > \binom{d+3}{3} - 16$$

The first d such that this happens is $d = 18$. See the code below. Note that even in $d = 5, 6$ we may have (9.4).

```
LIB "general.lib";
for (int d=4; d<=20; d=d+1){d, binomial(d-1,3), binomial(d+3,3)-16, 2*binomial(d-1,3)-binomial(d+3,3)+16;}
4 1 19 -17
5 4 40 -32
6 10 68 -48
7 20 104 -64
8 35 149 -79
9 56 204 -92
10 84 270 -102
11 120 348 -108
12 165 439 -109
13 220 544 -104
14 286 664 -92
15 364 800 -72
16 455 953 -43
17 560 1124 -4
18 680 1314 46
19 816 1524 108
20 969 1755 183
```

9.3 Quintic Fermat surfaces

For the quintic Fermat surface X_2^5, the matrix $[\mathsf{p}_{i+j}]$ is 4×40. Special Hodge cycles δ has the ν invaraint with

$$\nu(\delta)=2,3$$

Proposition 9.3.1 *There are finite number of special Hodge cycles* $\delta\in \mathrm{Hodge}_2(X_2^5,\mathbb{Z})$.

9.4 Smoothnes and reducedness of components of the Hodge loci

Let us denote by $(x_1,x_2,\cdots,x_n)$ be a coordinate system for $(\mathbb{C}^n,0)$ and let $\mathscr{O}_{\mathbb{C}^n,0}$ be the ring of holomorphic functions in $(\mathbb{C}^n,0)$.

Proposition 9.4.1 *Let us be given* $f,f_1,f_2,\ldots,f_k\in\mathscr{O}_{\mathbb{C}^n,0}$*, all vanishing at* 0*, and assume that the linear part of* $f_1,f_2,\ldots,f_k$ *together with* $x_{k+1},\ldots,x_n$ *are linearly independent over* $\mathbb{C}$*. Then* f *belongs to the ideal generated by* $f_1,f_2,\ldots,f_k$ *if and only if*

$$\left[\frac{\partial f}{\partial x_{k+1}},\frac{\partial f}{\partial x_{k+2}},\cdots,\frac{\partial f}{\partial x_n}\right]=\left[\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\cdots,\frac{\partial f}{\partial x_k}\right]\begin{bmatrix}\frac{\partial f_1}{\partial x_1}&\frac{\partial f_1}{\partial x_2}&\cdots&\frac{\partial f_1}{\partial x_k}\\ \frac{\partial f_2}{\partial x_1}&\frac{\partial f_2}{\partial x_2}&\cdots&\frac{\partial f_2}{\partial x_k}\\ \vdots&\vdots&\ddots&\vdots\\ \frac{\partial f_k}{\partial x_1}&\frac{\partial f_k}{\partial x_2}&\cdots&\frac{\partial f_k}{\partial x_k}\end{bmatrix}^{-1}\begin{bmatrix}\frac{\partial f_1}{\partial x_{k+1}}&\frac{\partial f_1}{\partial x_{k+2}}&\cdots&\frac{\partial f_1}{\partial x_n}\\ \frac{\partial f_2}{\partial x_{k+1}}&\frac{\partial f_{k+2}}{\partial x_2}&\cdots&\frac{\partial f_2}{\partial x_n}\\ \vdots&\vdots&\ddots&\vdots\\ \frac{\partial f_k}{\partial x_{k+1}}&\frac{\partial f_k}{\partial x_{k+2}}&\cdots&\frac{\partial f_k}{\partial x_n}\end{bmatrix}\tag{9.5}$$

Proof. Let $B_2=B_1A_1^{-1}A_2$ be the above equaliy. We take $f_{k+1},\cdots,f_n\in\mathscr{O}_{\mathbb{C}^n,0}$ which vanish at 0 and such that the derivative of $f=(f_1,f_2,\ldots,f_n)$ at 0 has non-zero determinant, and hence, we can regard f as a new coordinate system in $(\mathbb{C}^n,0)$. Now, f belongs to the ideal generated by $f_1,f_2,\ldots,f_k$ if and only if its derivative with respect to $f_{k+1},\cdots,f_n$ are zero. Using the chain rule and inverse function theorem:

