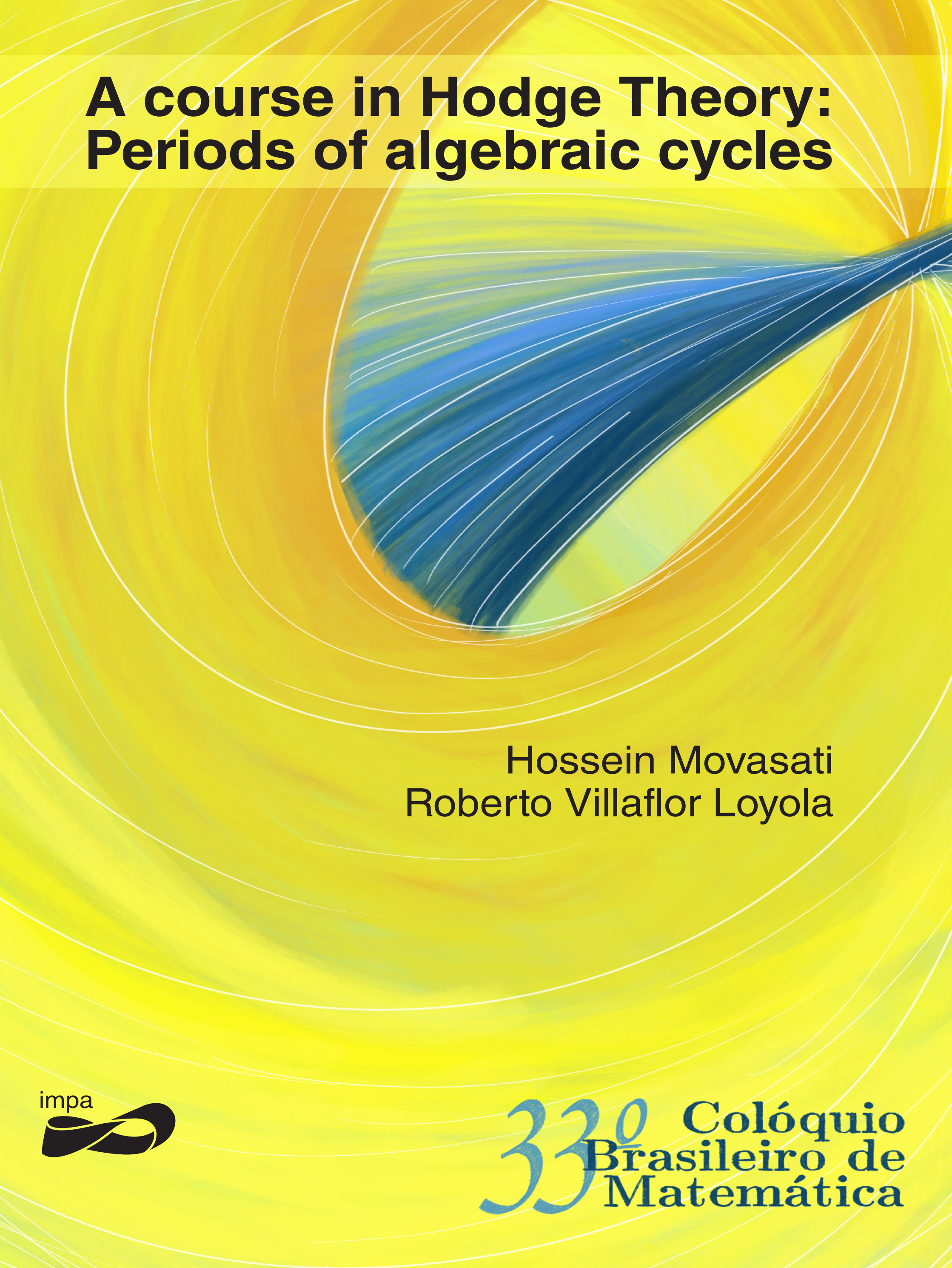

A course in Hodge Theory: Periods of algebraic cycles

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در آتش خویش چون دمی جوش کنم
خواهم که دمی ترا فراموش کنم
گیرم جانی که عقل بیهوش کند
در جام درآئی و ترا نوش کنم

If I boil in the fire of my existence for a while,
that is because I want to forget you for a while,
to get a new soul and put away my wisdom,
and then you become the wine of my glass.

Quatrain № 1215 by Rumi (Jalal al-Din Muhammad Balkhi) published in *Kulliyat-e Shams-e Tabrizi*, with translation by the first author, Hossein Movasati.

Contents

Preface

The present book is a sequel to the first author's book "A Course in Hodge Theory: With Emphasis on Multiple Integrals". The first book focuses mainly on affine hypersurfaces and the study of the Hodge locus through the Fermat variety and uses techniques from singularity theory, such as Brieskorn modules and vanishing cycles. This book intends to tell us the Hodge theory of smooth projective varieties and their properties inside families. This is the study of Čech cohomology, hypercohomology, Algebraic de Rham cohomology, Gauss–Manin connection, infinitesimal variation of Hodge structures and Hodge loci. Despite this, in order to do concrete computations, we will be back to our favorite example of hypersurfaces. Both books intend to make Hodge theory as computational as possible, either by hand or by computer. Together with other classical books in Hodge theory such as Voisin's two volume books, they can be used in a Graduate course. It is mainly for students and researchers who want to study the Hodge conjecture in families. We assume a basic knowledge in both Algebraic Topology and Algebraic Geometry.

Hossein Movasati, Roberto Villaflor Loyola
May 2021, Rio de Janeiro, RJ, Brazil

1 Introduction

The first draft of the present book was the lecture notes of a second course in Hodge theory presented by the first author in 2015 to the second author. These notes were developed into the second author's Ph.D. thesis and the present text is the outcome of this collaboration.

1.1 Computational Hodge theory

In order to solve a mathematical problem one may generalize it until a solution comes out by itself and this method is, for instance, present in Grothendieck's philosophy. A completely different approach must be adopted if one has the feeling that the Hodge conjecture is wrong. Instead of generalizations, one has to study so many particular examples, and one has to compute so many well-known theoretical data, such that the counterexample comes out by itself. Once you are in the ocean without compass, all directions might lead you to a land. If such a counterexample is found then it would be like a Columbus' egg and there will be an explosion of other counterexamples. Even if the Hodge conjecture is true, the belief that it is false makes us to take a more computational approach, and at least this makes the Hodge theory accessible to broader class of mathematicians, and in particular those who love computational mathematics, either by hand or by computer. The first author's book Movasati (2021) is the first attempt in this direction. Since the main focus of this book is smooth hypersurfaces, we feel that one has to prepare the computational ground for arbitrary smooth projective varieties and the present text is the output of this attempt.

1.2 Some missing details

In an attempt to make a piece of mathematics more computational, one might think that all the credit belongs to theory makers and one does only hard exercises of these theories, and therefore, this kind of mathematics might be banned from publication in "prestigious journals". The amount of time and effort needed for this purpose is usually higher than if we wanted to contribute for producing more theories. One may also find some missing details in the work of theory makers which are as important as the main body of such theories. Here, we would like to highlight one example. In a personal communication (November 09, 2018) with P. Deligne, the first author posed the following question: " I got motivated to write this email after seeing your talk 'what do we mean by equal', and after getting the feeling that in the foundation of Hodge theory not every detail is explained. Let me explain this. For a smooth projective variety over $\mathbb{C}$ we know that there is a concrete canonical isomorphism between the usual de Rham cohomology by C^{∞} forms and the algebraic de Rham cohomology. Therefore, all the concepts in the topological side, such as cup product, cohomology class of cycles etc. can be transported to the algebraic side and can be defined in a purely algebraic fashion. However, my impression is that nobody has verified concretely that the algebraic objects are the exact transportation of the corresponding topological objects. For instance, Grothendieck's definition of a class of an algebraic cycle must corresponds to the one defined by integration, but I do not see if it is written somewhere. In particular, when the algebraic cycle is singular, the only rigorous proof that I see is the resolution of singularities. Anyway, with my student Roberto Villaflor, we are trying to write a book containing the maximum details, however, we are stuck in this kind of issue. I would be grateful if you clarify this for us." The answer came few days later. "I have not thought about proving the compatibility between Grothendieck's definition of the class of a cycle and integration, because I never used the latter. I like to use <u>co</u>homology, not homology, and I like definitions which are uniform across cohomology theories (motivic philosophy)...", (P. Deligne, personal communication November 13, 2018). As an idiom says "the devil is in the detail" and such missing details in the literature took more than 4 years of both authors.

1.3 The organization of the text

The emphasis of the first book Movasati (ibid.) was mainly on hypersurfaces and the study of the Hodge locus through the Fermat variety. This book intends to tell us the Hodge theory of smooth projective varieties and their properties inside families. This is the study of Čech cohomology, hypercohomology, Gauss–Manin connection, infinitesimal variation of Hodge structures, Hodge loci etc. Despite this, in order to do concrete computations, we will be back to our favorite example of hypersurfaces. A synopsis of each chapter is explained below.

In Chapter 2 we present a minimum amount of material so that the reader gets familiar with Čech cohomology. We need to represent elements of cohomologies with concrete data and we do this using an acyclic covering. In Chapter 3 we aim to define hypercohomology of a complex of sheaves relative to an acyclic covering, and hence, we describe elements of a hypercohomology with concrete data. We discuss quasi-isomorphisms, filtrations etc., adapted for computations. This chapter is presented for general sheaves, however, our main example for this is the sheaf of differential forms. In Chapter 4 we prove the Atiyah–Hodge theorem which says that the elements of the de Rham cohomology of an affine variety is given by algebraic differential forms. This paves the road for the definition of algebraic de Rham cohomology in the next chapter. Chapter 5 is fully dedicated to algebraic de Rham cohomology and the fact that it is isomorphic to the classical de Rham cohomology. We need to describe this isomorphism as explicitly as possible because we want to transport the integration of C^∞ forms to the algebraic side, where integration becomes a purely algebraic operation. The objective of this chapter is to collect all necessary material for computing the integration of elements of algebraic de Rham cohomologies over algebraic cycles. In Chapter 6 we capture the cohomology of affine varieties by using logarithmic differential forms. This is needed in order to take residues. This is not possible using only Atiyah–Hodge theorem. This theorem for the complement of smooth hypersurfaces turns out to be the Griffiths theorem which also finds a basis of such de Rham cohomologies. This is explained in Chapter 7. These are used in order to integrate elements of algebraic de Rham cohomology of hypersurfaces over complete intersection algebraic cycles which is done in Chapter 8. Chapter 9 is devoted to the description of Gauss–Manin connection of families of algebraic varieties. In this chapter we work on the general context of arbitrary families of projective varieties, whereas

Movasati (2021) focused on the computation of Gauss–Manin connection for tame polynomials, and in particular families of hypersurfaces. One of the main theorems proved in this chapter is Griffiths transversality. It relates the Gauss–Manin connection to the underlying Hodge filtrations. We do not give concrete applications of the Gauss–Manin connection in Algebraic Geometry, however, its partial data, namely the infinitesimal variation of Hodge structures (IVHS) has successful applications. This includes the famous Noether–Lefschetz theorem which says that a generic surface in the projective space of dimension three has Picard rank one. Chapter 10 is dedicated to this topic. In Chapter 11 we observe that Hodge cycles of smooth hypersurfaces give us Artinian Gorenstein rings, and in this way, many topological problems can be reduced into commutative algebra problems. This will be elaborated more in Chapter 12 in which we explain many well known components of the Hodge locus for hypersurfaces.

2 Čech cohomology

Il faut faisceautiser. (The motto of the french revolution in algebraic and complex geometry, see Remmert (1995, page 6)).

2.1 Introduction

The first examples of cohomology theories were constructed in the first half of 20th century, being the first one of them the singular cohomology. Almost at the same time, other cohomology theories, such as De Rham and Čech cohomology, were constructed and intensively studied. In this chapter we aim to introduce Čech cohomology, as an explicit construction of the so called sheaf cohomology.

Similarly to the case of singular cohomology, in most situations we only need to know a bunch of properties of sheaf cohomology in order to compute it. Thus, we might have wanted to give an abstract approach to sheaf cohomology (as we did in Movasati (2021, Chapter 4) by giving the Eilenberg–Steenrod axioms for singular cohomology). It turns out that the right abstract approach to introducing sheaf cohomology is through category theory, as derived functors. Although that approach has the advantage of providing natural proofs for several properties of sheaf cohomology, in some occasions we need to have a concrete description of these groups and their elements. For instance when we want to formulate obstructions as elements of some sheaf cohomology group. Since we are mainly interested in computing elements of the cohomology groups, we decided to introduce Čech cohomology and prove directly from its constructive definition the main properties of sheaf cohomology. This may have enlarged a little the proofs of some properties, but it has the advantage of saving us the trouble of introducing the language of derived functors, making the exposition more elementary. We assume that the reader is familiar with sheaves of abelian groups on topological spaces. The interested reader can consult other books like Bott and Tu (1982, Section 10), Voisin (2002, Section 4.3), Godement (1973) or Bredon (1997), for a categorical and fairly more involved presentation of sheaf cohomology theory.

As this is a course of Hodge theory, we will be interested in sheaves related to the complex structure of a complex manifold or variety. In what follows, we list some of the main examples of sheaves we will study over a complex manifold. Consider a complex manifold X, we denote by $\mathcal{O}_{X^{\mathrm{an}}}$ the sheaf of holomorphic functions on X, by $\Omega^p_{X^{\mathrm{an}}}$ the sheaf of holomorphic p-forms on X, by $\Omega^{p,q}_X$ the sheaf of C^∞ (p,q)-forms on X, by $\Omega^p_{X^\infty}$ the sheaf of C^∞ p-forms on X, by $\mathcal{I}_{Y^{\mathrm{an}}}$ the analytic ideal sheaf of a subvariety $i : Y \hookrightarrow X$ and by $\mathcal{O}^*_{X^{\mathrm{an}}}$ the sheaf of invertible holomorphic functions. Note that the group structure in all these examples is given by addition, except for the last one in which it is given by multiplication. Note also that the sheaves of C^∞ forms, such as $\Omega^{p,q}_X$ and $\Omega^p_{X^\infty}$, are $\mathcal{C}^\infty_X$-modules, where $\mathcal{C}^\infty_X$ denotes the sheaf of C^∞ complex valued functions defined on X. On the other hand, $\mathcal{O}_{X^{\mathrm{an}}}$, $\Omega^p_{X^{\mathrm{an}}}$ and $\mathcal{I}_{Y^{\mathrm{an}}}$ are $\mathcal{O}_{X^{\mathrm{an}}}$-modules, also called **analytic sheaves**. Some important examples of short exact sequences of sheaves are

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2\pi i} \mathcal{O}_{X^{\mathrm{an}}} \xrightarrow{\exp} \mathcal{O}^*_{X^{\mathrm{an}}} \to 0, \tag{2.1}$$

$$0 \to \mathcal{I}_{Y^{\mathrm{an}}} \to \mathcal{O}_{X^{\mathrm{an}}} \to i_*\mathcal{O}_{Y^{\mathrm{an}}} \to 0. \tag{2.2}$$

Later, in Chapter 4, we will introduce on a smooth complex algebraic variety X, the algebraic analogues of the analytic sheaves mentioned above. We will keep the same notation for algebraic sheaves, dropping the superscript X^{an} which indicates the analytic structure. We reserve the clean notation for algebraic sheaves, since after we have introduced them, we will continue working almost exclusively with them.

2.2 The first cohomology group

Before going to the general definition, let us explain the basic idea behind the definition of the first cohomology group. Recall that a **sheaf of abelian groups** $\mathcal{S}$

on a topological space X is a collection of abelian groups

$$\mathcal{S}(U), \ U \subset X \text{ open},$$

called the sections of $\mathcal{S}$ defined over U, together with restriction maps of sections such that every section is uniquely determined by its restrictions to any open cover of its domain. The group $\mathcal{S}(X)$ is called the set of **global sections** of $\mathcal{S}$. Some other equivalent notations for this are

$$\mathcal{S}(X) = \Gamma(X, \mathcal{S}) = H^0(X, \mathcal{S}).$$

For every point $x \in X$, the **stalk** of $\mathcal{S}$ at x is denoted $\mathcal{S}_x$ and corresponds to the group of germs of sections of $\mathcal{S}$ defined over some neighbourhood of $x \in X$. A short sequence of sheaves

$$0 \to \mathcal{S}_1 \to \mathcal{S}_2 \to \mathcal{S}_3 \to 0,$$

is **exact** if and only if it is exact at its stalks

$$0 \to \mathcal{S}_{1,x} \to \mathcal{S}_{2,x} \to \mathcal{S}_{3,x} \to 0,$$

for all $x \in X$. It is not difficult to see that every exact sequence of sheaves of abelian groups

$$0 \to \mathcal{S}_1 \to \mathcal{S}_2 \to \mathcal{S}_3 \to 0 \tag{2.3}$$

induces an exact sequence in global sections

$$0 \to \mathcal{S}_1(X) \to \mathcal{S}_2(X) \to \mathcal{S}_3(X), \tag{2.4}$$

where the last map is not necessarily surjective. The sheaf cohomology groups $H^i(X, \mathcal{S})$ are groups completing (2.4) into a long exact sequence of the form

$$0 \to H^0(X, \mathcal{S}_1) \to H^0(X, \mathcal{S}_2) \to H^0(X, \mathcal{S}_3) \to H^1(X, \mathcal{S}_1) \to H^1(X, \mathcal{S}_2) \to$$
$$H^1(X, \mathcal{S}_3) \to H^2(X, \mathcal{S}_1) \to H^2(X, \mathcal{S}_2) \to \cdots$$

with $H^0(X, \mathcal{S}) = \mathcal{S}(X)$. For this reason, the elements of $H^1(X, \mathcal{S}_1)$ can be regarded as obstructions to the surjectivity of $H^0(X, \mathcal{S}_2) \to H^0(X, \mathcal{S}_3)$. The idea to construct these obstructions is as follows. Suppose there exists an open covering $\mathcal{U} = \{U_i, \ i \in I\}$ of X such that the exact sequence (2.3) remains exact when taking the sections defined over U_i and $U_i \cap U_j$ for every $i, j \in I$. For instance, we have

$$0 \to \mathcal{S}_1(U_i) \to \mathcal{S}_2(U_i) \to \mathcal{S}_3(U_i) \to 0.$$

Remark 2.1. In general such covering might not exist. In fact, the surjectivity of $\mathcal{S}_{2,x} \to \mathcal{S}_{3,x}$ does not imply the surjectivity of $\mathcal{S}_2(U) \to \mathcal{S}_3(U)$ for some open neighborhood U of x. However, this will be the case in most of the examples in this book and (in particular for (2.1) and (2.2)).

For every $f \in \mathcal{S}_3(X)$ we can take $f_i \in \mathcal{S}_2(U_i)$, $i \in I$ such that f_i is mapped to $f|_{U_i}$ under $\mathcal{S}_2(U_i) \to \mathcal{S}_3(U_i)$. This implies that the elements $f_j|_{U_i \cap U_j} - f_i|_{U_i \cap U_j} \in \mathcal{S}_2(U_i \cap U_j)$ are mapped to zero in $\mathcal{S}_3(U_i \cap U_j)$ and so there are elements $f_{ij} \in \mathcal{S}_1(U_i \cap U_j)$ which are mapped to $f_j|_{U_i \cap U_j} - f_i|_{U_i \cap U_j}$. If we consider a different choice $\widetilde{f_i} \in \mathcal{S}_2(U_i)$ mapped to $f|_{U_i}$ under $\mathcal{S}_2(U_i) \to \mathcal{S}_3(U_i)$, then

$$g_i := f_i - \widetilde{f_i} \in \mathcal{S}_2(U_i)$$

is mapped to zero in $\mathcal{S}_3(U_i)$ and so there exists $\widetilde{g_i} \in \mathcal{S}_1(U_i)$ that is mapped to g_i. In consequence

$$\widetilde{g_j}|_{U_i \cap U_j} - \widetilde{g_i}|_{U_i \cap U_j} - f_{ij} + \widetilde{f_{ij}} = 0, \tag{2.5}$$

since it is mapped to zero in $\mathcal{S}_2(U_i \cap U_j)$. This leads us to define the **first cohomology group** of $\mathcal{U}$ with coefficients in $\mathcal{S}_1$ as

$$H^1(\mathcal{U}, \mathcal{S}_1) := \{(f_{ij}, i, j \in I) : f_{ij} \in \mathcal{S}_1(U_i \cap U_j), f_{ii} = 0,\ f_{ij} + f_{jk} + f_{ki} = 0 \text{ in } U_i \cap U_j \cap U_k\}/\sim$$

modulo the equivalence relation given by (2.5), that is,

$$(f_{ij},\ i, j \in I) \sim (\widetilde{f_{ij}},\ i, j \in I)$$

if there exist $\widetilde{g_i} \in \mathcal{S}_1(U_i)$ such that (2.5) holds.

2.3 Coverings and Čech cohomology

After introducing the first sheaf cohomology group relative to a covering, we turn now to the general definition of the so called Čech cohomology groups. As usual, these groups correspond to the cohomology groups associated to a complex of abelian groups that is defined as follows.