$$\left[\frac{\partial f}{\partial f_1}, \frac{\partial f}{\partial f_2}, \cdots, \frac{\partial f}{\partial f_n}\right] = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n}\right] \begin{bmatrix} \frac{\partial x_1}{\partial f_1} & \frac{\partial x_1}{\partial f_2} & \cdots & \frac{\partial x_1}{\partial f_n} \\ \frac{\partial x_2}{\partial f_1} & \frac{\partial x_2}{\partial f_2} & \cdots & \frac{\partial x_2}{\partial f_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial f_1} & \frac{\partial x_n}{\partial f_2} & \cdots & \frac{\partial x_n}{\partial f_n} \end{bmatrix} \tag{9.6}$$

$$= \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n}\right] \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}^{-1} \tag{9.7}$$

$$= [B_1, B_2] \begin{bmatrix} A_1 & A_2 \\ 0 & I \end{bmatrix}^{-1} \tag{9.8}$$

$$= [B_1, B_2] \begin{bmatrix} A_1^{-1} & -A_1^{-1}A_2 \\ 0 & I \end{bmatrix} \tag{9.9}$$

9.5 Intersection of vanishing and algebraic cycles

We work in the affine chart $x_0 = 1$. The sets Z_1 and δ_β intersect each other at the point $p := (1, 0, \cdots, 0, 0) \in \mathbb{C}^{n+1}$. This point is not a smooth point of δ_β and this make our analysis of its intersection number with Z_1 harder.

Proposition 9.5.1 *For d an even number, the intersection number of Z_1 with Γ_β at p is given by $(-1)^a$, where a is the number of $2i$'s such that*

$$\mathrm{Im}(\zeta_{2d}^{1+2\beta_{2i+1}-2\beta_{2i}}) < 0.$$

Proof. Using the action of G on U we can assume that $\beta = 0$ and Z_1 is given by

$$Z_1 : x_0 - x_1 = x_2 - a_1 x_3 = x_4 - a_2 x_5 = \cdots = x_n - a_{\frac{n}{2}} x_{n+1} = 0.$$

where $a_i = \zeta_{2d}^{1+2\beta_{2i+1}-2\beta_{2i}}$. The tangent space of the Fermat variety X at the point p, is the linear subspace of $\mathbb{C}^{n+1}$ given by $x_1 = 0$. Therefore, the orientation of X at p is given by

$$\alpha := d\mathrm{Re}(x_2) \wedge d\mathrm{Im}(x_2) \wedge \cdots \wedge d\mathrm{Re}(x_{n+1}) \wedge d\mathrm{Im}(x_{n+1}).$$

Here, by abuse of classical notations, we consider $d(\mathrm{Re}(x_2))$ and $d(\mathrm{Im}(x_2))$ the vector in the real tangent space of X at p given by $(0, 1, 0 \cdots, 0)$ and $(0, i, 0, 1, \cdots, 0)$, respectively. The orientation of Γ_β, $\beta = 0$ at p is given by

$$\beta := d\mathrm{Re}(x_2) \wedge \cdots \wedge d\mathrm{Re}(x_{n+1}).$$

The tangent space of Z_1 at p has a basis of the form $(0,a_1,1,0,\cdots,0)$, $(0,ia_1,i,0,\cdots,0),\cdots$. Let us make wedge γ of all these vectors and compute $\beta\wedge\gamma$:

$$\beta\wedge\gamma := \beta\wedge \mathrm{Im}(a_1)d\mathrm{Im}(x_2)\wedge d\mathrm{Im}(x_3)\wedge\cdots\wedge \mathrm{Im}(a_{\frac{n}{2}})d\mathrm{Im}(x_n)\wedge d\mathrm{Im}(x_{n+1}).$$

Comparing α with $\beta\wedge\gamma$, the proof of the proposition follows.

Remark 9.5.1 Proposition 9.5.4 and its proof fails for d odd, because in this case Γ_β is not transversal to Z_1 if for some i

$$2(\beta_{2i+1}-\beta_{2i})+1\equiv_{2d} d.$$

Proposition 9.5.2 *For d an even number we have*

$$\langle Z_1,\delta_\beta\rangle=\begin{cases}(-1)^{\sum_{i=1}^{n+1}\beta_i} & \text{if } \beta_1=0\\ 0 & \text{otherwise.}\end{cases}$$

After many days of strygling I failed to compute this intersection number. Let me just report what I did. We know that only δ_β with $\beta_1=0$ passes through p. A face $\Gamma_{\beta+a}$ of δ_β with $a_1=0$ passes also through p. Two such faces have common edge if an only if the corresponding a's differs only in one entry. For simplicity, one may take the case $n=2$, where we are dealing with vanishing cycles which are two dimensional spheres. In this case we have four faces of δ_β passing through p. They have different intersection numbers with Z_1 and I do not know how the total intersection number depends on these four numbers.