Definition 2.1. Let X be a topological space, $\mathcal{S}$ be a sheaf of abelian groups on X and $\mathcal{U} = \{U_i,\ i \in I\}$ be a covering of X by open sets. Let $\mathcal{U}^p$ be the set of $(p+1)$-tuples $\sigma = (U_{i_0}, \ldots, U_{i_p})$, $i_0, \ldots, i_p \in I$. Define for every $\sigma \in \mathcal{U}^p$

$$|\sigma| := U_{i_0 \cdots i_p} := \cap_{j=0}^{p} U_{i_j}.$$

A p-**cochain** $f = (f_\sigma)_{\sigma\in\mathcal{U}^p}$ is an element of

$$\prod_{\sigma\in\mathcal{U}^p} H^0(|\sigma|, \mathcal{S}).$$

Let π be an element in the permutation group of the set $\{0, 1, 2, \dots, p\}$. It acts on $\mathcal{U}^p$ in a canonical way: for $\sigma = (U_{i_0}, \dots, U_{i_p})$ the action of π is given by $\pi\sigma := (U_{i_{\pi(0)}}, \dots, U_{i_{\pi(p)}})$. We say that a p-cochain f is **skew-symmetric** if

$$f_{\pi\sigma} = \operatorname{sign}(\pi) f_\sigma \qquad \forall\sigma \in \mathcal{U}^p.$$

These cochains form an abelian group denoted by

$$C^p(\mathcal{U}, \mathcal{S}) := \left\{ f \in \prod_{\sigma\in\mathcal{U}^p} H^0(|\sigma|, \mathcal{S}) : f \text{ is skew-symmetric} \right\}.$$

For $\sigma \in \mathcal{U}^p$ and $j = 0, 1, \dots, p$ denote by σ_j the element in $\mathcal{U}^{p-1}$ obtained by removing the j-th entry of σ. Since $|\sigma| \subset |\sigma_j|$, the restriction maps from $H^0(|\sigma_j|, \mathcal{S})$ to $H^0(|\sigma|, \mathcal{S})$ are well-defined. We define the **boundary map** as

$$\delta : \prod_{\sigma\in\mathcal{U}^p} H^0(|\sigma|, \mathcal{S}) \to \prod_{\sigma\in\mathcal{U}^{p+1}} H^0(|\sigma|, \mathcal{S}), \ (\delta f)_\sigma := \sum_{j=0}^{p+1} (-1)^j f_{\sigma_j}|_{|\sigma|}.$$

Proposition 2.1. *The boundary map defines a map of skew-symmetric chains, i.e.*

$$\delta(C^p(\mathcal{U}, \mathcal{S})) \subseteq C^{p+1}(\mathcal{U}, \mathcal{S}).$$

We omit the proof of Proposition 2.1, which can be easily seen from the following simplification of our notation. From now on we identify σ with $i_0 i_1 \cdots i_p$ and write a p-cochain as $f = (f_{i_0 i_1 \cdots i_p},\ i_j \in I)$. For simplicity we also write

$$(\delta f)_{i_0 i_1 \cdots i_{p+1}} := \sum_{j=0}^{p+1} (-1)^j f_{i_0 i_1 \cdots i_{j-1} \hat{i}_j i_{j+1} \cdots i_{p+1}},$$

where $\hat{i}_j$ means that i_j has been removed.

Proposition 2.2. *The boundary map restricted to skew-symmetric chains defines a complex of abelian groups, i.e.*

$$\delta \circ \delta = 0.$$

Proof. Let $f \in C^p(\mathcal{U}, \mathcal{S})$. We have

$$
\begin{aligned}
(\delta^2 f)_{i_0 i_1 \cdot i_{p+2}} &= \sum_{j=0}^{p+2} (-1)^j (\delta f)_{i_0 i_1 \cdots \hat{i}_j \cdot i_{p+2}} \\
&= \sum_{j=0}^{p+2} \left(\sum_{k=0}^{j-1} (-1)^{j+k} f_{i_0 i_1 \cdots \hat{i}_k \cdots \hat{i}_j \cdots i_{p+2}} + \right. \\
&\qquad \left. + \sum_{l=j+1}^{p+2} (-1)^{j+l-1} f_{i_0 i_1 \cdots \hat{i}_j \cdots \hat{i}_l \cdots i_{p+2}} \right) \\
&= \sum_{k<j} (-1)^{j+k} f_{i_0 i_1 \cdots \hat{i}_k \cdots \hat{i}_j \cdots i_{p+2}} + \\
&\qquad + \sum_{j<l} (-1)^{j+l-1} f_{i_0 i_1 \cdots \hat{i}_j \cdots \hat{i}_l \cdots i_{p+2}} = 0.
\end{aligned}
$$

□

Definition 2.2. The **Čech cohomology** of $\mathcal{U}$ with coefficients in the sheaf $\mathcal{S}$ corresponds to the cohomology associated to the following cochain complex

$$0 \to C^0(\mathcal{U}, \mathcal{S}) \xrightarrow{\delta} C^1(\mathcal{U}, S) \xrightarrow{\delta} C^2(\mathcal{U}, \mathcal{S}) \xrightarrow{\delta} C^3(\mathcal{U}, \mathcal{S}) \xrightarrow{\delta} \cdots . \tag{2.6}$$

In other words, the **p-th Čech cohomology group** of $\mathcal{U}$ with coefficients in $\mathcal{S}$ is

$$H^p(\mathcal{U}, \mathcal{S}) := \frac{\text{Kernel}(C^p(\mathcal{U}, \mathcal{S}) \xrightarrow{\delta} C^{p+1}(\mathcal{U}, \mathcal{S}))}{\text{Image}(C^{p-1}(\mathcal{U}, \mathcal{S}) \xrightarrow{\delta} C^p(\mathcal{U}, \mathcal{S}))}.$$

The above definition depends on the covering $\mathcal{U}$. We want to construct cohomology groups $H^p(X, \mathcal{S})$ depending only on X and $\mathcal{S}$. In order to do this, we analyze the behaviour of the Čech cohomology groups under refinements.

Definition 2.3. For two coverings $\mathcal{U}_i = \{U_{i,j},\ j \in I_i\}$, $i = 1, 2$ we write $\mathcal{U}_1 \leqslant \mathcal{U}_2$ and say that $\mathcal{U}_1$ is a **refinement** of $\mathcal{U}_2$, if there is a map from $\phi : I_1 \to I_2$ such that $U_{1,i} \subset U_{2,\phi(i)}$ for all $i \in I_1$.

The set of all open coverings of X together with the refinement relation is a directed set, i.e. for two coverings $\mathcal{U}_1$ and $\mathcal{U}_2$ there is another covering $\mathcal{U}_3$ such that $\mathcal{U}_3 \leqslant \mathcal{U}_1$ and $\mathcal{U}_3 \leqslant \mathcal{U}_2$.

Every refinement $\mathcal{U}_1 \leqslant \mathcal{U}_2$ induced by a map $\phi : I_1 \to I_2$ defines a cochain map

$$\Phi : C^p(\mathcal{U}_2, \mathcal{S}) \to C^p(\mathcal{U}_1, \mathcal{S}), \quad \Phi(s)_{i_0\cdots i_p} := s_{\phi(i_0)\cdots\phi(i_p)}|U_{i_0\cdots i_p}.$$

Since this map commutes with the boundary map, it induces a map in Čech cohomology.

Proposition 2.3. *The induced map in cohomology*

$$\Phi : H^p(\mathcal{U}_2, \mathcal{S}) \to H^p(\mathcal{U}_1, \mathcal{S}).$$

is independent of the considered refinement map $\phi : I_1 \to I_2$.

Proof. If $\psi : I_1 \to I_2$ is another refinement, the induced map

$$\Psi : C^p(\mathcal{U}_2, \mathcal{S}) \to C^p(\mathcal{U}_1, \mathcal{S})$$

is homotopic to Φ. In fact, the homotopy $H : C^{p+1}(\mathcal{U}_2, \mathcal{S}) \to C^p(\mathcal{U}_1, \mathcal{S})$ is given by

$$H(s)_{i_0\cdots i_p} := \sum_{l=0}^{p} (-1)^l s_{\psi(i_0)\cdots\psi(i_l)\phi(i_l)\cdots\phi(i_p)}|U_{i_0\cdots i_p}. \tag{2.7}$$

It is left to the reader the verification of the equality $\Phi - \Psi = \delta \circ H + H \circ \delta$. □

The previous proposition implies that for any topological space X, the p-th Čech cohomology groups of a covering form a directed system of abelian groups under refinements.

Definition 2.4. The **p-th Čech cohomology group** of X with coefficients in $\mathcal{S}$ is the group

$$H^p(X, \mathcal{S}) := \text{dir}\lim_{\mathcal{U}} H^p(\mathcal{U}, \mathcal{S}).$$

Recall that the direct limit $H^p(X, \mathcal{S})$ can be realized as the union of all $H^p(\mathcal{U}, \mathcal{S})$ for all coverings $\mathcal{U}$, quotient by the following equivalence relation: two elements $\alpha \in H^p(\mathcal{U}_1, \mathcal{S})$ and $\beta \in H^p(\mathcal{U}_2, \mathcal{S})$ are equivalent if there exists a covering $\mathcal{U}_3 \leqslant \mathcal{U}_1$ and $\mathcal{U}_3 \leqslant \mathcal{U}_2$ such that α and β are mapped to the same element in $H^p(\mathcal{U}_3, \mathcal{S})$.

2.4 Acyclic sheaves

Usually, we do not deal with the direct limit of Definition 2.4. This is due to the fact that most sheaves admit good coverings attaining the limit, the so called acyclic coverings. In order to prove such a result we have to introduce first acyclic sheaves.

Definition 2.5. A sheaf $\mathcal{S}$ of abelian groups on a topological space X is called **acyclic** if

$$H^k(X, \mathcal{S}) = 0, \; k = 1, 2, \ldots$$

Definition 2.6. Let $\mathcal{A}$ be a sheaf of rings over a topological space X. Every sheaf $\mathcal{F}$ of $\mathcal{A}$-modules is called **fine** if $\mathcal{A}$ satisfies the following condition: For every locally finite open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X, there exists a **partition of unity** $f_i \in \mathcal{A}(X)$ with $\operatorname{Supp} f_i \subseteq U_i$ and $\sum_{i \in I} f_i = 1$ (where this last sum is well defined locally and define a global section since $\mathcal{A}$ is a sheaf).

Example 2.1. The main examples of fine sheaves that we have in mind are the following: Let X be a C^∞ manifold, and denote $\mathcal{C}_X^\infty$ the sheaf of C^∞ functions with real or complex values. Then all the $\mathcal{C}_X^\infty$-modules are fine, such as $\Omega^i_{X^\infty}$ of C^∞ differential i-forms on X. If X is a complex manifold then the sheaves $\Omega_X^{p,q}$ of C^∞ differential (p, q)-forms on X are also fine.

Proposition 2.4. *Let $\mathcal{S}$ be any fine sheaf over a topological space X. Then for every locally finite open covering $\mathcal{U}$ of X and every $k \geqslant 1$*

$$H^k(\mathcal{U}, \mathcal{S}) = 0.$$

In particular, every fine sheaf on a paracompact topological space is acyclic.

Proof. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be the locally finite open covering of X. Then for every $k \geqslant 1$ and $\sigma \in C^k(\mathcal{U}, \mathcal{S})$ with $\delta\sigma = 0$ let us define $\tau \in C^{k-1}(\mathcal{U}, \mathcal{S})$ as

$$\tau_{i_0 \cdots i_{k-1}} := \sum_{i \in I} f_i \sigma_{i i_0 \cdots i_{k-1}},$$

for some partition of unity $\{f_i\}_{i \in I}$ subordinated to $\mathcal{U}$. Then

$$(\delta\tau)_{i_0 \cdots i_k} = \sum_{l=0}^{k} (-1)^l \sum_{i \in I} f_i \sigma_{i i_0 \cdots \widehat{i_l} \cdots i_k} = \sum_{\in I} f_i \sum_{l=0}^{k} (-1)^l \sigma_{i i_0 \cdots \widehat{i_l} \cdots i_k} = \sigma_{i_0 \cdots i_k}.$$

In other words $\sigma = \delta\tau$, and so $H^k(\mathcal{U}, \mathcal{S}) = 0$, $\forall k \geqslant 1$. $\square$

Definition 2.7. A sheaf $\mathcal{S}$ is said to be **flasque** if for every pair of open sets $V \subset U$, the restriction map $\mathcal{S}(U) \to \mathcal{S}(V)$ is surjective.

Proposition 2.5. *Let $\mathcal{S}$ be any flasque sheaf over a topological space X. Then for every open covering $\mathcal{U}$ of X and every $k \geqslant 1$*

$$H^k(\mathcal{U}, \mathcal{S}) = 0.$$

In particular, every flasque sheaf is acyclic.

Proof. Let $\mathcal{S}$ be a flasque sheaf and $\sigma \in C^k(\mathcal{U}, \mathcal{S})$ such that $\delta\sigma = 0$. Let us define $\tau \in C^{k-1}(\mathcal{U}, \mathcal{S})$. For simplicity let us assume that the covering $\mathcal{U} = \{U_i\}_{i \in I}$ is finite (this assumption might be removed by a transfinite induction argument). Inductively on the set of multi-indexes $(i_0, \ldots, i_k)$ with $i_0 < \cdots < i_k$ ordered lexicographically, we define

$$\tau_{i_1 \cdots i_k} := \begin{cases} \text{any element of } \mathcal{S}(U_{i_1 \cdots i_k}) & \text{if } i_1 = 0 \\ \text{any extension of } \widetilde{\sigma}_{i_1 \cdots i_k} & \text{if } i_1 > 0 \end{cases},$$

where $\widetilde{\sigma}_{i_1 \cdots i_k} \in \mathcal{S}(\cup_{j=0}^{i_1 - 1} U_{j i_1 \cdots i_k})$ is locally defined as

$$\widetilde{\sigma}_{i_1 \cdots i_k}|_{U_{j i_1 \cdots i_k}} := \sigma_{j i_1 \cdots i_k} - \sum_{l=1}^{k} (-1)^l \tau_{j i_1 \cdots \widehat{i_l} i_k}|_{U_{j i_1 \cdots i_k}}.$$

It is routine to check $\tau \in C^{k-1}(\mathcal{U}, \mathcal{S})$ is well defined and $\delta\tau = \sigma$. $\square$

Example 2.2. Let X be a topological space and G be any abelian group. The **skyscraper sheaf** supported in a point $x \in X$ with values in G is the sheaf $i_x(G)$ given by

$$i_x(G)(U) := \begin{cases} G & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

with restrictions the identity or the zero map. This an example of flasque sheaf.

Example 2.3. Let G be any abelian group. The **constant sheaf** G is the sheaf of locally constant functions from a topological space X to G. This sheaf is not flasque in general. In fact, when X is an irreducible topological space, G will be flasque (for instance if X is an irreducible algebraic variety with its Zariski topology). But as we will see later, when X is a smooth manifold and $G = \mathbb{R}$, $H^i(X, \mathbb{R}) = H^i_{\mathrm{dR}}(X)$ and so $\mathbb{R}$ is not acyclic in general.

Example 2.4. Every sheaf $\mathcal{S}$ over a topological space X is a subsheaf of a flasque sheaf. In fact, it is enough to define $\mathcal{F}$ as the **sheaf of discontinuous sections** of $\mathcal{S}$, i.e.

$$\mathcal{F}(U) = \prod_{x \in U} \mathcal{S}_x.$$

This construction was used by Godement in order to produce acyclic resolutions of any sheaf of abelian groups, also called **Godement resolution**. Such resolution consists of a long exact sequence of sheaves

$$0 \to \mathcal{S}_0 \to \mathcal{S}_1 \to \mathcal{S}_2 \to \cdots$$

where $\mathcal{S}_0 := \mathcal{S}$, $\mathcal{S}_1 := \mathcal{F}$ and $\mathcal{S}_{i+1}$ is the sheaf of discontinuous sections of $\operatorname{Coker}(\mathcal{S}_{i-1} \to \mathcal{S}_i)$. In particular $\mathcal{S}_i$ is flasque for all $i \geqslant 1$.

2.5 Short exact sequences

Going back to our motivations we will justify that Čech cohomology is a good candidate for sheaf cohomology since it fulfills one of our main goals. Namely, for every exact sequence of sheaves of abelian groups

$$0 \to \mathcal{S}_1 \to \mathcal{S}_2 \to \mathcal{S}_3 \to 0$$

we will show the existence of a long exact sequence in cohomology

$$\cdots \to H^i(X, \mathcal{S}_1) \to H^i(X, \mathcal{S}_2) \to H^i(X, \mathcal{S}_3) \to H^{i+1}(X, \mathcal{S}_1) \to \cdots \tag{2.8}$$

All the maps in the above sequence are canonical except those from i-dimensional cohomology to $(i+1)$-dimensional cohomology. In this section we explain how to construct this map.

Proposition 2.6. *Let X be a topological space and*

$$0 \to \mathcal{S}_1 \xrightarrow{f} \mathcal{S}_2 \xrightarrow{g} \mathcal{S}_3 \to 0$$

be a short exact sequence of sheaves. If $H^1(X, \mathcal{S}_1) = 0$, then the induced sequence in global sections

$$0 \to \mathcal{S}_1(X) \xrightarrow{f} \mathcal{S}_2(X) \xrightarrow{g} \mathcal{S}_3(X) \to 0$$

is exact.

Proof. We want to show the surjectivity of g in global sections. For any $\omega \in \mathcal{S}_3(X)$, there exist an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X such that for every $i \in I$ there exist some $\mu_i \in \mathcal{S}_2(U_i)$ such that $g_{U_i}(\mu_i) = \omega|_{U_i}$. In particular, for every $i, j \in I$

$$\delta(\mu)_{ij} = \mu_j - \mu_i \in \operatorname{Ker} g_{U_{ij}}.$$

In consequence, there exist $\eta_{ij} \in \mathcal{S}_1(U_{ij})$ such that $f_{U_{ij}}(\eta_{ij}) = \delta(\mu)_{ij}$. Then

$$f_{U_{ijk}}((\delta\eta)_{ijk}) = 0,$$

and since $f_{U_{ijk}}$ is injective it follows that $\delta\eta = 0$. Since $H^1(X, \mathcal{S}_1) = 0$, there exist some refinement $\mathcal{U}' \leqslant \mathcal{U}$ and $\zeta \in C^0(\mathcal{U}', \mathcal{S}_1)$ such that $\eta|_{\mathcal{U}'} = \delta\zeta$. In consequence $\mu|_{\mathcal{U}'} - f(\zeta) \in C^0(\mathcal{U}', \mathcal{S}_2)$ and $\delta(\mu|_{\mathcal{U}'} - f(\zeta)) = 0$, so it defines a global section $\widetilde{\mu} \in \mathcal{S}_2(X)$ such that $g(\widetilde{\mu}) = \omega$. $\square$

Corollary 2.1. *Consider over a topological space X a short exact sequence of sheaves*

$$0 \to \mathcal{S}_1 \xrightarrow{f} \mathcal{S}_2 \xrightarrow{g} \mathcal{S}_3 \to 0$$

and an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ such that $H^1(\cap_{l=1}^k U_{i_l}, \mathcal{S}_1) = 0$ for every $\{i_0 < \cdots < i_k\} \subseteq I$ finite not empty. Then there exist a long exact sequence

$$\cdots \to H^i(\mathcal{U}, \mathcal{S}_1) \to H^i(\mathcal{U}, \mathcal{S}_2) \to H^i(\mathcal{U}, \mathcal{S}_3) \to H^{i+1}(\mathcal{U}, \mathcal{S}_1) \to \cdots \tag{2.9}$$

Proof. By Proposition 2.6 we have a short exact sequence of complexes of abelian groups

$$0 \to C^\bullet(\mathcal{U}, \mathcal{S}_1) \to C^\bullet(\mathcal{U}, \mathcal{S}_2) \to C^\bullet(\mathcal{U}, \mathcal{S}_3) \to 0$$

which induces the long exact sequence in cohomology (2.9). $\square$

After Corollary 2.1, we can give an explicit description of the coboundary map

$$H^i(\mathcal{U}, \mathcal{S}_3) \to H^{i+1}(\mathcal{U}, \mathcal{S}_1),$$

for $\mathcal{U}$ an acyclic open cover with respect to $\mathcal{S}_1$. In fact, given any $\omega \in C^i(\mathcal{U}, \mathcal{S}_3)$, there exist some $\mu \in C^i(\mathcal{U}, \mathcal{S}_2)$ given by Proposition 2.6 such that $\omega = g(\mu)$. If $\delta\omega = 0$, then $g(\delta\mu) = 0$ and so $\delta\mu = f(\eta)$ for some $\eta \in C^{i+1}(\mathcal{U}, \mathcal{S}_1)$. Since f is injective and $f(\delta\eta) = 0$, it follows that $\delta\eta = 0$ and so the image of $\omega \in H^i(\mathcal{U}, \mathcal{S}_3)$ under the coboundary map is $\eta \in H^{i+1}(\mathcal{U}, \mathcal{S}_1)$.

Now it is easy to describe the coboundary map

$$H^i(X, \mathcal{S}_3) \to H^{i+1}(X, \mathcal{S}_1),$$

in general. Take any open cover $\mathcal{U}$ of X and any $\omega \in C^i(\mathcal{U}, \mathcal{S}_3)$. There exist some refinement $\mathcal{U}' \leqslant \mathcal{U}$ such that there exist $\mu \in C^i(\mathcal{U}', \mathcal{S}_2)$ such that $g(\mu) = \omega|_{\mathcal{U}'}$. Imitating the construction of the coboundary map explained above, we produce the image of $\omega \in H^i(\mathcal{U}, \mathcal{S}_3)$ under the coboundary map as $\eta \in H^{i+1}(\mathcal{U}', \mathcal{S}_1)$. It is routine to check that this construction is independent of the choices made, and is compatible with restrictions to refinements, thus it determines the coboundary map in Čech cohomology. Now it is an exercise to prove that the long sequence induced is exact.

Proposition 2.7. *For every short exact sequence of sheaves over a topological space X*

$$0 \to \mathcal{S}_1 \xrightarrow{f} \mathcal{S}_2 \xrightarrow{g} \mathcal{S}_3 \to 0,$$

there exist a long exact sequence in Čech cohomology

$$\cdots \to H^i(X, \mathcal{S}_1) \to H^i(X, \mathcal{S}_2) \to H^i(X, \mathcal{S}_3) \to H^{i+1}(X, \mathcal{S}_1) \to \cdots \quad (2.10)$$

2.6 How to compute Čech cohomology groups

Now we are in position to prove Leray's lemma, which allows to compute Čech cohomology groups in terms of acyclic coverings.

Definition 2.8. The covering $\mathcal{U}$ is called **acyclic** with respect to $\mathcal{S}$ if

$$H^p(U_{i_1} \cap \cdots \cap U_{i_k}, \mathcal{S}) = 0$$

for all $U_{i_1}, \ldots, U_{i_k} \in \mathcal{U}$ and $p, k \geqslant 1$.

For a real manifold X of dimension n we have a covering of X such that all the non-empty intersections $U_{i_1} \cap \cdots \cap U_{i_k}$ are diffeomorphic to $\mathbb{R}^n$ (actually we will only need that they are contractible to points). This is also called a **good cover** and its existence follows after constructing a Riemann metric on X, see for instance Bott and Tu (1982, Theorem 5.1, page 42). A good cover is acyclic for constant sheaves.