I have computed the dimension of the Hodge loci passing through the algebraic cycle in Proposition **??**. For the singular code see here

Intersection of Aoki-Shioda cycle with vanishing cycles

For the intersection of the Aoki cycle with δ_β, we did not find a closed formula and so we had to compute it by hand.

Definition 9.5.1 A triple $(\beta_1,\beta_2,\beta_3)\in\{0,1,2,3,4,5\}^3$ is called admissible if it is of the format $(a_1+3b_1,a_2+3b_2,a_3+3b_3)$, where a and b varies in

$$\{a_1,a_2,a_3\}=\{0,1,2\},\ b_i\in\{0,1\},\ b_1+b_2+b_3=1 \text{ or } 3 \tag{9.10}$$

In othwer words, it belongs to the following set of 24 elements:

****

Proposition 9.5.3 *We have*

$$\langle Z_2,\delta_\beta\rangle=\sum_{a\in\{0,1\}^3}(-1)^{\sum_{i=1}^{3}(1-a_i)}\varepsilon(\beta_3+a_1,\beta_4+a_2,\beta_5+a_3).$$

where

$$\varepsilon(\beta_1,\beta_2,\beta_3)=\begin{cases}+1 & \text{if } \beta \text{ is admissible and } \mathrm{Im}((\zeta^{4\beta_2}-\zeta^{4\beta_3})(\zeta^{2\beta_3}-\zeta^{2\beta_1}))>0\\ -1 & \text{if } \beta \text{ is admissible and } \mathrm{Im}((\zeta^{4\beta_2}-\zeta^{4\beta_3})(\zeta^{2\beta_3}-\zeta^{2\beta_1}))<0\\ 0 & \text{otherwise.}\end{cases}$$

$\varepsilon(B)=1$ *if B is admissible and* $=0$ *otherwise.*

Proof. Let us define

$$g := -x_1^6-x_2^6$$
$$f := -1+x_3^6+x_4^6+x_5^6$$

A vanishing cycle δ_β is a join of a vanishing cycles δ_1 of $f=t$ with a vanishing cycle δ_2 of $g=t$, see [Mov11] §6.5. We have

$$U=\cup_{t\in\mathbb{C}} f^{-1}(t)\times g^{-1}(t)$$

and in $f^{-1}(0)$ and $g^{-1}(0)$ we have algebraic cycles

$$W_1 : x_3^2+x_4^2+x_5^2=1-\sqrt{3}x_3x_4x_5=0$$
$$W_2 : x_1+ix_2=0,$$

respectively. By definition $Z_2=W_1\times W_2$. The number of intersection points of Z_2 with δ_β is in one to one correpondance with the number of intersection points of W_1 with δ_1. We compute the latter intersections.

Let us take a face Γ of δ_1 which is parameterized by $(\zeta_1 t_1^{\frac{1}{6}},\zeta_2 t_2^{\frac{1}{6}},\zeta_3 t_3^{\frac{1}{6}})$, where ζ_i's are 6-th roots of unity and $t_1+t_2+t_3=1, t_i\geq 0$. The intersection of W_1 with Γ is given by the equations

$$\begin{cases}\zeta_1^2 t_1^{\frac{1}{3}}+\zeta_2^{\frac{1}{3}} t_2^{\frac{1}{3}}+\zeta_3^2 t_3^{\frac{1}{3}}=0\\ (t_1t_2t_3)^{\frac{1}{6}}=(\sqrt{3}\zeta_1\zeta_2\zeta_3)^{-1}\\ t_1+t_2+t_3=1.\end{cases}$$

This implies that $t_1=t_2=t_3=\frac{1}{3}$ and

$$\zeta_1^2+\zeta_2^2+\zeta_3^2=0,\quad \zeta_1\zeta_2\zeta_3=1.$$

We conclude that ζ_i's must be of the form:

$$(\zeta_1,\zeta_2,\zeta_3)=(\zeta^{a_1+3b_1},\zeta^{a_2+3b_2},\zeta^{a_3+3b_3})$$

where a and b varie in (9.11). These are 24 intersection points with the face Γ. Note that we have still argue that the intersections are transversal. We have to discuss the signs too.

Remark 9.5.2 The set theoretic intersection of Z_2 with a δ_β is a a union of real one dimensional curves. Let us take a face of δ which is parameterized by $(\zeta_1 t_1^{\frac{1}{6}}, \cdots, \zeta_5 t_5^{\frac{1}{6}})$, where ζ_i's are 6-th roots of unity. The last equation of Z_2 implies that $t_1 = t_2$ and $\zeta_1 + i\zeta_2 = 0$. The other equations of Z_2 result in discrete values for other t_i's and ζ_i's.

Remark 9.5.3 The algebraic cycle

$$-x_0 + x_1 = x_2^2 + x_3^2 + x_4^2 = x_5^3 - \sqrt{-3}x_2x_3x_4.$$

intersects the vanishing cycle $\delta_\beta,\ \beta_1 = 0$ in just one point and for $\beta_1 \neq 0$ it does not intersects it. This intersection is not a smooth point of δ_β and hence, even after taking a smooth cycle homotop to δ_β, it may not be transversal and so the intersection point might be multiple.

With the action of the group G and S_6 we can produce more algebraic cycles. For this topic see the draft May 2015. A singular code can be found here.

Intersection of linear vanishing cycles

In the Singular code below, still we have to inset the algebraic numbers contributed by B_β. For $d = 3$ these numbers can be computed easily.

```
LIB "foliation.lib";
 intvec mlist=4,4,4; int n=size(mlist)-1; int d=lcm(mlist);
 list wlist;  //weights of variables
 for (int i=1; i<=size(mlist); i=i+1)
     { wlist=insert(wlist, (d div mlist[i]), size(wlist));}
 ring r=(0,z), (x(1..n+1)),wp(wlist[1..n+1]);
 poly cp=cyclotomic(2*d);  cp=subst(cp, x(1),par(1));
 minpoly =number(cp);
 list komak=MixedHodgeFermat(mlist);
 list mhf=komak[1]; list I=komak[3];
     //mhf serves as the elements of the cohomology
     //I serves as the elements of the homology
  list deX; for (i=1; i<=n+1; i=i+1)
                { deX=deX+mhf[i];}
  int cmu=size(deX);int mu=size(I); list Il=I[1..cmu];
  matrix intermat=IntersectionMatrix(Il);
  matrix permat=PeriodMatrix(deX,Il, par(1)^2);

  intvec aa=0,0,0,0;  intvec pp=0,1,2,3;
  matrix Per=PeriodsLinearCycle(mlist, aa, pp,z);
  int honu=(cmu-ncols(Per)) div 2;
  matrix Per2[1][cmu]; Per2[1,honu+1..honu+ncols(Per)]=Per;



  matrix myinter=Per2*inverse(permat)*transpose(intermat);
```

Algebraic cycle I

In this section we assume that n is even and we describe a set of algebraic cycles of dimension $\frac{n}{2}$ which induce elements in $H_n(X,\mathbb{Z})$. A trivial algebraic cycle in X is obtained by an intersection of $\mathbb{P}^{\frac{n}{2}+1}$ with X.

$$Z : x_0 = x_2 = \cdots = x_{n-4} = x_{n-2} = 0.$$

In other words, the algebraic cycle Z is induced by the polarization $X \subset \mathbb{P}^{n+1}$. We have

$$\langle Z, Z\rangle = d.$$

For an algebraic cycle Z_i of dimension $\frac{n}{2}$ in X let us define

$$\check{Z}_i := Z_i - \frac{\langle Z_2, Z\rangle}{\langle Z, Z\rangle} Z$$

This is charactrized by the fact that $\langle \check{Z}_i, Z\rangle = 0$ and so $\check{Z}_i$ induces an element $[\check{Z}_i] \in H_n(X,\mathbb{Z})_0$. The very special format of the Fermat variety gives us also projective varieties $\mathbb{P}^{\frac{n}{2}}$ inside X:

$$Z_1 : x_0 - x_1 = x_2 - \zeta_{2d} x_3 = x_4 - \zeta_{2d} x_5 = \cdots = x_n - \zeta_{2d} x_{n+1} = 0.$$