Theorem 2.1 (Leray's lemma)**.** *Let $\mathcal{U}$ be an acyclic covering of a variety X with respect to a sheaf $\mathcal{S}$. There is a natural isomorphism*

$$H^k(\mathcal{U}, \mathcal{S}) \cong H^k(X, \mathcal{S}).$$

Proof. Let $\mathcal{S}$ be a sheaf over X, and $\mathcal{U}$ an acyclic open cover of X with respect to $\mathcal{S}$. Let $\mathcal{F}$ be the Godement's flasque sheaf associated to $\mathcal{S}$ given in Example 2.4. And let $\mathcal{G} := \mathcal{F}/\mathcal{S}$. We will show by induction on k that the natural map

$$H^k(\mathcal{U}, \mathcal{S}) \to H^k(X, \mathcal{S})$$

is an isomorphism. This is clear for $k = 0$. For $k \geqslant 1$, it follows from Proposition 2.5 that $H^m(\mathcal{U}, \mathcal{F}) = H^m(X, \mathcal{F}) = 0$ for all $m \geqslant 1$. In order to use the induction hypothesis for $\mathcal{G}$, we need to show that $\mathcal{U}$ is an acyclic cover with respect to $\mathcal{G}$. Considering the long exact sequence in Čech cohomology associated to the short exact sequence

$$0 \to \mathcal{S} \to \mathcal{F} \to \mathcal{G} \to 0$$

restricted to each intersection $U_{i_0 \cdots i_p} = \cap_{l=0}^{p} U_{i_l}$, and using the fact that $\mathcal{F}|U_{i_0 \cdots i_p}$ is flasque, we get that

$$H^m(U_{i_0 \cdots i_p}, \mathcal{G}) \simeq H^{m+1}(U_{i_0 \cdots i_p}, \mathcal{S}) = 0, \qquad \forall m \geqslant 1, \forall p \geqslant 0,$$

and so $\mathcal{U}$ is an acyclic cover with respect to $\mathcal{G}$. By induction hypothesis we have the following commutative diagram with exact rows (2.9) and (2.10)

$$\begin{tikzcd}
H^{k-1}(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d, "\wr"] & H^{k-1}(\mathcal{U}, \mathcal{G}) \arrow[r] \arrow[d, "\wr"] & H^k(\mathcal{U}, \mathcal{S}) \arrow[r] \arrow[d] & H^k(\mathcal{U}, \mathcal{F}) = 0 \arrow[r] \arrow[d, "\wr"] & 0 \arrow[d, "\wr"] \\
H^{k-1}(X, \mathcal{F}) \arrow[r] & H^{k-1}(X, \mathcal{G}) \arrow[r] & H^k(X, \mathcal{S}) \arrow[r] & H^k(X, \mathcal{F}) = 0 \arrow[r] & 0
\end{tikzcd}$$

and the result follows from the five lemma. □

For a sheaf of abelian groups $\mathcal{S}$ over a topological space X, we will mainly use $H^1(X, \mathcal{S})$. Recall that for an acyclic covering $\mathcal{U}$ of X an element of $H^1(X, \mathcal{S})$ is represented by

$$f_{ij} \in \mathcal{S}(U_i \cap U_j),\ i, j \in I$$

$$f_{ij} + f_{jk} + f_{ki} = 0,\ f_{ij} = -f_{ji},\ i, j, k \in I$$

It is zero in $H^1(X, \mathcal{S})$ if and only if there are $f_i \in \mathcal{S}(U_i)$, $i \in I$ such that $f_{ij} = f_j - f_i$.

Remark 2.2. For sheaves of abelian groups $\mathcal{S}_i$, $i = 1, 2, \ldots, k$ over a variety X it is easy to see that

$$H^p(\mathcal{U}, \bigoplus_{i \geqslant 1} \mathcal{S}_i) = \bigoplus_{i \geqslant 1} H^p(\mathcal{U}, \mathcal{S}_i), \; p = 0, 1, \ldots$$

for every open cover $\mathcal{U}$ of X. In particular,

$$H^p(X, \bigoplus_{i \geqslant 1} \mathcal{S}_i) = \bigoplus_{i \geqslant 1} H^p(X, \mathcal{S}_i), \; p = 0, 1, \ldots$$

2.7 Homology

Our main example of an abelian sheaf $\mathcal{S}$ is the constant sheaf $\mathbb{Z}$, and in this case it is natural to talk about homologies as these are more adequate to intuition and the historical point of view. In this section we are going to describe the construction of homologies which is not the original one using simplexes, and hence, the names of universal coefficients in homology and cohomology are interchanged in our case. For the original construction see for instance Massey (1991). For a fast overview of homology theories see Movasati (2021, Chapter 4).

Let us fix an abelian group G. For any other abelian group $\tilde{G}$, let $\tilde{G}^\vee$ be its dual abelian group:

$$\tilde{G}^\vee := \left\{ f : \tilde{G} \to G \text{ morphism of abelian groups} \right\}.$$

Our notation of dual is inspired by the case $G = \mathbb{Z}$. For a morphism of two abelian groups $\delta : G_1 \to G_2$, its dual $\delta^\vee : G_2^\vee \to G_1^\vee$ is defined in a natural way. We take the dual of the complex (2.6) and define the set of **p-chains** as $C_i(\mathcal{U}, \mathcal{S}) := C^i(\mathcal{U}, \mathcal{S})^\vee$. We arrive at a new complex

$$0 \leftarrow C_0(\mathcal{U}, \mathcal{S}) \overset{\delta^\vee}{\leftarrow} C_1(\mathcal{U}, \mathcal{S}) \overset{\delta^\vee}{\leftarrow} C_2(\mathcal{U}, \mathcal{S}) \overset{\delta^\vee}{\leftarrow} C_3(\mathcal{U}, \mathcal{S}) \overset{\delta^\vee}{\leftarrow} \cdots \tag{2.11}$$

Note that for simplicity, we have dropped s from the set of p-chains. We also do not mention G in our notations. It is not hard to see that $\delta^\vee \circ \delta^\vee = 0$ and so we have a complex with decreasing indices. The **p-th homology with coefficients in** $\mathcal{S}$ is defined as

$$H_p(\mathcal{U}, \mathcal{S}) := \frac{\text{Kernel}(C_p(\mathcal{U}, \mathcal{S}) \overset{\delta^\vee}{\to} C_{p-1}(\mathcal{U}, \mathcal{S}))}{\text{Image}(C_{p+1}(\mathcal{U}, \mathcal{S}) \overset{\delta^\vee}{\to} C_p(\mathcal{U}, \mathcal{S}))}.$$

The reader is invited to analyze the next theorem just in the case $\mathcal{S} = \mathbb{Z}, \mathbb{Q}$ and $G = \mathbb{Z}, \mathbb{Q}$.

Theorem 2.2 (Universal coefficients theorem in homology)**.** *Let G be an abelian group. We have the following short exact sequence relating homology with cohomology*

$$0 \to \mathrm{Ext}(H^{p+1}(X, \mathcal{S}), G) \to H_p(X, \mathcal{S}) \to \mathrm{Hom}(H^p(X, \mathcal{S}), G) \to 0.$$

It might be also interesting to relate $H^p(X, \mathrm{Hom}(\mathcal{S}, G))$ to $H_p(X, \mathcal{S})$. The proof of Theorem 2.2 is similar to the proof of Massey (1991, Theorem 4.4 page 314). The only difference is that we have started from cohomology and then we have defined homology, whereas Massey has done the other way around. The definition of the functor Ext can be found in Massey (ibid., page 313). The reader might carry in mind that $\mathrm{Ext}(H^{p+1}(X, \mathbb{Z}), \mathbb{Q}) = 0$, and hence,

$$H_p(X, \mathbb{Q}) = H^p(X, \mathbb{Q})^\vee.$$

For a sheaf $\mathcal{S}$ of abelian groups and an abelian group G it is also natural to consider the sheaf of abelian groups $\mathcal{S} \otimes_{\mathbb{Z}} G$ and its cohomologies.

Theorem 2.3 (Universal coefficient theorem in cohomology)**.** *We have the short exact sequence*

$$0 \to H^p(X, \mathcal{S}) \otimes_{\mathbb{Z}} G \to H^p(X, \mathcal{S} \otimes_{\mathbb{Z}} G) \to \mathrm{Tor}(H^{p+1}(X, \mathcal{S}), G) \to 0.$$

Again the proof is similar to the proof of Massey (ibid., Theorem 6.2, page 271).

2.8 Relative Čech cohomology

We consider a pair of topological spaces $Y \subset X$ and a sheaf of abelian groups $\mathcal{S}$ on Y. The **trivial extension** $\widetilde{\mathcal{S}}$ of the sheaf $\mathcal{S}$ to X is defined as follows: for a connected open set $U \subset X$ we have $\widetilde{\mathcal{S}} = \{0\}$ if U does not intersects Y and $= \mathcal{S}(U \cap Y)$ if U intersects Y. If we denote by $i : Y \hookrightarrow X$ the inclusion map, then the trivial extension is just $\widetilde{\mathcal{S}} = i_* \mathcal{S}$. The proof of the next proposition is left to the reader.

Proposition 2.8. *We have $H^p(X, \widetilde{\mathcal{S}}) = H^p(Y, \mathcal{S})$ for all $p \geqslant 0$.*

We usually omit the tilde and simply write $\mathcal{S}$, being clear in the context whether $\mathcal{S}$ is a sheaf on X or Y. Now let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two sheaves of abelian groups over X and Y. We consider the trivial extension of $\mathcal{S}_2$ to X and a surjective morphism $\mathcal{S}_1 \to \mathcal{S}_2$ of abelian sheaves. Our main example for this is $\mathcal{S}_1 = \mathbb{Z}$ $\mathcal{S}_2 = \mathbb{Z}$ and $\mathcal{S}_1 \to \mathcal{S}_2$ is the restriction map. Let $\mathcal{S}$ be the kernel of this map, and hence, we have a short exact sequence

$$0 \to \mathcal{S} \to \mathcal{S}_1 \to \mathcal{S}_2 \to 0. \tag{2.12}$$

In our main example of constant sheaves $\mathcal{S}_1 = \mathbb{Z}$ and $\mathcal{S}_2 = \mathbb{Z}$ we have

$$\mathcal{S}(U) = \begin{cases} \mathbb{Z}, & \text{if } U \cap Y = \emptyset \\ 0 & \text{otherwise} \end{cases} . \tag{2.13}$$

for connected open sets U.

Definition 2.9. We define the **relative cohomology** as

$$H^p(X, Y, \mathbb{Z}) := H^p(X, \mathcal{S}),$$

where $\mathcal{S}$ is the sheaf (2.13).

Let us consider a covering $\mathcal{U}$ of X. Such a covering gives us immediately a covering of Y: $\hat{\mathcal{U}} := \{\hat{U}_i := U_i|_Y\}_{i \in I}$. We would like to have an acyclic covering for the sheaf $\mathcal{S}$. This occurs in our main example (2.13) if all the pairs $(U_{i_0 i_1 \dots i_p}, U_{i_0 i_1 \dots i_p} \cap Y)$ are contractible to points.

Proposition 2.9. *For a pair of topological spaces $Y \subset X$ we have an induced long exact sequence in cohomology*

$$\cdots \to H^{p+1}(Y, \mathbb{Z}) \to H^p(X, Y, \mathbb{Z}) \to H^p(X, \mathbb{Z}) \to H^p(Y, \mathbb{Z}) \to \cdots$$

Proof. It is induced by the short exact sequence of sheaves (2.12). □

The reader who for the first time encounter the notion of Čech cohomology is invited to compute the cohomologies of the n-dimensional sphere:

$$H^m(\mathbb{S}^n, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } m = 0, n \\ 0 & \text{otherwise} \end{cases} , \tag{2.14}$$

$$H^m(\mathbb{B}^{n+1}, \mathbb{S}^n, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } m = n \\ 0, & \text{otherwise} \end{cases} . \tag{2.15}$$

For examples of such a computation see Bott and Tu (1982, page 100).

2.9 Resolution of sheaves

Now we turn to a more conceptual way to compute cohomology groups in terms of acyclic resolutions. Recall that a complex of sheaves of abelian groups is the following data:

$$\mathcal{S}^\bullet : \mathcal{S}^0 \overset{d_0}{\to} \mathcal{S}^1 \overset{d_1}{\to} \cdots \overset{d_{k-1}}{\to} \mathcal{S}^k \overset{d_k}{\to} \cdots$$

where $\mathcal{S}^k$'s are sheaves of abelian groups and $\mathcal{S}^k \to \mathcal{S}^{k+1}$ are morphisms of sheaves of abelian groups such that the composition of two consecutive morphism is zero, i.e

$$d_{k-1} \circ d_k = 0, \ k = 1, 2, \ldots.$$

Definition 2.10. A complex $\mathcal{S}^\bullet$ is called a **resolution** of $\mathcal{S}$ if

$$\mathrm{Im}(d^k) = \ker(d^{k+1}), \ k = 0, 1, 2, \ldots$$

and there exists an injective morphism $i : \mathcal{S} \to \mathcal{S}^0$ such that $\mathrm{Im}(i) = \ker(d^0)$. We write this simply in the form

$$\mathcal{S} \to \mathcal{S}^\bullet$$

For simplicity we will write $d = d_k$, being clear from the context the domain of the map d.

Definition 2.11. A resolution $\mathcal{S} \to \mathcal{S}^\bullet$ is called **acyclic** if all $\mathcal{S}^k$ are acyclic.

Proposition 2.10. *If* $0 \to \mathcal{S}^\bullet$ *is an acyclic resolution, then*

$$0 \to \Gamma(\mathcal{S}^0) \overset{d_0}{\to} \Gamma(\mathcal{S}^1) \overset{d_1}{\to} \cdots \overset{d_{k-1}}{\to} \Gamma(\mathcal{S}^k) \overset{d_k}{\to} \cdots$$

is exact.

Proof. For every $i \geqslant 0$ define the sheaf $\mathcal{F}^i := \mathrm{Im}\, d_i$. We get short exact sequences for every $i \geqslant 1$

$$0 \to \mathcal{F}^{i-1} \to \mathcal{S}^i \to \mathcal{F}^i \to 0.$$

We claim that $\mathcal{F}^i$ is acyclic for every $i \geqslant 0$. In fact, $\mathcal{F}^0 = \mathcal{S}^0$ is acyclic. And for $i \geqslant 1$, it follows from the long exact sequence in Čech cohomology and the fact that $\mathcal{S}^i$ is acyclic, that

$$H^k(X, \mathcal{F}^i) \simeq H^{k+1}(X, \mathcal{F}^{i-1}) = 0, \qquad \forall k \geqslant 1.$$

Then, by Proposition 2.6 we obtain that

$$0 \to \Gamma(\mathcal{F}^{i-1}) \to \Gamma(\mathcal{S}^i) \to \Gamma(\mathcal{F}^i) \to 0$$

is exact, and the result follows. □

The next result tells us how to compute Čech cohomology in terms of the cohomology in global sections of an acyclic resolution of the sheaf. For the reader acquainted with derived functors, this theorem shows that the Čech cohomology is isomorphic to the derived functor of the global sections functor. In other words Čech cohomology is isomorphic to sheaf cohomology.

Theorem 2.4. *Let $\mathcal{S}$ be a sheaf of abelian groups on a topological space X and $\mathcal{S} \to \mathcal{S}^\bullet$ be an acyclic resolution of $\mathcal{S}$ then*

$$H^n(X, \mathcal{S}) \cong H^n(\Gamma(X, \mathcal{S}^\bullet), d), \ n = 0, 1, 2, \dots. \tag{2.16}$$

where

$$\Gamma(X, \mathcal{S}^\bullet) : \Gamma(\mathcal{S}^0) \overset{d_0}{\to} \Gamma(\mathcal{S}^1) \overset{d_1}{\to} \cdots \overset{d_{n-1}}{\to} \Gamma(\mathcal{S}^n) \overset{d_n}{\to} \cdots$$

and

$$H^n(\Gamma(X, \mathcal{S}^\bullet), d) := \frac{\ker(d_n)}{\operatorname{Im}(d_{n-1})}.$$

Proof. We will prove it assuming there exist an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X which is acyclic with respect to $\mathcal{S}$ and all $\mathcal{S}^i, \ i \geqslant 1$. Moreover, all the cohomologies of the complex of global sections of $\mathcal{S}^\bullet$ over intersections of open sets in $\mathcal{U}$ are zero. A more abstract proof can be given similar to the proof of Proposition 2.10, see for instance Gunning 1990c, Corollary D5. Let

$$\mathcal{S}^i_j := C^j(\mathcal{U}, \mathcal{S}^i), \ \ \mathcal{S}_j := C^j(\mathcal{U}, \mathcal{S}), \ \ \Gamma(\mathcal{S}^i) := \Gamma(X, \mathcal{S}^i).$$

Consider the double complex

$$
\begin{array}{ccccccccccccccc}
 & & & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
0 & \to & \mathcal{S}_n & \to & \mathcal{S}_n^0 & \to & \mathcal{S}_n^1 & \to & \mathcal{S}_n^2 & \to & \cdots & \to & \mathcal{S}_n^n & \to & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
0 & \to & \mathcal{S}_{n-1} & \to & \mathcal{S}_{n-1}^0 & \to & \mathcal{S}_{n-1}^1 & \to & \mathcal{S}_{n-1}^2 & \to & \cdots & \to & \mathcal{S}_{n-1}^n & \to & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
0 & \to & \mathcal{S}_2 & \to & \mathcal{S}_2^0 & \to & \mathcal{S}_2^1 & \to & \mathcal{S}_2^2 & \to & \cdots & \to & \mathcal{S}_2^n & \to & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
0 & \to & \mathcal{S}_1 & \to & \mathcal{S}_1^0 & \to & \mathcal{S}_1^1 & \to & \mathcal{S}_1^2 & \to & \cdots & \to & \mathcal{S}_1^n & \to & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
0 & \to & \mathcal{S}_0 & \to & \mathcal{S}_0^0 & \to & \mathcal{S}_0^1 & \to & \mathcal{S}_0^2 & \to & \cdots & \to & \mathcal{S}_0^n & \to & \\
 & & & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
 & & & & \Gamma(\mathcal{S}^0) & \to & \Gamma(\mathcal{S}^1) & \to & \Gamma(\mathcal{S}^2) & \to & \cdots & \to & \Gamma(\mathcal{S}^n) & \to & \\
 & & & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
 & & & & 0 & & 0 & & 0 & & & & 0 & &
\end{array}
\tag{2.17}
$$

The up arrows are δ and the left arrow at $\mathcal{S}_q^p$ is $(-1)^q d$, that is, we have multiplied the map d with $(-1)^q$, where q denotes the index related to Čech cohomology. Let us define the map $A : H^n(\Gamma(X, \mathcal{S}^\bullet), d) \to H^n(X, \mathcal{S})$. It sends a d-closed global section ω of $\mathcal{S}^n$ to a δ-closed cocycle $\alpha \in C^n(\mathcal{U}, \mathcal{S})$ and the recipe is sketched here: We can construct the following sequence of elements in the diagram (2.17) starting from $\omega \in \Gamma(\mathcal{S}^n)$

$$
\begin{array}{ccccccccccccc}
 & & 0 & & & & & & & & & & \\
 & & \uparrow & & & & & & & & & & \\
\alpha & \to & \omega^0 & \to & 0 & & & & & & & & \\
 & & \uparrow & & \uparrow & & & & & & & & \\
 & & \eta^0 & \to & \omega^1 & \to & 0 & & & & & & \\
 & & & & \uparrow & & & & & & & & \\
 & & & & \eta^1 & & \ddots & & \ddots & & & & \\
 & & & & & & \ddots & & \omega^{n-1} & \to & 0 & & \\
 & & & & & & & & \uparrow & & \uparrow & & \\
 & & & & & & & & \eta^{n-1} & \to & \omega^n & \to & 0 \\
 & & & & & & & & & & \uparrow & & \\
 & & & & & & & & & & \omega & &
\end{array}
\tag{2.18}
$$

Where an arrow $a \to b$ means that a is mapped to b under the corresponding map in (2.17), and ω^n is the restriction of ω to opens sets U_i's. The same diagram (2.18) can be used to explain the map $B : H^n(X, \mathcal{S}) \to H^n(\Gamma(X, \mathcal{S}^\bullet), d)$. In this case we start from α and we reach ω. If we check that A and B are well-defined maps, it follows immediately that one is the inverse of the other. Let us check that A is well-defined (for B is similar). Consider a different choice in the diagram

$$
\begin{array}{ccccccccccccc}
 & & 0 & & & & & & & & & & \\
 & & \uparrow & & & & & & & & & & \\
\widetilde{\alpha} & \to & \widetilde{\omega}^0 & \to & 0 & & & & & & & & \\
 & & \uparrow & & \uparrow & & & & & & & & \\
 & & \widetilde{\eta}^0 & \to & \widetilde{\omega}^1 & \to & 0 & & & & & & \\
 & & & & \uparrow & & & & & & & & \\
 & & & & \widetilde{\eta}^1 & & \ddots & & \ddots & & & & \\
 & & & & & & \ddots & & \widetilde{\omega}^{n-1} & \to & 0 & & \\
 & & & & & & & & \uparrow & & \uparrow & & \\
 & & & & & & & & \widetilde{\eta}^{n-1} & \to & \omega^n & \to & 0 \\
 & & & & & & & & & & \uparrow & & \\
 & & & & & & & & & & \omega & &
\end{array}
\tag{2.19}
$$

We have to show that $\alpha = \widetilde{\alpha} \in H^n(X, \mathcal{S})$, in other words that $\alpha - \widetilde{\alpha} = \delta\beta$ for some $\beta \in C^{n-1}(\mathcal{U}, \mathcal{S})$. In order to do this, we have to show first that

$$
\eta^i - \widetilde{\eta}^i = \delta\beta^i + d\gamma^i
$$

for some $\beta^i \in C^{n-2-i}(\mathcal{U}, \mathcal{S}^i)$ and $\gamma^i \in C^{n-1-i}(\mathcal{U}, \mathcal{S}^{i-1})$ for every $i = 0, \dots, n-1$. This is not hard using the following diagram obtained after subtracting (2.18) and (2.19)

$$
\begin{array}{cccccccccccc}
 & & 0 & & & & & & & & & \\
 & & \uparrow & & & & & & & & & \\
\alpha-\widetilde{\alpha} & \to & \omega^0-\widetilde{\omega}^0 & \to & 0 & & & & & & & \\
 & & \uparrow & & \uparrow & & & & & & & \\
 & & \eta^0-\widetilde{\eta}^0 & \to & \omega^1-\widetilde{\omega}^1 & \to & 0 & & & & & \\
 & & & & \uparrow & & & & & & & \\
 & & & & \eta^1-\widetilde{\eta}^1 & & \ddots & \ddots & & & & \\
 & & & & & & \ddots & \omega^{n-1}-\widetilde{\omega}^{n-1} & \to & 0 & & \\
 & & & & & & & \uparrow & & \uparrow & & \\
 & & & & & & & \eta^{n-1}-\widetilde{\eta}^{n-1} & \to & 0 & \to & 0 \\
 & & & & & & & & & \uparrow & & \\
 & & & & & & & & & 0 & &
\end{array}
\tag{2.20}
$$

In fact, it follows from (2.20) that $\eta^{n-1}-\widetilde{\eta}^{n-1} = d\gamma^{n-1}$, and proceeding by decreasing induction on i we are able to show that $\eta^{i-1}-\widetilde{\eta}^{i-1}+(-1)^i\delta\gamma^i$ is d-closed, and so d-exact. In particular we get that $\eta^0-\widetilde{\eta}^0 = \delta\beta^0+\gamma^0$ for some $\gamma^0 \in C^{n-1}(\mathcal{U},\mathcal{S})$ and so $\delta\gamma^0 = \omega^0-\widetilde{\omega}^0 = \alpha-\widetilde{\alpha}$ as desired (note that in our notation we have identified α with ω^0 and $\widetilde{\alpha}$ with $\widetilde{\omega}^0$ by looking at $\mathcal{S}$ as a sub-sheaf of $\mathcal{S}^0$).