This cycle is obtained by the factorization

$$F = (x_0 - x_1) g_0(x_0, x_1) + (x_2 - \zeta_{2d} x_3) g_2(x_2, x_3) + (x_4 - \zeta_{2d} x_5) g_4(x_4, x_5) + \cdots + g_n(x_n, x_{n+1}).$$

where ζ_{2d} is the $2d$-th primitive root of unity. The algebraic cycles Z_1 intersects Z transversely in the point $[0;\cdots;0;0;1;\zeta_{2d}]$ (this is the intersection of $n+1$ codimension one linear sub space of $\mathbb{P}^{n+1}$) and so

$$\langle Z_1, Z\rangle = 1.$$

We work in the affine chart $x_0 = 1$. The sets Z_1 and δ_β intersect each other at the point $p := (1, 0, \cdots, 0, 0) \in \mathbb{C}^{n+1}$. This point is not a smooth point of δ_β and this make our analysis of its intersection number with Z_1 harder.

Proposition 9.5.4 *For d an even number, the intersection number of Z_1 with Γ_β at p is given by $(-1)^a$, where a is the number of $2i$'s such that*

$$\mathrm{Im}(\zeta_{2d}^{1+2\beta_{2i+1}-2\beta_{2i}}) < 0.$$

Proof. Using the action of G on U we can assume that $\beta = 0$ and Z_1 is given by

$$Z_1 : x_0 - x_1 = x_2 - a_1 x_3 = x_4 - a_2 x_5 = \cdots = x_n - a_{\frac{n}{2}} x_{n+1} = 0.$$

where $a_i = \zeta_{2d}^{1+2\beta_{2i+1}-2\beta_{2i}}$. The tangent space of the Fermat variety X at the point p, is the linear subspace of $\mathbb{C}^{n+1}$ given by $x_1 = 0$. Therefore, the orientation of X at p is given by

$$\alpha := d\mathrm{Re}(x_2) \wedge d\mathrm{Im}(x_2) \wedge \cdots \wedge d\mathrm{Re}(x_{n+1}) \wedge d\mathrm{Im}(x_{n+1}).$$

Here, by abuse of classical notations, we consider $d(\mathrm{Re}(x_2))$ and $d(\mathrm{Im}(x_2))$ the vector in the real tangent space of X at p given by $(0,1,0\cdots,0)$ and $(0,i,0,1,\cdots,0)$, respectively. The orientation of $\Gamma_\beta,\ \beta = 0$ at p is given by

$$\beta := d\mathrm{Re}(x_2) \wedge \cdots \wedge d\mathrm{Re}(x_{n+1}).$$

The tangent space of Z_1 at p has a basis of the form $(0,a_1,1,0,\cdots,0)$, $(0,ia_1,i,0,\cdots,0),\cdots$. Let us make wedge γ of all these vectors and compute $\beta \wedge \gamma$:

$$\beta \wedge \gamma := \beta \wedge \mathrm{Im}(a_1) d\mathrm{Im}(x_2) \wedge d\mathrm{Im}(x_3) \wedge \cdots \wedge \mathrm{Im}(a_{\frac{n}{2}}) d\mathrm{Im}(x_n) \wedge d\mathrm{Im}(x_{n+1}).$$

Comparing α with $\beta \wedge \gamma$, the proof of the proposition follows.

Remark 9.5.4 Proposition 9.5.4 and its proof fails for d odd, because in this case Γ_β is not transversal to Z_1 if for some i

$$2(\beta_{2i+1} - \beta_{2i}) + 1 \equiv_{2d} d.$$

Proposition 9.5.5 *For d an even number we have*

$$\langle Z_1, \delta_\beta \rangle = \begin{cases} (-1)^{\sum_{i=1}^{n+1} \beta_i} & \text{if } \beta_1 = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. After many days of strygling I failed to compute this intersection number. Let me just report what I did. We know that only δ_β with $\beta_1 = 0$ passes through p. A face $\Gamma_{\beta+a}$ of δ_β with $a_1 = 0$ passes also through p. Two such faces have common edge if an only if the corresponding a's differs only in one entry. For simplicity, one may take the case $n = 2$, where we are dealing with vanishing cycles which are two dimensional spheres. In this case we have four faces of δ_β passing through p. They have different intersection numbers with Z_1 and I do not know how the total intersection number depends on these four numbers.

I have computed the dimension of the Hodge loci passing through the algebraic cycle in Proposition **??**. For the singular code see here ******

Algebraic cycles II

Aoki in [Aok87] describes some new algebraic cycles for the Fermat variety. In the case of Fermat sextic fourfold this is:

$$Z_2 : x_3^2 + x_4^2 + x_5^2 = x_0^3 - \sqrt{3} x_3 x_4 x_5 = x_1 + i x_2 = 0.$$

The fact that Z_2 is inside X follows from:

$$\begin{aligned}-x_0^6+x_1^6+\cdots+x_5^6 = & x_1^6+x_2^6+\\ & (x_3^2)^3+(x_4^2)^3+(x_5^2)^3-3x_3^2x_4^2x_5^2+\\ & -(x_0^3)^2+(\sqrt{3}x_3x_4x_5)^2\end{aligned}$$

The basic idea behind this algebraic cycle is the following from high school algebra

$$x^3+y^3+z^3-3xyz=(x+y+z)(x+\zeta_3 y+\zeta_3^2 z)(x+\zeta_3^2 y+\zeta_3 z).$$

The algebraic cycles Z_2 intersects Z transversely at six points $[0;0;0;a_1;a_2;a_3]$, where $\{a_1,a_2,a_3\}=\{0,1,i\}$ (this set is a complete intersection of type $(1,1,1,2,3)$). Therefore,

$$\langle Z_2, Z\rangle = 6.$$

For the intersection of Z_2 with δ_β, we did not find a closed formula and so we had to compute it by hand.

Definition 9.5.2 A triple $(\beta_1,\beta_2,\beta_3)\in\{0,1,2,3,4,5\}^3$ is called admissible if it is of the format $(a_1+3b_1, a_2+3b_2, a_3+3b_3)$, where a and b varies in

$$\{a_1,a_2,a_3\}=\{0,1,2\},\ b_i\in\{0,1\},\ b_1+b_2+b_3=1 \text{ or } 3 \tag{9.11}$$

In othwer words, it belongs to the following set of 24 elements:

****

Proposition 9.5.6 *We have*

$$\langle Z_2,\delta_\beta\rangle=\sum_{a\in\{0,1\}^3}(-1)^{\sum_{i=1}^3(1-a_i)}\varepsilon(\beta_3+a_1,\beta_4+a_2,\beta_5+a_3).$$

where

$$\varepsilon(\beta_1,\beta_2,\beta_3)=\begin{cases}+1 & \textit{if } \beta \textit{ is admissible and } \mathrm{Im}((\zeta^{4\beta_2}-\zeta^{4\beta_3})(\zeta^{2\beta_3}-\zeta^{2\beta_1}))>0\\ -1 & \textit{if } \beta \textit{ is admissible and } \mathrm{Im}((\zeta^{4\beta_2}-\zeta^{4\beta_3})(\zeta^{2\beta_3}-\zeta^{2\beta_1}))<0\\ 0 & \textit{otherwise.}\end{cases}$$

$\varepsilon(B)=1$ *if B is admissible and* $=0$ *otherwise.*

Proof. Let us define

$$\begin{aligned} g &:= -x_1^6-x_2^6\\ f &:= -1+x_3^6+x_4^6+x_5^6\end{aligned}$$

A vanishing cycle δ_β is a join of a vanishing cycles δ_1 of $f=t$ with a vanishing cycle δ_2 of $g=t$, see [Mov11] §6.5. We have

$$U = \cup_{t\in\mathbb{C}} f^{-1}(t)\times g^{-1}(t)$$

and in $f^{-1}(0)$ and $g^{-1}(0)$ we have algebraic cycles

$$W_1 : x_3^2+x_4^2+x_5^2 = 1-\sqrt{3}x_3x_4x_5 = 0$$
$$W_2 : x_1+ix_2 = 0,$$

respectively. By definition $Z_2 = W_1\times W_2$. The number of intersection points of Z_2 with δ_β is in one to one correpondance with the number of intersection points of W_1 with δ_1. We compute the latter intersections.