In the general case we cannot apply Proposition 2.10 in order to construct the maps A and B. Nevertheless, we can proceed in the same way in order to construct A and B taking care of passing to adequate refinements at each step. It is still an exercise to check that these constructions are well-defined, and that one is the inverse of the other.

□

Remark 2.3. If we do not care about using d or $(-1)^q d$ we will still get isomorphisms A and B, however, they are defined up to multiplication by -1. The mines sign in $(-1)^q d$ is inserted so that $D := \delta + (-1)^q d$ becomes a differential operator, that is, $D \circ D = 0$. For further details see Bott and Tu (1982) Chapter 2. Another way to justify $(-1)^q$ is to see it in the double complex of differential (p,q)-forms in a complex manifold.

2.10 De Rham cohomology

Using different acyclic resolutions of a sheaf we can relate certain sheaf cohomology groups with other classical cohomology groups. Our first example is de Rham cohomology.

Definition 2.12. Let M be a (real) C^∞ manifold. The **de Rham cohomology** groups of M are defined as

$$H^i_{\mathrm{dR}}(M) := H^i(\Gamma(M, \Omega^\bullet_{M^\infty}), d) := \frac{\text{global } \textbf{closed } i\text{-forms on } M}{\text{global } \textbf{exact } i\text{-forms on } M}.$$

We denote by $\Omega^k_{M^\infty}$ the sheaf of C^∞ differential k-forms. Since every C^∞ map between smooth manifolds $f : X \to Y$, induces via pull-back of differential forms a morphism of sheaves $\Omega^i_{Y^\infty} \to f_* \Omega^i_{X^\infty}$ which commutes with the differential operator d. It follows that de Rham cohomology groups are functors from the category of smooth manifolds to the category of vector spaces, and so they are invariant under diffeomorphism. In fact, it is not hard to show that they are invariant under C^∞ homotopy equivalence.

Theorem 2.5. *Let $f, g : X \to Y$ be two homotopic C^∞ maps between smooth manifolds, i.e. such that there exists a C^∞ map $H : X \times \mathbb{R} \to Y$ such that*

$$H(x,t) = \begin{cases} f(x) & \text{if } t \geqslant 1, \\ g(x) & \text{if } t \leqslant 0. \end{cases}$$

Then

$$f^* = g^* : H^i_{\mathrm{dR}}(Y) \to H^i_{\mathrm{dR}}(X).$$

In particular, manifolds of the same homotopy type have isomorphic de Rham cohomology groups.

Proof. Define $s_0, s_1 : X \to X \times \mathbb{R}$ by $s_0(x) := (x, 0)$ and $s_1(x) := (x, 1)$. Then

$$f^* = s_1^* \circ H^* \quad \text{and} \quad g^* = s_0^* \circ H^*.$$

Thus it is enough to show that

$$s_0^* = s_1^* : H^i_{\mathrm{dR}}(X \times \mathbb{R}) \to H^i_{\mathrm{dR}}(X).$$

In fact we will show that both are inverse to π^* where $\pi : X \times \mathbb{R} \to X$ is the first projection. It is clear that $s_0^* \circ \pi^* = s_1^* \circ \pi^* = id_{H^i_{\mathrm{dR}}(X)}$. On the other hand in order to show that $\pi^* \circ s_0^*$ is the identity it is enough to find a map $K : \Gamma(\Omega^\bullet_{(X\times\mathbb{R})^\infty}) \to \Gamma(\Omega^{\bullet-1}_{(X\times\mathbb{R})^\infty})$ such that

$$id_{\Gamma(\Omega^i_{(X\times\mathbb{R})^\infty})} - \pi^* \circ s_0^* = dK + Kd.$$

The desired map is

$$K(\omega)(x,t) = \int_0^t \iota_{\frac{\partial}{\partial t}}(\omega)(x,s)ds.$$

$\square$

Corollary 2.2. *(Poincaré Lemma) If M is contractible (e.g. an open ball of $\mathbb{R}^n$) then*

$$H^i_{\mathrm{dR}}(M) = \begin{cases} \mathbb{R} & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$

Proof. The identity $id : M \to M$ is homotopic to the constant map $c : M \to M$ with $c(x) = x_0$ for all $x \in M$. Then $id^* = c^*$, but $c^*(H^i_{\mathrm{dR}}(M)) = 0$ for $i > 0$ and $c^*(H^0_{\mathrm{dR}}(M)) = \mathbb{R}$.

$\square$

Poincaré lemma implies that the complex of sheaves of differential forms is exact for $i > 0$.

Definition 2.13. Let M be a smooth manifold, the **de Rham resolution** of the constant sheaf $\mathbb{R} \to (\Omega^\bullet_{M^\infty}, d)$ is

$$0 \to \mathbb{R} \to \Omega^0_{M^\infty} \xrightarrow{d} \Omega^1_{M^\infty} \xrightarrow{d} \Omega^2_{M^\infty} \xrightarrow{d} \cdots$$

Theorem 2.6. *The de Rham resolution is acyclic. In consequence*

$$H^i(M, \mathbb{R}) \cong H^i_{\mathrm{dR}}(M).$$

Proof. In Example 2.1 we observed that that the sheaves $\Omega^i_{M^\infty}$ are are fine and so acyclic. The second statement follows from Theorem 2.4.

$\square$

Remark 2.4. In the case M is a complex smooth manifold we consider the sheaves of **complexified** differential forms $\Omega^i_{M^\infty} \otimes \mathbb{C}$. The de Rham cohomology groups with **complex coefficients** are defined as

$$H^i_{\mathrm{dR}}(M, \mathbb{C}) := H^i(\Gamma(\Omega^\bullet_{M^\infty} \otimes \mathbb{C}), d).$$

Since tensoring with $\mathbb{C}$ preserves exact sequences, it follows that

$$H^i_{\mathrm{dR}}(M, \mathbb{C}) \simeq H^i_{\mathrm{dR}}(M) \otimes \mathbb{C},$$

and so the complexified differential forms form the **(complexified) de Rham resolution** of the constant sheaf $\mathbb{C}$. Therefore

$$H^i(M, \mathbb{C}) \simeq H^i_{\mathrm{dR}}(M, \mathbb{C}).$$

From now on, whenever we are dealing with a complex manifold we will simply denote by $\Omega^i_{M^\infty}$ and $H^i_{\mathrm{dR}}(M)$ the complexified differential forms and the de Rham cohomology groups with complex coefficients respectively. In the rest of the text we will only work with complex manifolds and so we will forget to mention explicitly the underlying complexified structure of the C^∞ functions, differential forms and sheaves.

2.11 Singular cohomology

Another classical example of cohomology group, which can be related to a sheaf cohomology, is the singular cohomology. Recall from Movasati (2021, Chapter 4) that given G an abelian group and M a polyhedra, we have the singular cohomology groups $H^k(M, G)$, $k = 0, 1, 2, \ldots$ which satisfy the Eilenberg–Steenrod axioms. One way to relate the singular cohomology with the sheaf cohomology with coefficients in the constant sheaf G is by means of Eilenberg–Steenrod theorem.

Theorem 2.7. *In the category of polyhedra the Čech (or sheaf) cohomology of the constant sheaf G satisfies the Eilenberg–Steenrod axioms.*

Therefore, by uniqueness theorem the Čech cohomology of the sheaf of constants in G is isomorphic to the singular cohomology with coefficients in G. Thus there is no ambiguity in using the same notation to denote singular cohomology with coefficients in the abelian group G and Čech cohomology with coefficients in the constant sheaf G.

We will present another way to describe this isomorphism by means of a resolution of the constant sheaf G. In order to distinguish the singular cohomology groups we will denote them by $H^k_{\text{sing}}(M, G)$.

Definition 2.14. Let M be a topological space, we define the **pre-sheaf of singular cochains** with coefficients in the abelian group G as

$$C^k_{\text{sing}}(U) := \hom(C_k(U), G), \qquad \text{for each } U \subseteq M \text{ open},$$

where $C_k(U)$ denotes the free abelian group of singular chains of U (see for instance Movasati (ibid., Section 4.4)). The **singular cohomology groups** are

$$H^k_{\text{sing}}(M, G) := H^k(C^\bullet_{\text{sing}}(M), \delta),$$

where $\delta : C^k_{\text{sing}}(M) \to C^{k+1}_{\text{sing}}(M)$ is given by $\delta\alpha := \alpha \circ \partial$ and $\partial : C_{k+1}(M) \to C_k(M)$ is the boundary map. We denote by $\mathcal{C}^k_{\text{sing}}$ the **sheaf of singular cochains** which corresponds to the sheaf induced by C^k_{sing}. The operator δ naturally extends to a sheaf morphism $\delta : \mathcal{C}^k_{\text{sing}} \to \mathcal{C}^{k+1}_{\text{sing}}$.

Proposition 2.11. *Consider the sub-pre-sheaf of C^k_{sing} given by*

$$C^k_{\text{sing}}(U)_0 := \{\alpha \in C^k_{\text{sing}}(U) : \exists \mathcal{V} \text{ open cover of } U \text{ and } \alpha|_V = 0, \forall V \in \mathcal{V}\}.$$

Then

$$\mathcal{C}^k_{\text{sing}}(U) = C^k_{\text{sing}}(U)/C^k_{\text{sing}}(U)_0$$

and so $\mathcal{C}^k_{\text{sing}}$ is flasque.

Proof. Let $\mathcal{V} = \{V_i\}_{i\in I}$ be an open covering of some open set $U \subseteq M$. Given $\alpha_i \in C^k_{\text{sing}}(V_i)$ such that $\alpha_i|_{V_i\cap V_j} = \alpha_j|_{V_i \cap V_j}$ for all $i, j \in I$. It is clear that there exists a unique $\alpha \in \hom(C^{\mathcal{V}}_k(U), G)$ such that $\alpha|_{V_i} = \alpha_i$, where

$$C^{\mathcal{V}}_k(U) := \sum_{i\in I} C_k(V_i) \subset C_k(U)$$

is the group of $\mathcal{V}$-small singular chains of U. Therefore there exists a unique extension of α to $C^k_{\text{sing}}(U)$ modulo the subgroup

$$\{\alpha \in C^k_{\text{sing}}(U) : \alpha|_{C^{\mathcal{V}}_k(U)} = 0\}$$

and so the sheaf induced by C^k_{sing} is given in each open set U by $C^k_{\mathrm{sing}}(U)/C^k_{\mathrm{sing}}(U)_0$ as claimed. In order to see that $\mathcal{C}^k_{\mathrm{sing}}$ is flasque, it is enough to see that the restriction map $C^k_{\mathrm{sing}}(M) \to C^k_{\mathrm{sing}}(U)$ is surjective. □

Theorem 2.8. *Let M be a locally contractible topological space. Then the sheaf complex*

$$0 \to G \to \mathcal{C}^0_{\mathrm{sing}} \xrightarrow{\delta} \mathcal{C}^1_{\mathrm{sing}} \xrightarrow{\delta} \mathcal{C}^2_{\mathrm{sing}} \xrightarrow{\delta} \cdots$$

is exact, i.e. $(\mathcal{C}^\bullet_{\mathrm{sing}}, \delta)$ is a resolution of the constant sheaf G. In consequence

$$H^k(M, G) \simeq H^k_{\mathrm{sing}}(M, G).$$

Proof. In order to show the exactness of the sequence it is enough to show that the pre-sheaf sequence

$$0 \to G \to C^0_{\mathrm{sing}} \xrightarrow{\delta} C^1_{\mathrm{sing}} \xrightarrow{\delta} C^2_{\mathrm{sing}} \xrightarrow{\delta} \cdots$$

is exact at the stalks. Since M is locally contractible, it is enough to show that for every contractible open sets $U \subset M$ the sequence

$$0 \to G \to C^0_{\mathrm{sing}}(U) \xrightarrow{\delta} C^1_{\mathrm{sing}}(U) \xrightarrow{\delta} C^2_{\mathrm{sing}}(U) \xrightarrow{\delta} \cdots$$

is exact. This is equivalent to show that $H^k_{\mathrm{sing}}(U, G) = 0$ for $k > 0$ and $H^0_{\mathrm{sing}}(U, G) \simeq G$ which follows from the contractibility of U.

It follows that $(\mathcal{C}^\bullet_{\mathrm{sing}}, \delta)$ is a flasque resolution of G and so

$$H^k(M, G) \simeq H^k(\Gamma(\mathcal{C}^\bullet_{\mathrm{sing}}), \delta) = H^k(C^\bullet_{\mathrm{sing}}(M)/C^\bullet_{\mathrm{sing}}(M)_0, \delta).$$

In order to finish the proof it is enough to show that the natural projection

$$\pi : C^\bullet_{\mathrm{sing}}(M) \to C^\bullet_{\mathrm{sing}}(M)/C^\bullet_{\mathrm{sing}}(M)_0$$

is a quasi-isomorphism. Using the long exact sequence in cohomology we see that this is equivalent to show that $(C^\bullet_{\mathrm{sing}}(M)_0, \delta)$ is exact. Let $\alpha \in C^k_{\mathrm{sing}}(M)_0$ such that $\delta\alpha = 0$. Let $\mathcal{V}$ be an open covering of M such that $\alpha|_{C^{\mathcal{V}}_k(M)} = 0$. Given any singular chain $\gamma \in C_k(M)$ we can use barycentric subdivisions to produce a singular chain $\gamma_0 \in C^{\mathcal{V}}_k(M)$ such that

$$\gamma - \gamma_0 = \partial\lambda$$

for some $\lambda \in C_{k+1}(M)$. Thus $\alpha(\gamma) = \alpha(\gamma_0) + \delta\alpha(\lambda) = 0$, and so $\alpha = 0$. □

There is also a resolution for the relative cohomology. Recall that for a pair of topological spaces $Y \subseteq X$ and an abelian group G, if we denote by $i : Y \hookrightarrow X$ the inclusion then the relative cohomology is defined by

$$H^k(X, Y, G) := H^k(X, G_{X,Y}),$$

where $G_{X,Y}$ is a sheaf over X given by

$$0 \to G_{X,Y} \to G_X \to i_* G_Y \to 0,$$

with G_X and G_Y the constant sheaves over X and Y respectively.

Corollary 2.3. *Let X be a locally contractible topological space and $Y \subseteq X$ be also locally contractible. Denoting by $\mathcal{C}^k_{\text{sing}}(X)$ and $\mathcal{C}^k_{\text{sing}}(Y)$ the sheaves of singular cochains over X and Y respectively, consider*

$$(\mathcal{C}^\bullet_{\text{sing}}(X, Y), \delta) := \ker((\mathcal{C}^\bullet_{\text{sing}}(X), \delta) \to (i_* \mathcal{C}^\bullet_{\text{sing}}(Y), \delta)).$$

Then $(\mathcal{C}^\bullet_{\text{sing}}(X, Y), \delta)$ is an acyclic resolution of $G_{X,Y}$ and so

$$H^k(X, Y, G) \simeq H^k_{\text{sing}}(X, Y, G).$$

Proof. It is clear that the restriction morphism is an epimorphism and so we have the short exact sequence of complexes

$$0 \to \mathcal{C}^\bullet_{\text{sing}}(X, Y) \to \mathcal{C}^\bullet_{\text{sing}}(X) \to i_* \mathcal{C}^\bullet_{\text{sing}}(Y) \to 0.$$

Since the other two are resolutions of the respective constant sheaves it follows form the long exact sequence (of sheaves) in cohomology that $\mathcal{C}^\bullet_{\text{sing}}(X, Y)$ is a resolution of $G_{X,Y}$. On the other hand taking the long exact sequence in sheaf cohomology induced by the short exact sequence of sheaves we have that $\mathcal{C}^k_{\text{sing}}(X, Y)$ is acyclic for all $k \geqslant 0$. Note that for each open set $U \subseteq X$ we have the short exact sequence

$$0 \to \Gamma(U, \mathcal{C}^\bullet_{\text{sing}}(X, Y)) \to \Gamma(U, \mathcal{C}^\bullet_{\text{sing}}(X)) \to \Gamma(U \cap Y, \mathcal{C}^\bullet_{\text{sing}}(Y)) \to 0.$$

Taking the long exact sequences induced by the following diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}^\bullet_{\text{sing}}(X, Y) \arrow[r] \arrow[d, "f"] & \mathcal{C}^\bullet_{\text{sing}}(X) \arrow[r] \arrow[d, "g"] & \mathcal{C}^\bullet_{\text{sing}}(Y) \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & \Gamma(\mathcal{C}^\bullet_{\text{sing}}(X, Y)) \arrow[r] & \Gamma(\mathcal{C}^\bullet_{\text{sing}}(X)) \arrow[r] & \Gamma(\mathcal{C}^\bullet_{\text{sing}}(Y)) \arrow[r] & 0
\end{tikzcd}$$

it follows that f is a quasi-isomorphism (since g and h are both quasi-isomorphisms). Thus, we conclude that

$$H^k(X,Y,G) \simeq H^k(\Gamma(\mathcal{C}^\bullet_{\text{sing}}(X,Y)),\delta) \simeq H^k(C^\bullet_{\text{sing}}(X,Y),\delta) = H^k_{\text{sing}}(X,Y,G).$$

□

In the case $G = \mathbb{R}$ (or $\mathbb{C}$) we have the following isomorphisms for M a manifold

$$H^k_{\text{dR}}(M) \simeq H^k(M,\mathbb{R}) \simeq H^k_{\text{sing}}(M,\mathbb{R}).$$

This isomorphism can be described directly in terms of integration

$$H_k^{\text{sing}}(M,\mathbb{Z}) \times H^k_{\text{dR}}(M) \to \mathbb{R},\ (\delta,\omega) \mapsto \int_\delta \omega$$

This gives us

$$H^k_{\text{dR}}(M) \to H_k^{\text{sing}}{}^{\check{}}(M,\mathbb{R}) \cong H^k_{\text{sing}}(M,\mathbb{R})$$

where ˇ means dual of vector space.

Theorem 2.9. *The integration map gives us an isomorphism*

$$H^k_{\text{dR}}(M) \cong H^k_{\text{sing}}(M,\mathbb{R})$$

Under this isomorphism the cup product corresponds to

$$H^i_{\text{dR}}(M) \times H^j_{\text{dR}}(M) \to H^{i+j}_{\text{dR}}(M),\ (\omega_1,\omega_2) \mapsto \omega_1 \wedge \omega_2,\ i,j = 0,1,2,\dots$$

where $\wedge$ is the wedge product of differential forms.

If M is an oriented manifold of dimension n then we have the following bilinear map

$$H^i_{\text{dR}}(M) \times H^{n-i}_{\text{dR}}(M) \to \mathbb{R},\ (\omega_1,\omega_2) \mapsto \int_M \omega_1 \wedge \omega_2,\ i = 0,1,2,\dots$$

2.12 Dolbeault cohomology

Let M be a complex manifold and $\Omega_M^{p,q}$ be the sheaf of C^∞ differential (p,q)-forms on M. We have the complex

$$\Omega_M^{p,0} \xrightarrow{\bar{\partial}} \Omega_M^{p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Omega_M^{p,q} \xrightarrow{\bar{\partial}} \cdots$$

and the **Dolbeault cohomology** of M is defined to be

$$H_{\bar{\partial}}^{p,q}(M) := H^q(\Gamma(M, \Omega_M^{p,\bullet}), \bar{\partial}) = \frac{\text{global } \bar{\partial}\text{-closed } (p,q)\text{-forms on } M}{\text{global } \bar{\partial}\text{-exact } (p,q)\text{-forms on } M}.$$

The proof of the following classical theorem due to Dolbeault can be found in Gunning (1990a, Chapter E, Theorem 5).

Theorem 2.10 (Dolbeault Lemma). *If M is the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the complex line $\mathbb{C}$ or a product of one dimensional disks and complex lines then $H_{\bar{\partial}}^{p,q}(M) = 0$ for $q > 0$.*

Let $\Omega_{M^{\mathrm{an}}}^p$ be the sheaf of holomorphic p-forms on M. By Example 2.1 we know that $\Omega_M^{p,q}$'s are fine sheaves and so we have the resolution of $\Omega_{M^{\mathrm{an}}}^p$:

$$\Omega_{M^{\mathrm{an}}}^p \to \Omega_M^{p,\bullet}.$$

By Theorem 2.4 we conclude that:

Theorem 2.11 (Dolbeault theorem). *For M a complex manifold*

$$H^q(M, \Omega_{M^{\mathrm{an}}}^p) \cong H_{\bar{\partial}}^{p,q}(M).$$

There are examples of domains D in $\mathbb{C}^n$ such that $H_{\bar{\partial}}^{0,1}(D) \neq 0$. See Gunning (1990b, end of Chapter E).

2.13 Čech resolution of a sheaf

The relation between Čech cohomology and sheaf cohomology can also be expressed in terms of a resolution, as we explain in this section. Let $\mathcal{S}$ be a sheaf of abelian groups over a topological space X and $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X. For every open subset $U \subseteq X$ we define the covering $\mathcal{U}|_U := \{U_i \cap U\}_{i \in I}$ of U. The **sheaf of Čech cochains** is defined as

$$\mathcal{C}^k(\mathcal{U}, \mathcal{S})(U) := C^k(\mathcal{U}|_U, \mathcal{S}),$$

with the natural restriction maps. The boundary map induces a sheaf morphism

$$\delta : \mathcal{C}^k(\mathcal{U}, \mathcal{S}) \to \mathcal{C}^{k+1}(\mathcal{U}, \mathcal{S}).$$

Proposition 2.12. *The sheaf complex* $(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{S}), \delta)$ *is a resolution of the sheaf* $\mathcal{S}$.

Proof. Certainly $\ker(\delta : \mathcal{C}^0(\mathcal{U}, \mathcal{S}) \to \mathcal{C}^1(\mathcal{U}, \mathcal{S})) = \mathcal{S}$. For $k \geqslant 1$, let us show that

$$\mathcal{C}^{k-1}(\mathcal{U}, \mathcal{S}) \xrightarrow{\delta} \mathcal{C}^k(\mathcal{U}, \mathcal{S}) \xrightarrow{\delta} \mathcal{C}^{k+1}(\mathcal{U}, \mathcal{S})$$

is exact. In order to do this it is enough to show that for every $x \in X$ and every small enough open neighbourhood U of x,

$$C^{k-1}(\mathcal{U}|_U, \mathcal{S}) \xrightarrow{\delta} C^k(\mathcal{U}|_U, \mathcal{S}) \xrightarrow{\delta} C^{k+1}(\mathcal{U}|_U, \mathcal{S})$$

is exact. Let $x \in X$, then $x \in U_i$ for some $i \in I$. Consider any open neighbourhood U of x such that $U \subseteq U_i$ and let $\sigma \in \ker(\delta : C^k(\mathcal{U}|_U, \mathcal{S}) \to C^{k+1}(\mathcal{U}|_U, \mathcal{S}))$. Then

$$\sigma_{i_0 \cdots i_k}|_{U_{i i_0 \cdots i_k} \cap U} = \sum_{l=0}^{k} (-1)^l \sigma_{i i_0 \cdots \widehat{i_l} \cdots i_k}|_{U_{i i_0 \cdots i_k} \cap U}.$$

Since $U \subseteq U_i$ it follows that $U_{i i_0 \cdots i_k} \cap U = U_{i_0 \cdots i_k} \cap U$. Thus defining $\tau_{i_0 \cdots i_{k-1}} := \sigma_{i i_0 \cdots i_{k-1}}|_{U_{i_0 \cdots i_{k-1}} \cap U}$ it follows that $\delta\tau = \sigma$. $\square$

As a consequence we see that the Čech resolution is an acyclic resolution of $\mathcal{S}$ if and only if the covering $\mathcal{U}$ is acyclic with respect to $\mathcal{S}$, and so in such case we have

$$H^k(X, \mathcal{S}) \simeq H^k(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{S}), \delta) = H^k(\mathcal{U}, \mathcal{S}).$$

Another consequence of the Čech resolution is the sheaf cohomology version of Mayer–Vietoris sequence.

Proposition 2.13. *Let X be a topological space and $\mathcal{S}$ be a sheaf over X. Given two open sets $U, V \subseteq X$ such that $U \cup V = X$ we have a Mayer–Vietoris long exact sequence*

$$\cdots \to H^k(X,\mathcal{S}) \to H^k(U,\mathcal{S}) \oplus H^k(V,\mathcal{S}) \to H^k(U \cap V,\mathcal{S}) \to H^{k+1}(X,\mathcal{S}) \to \cdots$$

Proof. Just note that taking the covering $\mathcal{U} = \{U, V\}$, the Čech resolution gives us a short exact sequence of sheaves

$$0 \to \mathcal{S} \to \mathcal{C}^0(\mathcal{U},\mathcal{S}) = \mathcal{S}|_U \oplus \mathcal{S}|_V \to \mathcal{C}^1(\mathcal{U},\mathcal{S}) = \mathcal{S}|_{U\cap V} \to 0,$$

and the long sequence induced in cohomology is Mayer–Vietoris sequence. □

2.14 Cohomology of manifolds

The first natural sheaves are constant sheaves. For an abelian group G, the sheaf of constants on X with coefficients in G is a sheaf such that for any connected open set it associates G and the restriction maps are the identity. We also denote by G the corresponding sheaf. Our main examples are $(R, +)$, $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$. For a smooth manifold X, the cohomology groups $H^i(X, G)$ are isomorphic in a natural way to singular cohomology and de Rham cohomology groups, see Theorem 2.8 and Theorem 2.6 respectively. We will need the following topological statements.

Proposition 2.14. *Let X be a topological space which is contractible to a point. Then $H^p(X, G) = 0$ for all $p > 0$.*

This statement follows from another statement which says that two homotopic maps induce the same map in cohomology.

Proposition 2.15. *Let X be a manifold of dimension n. Then X has a covering $\mathcal{U} = \{U_i,\ i \in I\}$ such that*

1. *all U_i's and their intersections are contractible to points.*

2. *The intersection of any $n + 2$ open sets U_i is empty.*

Proof. The first part is proved in Bott and Tu (1982, Theorem 5.1 page 42). □

Using both propositions we get an acyclic covering of a manifold and we prove.

Proposition 2.16. *Let M be an orientable manifold of dimension m.*

1. *We have $H^i(M, \mathbb{Z}) = 0$ for $i > m$.*
2. *If M is not compact then the top cohomology $H^m(M, \mathbb{Z})$ is zero.*
3. *If M is compact then we have a canonical isomorphism $H^m(M, \mathbb{Z}) \cong \mathbb{Z}$ given by the orientation of M.*

If M is a complex manifold of dimension n, then it has a canonical orientation given by the orientation of $\mathbb{C}$ and so we can apply the above proposition in this case. Note that M is of real dimension $m = 2n$.

3 Hypercohomology

My mathematics work is proceeding beyond my wildest hopes, and I am even a bit worried - if it's only in prison that I work so well, will I have to arrange to spend two or three months locked up every year? (André Weil writes from Rouen prison, O'Connor and Robertson (2016)).

3.1 Introduction

After a fairly complete understanding of singular homology and de Rham cohomology of manifolds and the invention of Čech cohomology, a new wave of abstraction in mathematics started. Cartan and Eilenberg (1956) in their foundational book called *Homological Algebra* took many ideas from topology and replaced it with categories and functors. Grothendieck (1957) took this into a new level of abstraction and the by-product of his effort was the creation of many cohomology theories, such as Étale and algebraic de Rham cohomology. Étale cohomology was mainly created in order to solve Weil conjectures, however, the relevant one to integrals is the algebraic de Rham cohomology. Its main ingredient is the concept of hypercohomology of complexes of sheaves which soon after its creation was replaced with derived functors and derived categories. This has made it an abstract concept far beyond concrete computations and the situation is so that the introduction of C. A. Weibel's book, Weibel (1994), starts with: "Homological algebra is a tool used to prove nonconstructive existence theorems in algebra (and in algebraic topology)." In this chapter we aim to introduce hypercohomology without going into categorical approach, and the main reason for this is that we would like to emphasize that its elements can be computed and in the case of complex of differential forms, they can be integrated over topological cycles. For further information the reader is referred to EGA III 6.2.2, Brylinski (2008), Grothendieck (1966), and Voisin (2003).

3.2 Hypercohomology of complexes

Let us be given a complex of sheaves of abelian groups on a topological space X:

$$\mathcal{S}^\bullet: \quad \mathcal{S}^0 \xrightarrow{d} \mathcal{S}^1 \xrightarrow{d} \mathcal{S}^2 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{S}^n \xrightarrow{d} \cdots, \quad d \circ d = 0. \tag{3.1}$$

We would like to associate to $\mathcal{S}^\bullet$ a cohomology which encodes all the Čech cohomologies of individual $\mathcal{S}^i$ together with the differential operators d. We first explain this cohomology using a covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X.

Consider the double complex

$$\begin{array}{ccccccccccc}
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
\mathcal{S}^0_n & \to & \mathcal{S}^1_n & \to & \mathcal{S}^2_n & \to & \cdots & \to & \mathcal{S}^n_n & \to & \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
\mathcal{S}^0_{n-1} & \to & \mathcal{S}^1_{n-1} & \to & \mathcal{S}^2_{n-1} & \to & \cdots & \to & \mathcal{S}^n_{n-1} & \to & \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
\vdots & & \vdots & & \vdots & & & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
\mathcal{S}^0_2 & \to & \mathcal{S}^1_2 & \to & \mathcal{S}^2_2 & \to & \cdots & \to & \mathcal{S}^n_2 & \to & \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
\mathcal{S}^0_1 & \to & \mathcal{S}^1_1 & \to & \mathcal{S}^2_1 & \to & \cdots & \to & \mathcal{S}^n_1 & \to & \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
\mathcal{S}^0_0 & \to & \mathcal{S}^1_0 & \to & \mathcal{S}^2_0 & \to & \cdots & \to & \mathcal{S}^n_0 & \to &
\end{array} \tag{3.2}$$

where

$$\mathcal{S}^i_j := C^j(\mathcal{U}, \mathcal{S}^i).$$

The horizontal arrows are the usual differential operators d of $\mathcal{S}^i$'s and the vertical arrows are the differential operators δ in the sense of Čech cohomology. The m-th piece of the total chain of (3.2) is

$$\mathcal{L}^m := \oplus_{i=0}^m \mathcal{S}_{m-i}^i$$

with the differential operator D which is defined on $\mathcal{S}_j^i$ by:

$$D = \delta + (-1)^j d. \tag{3.3}$$

Remark 3.1. Our convention for D is compatible with the one used in Bott and Tu (1982, page 90) and Brylinski (2008, page 14). In other references such as Voisin (2003), we also find $D' = (-1)^i \delta + d$. This difference produces a sign ambiguity. In order to correct this ambiguity it is necessary to identify both hypercohomologies via the isomorphism $H^k(\mathcal{L}^\bullet, D) \simeq H^k(\mathcal{L}^\bullet, D')$ induced by the map

$$\varphi : \mathcal{L}^k \to \mathcal{L}^k,$$

defined over each $\omega = \sum_{i=0}^k \omega^i \in \oplus_{i=0}^k \mathcal{S}_j^i$ as

$$\varphi(\omega^i) := (-1)^{i(k-i)} \omega^i.$$

Exercise 3.1. *For the operator D in* (3.3) *of the double complex* (3.2) *show that* $D \circ D = 0$.

This also justifies the appearance of the sign $(-1)^j$ in the definition of D.

Definition 3.1. The **hypercohomology of the complex $\mathcal{S}^\bullet$ relative to the covering $\mathcal{U}$** is defined as

$$\mathbb{H}^m(\mathcal{U}, \mathcal{S}^\bullet) := H^m(\mathcal{L}^\bullet, D) = \frac{\ker(\mathcal{L}^m \to \mathcal{L}^{m+1})}{\operatorname{Im}(\mathcal{L}^{m-1} \to \mathcal{L}^m)}.$$

Remark 3.2. As in the case of Čech cohomology, given $\mathcal{U}_1 \leqslant \mathcal{U}_2$, there is a well-defined map

$$\mathbb{H}^m(\mathcal{U}_2, \mathcal{S}^\bullet) \to \mathbb{H}^m(\mathcal{U}_1, \mathcal{S}^\bullet).$$

In fact, if $\mathcal{U}_i = \{U_{i,j}, j \in I_i\}$ for $i = 1, 2$, given any refinement map $\phi : I_1 \to I_2$, we have an induced map

$$\Phi : \oplus_{i=0}^m C^i(\mathcal{U}_2, \mathcal{S}^{m-i}) \to \oplus_{i=0}^m C^i(\mathcal{U}_1, \mathcal{S}^{m-i}).$$

This map induces a map in hypercohomology

$$\tilde{\Phi} : \mathbb{H}^m(\mathcal{U}_2, \mathcal{S}^\bullet) \to \mathbb{H}^m(\mathcal{U}_1, \mathcal{S}^\bullet)$$

which does not depend on the refinement map. In fact, given other refinement map $\psi : I_1 \to I_2$, there exists a homotopy

$$H : \oplus_{i=0}^{m+1} C^i(\mathcal{U}_2, \mathcal{S}^{m+1-i}) \to \oplus_{i=0}^{m} C^i(\mathcal{U}_1, \mathcal{S}^{m-i})$$

such that

$$\Phi - \Psi = D \circ H + H \circ D.$$

It is an exercise left to the reader to check that this homotopy map is the one induced by the homotopy map (2.7) described for Čech cohomology (it is enough to see that the homotopy commutes with the differentials d).

Definition 3.2. The **hypercohomology** $\mathbb{H}^m(X, \mathcal{S}^\bullet)$ is defined to be the direct limit of the total cohomology of the double complex (3.2), i.e.

$$\mathbb{H}^m(X, \mathcal{S}^\bullet) \cong \operatorname{dirlim}_{\mathcal{U}} \mathbb{H}^m(\mathcal{U}, \mathcal{S}^\bullet).$$

From a computational point of view this definition is not very useful. We have to look for coverings $\mathcal{U}$ such that $\mathbb{H}^m(\mathcal{U}, \mathcal{S}^\bullet)$ becomes the hypercohomology itself.

Theorem 3.1. *If the covering $\mathcal{U}$ is acyclic with respect to all abelian sheaves $\mathcal{S}^i$'s, that is,*

$$H^k(U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_r}, \mathcal{S}^i) = 0, \ k, r = 1, 2, \ldots, \ i = 0, 1, 2, \ldots . \tag{3.4}$$

then

$$\mathbb{H}^m(X, \mathcal{S}^\bullet) \cong \mathbb{H}^m(\mathcal{U}, \mathcal{S}^\bullet). \tag{3.5}$$

Definition 3.3. A covering $\mathcal{U}$ is said to be a **good covering** with respect to a complex of sheaves $\mathcal{S}^\bullet$, if it satisfies (3.4).

By definition of the direct limit, we have already a map

$$\mathbb{H}^m(\mathcal{U}, \mathcal{S}^\bullet) \to \mathbb{H}^m(X, \mathcal{S}^\bullet)$$

which sends an α to its equivalence class. We have to show that it is a bijection. The proof of this result will be given in Section 3.5.

Before proceeding further, let us mention one of our main examples in this book. We take a smooth projective variety $X \subset \mathbb{P}^N$ of dimension n over an algebraically closed field k. In the next chapter we will introduce the complex $\Omega_X^\bullet$ of algebraic differential forms on X. These sheaves are coherent and so by Theorem 4.6 affine coverings are good coverings.

Proposition 3.1. *There is a covering of X with $n+1$ affine Zariski open subsets.*

Proof. The covering is going to be

$$U_i := \{g_i \neq 0\}, \quad i = 0, 1, 2, \dots, n,$$

where g_i's are linear homogeneous polynomials in $x_0, x_1, \dots, x_N$ and we have assumed that the projective space $\mathbb{P}^{N-n-1} \subset \mathbb{P}^N$ given by $g_0 = g_1 = \cdots = g_{n+1} = 0$ does not intersect X. This happens for a generic $\mathbb{P}^{N-n-1}$. For instance for a generic linear $\mathbb{P}^{N-n}$ intersects X in $\deg(X)$ distinct points. This is the definition of the degree of a projective variety. Now we can take $\mathbb{P}^{N-n-1} \subset \mathbb{P}^{N-n}$ such that it does not cross the mentioned $\deg(X)$ points. □

For a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ given by the homogeneous degree d polynomial $f(x_0, x_1, \dots, x_{n+1})$, we have also another useful covering given by

$$U_i := \left\{ \frac{\partial f}{\partial x_i} \neq 0 \right\}, \quad i = 0, 1, 2, \dots, n+1$$

which is called the **Jacobian covering** of X. Note that for this covering we use the fact that X is smooth. It has $n+2$ open sets. For a fixed $k = 0, 1, 2, \dots, n+1$ the open sets $U_0, U_1, \dots, U_{k-1}, U_{k+1}, \dots, U_{n+1}$ cover X if X is smooth and $x_k = 0$ intersects X transversely.

3.3 An element of hypercohomology

The reader who is mainly interested in computational aspects of hypercohomology may take (3.5) as the definition of hypercohomology. In this way we can describe its elements explicitly. An element of $\mathbb{H}^m(X, \mathcal{S}^\bullet)$ is represented by

$$\omega = \omega^0 + \omega^1 + \cdots + \omega^m, \quad \omega^j \in C^{m-j}(\mathcal{U}, \mathcal{S}^j),$$

where $\mathcal{U}$ is a good covering with respect to $\mathcal{S}^\bullet$. Each ω^j itself is the following data:

$$\omega^j_{i_0 i_1 \cdots i_{m-j}} \in \mathcal{S}^j(U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_{m-j}})$$

for all $i_0, i_1, \cdots, i_{m-j} \in I$. Such an ω is D-closed, that is, $D(\omega) = 0$, if and only if, the following equalities hold

$$
\begin{array}{rcl}
0 & = & \delta\omega^0, \\
(-1)^{m-1} d\omega^0 & = & \delta(\omega^1), \\
(-1)^{m-2} d\omega^1 & = & \delta(\omega^2), \\
 & \vdots & \\
d\omega^{m-1} & = & \delta(\omega^m), \\
d\omega^m & = & 0.
\end{array}
\tag{3.6}
$$

Such an ω is D-exact, or equivalently it is zero in $\mathbb{H}^m(X, \mathcal{S}^\bullet)$, if and only if there is $\eta = \sum_{j=0}^{m-1} \eta^j, \quad \eta^j \in C^{m-1-j}(\mathcal{U}, \mathcal{S}^j)$ such that

$$
\begin{array}{rcl}
\omega^0 & = & \delta(\eta^0), \\
\omega^1 & = & (-1)^{m-1} d(\eta^0) + \delta(\eta^1), \\
 \vdots & & \\
\omega^{m-1} & = & -d(\eta^{m-2}) + \delta(\eta^{m-1}), \\
\omega^m & = & d\eta_{m-1}.
\end{array}
\tag{3.7}
$$

In order to memorize better these equalities the diagram below can be helpful

$$
\begin{array}{cccccc}
0 & & & & & \\
\omega^0 & 0 & & & & \\
\eta^0 & \omega^1 & 0 & & & \\
 & \eta^1 & \ddots & \ddots & & \\
 & & \ddots & \omega^{m-1} & 0 & \\
 & & & \eta^{m-1} & \omega^m & 0
\end{array}
\tag{3.8}
$$

Recall that one must use $(-1)^j d$ for horizontal map and δ for vertical maps. It is instructive to consider the cases $m = 0, 1, 2$ separately.

- ($m = 0$) We have

$$
\mathbb{H}^0(X, \mathcal{S}^\bullet) \cong \left\{\omega \in \mathcal{S}^0(X) \mid d\omega = 0\right\}.
$$

- ($m = 1$) The first hypercohomology $\mathbb{H}^1(X, \mathcal{S}^\bullet)$ is the set of pairs (ω^0, ω^1), where ω^0 consists of $\omega^0_{i_0 i_1} \in \mathcal{S}^0(U_{i_0} \cap U_{i_1}),\ \ i_0, i_1 \in I$ and ω^1 consists of $\omega^1_{i_0} \in \mathcal{S}^1(U_{i_0}),\ \ i_0 \in I$ which satisfy the relation

$$\omega^1_{i_1} - \omega^1_{i_0} = d\omega^0_{i_0 i_1}.$$

  Such an ω is taken modulo those of the form $(f_{i_1} - f_{i_0}, df_{i_0})$.

- ($m = 2$) The second hypercohomology $\mathbb{H}^2(X, \mathcal{S}^\bullet)$ is the set of triples $\omega = (\omega^0, \omega^1, \omega^2)$, where ω^0 consists of $\omega^0_{i_0 i_1 i_2} \in \mathcal{S}^0(U_{i_0} \cap U_{i_1} \cap U_{i_2}),\ \ i_0, i_1, i_2 \in I$, ω^1 consists of $\omega^1_{i_0 i_1} \in \mathcal{S}^1(U_{i_0} \cap U_{i_1}),\ \ i_0, i_1 \in I$ and ω^2 consists of $\omega^2_{i_0} \in \mathcal{S}^2(U_{i_0}),\ \ i_0 \in I$. They satisfy the relations

$$\begin{aligned}
&\omega^0_{i_1 i_2 i_3} - \omega^0_{i_0 i_2 i_3} + \omega^0_{i_0 i_1 i_3} - \omega^0_{i_0 i_1 i_2} = 0,\\
&\omega^1_{i_1 i_2} - \omega^1_{i_0 i_2} + \omega^1_{i_0 i_1} \ = -d\omega^0_{i_0 i_1 i_2},\\
&\omega^2_{i_1} - \omega^2_{i_0} = d\omega^1_{i_1 i_0}.
\end{aligned}$$

  Such an ω is zero in $\mathbb{H}^2(X, \mathcal{S}^\bullet)$ if

$$\omega^0_{i_0 i_1 i_2} = \eta^0_{i_1 i_2} - \eta^0_{i_0 i_2} + \eta^0_{i_0 i_1}, \tag{3.9}$$

$$\omega^1_{i_0 i_1} = -d\eta^0_{i_0 i_1} + \eta^1_{i_1} - \eta^1_{i_0}, \tag{3.10}$$

$$\omega^2_{i_0} = d\eta^1_{i_0}, \tag{3.11}$$

  for some η's whose type can be determined by their indices.

When the covering $\mathcal{U} := \{U_0, U_1, \ldots, U_n\}$ consists of n open sets and $\mathcal{S}^m = 0,\ \ m > n$ then by definition

$$\mathbb{H}^m(X, \mathcal{S}^\bullet) = 0,\ \ m > 2n$$

Moreover, if we define $\check{U}_i := U_0 \cap U_1 \cap \cdots \cap U_{i-1} \cap U_{i+1} \cap \cdots \cap U_n$ then

$$\mathbb{H}^{2n}(X, \mathcal{S}^\bullet) \cong \frac{\mathcal{S}^n(U_0 \cap U_1 \cap \cdots \cap U_n)}{d\mathcal{S}^{n-1}(U_0 \cap U_1 \cap \cdots \cap U_n) + \mathcal{S}^n(\check{U}_0) + \mathcal{S}^n(\check{U}_1) + \cdots + \mathcal{S}^n(\check{U}_n)}. \tag{3.12}$$

For simplicity, we have not written the restriction maps. Note that the last $n+1$ terms in the denominator of (3.12) form the set $\{\delta\eta^n = \eta^n_0 - \eta^n_1 + \cdots + (-1)^n \eta^n_n \mid \eta^n_i \in \mathcal{S}^n(\check{U}_i)\}$. For $n = 1$ we get

$$\mathbb{H}^2(X, \mathcal{S}^\bullet) \cong \frac{\mathcal{S}^1(U_0 \cap U_1)}{d\mathcal{S}^0(U_0 \cap U_1) + \mathcal{S}^1(U_1) + \mathcal{S}^1(U_0)}.$$

3.4 Acyclic sheaves and hypercohomology

As we saw in the previous chapter, an important tool to compute Čech or sheaf cohomology is by means of an acyclic resolution. In this section we will extend this fact to compute hypercohomology.