Let us take a face Γ of δ_1 which is parameterized by $(\zeta_1 t_1^{\frac{1}{6}}, \zeta_2 t_2^{\frac{1}{6}}, \zeta_3 t_3^{\frac{1}{6}})$, where ζ_i's are 6-th roots of unity and $t_1+t_2+t_3 = 1, t_i \geq 0$. The intersection of W_1 with Γ is given by the equations

$$\begin{cases} \zeta_1^2 t_1^{\frac{1}{3}} + \zeta_2^{\frac{1}{3}} t_2^{\frac{1}{3}} + \zeta_3^2 t_3^{\frac{1}{3}} = 0 \\ (t_1t_2t_3)^{\frac{1}{6}} = (\sqrt{3}\zeta_1\zeta_2\zeta_3)^{-1} \\ t_1+t_2+t_3 = 1. \end{cases}$$

This implies that $t_1 = t_2 = t_3 = \frac{1}{3}$ and

$$\zeta_1^2+\zeta_2^2+\zeta_3^2 = 0,\ \ \zeta_1\zeta_2\zeta_3 = 1.$$

We conclude that ζ_i's must be of the form:

$$(\zeta_1,\zeta_2,\zeta_3) = (\zeta^{a_1+3b_1}, \zeta^{a_2+3b_2}, \zeta^{a_3+3b_3})$$

where a and b varie in (9.11). These are 24 intersection points with the face Γ. Note that we have still argue that the intersections are transversal. We have to discuss the signs too.

Remark 9.5.5 The set theoretic intersection of Z_2 with a δ_β is a a union of real one dimensional curves. Let us take a face of δ which is parameterized by $(\zeta_1 t_1^{\frac{1}{6}},\cdots,\zeta_5 t_5^{\frac{1}{6}})$, where ζ_i's are 6-th roots of unity. The last equation of Z_2 implies that $t_1 = t_2$ and $\zeta_1 + i\zeta_2 = 0$. The other equations of Z_2 result in discrete values for other t_i's and ζ_i's.

Remark 9.5.6 The algebraic cycle

$$-x_0+x_1 = x_2^2+x_3^2+x_4^2 = x_5^3-\sqrt{-3}x_2x_3x_4.$$

intersects the vanishing cycle $\delta_\beta,\ \beta_1 = 0$ in just one point and for $\beta_1 \neq 0$ it does not intersects it. This intersection is not a smooth point of δ_β and hence, even after taking a smooth cycle homotop to δ_β, it may not be transversal and so the intersection point might be multiple.

With the action of the group G and S_6 we can produce more algebraic cycles.

More on v invariant

Theorem 9.5.1 *There are infinite number of components of the Noether-Lefschetz locus passing through the Fermat point. The union of all these components is dense in* T *both in the sense of Zariski and usuall topology.*

Proof of Theorem **??** and Theorem **??**

Theorem **??** follows from Green-Voisin's theorem, see [Gre88, Gre89, Voi88], which says that the set of surfaces $X \subset \mathbb{P}^3$ containing a linear $\mathbb{P}^1$ is a component of the Noether-Lefschetz loci and it is the only component attaining the minimum codimension $d-3$. In order to see this we take a fixed $\mathbb{P}^1 \subset X$ and assume that it is parameterized by $0 \in \mathsf{T}$. We consider the local Noether-Lefschetz loci A in a neighborhood of $0 \in \mathsf{T}$ corresponding to Hodge cycle deformations of $\delta_0 = [\mathbb{P}^1] \in H_2(X,\mathbb{Z})$. From the above mentioned result of Green and voisin it follows that any component A_i of A parameterizes surfaces with a linear rational curve. For each A_i we have an analytic family of rational curves $\mathbb{P}^1_t \subset X_t$ for $t \in A_i$. We put $t = 0$ and for each i we get a linear ratinal curve $\mathbb{P}^1_i \subset X_0$ which are all homologous to the original $\mathbb{P}^1$. Furthermore, they are distinct. In order to finish the proof it remains to prove that: Let $X \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 4$ which contains two distinct linear rational curves $\mathbb{P}^1_1, \mathbb{P}^1_2$. Then the homology classes $[\mathbb{P}^1_1], [\mathbb{P}^1_2] \in H_2(X,\mathbb{Z})$ are also distinct. Two such line have positive intersection, whereas by adjunction formula, the self intersection of $\mathbb{P}^1_i$'s are negative.

Voisin in [Voi89] has shown that for $d \geq 5$ the second biggest component of NLL_d is of codimension $2d-7$, which consists of those surfaces containing a conic. This gives an affirmative answer to Conjecture **??** for $n = 2$ and $d_1 = 1, d_2 = 2$. Further partial results in this direction are due to Cox [Cox90] for elliptic surfaces, Debarre and Laszlo [DL90] for abelian varieties and Voisin [Voi90] for surfaces of degree $d = 6,7$.