Proposition 3.2. *If all the sheaves $\mathcal{S}^i$ are acyclic, that is, $H^j(X, \mathcal{S}^i) = 0$, $j = 1, 2, \dots$ then for every good covering $\mathcal{U}$ with respect to $\mathcal{S}^\bullet$*

$$\mathbb{H}^m(\mathcal{U}, \mathcal{S}^\bullet) \cong H^m(H^0(X, \mathcal{S}^\bullet), d) := \frac{\ker(H^0(X, \mathcal{S}^m) \to H^0(X, \mathcal{S}^{m+1}))}{\operatorname{Im}(H^0(X, \mathcal{S}^{m-1}) \to H^0(X, \mathcal{S}^m))}.$$

Proof. We have already a map

$$f : H^m(H^0(X, \mathcal{S}^\bullet), d) \to \mathbb{H}^m(\mathcal{U}, \mathcal{S}^\bullet) \tag{3.13}$$

which is obtained by restricting a global section of $\mathcal{S}^m$ to the open sets of the covering $\mathcal{U}$. We have to define its inverse

$$f^{-1} : \mathbb{H}^m(\mathcal{U}, \mathcal{S}^\bullet) \to H^m(H^0(X, \mathcal{S}^\bullet), d). \tag{3.14}$$

Let us take an element $\omega = \sum_{j=0}^m \omega^j \in \mathcal{L}^m$ which is D-closed. We have written the ingredient equalities derived from this in (3.6). In particular, $\delta\omega^0 = 0$ and by our hypothesis $\omega^0 = \delta\eta^0$ for some $\eta^0 \in \mathcal{S}^0_{m-1}$ (here we are using Leray's lemma, since $\mathcal{U}$ is an acyclic covering with respect to $\mathcal{S}^0$, and that $H^m(X, \mathcal{S}^0) = 0$). The elements ω and $\omega - D\eta^0$ represent the same object in $\mathbb{H}^m(\mathcal{U}, \mathcal{S}^\bullet)$, and so we can assume that $\omega^0 = 0$. This process continues and finally we get an element in $\mathcal{S}^m_0$ which is is equivalent to ω in $\mathbb{H}^m(\mathcal{U}, \mathcal{S}^\bullet)$, and moreover, it is both δ and d-closed. This gives us a global section $f^{-1}(\omega) \in H^0(X, \mathcal{S}^m)$. We have to check that f^{-1} is well-defined, and it is the inverse of f. These details are left to the reader. □

Using the same argument, but taking care of passing through the corresponding refinements it is an exercise to prove the following result:

Proposition 3.3. *If $\mathcal{S}^\bullet$ is a complex of acyclic sheaves, then*

$$\mathbb{H}^m(X, \mathcal{S}^\bullet) \cong H^m(H^0(X, \mathcal{S}^\bullet), d).$$

3.5 Long exact sequence in hypercohomology

As short exact sequences of sheaves induce long exact sequences in cohomology, short exact sequences of sheaf complexes induce long exact sequences in hypercohomology.

Proposition 3.4. *Consider*

$$0 \to \mathcal{S}_1^\bullet \xrightarrow{f} \mathcal{S}_2^\bullet \xrightarrow{g} \mathcal{S}_3^\bullet \to 0$$

a short exact sequence of complexes of sheaves. If $\mathcal{U} = \{U_i\}_{i \in I}$ is an open covering such that $H^1(U_{i_0 \cdots i_k}, \mathcal{S}_1^p) = 0$ for every $k, p \geqslant 0$, then there exists a long exact sequence in hypercohomology

$$\cdots \to \mathbb{H}^m(\mathcal{U}, \mathcal{S}_1^\bullet) \to \mathbb{H}^m(\mathcal{U}, \mathcal{S}_2^\bullet) \to \mathbb{H}^m(\mathcal{U}, \mathcal{S}_3^\bullet) \to \mathbb{H}^{m+1}(\mathcal{U}, \mathcal{S}_1^\bullet) \to \cdots . \tag{3.15}$$

Proof. By the hypothesis, for every $p \geqslant 0$, we have a short exact sequence of complexes

$$0 \to C^\bullet(\mathcal{U}, \mathcal{S}_1^p) \to C^\bullet(\mathcal{U}, \mathcal{S}_2^p) \to C^\bullet(\mathcal{U}, \mathcal{S}_3^p) \to 0.$$

This induces a short exact sequence of complexes of abelian groups

$$0 \to \oplus_{p+q=\bullet} C^q(\mathcal{U}, \mathcal{S}_1^p) \to \oplus_{p+q=\bullet} C^q(\mathcal{U}, \mathcal{S}_2^p) \to \oplus_{p+q=\bullet} C^q(\mathcal{U}, \mathcal{S}_3^p) \to 0$$

whose induced long exact sequence corresponds to (3.15). □

We can give an explicit description of the coboundary map in hypercohomology

$$\mathbb{H}^m(\mathcal{U}, \mathcal{S}_3^\bullet) \to \mathbb{H}^{m+1}(\mathcal{U}, \mathcal{S}_1^\bullet),$$

for $\mathcal{U}$ a good cover with respect to $\mathcal{S}_1^\bullet$. In fact, given any $\omega \in \oplus_{p+q=m} C^q(\mathcal{U}, \mathcal{S}_3^p)$, there exists some $\mu \in \oplus_{p+q=m} C^q(\mathcal{U}, \mathcal{S}_2^p)$ such that $\omega = g(\mu)$. If $D\omega = 0$, then $g(D\mu) = 0$ and so $D\mu = f(\eta)$ for some $\eta \in \oplus_{p+q=m+1} C^q(\mathcal{U}, \mathcal{S}_1^p)$. Since f is injective and $f(D\eta) = 0$, it follows that $D\eta = 0$ and so the image of $\omega \in \mathbb{H}^m(\mathcal{U}, \mathcal{S}_3^\bullet)$ under the coboundary map is $\eta \in \mathbb{H}^{m+1}(\mathcal{U}, \mathcal{S}_1^\bullet)$.

Now it is easy to describe the coboundary map

$$\mathbb{H}^m(X, \mathcal{S}_3^\bullet) \to \mathbb{H}^{m+1}(X, \mathcal{S}_1^\bullet),$$

in general. Take any open cover $\mathcal{U}$ of X and any $\omega \in \oplus_{p+q=m} C^q(\mathcal{U}, \mathcal{S}_3^p)$. There exists some refinement $\mathcal{U}' \leqslant \mathcal{U}$ such that there exists $\mu \in \oplus_{p+q=m} C^q(\mathcal{U}, \mathcal{S}_2^p)$ such that $g(\mu) = \omega|_{\mathcal{U}'}$. Imitating the construction of the coboundary map explained above, we produce the image of $\omega \in \mathbb{H}^m(\mathcal{U}, \mathcal{S}_3^\bullet)$ under the coboundary map as $\eta \in \mathbb{H}^{m+1}(\mathcal{U}, \mathcal{S}_1^\bullet)$. It is routine to check that this construction is independent of the choices made, and is compatible with restrictions to refinements, thus it determines the coboundary map in hypercohomology. Now it is an exercise to prove that the long sequence induced is exact.

Proposition 3.5. *Consider*

$$0 \to \mathcal{S}_1^\bullet \xrightarrow{f} \mathcal{S}_2^\bullet \xrightarrow{g} \mathcal{S}_3^\bullet \to 0$$

a short exact sequence of complexes of sheaves. Then there exists a long exact sequence in hypercohomology

$$\cdots \to \mathbb{H}^m(X, \mathcal{S}_1^\bullet) \to \mathbb{H}^m(X, \mathcal{S}_2^\bullet) \to \mathbb{H}^m(X, \mathcal{S}_3^\bullet) \to \mathbb{H}^{m+1}(X, \mathcal{S}_1^\bullet) \to \cdots . \tag{3.16}$$

Now we are in position to give a proof of Theorem 3.1. But first we need an algebraic lemma about sheaves.

Lemma 3.1. *Let $\mathcal{S}^\bullet$ be a bounded complex of sheaves, i.e. $\mathcal{S}^m = 0$ for $m >> 0$. Then we can embed $\mathcal{S}^\bullet$ in a complex of sheaves $\mathcal{I}^\bullet$ such that $\mathcal{H}^{k-1} :=$ Ker $(\mathcal{I}^{k-1} \to \mathcal{I}^k)$ is flasque and $H^k(\mathcal{I}^\bullet) = 0$ for every $k > 0$.*

Exercise 3.2. *Prove Lemma 3.1 using Godement's construction of Example 2.4.*

Proof of Theorem 3.1. Let $\mathcal{U}$ be a good covering with respect to $\mathcal{S}^\bullet$. we want to show that for every $m \geqslant 0$ the natural map

$$\mathbb{H}^m(\mathcal{U}, \mathcal{S}^\bullet) \to \mathbb{H}^m(X, \mathcal{S}^\bullet)$$

is an isomorphism. Fixing m, we can truncate the complex at some $n >> m$ and reduce ourselves to the case $\mathcal{S}^\bullet$ is bounded. Let us assume then that $\mathcal{S}^\bullet$ is bounded. Let $\mathcal{S}^\bullet \hookrightarrow \mathcal{I}^\bullet$ be given by Lemma 3.1 and define $\mathcal{G}^\bullet := \mathcal{I}^\bullet/\mathcal{S}^\bullet$. We will prove the result by induction on m. For $m = 0$ it is clear that $\mathbb{H}^0(\mathcal{U}, \mathcal{S}^\bullet) = H^0(\mathcal{U}, \mathcal{S}^0) = H^0(X, \mathcal{S}^0) = \mathbb{H}^0(X, \mathcal{S}^\bullet)$ independently of the fact that $\mathcal{U}$ is a good covering. For $m > 0$, it follows from Lemma 3.1 that for every $k \geqslant 0$ we have a short exact sequence

$$0 \to \mathcal{H}^k \to \mathcal{I}^k \to \mathcal{H}^{k+1} \to 0. \tag{3.17}$$

Since every $\mathcal{H}^k$ is flasque, it follows that every $\mathcal{I}^k$ is acyclic and that

$$0 \to \Gamma(\mathcal{H}^k) \to \Gamma(\mathcal{I}^k) \to \Gamma(\mathcal{H}^{k+1}) \to 0 \tag{3.18}$$

is exact for every $k \geqslant 0$. In consequence $H^k(\Gamma(\mathcal{I}^\bullet)) = 0$ for every $k > 0$. Furthermore (3.17) remains exact when restricting to any open set $U \subseteq X$, and so $\mathcal{I}^k|_U$ is acyclic for every $k \geqslant 0$. In consequence, every covering of X is a good covering with respect to $\mathcal{I}^\bullet$. By Proposition 3.2 we get that $\mathbb{H}^k(\mathcal{U}', \mathcal{I}^\bullet) \simeq H^k(\Gamma(\mathcal{I}^\bullet)) = 0$ for every covering $\mathcal{U}'$ of X and every $k > 0$. In particular $\mathbb{H}^k(\mathcal{U}, \mathcal{I}^\bullet) = \mathbb{H}^k(X, \mathcal{I}^\bullet) = 0$ for all $k > 0$.

In order to apply the induction hypothesis to $\mathcal{G}^\bullet$, we need to check that $\mathcal{U}$ is a good covering with respect to $\mathcal{G}^\bullet$. This follows from the long exact sequence in Čech cohomology associated to the short exact sequence

$$0 \to \mathcal{S}^p|_{U_{i_0 \cdots i_k}} \to \mathcal{I}^p|_{U_{i_0 \cdots i_k}} \to \mathcal{G}^p|_{U_{i_0 \cdots i_k}} \to 0$$

together with the fact that $\mathcal{U}$ is a good covering with respect to $\mathcal{S}^\bullet$ and that $\mathcal{I}^p|_{U_{i_0 \cdots i_k}}$ is flasque.

Finally, by Proposition 3.4 and Proposition 3.5 we have the following commutative diagram

$$\begin{array}{ccccccccc}
\mathbb{H}^{m-1}(\mathcal{U}, \mathcal{I}^\bullet) & \to & \mathbb{H}^{m-1}(\mathcal{U}, \mathcal{G}^\bullet) & \longrightarrow & \mathbb{H}^m(\mathcal{U}, \mathcal{S}^\bullet) & \to & \mathbb{H}^m(\mathcal{U}, \mathcal{I}^\bullet) = 0 & \longrightarrow & 0 \\
\downarrow \wr & & \downarrow \wr & & \downarrow & & \downarrow \wr & & \downarrow \wr \\
\mathbb{H}^{m-1}(X, \mathcal{I}^\bullet) & \to & \mathbb{H}^{m-1}(X, \mathcal{G}^\bullet) & \longrightarrow & \mathbb{H}^m(X, \mathcal{S}^\bullet) & \to & \mathbb{H}^m(X, \mathcal{I}^\bullet) = 0 & \longrightarrow & 0
\end{array}$$

and the result follows from the five lemma. □

3.6 Quasi-isomorphism and hypercohomology

A morphism between two complexes $\mathcal{S}^\bullet$ and $\check{\mathcal{S}}^\bullet$ is the following commutative diagram:

$$\begin{array}{ccccccccc}
\cdots & \to & \mathcal{S}^{n-1} & \to & \mathcal{S}^n & \to & \mathcal{S}^{n+1} & \to & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \check{\mathcal{S}}^{n-1} & \to & \check{\mathcal{S}}^n & \to & \check{\mathcal{S}}^{n+1} & \to & \cdots
\end{array}$$

It induces a canonical map in the hypercohomologies

$$\mathbb{H}^m(X, \mathcal{S}^\bullet) \to \mathbb{H}^m(X, \check{\mathcal{S}}^\bullet).$$

For a morphism $f : \mathcal{S}^\bullet \to \check{\mathcal{S}}^\bullet$ of complexes we have a canonical morphism

$$H^m(f) : H^m(\mathcal{S}^\bullet) \to H^m(\check{\mathcal{S}}^\bullet)$$

where for a complex $\mathcal{S}^\bullet$ we have defined

$$H^m(\mathcal{S}^\bullet) := \text{The sheaf constructed from the presheaf } \frac{\ker(d^m)}{\operatorname{Im}(d^{m-1})}, \quad m \in \mathbb{Z}$$

Definition 3.4. A morphism f of complexes is called a **quasi-isomorphism** if the induced morphisms $H^m(f)$, $m \in \mathbb{Z}$ are isomorphisms.

Example 3.1. Let X be a smooth variety over the field of complex numbers and let X^∞ be the underlying C^∞ manifold. Let $\Omega^\bullet_X$ be the complex of algebraic differential forms in X (see Definition 4.10). We also consider the complex $\Omega^\bullet_{X^\infty}$ of C^∞ differential forms in X for the Zariski topology of X^∞. We will only need to consider Zariski open sets. In Chapter 5 we are going to show that the natural inclusion $\Omega^\bullet_X \to \Omega^\bullet_{X^\infty}$ is a quasi-isomorphism. By now we will prove that the natural inclusion $\Omega^\bullet_{X^{\mathrm{an}}} \to \Omega^\bullet_{X^\infty}$ of complexes of sheaves over the analytic topology is a quasi-isomorphism. In order to do this we need the following algebraic lemma.

Lemma 3.2. *Let $A^\bullet$ be a left bounded complex in an abelian category. Let $(I^{\bullet,\bullet}, d_1, d_2)$ be a double complex such that for every $p \geqslant 0$ we have a resolution $A^p \hookrightarrow (I^{p,\bullet}, d_2)$ which induces a morphism of complexes $A^\bullet \hookrightarrow I^{\bullet,\bullet}$. Let $(I^\bullet, D)$ be the* ***simple complex*** *associated to the double complex $I^{\bullet,\bullet}$, with*

$$I^k := \bigoplus_{p+q=k} I^{p,q}$$

and $D|_{I^{p,q}} := d_2 + (-1)^q d_1$. Then the natural inclusion of complexes $A^\bullet \hookrightarrow I^\bullet$ is a quasi-isomorphism.

Proof. Let us construct the inverse map

$$H^k(I^\bullet, D) \to H^k(A^\bullet, d).$$

Consider an element $\eta = \sum_{p=0}^k \eta^p \in I^k$ with $\eta^p \in I^{p,q}$, such that $D\eta = 0$. Then $d_2\eta^0 = 0$. Since $(I^{0,\bullet}, d_2)$ is a resolution of A^0, there exists some

$\mu^0 \in I^{0,k-1}$ such that $d_2\mu^0 = \eta^0$. Thus subtracting $D\mu^0$ we can find another representative of η in $H^k(I^\bullet, D)$ with $\eta^0 = 0$. Repeating this argument we can find a representative of η with $\eta^0 = \eta^1 = \cdots = \eta^{k-1} = 0$, and so $\eta = \eta^k \in I^{k,0}$. Since $D\eta = 0$, it follows that $d_1\eta^k = 0$ and $d_2\eta^k = 0$. Again since $(I^{k,\bullet}, d_2)$ is a resolution of A^k we conclude that $\eta^k \in A^k$ and $d_1\eta^k = d\eta^k = 0$. The desired inverse map takes the class of η in $H^k(I^\bullet, D)$ to the class of η^k in $H^k(A^\bullet, d)$. The verification that this map is a well-defined inverse is left to the reader. □

Corollary 3.1. *Let X be a complex manifold. The natural inclusion of complexes of sheaves*

$$(\Omega^\bullet_{X^{\mathrm{an}}}, \partial) \hookrightarrow (\Omega^\bullet_{X^\infty}, d)$$

is a quasi-isomorphism.

Proof. By Dolbeault lemma (Theorem 2.10) we have a resolution

$$\Omega^p_{X^{\mathrm{an}}} \hookrightarrow (\Omega^{p,\bullet}_X, \bar\partial)$$

Applying Lemma 3.2 to the double complex $(\Omega^{\bullet,\bullet}_X, (-1)^q\partial, \bar\partial)$ we get the result. □

Proposition 3.6. *Let $f : \mathcal{S}^\bullet \to \check{\mathcal{S}}^\bullet$ be a quasi-isomorphism then the induced map in hypercohomology $f_* : \mathbb{H}^m(\mathcal{U}, \mathcal{S}^\bullet) \to \mathbb{H}^m(\mathcal{U}, \check{\mathcal{S}}^\bullet)$ is an isomorphism for every open covering $\mathcal{U}$ of X. In consequence $\mathbb{H}^m(\mathcal{U}, \mathcal{S}^\bullet) \simeq \mathbb{H}^m(\mathcal{U}, \check{\mathcal{S}}^\bullet)$.*

Proof. We have to define the inverse map

$$f_*^{-1} : \mathbb{H}^m(\mathcal{U}, \check{\mathcal{S}}^\bullet) \to \mathbb{H}^m(\mathcal{U}, \mathcal{S}^\bullet) \tag{3.19}$$

Let $\check\omega = \sum_{j=0}^m \check\omega^j \in \mathbb{H}^m(\mathcal{U}, \check{\mathcal{S}}^\bullet)$. We start by looking at $\check\omega^m$. We have $d(\check\omega^m) = 0$ and so there is

$$\omega^m \in \mathcal{S}^m_0, d\omega^m = 0$$

such that

$$\check\omega^m - f_*\omega^m = d\check\eta^{m-1}$$

for some $\check\eta^{m-1} \in \check{\mathcal{S}}^{m-1}_0$. We replace $\check\omega$ with $\check\omega - D\check\eta^{m-1}$ and in this way we can assume that

$$\check\omega^m = f_*\omega^m.$$

Now we look at the $(m-1)$-level in which we have

$$d\check\omega^{m-1} = \delta\check\omega^m = \delta f_*\omega^m = f_*\left(\delta\omega^m\right). \tag{3.20}$$

Then $\delta\omega^m$ is a d-closed element that it is mapped to zero under $H^m(f)$. Therefore, using that f is a quasi-isomorphism, we conclude that $\delta\omega^m$ is d-exact, that is, there exists $\nu^{m-1} \in \mathcal{S}_1^{m-1}$ such that

$$\delta\omega^m - d\nu^{m-1} = 0. \tag{3.21}$$

Combining (3.20) and (3.21) we get

$$d\left(\check{\omega}^{m-1} - f_*\nu^{m-1}\right) = 0. \tag{3.22}$$

We are now in a similar situation as in the beginning. Since f is a quasi-isomorphism we have

$$\check{\omega}^{m-1} - f_*\nu^{m-1} - f_*\mu^{m-1} = d\check{\eta}^{m-2},$$

for some $\check{\eta}^{m-2} \in \mathcal{S}_1^{m-2}$, $\mu^{m-1} \in \mathcal{S}_1^{m-1}$ and $d\mu^{m-1} = 0$. Replacing $\check{\omega}$ by $\check{\omega} + D\check{\eta}^{m-2}$ and defining $\omega^{m-1} := \nu^{m-1} + \mu^{m-1} \in \mathcal{S}_1^{m-1}$ we obtain

$$\check{\omega}^{m-1} = f_*\omega^{m-1} \text{ and } \delta\omega^m = d\omega^{m-1}.$$

At the $(m-2)$-the level we get $\omega^m \in \mathcal{S}_0^m$, $\omega^{m-1} \in \mathcal{S}_1^{m-1}$, $\nu^{m-2} \in \mathcal{S}_2^{m-2}$ with the following identities

$$\check{\omega}^m = f_*\omega^m,\ \check{\omega}^{m-1} = f_*\omega^{m-1},\ -d\omega^{m-1} + \delta\omega^m = 0,$$

and

$$d\left(\check{\omega}^{m-2} - f_*\nu^{m-2}\right) = 0. \tag{3.23}$$

This process stops at $(m+1)$-th step and we get $\omega = \sum_{i=0}^m \omega^m$ which is D-closed and $f_*\omega = \check{\omega}$. We define $f_*^{-1}(\check{\omega})$ to be equal to ω. It is left to the reader to prove that f_*^{-1} is well-defined, i.e. it is independent of all choices made, and it is inverse to f_*. □

Exercise 3.3. *Complete the details of the proof of Proposition 3.6.*

Proposition 3.7. *Let $\mathcal{S} \to \mathcal{S}^\bullet$ be a resolution of $\mathcal{S}^\bullet$. Then*

$$\mathbb{H}^k(X, \mathcal{S}^\bullet) \cong H^k(X, \mathcal{S})$$

Proof. By hypothesis the complex $\cdots \to 0 \to \mathcal{S} \to 0 \to \cdots$ with $\mathcal{S}$ in the 0-th place, is quasi-isomorphic to the complex $\mathcal{S}^\bullet$ and so the result follows from Proposition 3.6. □

3.7 A description of an isomorphism

Let us now be given a quasi-isomorphism of complexes $\mathcal{S}^\bullet \to \check{\mathcal{S}}^\bullet$ and assume that all the abelian sheaves $\check{\mathcal{S}}^m$ are acyclic. We use Proposition 3.2 and Proposition 3.6 and we get an isomorphism

$$\mathbb{H}^m(X, \mathcal{S}^\bullet) \cong H^m(\check{\mathcal{S}}^\bullet(X), d).$$

For later applications we need to describe the two maps

$$\begin{aligned} A: \qquad & \mathbb{H}^m(X, \mathcal{S}^\bullet) \to H^m(\check{\mathcal{S}}^\bullet(X), d) \\ A^{-1}: \qquad & H^m(\check{\mathcal{S}}^\bullet(X), d) \to \mathbb{H}^m(X, \mathcal{S}^\bullet). \end{aligned}$$

such that both $A \circ A^{-1}$ and $A^{-1} \circ A$ are identity maps. We take an acyclic covering $\mathcal{U} = \{U_i\}_{i \in I}$ with respect to all sheaves $\mathcal{S}^i$ and $\check{\mathcal{S}}^i$. The map A is obtained by the composition of the maps

$$\mathbb{H}^m(X, \mathcal{S}^\bullet) \to \mathbb{H}^m(X, \check{\mathcal{S}}^\bullet) \to H^m(\check{\mathcal{S}}^\bullet(X), d)$$

The first map is simply induced by the quasi-isomorphism. The second map is the map (3.14). Its explicit description is given in the proof of Proposition 3.2. The map A^{-1} is the composition of

$$H^m(\check{\mathcal{S}}^\bullet(X), d) \to \mathbb{H}^m(X, \check{\mathcal{S}}^\bullet) \to \mathbb{H}^m(X, \mathcal{S}^\bullet)$$

The first map is obtained by restricting a global section of $\check{\mathcal{S}}^m$ to the open sets of the covering $\mathcal{U}$. The second map is (3.19) and its explicit description is given in the proof of Proposition 3.6.

3.8 Filtrations

For a complex $\mathcal{S}^\bullet$ and $k \in \mathbb{Z}$ we define the **truncated complexes**

$$\mathcal{S}^{\leqslant k}: \quad \cdots \to \mathcal{S}^{k-1} \to \mathcal{S}^k \to 0 \to 0 \to \cdots$$

and

$$\mathcal{S}^{\geqslant k}: \quad \cdots \to 0 \to 0 \to \mathcal{S}^k \to \mathcal{S}^{k+1} \to \cdots$$

We have canonical morphisms of complexes:

$$\mathcal{S}^{\leqslant k} \to \mathcal{S}^\bullet, \; \mathcal{S}^{\geqslant k} \to \mathcal{S}^\bullet.$$

Assume that $\mathcal{S}^\bullet$ is a **left-bounded complex**, that is,

$$\mathcal{S}^\bullet : \;\; 0 \to \mathcal{S}^0 \to \mathcal{S}^1 \to \cdots$$

The morphism $\mathcal{S}^{\geqslant i} \to \mathcal{S}^\bullet$ induces a map in hypercohomologies and we define

$$F^i := \mathrm{Im}(\mathbb{H}^m(X, \mathcal{S}^{\geqslant i}) \to \mathbb{H}^m(X, \mathcal{S}^\bullet)). \tag{3.24}$$

This gives us the **naive filtration** in hypercohomology

$$\cdots \subset F^i \subset F^{i-1} \subset \cdots \subset F^1 \subset F^0 := \mathbb{H}^m(X, \mathcal{S}^\bullet).$$

We denote the p**-th graded piece** induced by the filtration $F^\bullet$ by

$$Gr_F^p \mathbb{H}^m(X, \mathcal{S}^\bullet) := F^p / F^{p+1}.$$

An element of F^i in an acyclic covering $\mathcal{U}$ of X is given by

$$\omega = \omega^i + \omega^{i+1} + \cdots + \omega^m, \quad \omega^j \in C^{m-j}(\mathcal{U}, \mathcal{S}^j)$$

where ω^j itself is the collection of $\omega^j_{i_0 i_1 \cdots i_{m-j}} \in \mathcal{S}^j(U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_{m-j}})$, for all $i_0, i_1, \cdots, i_{m-j} \in I$. Note that a priori the element ω might be D-exact in $\mathbb{H}^m(X, \mathcal{S}^\bullet)$, but as an element of $\mathbb{H}^m(X, \mathcal{S}^{\geqslant i})$ it might not be D-exact. In other words, the map in (3.24) might not be injective. This phenomenon is related to the degeneration at E_1 of the spectral sequence associated to the naive filtration.

3.9 Subsequent quotients and a spectral sequence

The differential map $d : \mathcal{S}^i \to \mathcal{S}^{i+1}$ induces the maps

$$d_1 : H^j(X, \mathcal{S}^i) \to H^j(X, \mathcal{S}^{i+1}), \; j \in \mathbb{N}_0.$$

If all these maps are zero then we can define the maps

$$d_2 : H^j(X, \mathcal{S}^i) \to H^{j-1}(X, \mathcal{S}^{i+2}), \; j \in \mathbb{N}_0$$

which are defined as follows: we take $\omega^i \in H^j(X, \mathcal{S}^i)$ and since d_1 is zero we have $\omega^{i+1} \in \mathcal{S}^{i+1}_{j-1}$ such that $\delta\omega^{i+1} + (-1)^j d\omega^i = 0$. Under d_2 the element ω^i is mapped to $d\omega^{i+1}$. In a similar way, we define

$$d_r : H^j(X, \mathcal{S}^i) \to H^{j-r+1}(X, \mathcal{S}^{i+r}),\ j \in \mathbb{N}_0.$$

and if $d_k,\ k \leqslant r$ are all zero we define d_{r+1}.

Theorem 3.2. *Assume that all d_r's constructed above are zero. Then we have canonical isomorphisms*

$$Gr^i_F \mathbb{H}^m(X, \mathcal{S}^\bullet) = F^i/F^{i+1} \cong H^{m-i}(X, \mathcal{S}^i). \tag{3.25}$$

The reader who is familiar with spectral sequences has noticed that the hypothesis in Theorem 3.2 is equivalent to say that the spectral sequence associated to the double complex (3.2) with respect to the naive filtration degenerates at E_1. In this section we will recall briefly how to construct this spectral sequence. For further details, and a treatment of spectral sequences associated to any complex of filtered abelian groups see Voisin (2002, Section 8.3).

Let $\mathcal{U}$ be a good cover with respect to $\mathcal{S}^\bullet$. Define for every $p, q, r \geqslant 0$ the abelian groups

$$Z^{p,q}_r := \{\omega \in \mathcal{L}^{p+q} : \omega^0 = \cdots = \omega^{p-1} = (D\omega)^p = \cdots = (D\omega)^{p+r-1} = 0\},$$

$$B^{p,q}_r := Z^{p+1,q-1}_{r-1} + DZ^{p-r+1,q+r-2}_{r-1} \subseteq Z^{p,q}_r,$$

and

$$E^{p,q}_r := \frac{Z^{p,q}_r}{B^{p,q}_r}.$$

Since $DZ^{p,q}_r \subseteq Z^{p+r,q-r+1}_r$ and $DB^{p,q}_r \subseteq B^{p+r,q-r+1}_r$, we have the induced map

$$d_r := D : E^{p,q}_r \to E^{p+r,q-r+1}_r$$

which turns $(E^{p+r\bullet,q-(r-1)\bullet}_r, d_r)$ into a complex. It is clear that $E^{p,q}_0 = \mathcal{S}^p_q$ and $d_0 = \delta$. On the other hand, since $DZ^{p,q}_{r+1} \subseteq Z^{p+r+1,q-r}_r \subseteq B^{p+r,q-r+1}_r$ there is a natural map

$$Z^{p,q}_{r+1} \to \ker\left(E^{p,q}_r \xrightarrow{d_r} E^{p+r,q-r+1}_r\right).$$

It is an exercise to check that this map induces the isomorphism

$$E_{r+1}^{p,q} \simeq \frac{\ker\left(E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-r+1}\right)}{\operatorname{Im}\left(E_r^{p-r,q+r-1} \xrightarrow{d_r} E_r^{p,q}\right)}.$$

In particular $E_1^{p,q} = H^q(X, \mathcal{S}^p)$ and $d_1 = (-1)^q d$. Furthermore, if we define

$$F^p\mathcal{L}^{p+q} := \{\omega \in \mathcal{L}^{p+q} : \omega^0 = \cdots = \omega^{p-1} = 0\},$$

then for $r > p + q + 1$

$$\begin{aligned} E_{r+1}^{p,q} &= \frac{\ker\left(D : F^p\mathcal{L}^{p+q} \to F^p\mathcal{L}^{p+q+1}\right)}{\ker\left(D : F^{p+1}\mathcal{L}^{p+q} \to F^{p+1}\mathcal{L}^{p+q+1}\right) + F^P\mathcal{L}^{p+q} \cap \operatorname{Im} D} \\ &\simeq F^p/F^{p+1} = Gr_F^p\mathbb{H}^{p+q}(X, \mathcal{S}^\bullet) =: E_\infty^{p,q}. \end{aligned}$$

The data $(E_r^{p,q}, d_r)$ is called the **spectral sequence** associated to the naive filtration.

Definition 3.5. We say that the spectral sequence $(E_r^{p,q}, d_r)$ **degenerates at** E_r if $d_k = 0$ for all $k \geqslant r$. In such case we have

$$E_r^{p,q} = E_k^{p,q} = E_\infty^{p,q} = F^p/F^{p+1}.$$

We say that the spectral sequence **abuts** to $\mathbb{H}^{p+q}(X, \mathcal{S}^\bullet)$, and it is denoted by

$$E_r^{p,q} \Rightarrow \mathbb{H}^{p+q}(X, \mathcal{S}^\bullet).$$

The principal question behind the isomorphism (3.25) is under which hypothesis can we assure that every $\omega^i \in \mathcal{S}^i_{m-i}$ such that $\delta\omega^i = 0$, can be extended to an $\omega = \omega^i + \cdots + \omega^m \in F^i\mathcal{L}^m$ such that $D\omega = 0$.

Definition 3.6. Let $\mathcal{U}$ be a good covering of X with respect to $\mathcal{S}^\bullet$. An element $\omega^i \in \mathcal{S}^i_{m-i}$ is said to be **extendable in hypercohomology** if there exist $\omega = \omega^i + \cdots + \omega^m \in F^i\mathcal{L}^m$ such that

$$D\omega = 0.$$

For every $r \geqslant 0$ we say that $\omega^i \in \mathcal{S}^i_{m-i}$ is **r-extendable** if there exist $\omega = \omega^i + \cdots + \omega^m \in F^i\mathcal{L}^m$ such that

$$D\omega \in F^{i+r}\mathcal{L}^{m+1}.$$

In particular, for $r > m+1-i$ being r-extendable is the same as being extendable in hypercohomology. Note that the hypothesis of Theorem 3.2 is the same as saying that every element 1-extendable is extendable in hypercohomology.

Proposition 3.8. *The spectral sequence $(E_r^{p,q}, d_r)$ degenerates at E_r if and only if every r-extendable element is extendable in hypercohomology.*

Proof. Let $\omega^i \in \mathcal{S}^i_{m-i}$ be an element r-extendable. Then there exist $\omega = \omega^i + \cdots + \omega^m \in F^i\mathcal{L}^m$ such that $D\omega \in F^{i+r}\mathcal{L}^{m+1}$. Then $\omega \in Z_r^{i,m-i}$, and since $d_r = 0$ we conclude that $D\omega \in B_r^{i+r,m-i-r+1} = Z_{r-1}^{i+r+1,m-i-r} + DZ_{r-1}^{i+1,m-i-1}$. In consequence, there exist $\eta = \eta^{i+1} + \cdots + \eta^m \in F^{i+1}\mathcal{L}^m$ such that $D(\omega - \eta) \in F^{i+r+1}\mathcal{L}^{m+1}$. This implies that ω^i is $(r+1)$-extendable. Inductively we show that ω^i is k-extendable for every $k \geqslant r$ and so it is extendable in hypercohomology.

Conversely, since every element k-extendable for $k \geqslant r$ is also r-extendable, it is enough to show that $d_r = 0$. Let $\omega = \omega^i + \cdots + \omega^m \in Z_r^{i,m-i}$, then $D\omega \in F^{i+r}\mathcal{L}^{m+1}$, and so ω^i is r-extendable. Therefore ω^i is extendable in hypercohomology, i.e. there exist $\eta = \omega^i + \eta^{i+1} + \cdots + \eta^m \in F^i\mathcal{L}^m$ such that $D\eta = 0$. Then $D\omega = D(\omega - \eta) \in DZ_{r-1}^{i+1,m-i-1} \subseteq B_r^{i+r,m-i-r+1}$, and so $d_r\omega = 0$. ☐

Proposition 3.9. *The spectral sequence $(E_r^{p,q}, d_r)$ degenerates at E_1 if and only if any of the following equivalent assertions hold:*

1. *For every $p \geqslant 0$ we have the isomorphism* (3.25)

$$Gr_F^p\mathbb{H}^{p+q}(X, \mathcal{S}^\bullet) = F^p/F^{p+1} \simeq H^q(X, \mathcal{S}^p).$$

2. *The natural map $\mathbb{H}^m(X, \mathcal{S}^{\geqslant p}) \to \mathbb{H}^m(X, \mathcal{S}^\bullet)$ is injective.*

Furthermore, when $\mathcal{S}^\bullet$ satisfies $H^k(\mathcal{S}^\bullet) = 0$ for $k > 0$, this is also equivalent to:

3. *For all $p, q \geqslant 0$ the map $H^q(X, \check{\mathcal{S}}^p) \to H^q(X, \mathcal{S}^p)$ is surjective, where $\check{\mathcal{S}}^p := \ker(d : \mathcal{S}^p \to \mathcal{S}^{p+1})$, or equivalently $d : H^q(X, \mathcal{S}^p) \to H^q(X, \check{\mathcal{S}}^{p+1})$ is zero.*

Proof. Suppose the spectral sequence degenerates at E_1, then

$$H^q(X, \mathcal{S}^p) \simeq E_1^{p,q} = E_\infty^{p,q} = F^p/F^{p+1},$$

this proves 1.

Consider now the map $\omega^i + \cdots + \omega^m \in \mathbb{H}^{p+q}(X, \mathcal{S}^{\geqslant p}) \mapsto \omega^i \in H^q(X, \mathcal{S}^p)$. It is easy to see that this map is well-defined, and its kernel is

$$F^{p+1}\mathbb{H}^{p+q}(X, \mathcal{S}^{\geqslant p}) := \operatorname{Im}\left(\mathbb{H}^{p+q}(X, \mathcal{S}^{\geqslant p+1}) \to \mathbb{H}^{p+q}(X, \mathcal{S}^{\geqslant p})\right).$$

We have the natural maps

$$\mathbb{H}^{p+q}(X, \mathcal{S}^{\geqslant p})/F^{p+1}\mathbb{H}^{p+q}(X, \mathcal{S}^{\geqslant p}) \twoheadrightarrow F^p/F^{p+1}, \tag{3.26}$$

and

$$\mathbb{H}^{p+q}(X, \mathcal{S}^{\geqslant p})/F^{p+1}\mathbb{H}^{p+q}(X, \mathcal{S}^{\geqslant p}) \hookrightarrow H^q(X, \mathcal{S}^p). \tag{3.27}$$

Assuming 1. we have that both maps are isomorphisms and so we conclude looking at the isomorphism (3.26) for decreasing values of p, that $\mathbb{H}^{p+q}(X, \mathcal{S}^{\geqslant p}) \simeq F^p$ for every $p \geqslant 0$. This proves 2.

Suppose now that 2. holds. We will show that the spectral sequence degenerates at E_1 by applying Proposition 3.8. Consider $\omega^i \in \mathcal{S}^i_{m-i}$ that is 1-extendable, i.e. $\delta\omega^i = 0$. Then $D\omega^i \in F^{i+1} \simeq \mathbb{H}^m(X, \mathcal{S}^{\geqslant i+1})$. Therefore, there exist some $\omega^{i+1} + \cdots + \omega^m \in F^{i+1}\mathcal{L}^m$ such that $D\omega^i = D(\omega^{i+1} + \cdots + \omega^m)$, and so $D(\omega^i - (\omega^{i+1} + \cdots + \omega^m)) = 0$, i.e. ω^i is extendable in hypercohomology.

Finally, it is clear that 3. implies that every 1-extendable $\omega^i \in \mathcal{S}^i_{m-i}$ is extendable in hypercohomology since $d\omega^i = \delta\omega^{i+1}$ for some $\omega^{i+1} \in \mathcal{S}^{i+1}_{m-i-1}$ with $d\omega^{i+1} = 0$. And conversely, every $\omega^i \in \mathcal{S}^i_{m-i}$ such that $\delta\omega^i = 0$ is 1-extendable, and so it is extendable in hypercohomology, i.e. there exist $\omega = \omega^i + \cdots + \omega^k \in F^i\mathcal{L}^m$ with $D\omega = 0$ for some $k \leqslant m$. We claim we can choose ω such that $k = i+1$. In fact, if we suppose $k > i+1$ we have $d\omega^k = 0$, and so there exists $\eta^{k-1} \in \mathcal{S}^{k-1}_{m-k}$ such that $(-1)^{m-k}d\eta^{k-1} = \omega^k$. Replacing ω by $\omega - D\eta^{k-1}$ we reduce the value of k. Repeating this process until $k = i+1$ we obtain 3. □

Remark 3.3 (Deligne's cohomology)**.** For the complex $\mathcal{S}^\bullet$ if the the kernel of $d : \mathcal{S}^0 \to \mathcal{S}^1$ is non-trivial then we can take any abelian subgroup B of $\ker(d)$ and form the new complex $B \to \mathcal{S}^\bullet$ and take its hypercohomology.

4 Atiyah–Hodge theorem

I'd come up to Cambridge at a time when the emphasis in geometry was on classical projective algebraic geometry of the old-fashioned type, which I thoroughly enjoyed. I would have gone on working in that area except that Hodge represented a more modern point of view- differential geometry in relation to topology; I recognized that. It was a very important decision for me. I could have worked in more traditional things, but I think that it was a wise choice, and by working with him I got much more involved with modern ideas. (Michael Atiyah in Minio (1984)).

4.1 Introduction

The algebraic de Rham cohomology was introduced by Grothendieck (1969) after many efforts in order to understand the de Rham cohomology of affine varieties. This is in some sense natural because the integration domain of integrals in Picard (1889), Picard and Simart (1897, 1906), and Poincaré (1887, 1895) are usually supported in affine varieties. The Atiyah–Hodge theorem is the final outcome of all these efforts.

After Hodge decomposition, Dolbeault theorem and Serre's GAGA correspondence we are able to recover each piece of the de Rham cohomology group of a smooth projective variety using only Cech (or sheaf) cohomology with coefficients in sheaves of algebraic differential forms. This is the first manifestation of an algebraic approach to de Rham cohomology. Unfortunately, this approach does not work for arbitrary smooth varieties, for instance for affine varieties. In this chapter we will introduce a result due to Atiyah and Hodge, which says that we can recover the de Rham cohomology groups in the case of affine varieties using only global algebraic differential forms. This theorem will allow us to introduce algebraic de Rham cohomology for any algebraic variety.

4.2 Analytic coherent sheaves

We begin recalling some facts about analytic varieties and coherent sheaves, our main reference is Gunning (1990c).

Definition 4.1. Let X be a complex manifold. The **analytic de Rham cohomology** groups are defined as

$$H^k_{\mathrm{dR}}(X^{\mathrm{an}}) := H^k(\Gamma(\Omega^\bullet_{X^{\mathrm{an}}}), \partial) = \frac{\ker(\partial : \Gamma(\Omega^k_{X^{\mathrm{an}}}) \to \Gamma(\Omega^{k+1}_{X^{\mathrm{an}}}))}{\mathrm{Im}(\partial : \Gamma(\Omega^{k-1}_{X^{\mathrm{an}}}) \to \Gamma(\Omega^k_{X^{\mathrm{an}}}))}.$$

In general, the analytic de Rham cohomology does not coincide with the usual de Rham cohomology, but sometimes it does.

Proposition 4.1. *Let X be a complex manifold such that $H^{p,q}_{\bar\partial}(X) = 0$ for $q > 0$. Then*

$$H^k_{\mathrm{dR}}(X^{\mathrm{an}}) \cong H^k_{\mathrm{dR}}(X).$$

Proof. By Corollary 3.1 the natural inclusion

$$(\Omega^\bullet_{X^{\mathrm{an}}}, \partial) \hookrightarrow (\Omega^\bullet_{X^\infty}, d)$$

is a quasi-isomorphism induced by the double complex resolution $(\Omega^\bullet_{X^{\mathrm{an}}}, \partial) \hookrightarrow (\Omega^{\bullet,\bullet}_X, (-1)^q\partial, \bar\partial)$. Since $H^{p,q}_{\bar\partial}(X) = 0$ for all $q > 0$, it follows that $(\Gamma(\Omega^\bullet_{X^{\mathrm{an}}}), \partial) \hookrightarrow (\Gamma(\Omega^{\bullet,\bullet}_X), (-1)^q\partial, \bar\partial)$ is again a double complex resolution, and so by Lemma 3.2 we have the quasi-isomorphism $(\Gamma(\Omega^\bullet_{X^{\mathrm{an}}}), \partial) \hookrightarrow (\Gamma(\Omega^\bullet_{X^\infty}), d)$. This means that $H^k_{\mathrm{dR}}(X^{\mathrm{an}}) \cong H^k_{\mathrm{dR}}(X)$. □

Definition 4.2. Let $R = (r_1, \ldots, r_n) \in (\mathbb{R}_{>0} \cup \{+\infty\})^n$. An **extended polydisc** centered at 0 is an open subset $\Delta \subseteq \mathbb{C}^n$ of the form

$$\Delta = \Delta(R) := \{z \in \mathbb{C}^n : |z_i| < r_i \text{ for } i = 1, \ldots, n\}.$$

Remark 4.1. By Dolbeault lemma (Theorem 2.10) and Proposition 4.1 we have for an extended polydisc that $H^k_{\mathrm{dR}}(\Delta^{\mathrm{an}}) = 0$ for all $k > 0$.

Definition 4.3. Let X be an analytic variety. A sheaf of $\mathcal{O}_{X^{\mathrm{an}}}$-modules is called an **analytic sheaf**. An analytic sheaf $\mathcal{S}$ is called **coherent** if it is **locally finitely generated**, i.e. around every point $x \in X$ there exists an open neighbourhood $U \subseteq X$ and an epimorphism of $\mathcal{O}_{X^{\mathrm{an}}}$-modules of the form

$$\mathcal{O}^m_{X^{\mathrm{an}}}|_U \to \mathcal{S}|_U \to 0,$$

and furthermore, for any such epimorphism, its kernel is also locally finitely generated. In particular for some open neighbourhood $V \subseteq X$ of x, $\mathcal{S}$ is finitely presented, i.e. there exists an exact sequence of the form

$$\mathcal{O}^m_{X^{\mathrm{an}}}|_V \to \mathcal{O}^m_{X^{\mathrm{an}}}|_V \to \mathcal{S}|_V \to 0.$$

In order to construct coherent analytic sheaves, we mainly use the following three results, whose proofs can be found in Gunning (ibid., Chapter B).

Theorem 4.1 (Oka's coherence theorem)**.** *Let X be an analytic variety, then $\mathcal{O}_{X^{\mathrm{an}}}$ is coherent.*

Theorem 4.2 (Cartan)**.** *Let X be an analytic variety and $Y \subseteq X$ an analytic subvariety. Then the ideal sheaf $\mathcal{I}_{Y^{\mathrm{an}}}$ of the subvariety Y is coherent.*

Theorem 4.3. *Let X be an analytic variety. The category of coherent sheaves over X is an abelian category.*

In order to compute sheaf cohomology groups with coefficients in coherent sheaves we ask ourselves how to construct acyclic coverings. It turns out that there is a plenty of such coverings, due to the following results.

Definition 4.4. Let X be a complex variety. We say that X is a **Stein variety** if every coherent sheaf over X is acyclic.

Remark 4.2. By Dolbeault Theorem 2.11 every Stein complex manifold X satisfies $H^{p,q}_{\bar{\partial}}(X) = 0$ for $q > 0$.

The main results we use to construct Stein varieties are summarized in the following theorem.

Theorem 4.4. *(i) If X is a Stein variety and $Y \subseteq X$ is an analytic subvariety, then Y is Stein.*

(ii) *If X is an analytic variety and $U, V \subseteq X$ are two Stein open subsets, then $U \cap V$ is Stein.*

(iii) *If X and Y are Stein varieties, then $X \times Y$ is also Stein.*

(iv) *The extended polydisc is a Stein variety.*

Proof. (i) Let $\mathcal{S}$ be a coherent sheaf over Y, and denote by $i : Y \hookrightarrow X$ the inclusion map. Then $i_*\mathcal{S}$ has a natural structure of $\mathcal{O}_{X^{\mathrm{an}}}$-module induced by the epimorphism $\mathcal{O}_{X^{\mathrm{an}}} \twoheadrightarrow i_*\mathcal{O}_{Y^{\mathrm{an}}}$. We claim that $i_*\mathcal{S}$ is a coherent $\mathcal{O}_{X^{\mathrm{an}}}$-module. In fact, since coherence is a local property we can assume there exists an exact sequence of the form

$$\mathcal{O}^n_{Y^{\mathrm{an}}} \to \mathcal{O}^m_{Y^{\mathrm{an}}} \to \mathcal{S} \to 0.$$

In consequence we have an exact sequence of $\mathcal{O}_{X^{\mathrm{an}}}$-modules of the form

$$i_*\mathcal{O}^n_{Y^{\mathrm{an}}} \to i_*\mathcal{O}^m_{Y^{\mathrm{an}}} \to i_*\mathcal{S} \to 0.$$

By Theorem 4.3, it is enough to show that $i_*\mathcal{O}_{Y^{\mathrm{an}}}$ is a coherent $\mathcal{O}_{X^{\mathrm{an}}}$-module. But this follows from Oka's coherence, Cartan's theorem and the following short exact sequence

$$0 \to \mathcal{I}_{Y^{\mathrm{an}}} \to \mathcal{O}_{X^{\mathrm{an}}} \to i_*\mathcal{O}_{Y^{\mathrm{an}}} \to 0.$$

Finally by Proposition 2.8 we get $H^q(Y, \mathcal{S}) = H^q(X, i_*\mathcal{S}) = 0$ for $q > 0$.

(ii), (iii), (iv) See Gunning (1990c, Chapter L, Corollary 10), Gunning (ibid., Chapter L, Corollary 9) and Gunning (ibid., Chapter H, Theorem 5).

□

Corollary 4.1. *(i) Every affine analytic variety is Stein.*

(ii) Every open covering given by Stein varieties is acyclic with respect to all coherent sheaves.

(iii) Every analytic variety admits a Stein open covering.

Remark 4.3. There are several characterizations of Stein varieties and each author chooses its favorite as definition. The one we choose here was not the originally used by Stein. As a consequence of choosing our definition we obtain for free an important theorem of Cartan which we state as follows.

Theorem 4.5 (Cartan B theorem)**.** *Let X be a Stein variety. Then*

$$H^i(X, \Omega^j_{X^{\mathrm{an}}}) = 0,\ i = 1, 2, \ldots,\ j = 0, 1, 2, \ldots.$$

4.3 Algebraic coherent sheaves

In this section we will recall the basic facts about algebraic coherent sheaves. Our main references are Grothendieck and Dieudonné (1961), Hartshorne (1977), and Serre (1955). For the basic definitions of algebraic varieties, regular maps, coordinate rings, etc, the reader is referred to any standard textbook such as Hartshorne (1977) and Shafarevich (1994).

Let $\mathsf{k} = \bar{\mathsf{k}}$ be an algebraically closed field, $\mathbb{A}^n_{\mathsf{k}} = \mathsf{k}^n$ be the affine space, and $X \subseteq \mathbb{A}^n_{\mathsf{k}}$ be an affine algebraic variety. We denote by

$$\mathsf{k}[X] \cong \mathsf{k}[x_1, \dots, x_n]/I(X)$$

the **coordinate ring** of X, which corresponds to the ring of **regular functions** $f : X \to \mathsf{k}$. For X any (abstract) algebraic variety over k, the regular functions over X form its **structure sheaf** $\mathcal{O}_X$, given on every affine open subset $U \subseteq X$ by

$$\mathcal{O}_X(U) = \mathsf{k}[U].$$

A sheaf over X is called **algebraic** if it is an $\mathcal{O}_X$-module.

Definition 4.5. Let X be an affine algebraic variety over k and M be a $\mathsf{k}[X]$-module. We define the $\mathcal{O}_X$-module $\widetilde{M}$ over each **principal open set** $X_f := \{x \in X : f(x) \neq 0\}$ for $f \in \mathsf{k}[X]$ as

$$\widetilde{M}(X_f) := M_f = \left\{ \frac{m}{f^k} : m \in M, k \geqslant 0 \right\}.$$

With the natural restrictions this induces an $\mathcal{O}_X$-module on X.

Definition 4.6. Let X be an algebraic variety over k. We say that an algebraic sheaf $\mathcal{S}$ over X is **quasi-coherent** if for every affine open subset $U \subseteq X$ there exists an $\mathcal{O}_X(U)$-module M such that $\mathcal{S}|_U \cong \widetilde{M}$. We say it is **coherent** if every such M is finitely generated.

Remark 4.4. In the more general context of schemes, there exists a notion of algebraic coherent sheaf analogous to our definition of analytic coherent sheaf (Definition 4.3). This notion coincides with Definition 4.6 when the scheme is Noetherian. Since we are working with algebraic varieties over a field, they are always Noetherian. One of the advantages of Definition 4.6 is that the following fact is an elementary consequence of the fact that the categories of A-modules and respectively finitely generated A-modules are abelian categories.

Proposition 4.2. *Let X be an algebraic variety. The categories of algebraic quasi-coherent sheaves and algebraic coherent sheaves are both abelian categories.*

Example 4.1. One advantage of working in the algebraic category is that several results about coherent sheaves are simpler than their analytic counterparts. For instance Oka's coherence Theorem 4.1 and Cartan's Theorem 4.2 follow directly from the definition of coherent sheaf. Therefore for X an algebraic variety and $Y \subseteq X$ an algebraic subvariety, $\mathcal{O}_X$, $\mathcal{I}_Y$ and every locally free sheaf is coherent. In particular, the following proposition has the same proof as Theorem 4.4 item (i).

Proposition 4.3. *Let X be an algebraic variety and $i : Y \hookrightarrow X$ be an algebraic subvariety. For every quasi-coherent (resp. coherent) sheaf $\mathcal{S}$ on Y, $i_*\mathcal{S}$ is a quasi-coherent (resp. coherent) sheaf on X and*

$$H^q(Y, \mathcal{S}) = H^q(X, i_*\mathcal{S}), \qquad \forall q \geqslant 0.$$

The following theorem of Serre (1955) shows that the algebraic analogues of Stein varieties are the affine algebraic varieties.

Theorem 4.6 (Serre). *Let X be an affine algebraic variety over k and $\mathcal{S}$ be a quasi-coherent algebraic sheaf. Then $\mathcal{S}$ is acyclic, i.e.*

$$H^q(X, \mathcal{S}) = 0, \qquad \forall q > 0.$$

Proof. Note first that by Proposition 4.3 it is enough to show the theorem for $X = \mathbb{A}^n_{\mathsf{k}}$ the affine space. Let $\mathcal{S} = \widetilde{M}$ for M an A-module with $A = \mathsf{k}[X] = \mathsf{k}[x_1, \dots, x_n]$. If we show that M admits a resolution of A-modules

$$0 \to M \to I_0 \to I_1 \to I_2 \to \cdots \tag{4.1}$$

such that each $\widetilde{I_j}$ is acyclic, then $\mathcal{S} = \widetilde{M} \hookrightarrow \widetilde{I_\bullet}$ is an acyclic resolution which remains exact in global sections (since we re-obtain (4.1)) and so by Theorem 2.4 $\mathcal{S}$ is acyclic. In order to prove this, it is enough to show the existence of one injection $M \hookrightarrow I$ with $\widetilde{I}$ acyclic. Consider G a divisible abelian group extending $M \hookrightarrow G$ (e.g. you can take $G = \hom_{\mathbb{Z}}(\hom_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$), then we have extensions of A-modules

$$M \hookrightarrow \hom_{\mathbb{Z}}(A, M) \hookrightarrow \hom_{\mathbb{Z}}(A, G).$$

Let $I := \hom_{\mathbb{Z}}(A, G)$. We will show that $\widetilde{I}$ is flasque, i.e. that for every open set $U \subseteq X$ the natural restriction map $I \to \widetilde{I}(U)$ is surjective. Let $s \in \widetilde{I}(U)$, if we write $U = X_{f_1} \cup X_{f_2} \cup \cdots \cup X_{f_k}$ as a finite union of principal open sets, then there exist for each $i = 1, \ldots, k$ some $\varphi_i \in I$ and some $n_i \in \mathbb{Z}_{>0}$ such that

$$s|_{X_{f_i}} = \frac{\varphi_i}{f_i^{n_i}}.$$

Consider the A-module

$$N := \{a = (a_1, \ldots, a_k) \in A^k : f_1^{n_1} \cdot a_1 = f_2^{n_2} \cdot a_2 = \cdots = f_k^{n_k} \cdot a_k\}$$

and the maps $f : N \to A$, $g : N \to G$ given by $f(a) := f_1^{n_1} \cdot a_1$ and $g(a) := \varphi_1(a_1)$. Since G is a divisible group and f is injective (since all $f_i \in \mathsf{k}[x_1, \ldots, x_k]$ are non-zero and so non-zero divisors), there exists some

$$\varphi : A \to G$$

such that $\varphi \circ f = g$. We claim that $\varphi \in I$ restricts to $s \in \widetilde{I}(U)$, in other words we claim that $\varphi|_{X_{f_i}} = s|_{X_{f_i}}$. In fact,

$$f_i^{n_i} \cdot \varphi(\alpha) = \frac{\varphi(f_1^{n_1} \cdots f_k^{n_k} \alpha)}{f_1^{n_1} \cdots f_k^{n_k} / f_i^{n_i}} = \frac{\varphi\left(f\left(\frac{f_1^{n_1} \cdots f_k^{n_k} \alpha}{f_1^{n_1}}, \ldots, \frac{f_1^{n_1} \cdots f_k^{n_k} \alpha}{f_k^{n_k}}\right)\right)}{f_1^{n_1} \cdots f_k^{n_k} / f_i^{n_i}}$$

$$= \frac{\varphi_1(f_1^{n_1} \cdots f_k^{n_k} \alpha / f_1^{n_1})}{f_1^{n_1} \cdots f_k^{n_k} / f_i^{n_i}} = \frac{f_i^{n_i} \varphi_1(\alpha)}{f_1^{n_1}} = \varphi_i(\alpha).$$

□

Remark 4.5. It follows from Serre's Theorem 4.6 that every affine open covering of an algebraic variety is an acyclic cover with respect to all quasi-coherent sheaves. In the particular case of a projective variety of dimension n, Proposition 3.1 shows that there exists a finite affine open covering with at most $n + 1$ open sets.

Let us recall some basic facts about algebraic coherent sheaves on projective varieties. Let $X \subseteq \mathbb{P}_{\mathsf{k}}^n$ be a projective variety. We denote by $S(X) := \frac{\mathsf{k}[x_0, \ldots, x_n]}{I(X)}$ the **ring of homogeneous coordinates** of X, where

$$I(X) = \langle \{P \in \mathsf{k}[x_0, \ldots, x_n] : P \text{ is homogeneous and } P|_X \equiv 0\} \rangle$$

is the **homogeneous ideal** of X. Letting $U_i := \{x \in \mathbb{P}^n_{\mathsf{k}} : x_i \neq 0\}$ we have

$$\mathcal{O}_X(X \cap U_i) = S(X)_{(x_i)} = \left\{ \frac{P}{x_i^l} : P \in S(X), \deg P = l \right\}.$$

Definition 4.7. For every $k \in \mathbb{Z}$ we define the **twisted sheaf** $\mathcal{O}_X(k)$ as

$$(\mathcal{O}_X(k))(X \cap U_i) := \left\{ \frac{P}{x_i^l} : P \in S(X), \deg P = l + k \right\}.$$

In general, if $\mathcal{S}$ is an $\mathcal{O}_X$-module we define $\mathcal{S}(k) := \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{O}_X(k)$.

Remark 4.6. The sheaves $\mathcal{O}_X(k)$ are coherent $\mathcal{O}_X$-modules, in fact they are locally free of rank 1 since

$$(\mathcal{O}_X(k))(X \cap U_i) \cong \mathcal{O}_X(X \cap U_i)$$

$$\frac{P}{x_i^l} \mapsto \frac{P}{x_i^{l+k}}.$$

In particular, if

$$0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$$

is a short exact sequence of $\mathcal{O}_X$-modules, then for every $k \in \mathbb{Z}$ we have a twisted short exact sequence of the form

$$0 \to \mathcal{S}'(k) \to \mathcal{S}(k) \to \mathcal{S}''(k) \to 0.$$

Proposition 4.4. *Consider $\mathbb{P}^n$ as an algebraic variety over* k*, then*

$$H^q(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^n}(k)) = \begin{cases} \mathsf{k}[x_0, \ldots, x_n]_k & \text{if } q = 0, \\ \mathsf{k}[x_0, \ldots, x_n]_{-k-n-1} & \text{if } q = n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Take $\mathcal{U} = \{U_i\}_{i=0}^n$ the standard covering of $\mathbb{P}^n$. By Serre's Theorem 4.6 $\mathcal{U}$ is an acyclic cover for $\mathcal{O}_{\mathbb{P}^n}(k)$ and so by Leray's Theorem 2.1, $H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \cong H^q(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n}(k))$. For $q = 0$ a global section $s \in H^0(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n}(k))$ is given by

$$s|_{U_i} = \frac{P_i}{x_i^{l_i}}$$

with $\deg P_i = l_i + k$, and

$$x_i^{l_i} \cdot P_j = x_j^{l_j} \cdot P_i$$

for $0 \leqslant i < j \leqslant n$. This last relation holds in the ring of polynomials $\mathsf{k}[x_0, \ldots, x_n]$ and so each $x_i^{l_i} | P_i$, therefore there exists some polynomial $P \in \mathsf{k}[x_0, \ldots, x_n]_k$ such that $P_i = P \cdot x_i^{l_i}$ for every $i = 0, \ldots, n$. In other words $s = P$. For $q = n$ consider some $s \in H^q(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n}(k))$. This element is given by some

$$s = \frac{P}{x_0^{l_0} \cdots x_n^{l_n}} \in (\mathcal{O}_{\mathbb{P}^n}(k))(U_0 \cap \cdots \cap U_n)$$

with $\deg P = l_0 + \cdots + l_n + k$. If some of the $l_i = 0$ we get that $s \in (\mathcal{O}_{\mathbb{P}^n}(k))(U_0 \cap \cdots \widehat{U_i} \cdots \cap U_n)$, there fore defining $t_j \in (\mathcal{O}_{\mathbb{P}^n}(k))(U_0 \cap \cdots \widehat{U_j} \cdots \cap U_n)$ by

$$t_j = \begin{cases} (-1)^i s & \text{if } j = i, \\ 0 & \text{otherwise} \end{cases}$$

we get that $s = \delta(t) = 0 \in H^n(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n}(k))$. Therefore expanding the sum of P, the non-zero terms must be of the form $\frac{1}{x_0^{i_0+1} \cdots x_n^{i_n+1}}$ with $i_0, \ldots, i_n$ non-negative integers. Since these terms are k-linearly independent, we get the isomorphism

$$\mathsf{k}[x_0, \ldots, x_n]_{-k-n-1} \cong H^n(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n}(k))$$

$$x_0^{i_0} \cdots x_n^{i_n} \mapsto \frac{1}{x_0^{i_0+1} \cdots x_n^{i_n+1}}.$$

Since the covering $\mathcal{U}$ has $n + 1$ open sets, it follows immediately that $H^q(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n}(k)) = 0$ for $q > n$. Finally for $0 < q < n$ consider any $s \in C^q(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n}(k))$ such that $\delta(s) = 0$. By Serre's Theorem 4.6, if we denote $\mathcal{U} \cap U_0 = \{U_i \cap U_0\}_{i=0}^n$ the covering of U_0 then

$$s|_{\mathcal{U} \cap U_0} \in C^q(\mathcal{U} \cap U_0, \mathcal{O}_{\mathbb{P}^n}(k)|_{U_0})$$

is exact, i.e. $s|_{\mathcal{U} \cap U_0} = \delta(t)$ for some $t \in C^{q-1}(\mathcal{U} \cap U_0, \mathcal{O}_{\mathbb{P}^n}(k)|_{U_0})$. In consequence for some $m > 0$, $x_0^m \cdot t \in C^{q-1}(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n}(k + m))$ and so

$$x_0^m \cdot s = \delta(x_0^m \cdot t) = 0 \in H^q(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n}(k + m)).$$

Finally it is enough to show that $H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \xrightarrow{\cdot x_0} H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k+1))$ is injective for any k. This follows by induction on q using the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(k) \xrightarrow{\cdot x_0} \mathcal{O}_{\mathbb{P}^n}(k+1) \to i_*\mathcal{O}_{\mathbb{P}^{n-1}}(k+1) \to 0 \tag{4.2}$$

for $i : \mathbb{P}^{n-1} = \{x_0 = 0\} \hookrightarrow \mathbb{P}^n$ the inclusion, and noting that (4.2) remains exact in global sections. $\square$

Theorem 4.7 (Serre)**.** *Let X be a projective algebraic variety over* k*, and $\mathcal{S}$ be a coherent sheaf over X. Then $\mathcal{S}$ is a quotient of a finite direct sum of twisted sheaves $\mathcal{O}_X(k)$. In particular, $\mathcal{S}(m)$ is globally generated for $m \gg 0$.*

Proof. Let $X \subseteq \mathbb{P}^n$ and $\mathcal{S}|_{X\cap U_i} = \widetilde{M_i}$ for $i = 0, \ldots, n$, with M_i a finitely generated $\mathsf{k}[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$-module. Let $m_{i,1}, \ldots, m_{i,k_i}$ be generators of M_i. For every i and $j \neq i$ we have $m_{i,l} \in (M_i)_{x_j} = (M_j)_{x_i}$ and so

$$x_i^{\alpha_{i,l}} \cdot m_{i,l} \in M_j$$

for some $\alpha_{i,l} \in \mathbb{N}$. Then for some $m \gg 0$

$$x_i^m \cdot m_{i,l} \in M_j, \qquad \forall i, j, l.$$

In consequence $x_i^m \cdot m_{i,l}$ is a global section for each i and l. These global sections generate $(\mathcal{S}(m))(X \cap U_i) = ((M_i)_{x_i})_n$ and so they induce an epimorphism of $\mathcal{O}_X$-modules

$$\bigoplus_{i,l} \mathcal{O}_X \to \mathcal{S}(m).$$

Twisting by $-m$ we conclude that $\mathcal{S}$ is a quotient of $\bigoplus_{i,l} \mathcal{O}_X(-m)$ as claimed. $\square$

Theorem 4.8 (Serre)**.** *Let X be a projective algebraic variety over* k *and $\mathcal{S}$ be a coherent sheaf over X. Then*

(i) $H^q(X, \mathcal{S})$ *is a finite dimensional* k*-vector space for all* $q \geqslant 0$.

(ii) $\exists n_0 \in \mathbb{N}, \forall n \geqslant n_0, \forall q > 0 : H^q(X, \mathcal{S}(n)) = 0.$

Proof. If $i : X \hookrightarrow \mathbb{P}^n$ is the inclusion map, $i_*\mathcal{S}$ is coherent on $\mathbb{P}^n$ and so we can reduce ourselves to the case $X = \mathbb{P}^n$. Note that by Proposition 4.4 the theorem is true in the case $\mathcal{S}$ is a finite sum of twisted sheaves. In order to prove it for any

$\mathcal{S}$ we use descending induction on q (for $q > n$ is trivially true since everything vanishes). By Theorem 4.7 there exists a short exact sequence

$$0 \to \mathcal{G} \to \mathcal{L} \to \mathcal{S} \to 0$$

with $\mathcal{L}$ a finite direct sum of twisted sheaves. Then we have an exact sequence

$$H^q(X, \mathcal{L}) \to H^q(X, \mathcal{S}) \to H^{q+1}(X, \mathcal{G})$$

with extremes of finite dimension (here we are using the induction hypothesis). It follows that $H^q(X, \mathcal{S})$ is finite dimensional proving (i). By Proposition 4.4 and the induction hypothesis there exists some $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$, $H^q(X, \mathcal{L}(n)) = H^q(X, \mathcal{G}(n)) = 0$. In consequence $H^q(X, \mathcal{S}(n)) = 0$ proving (ii). $\square$

4.4 Algebraic differential forms

In this section we recall algebraic differential forms on algebraic varieties. As we will see, these differential forms have nice algebraic descriptions that allow us to work with them and do computations explicitly in a wide range of cases.

Definition 4.8. Let R be a ring and S be an R-algebra. The **module of Kähler differentials of S over R** is the S-module generated by the set of symbols $\{df : f \in S\}$, subject to the relations

$$\begin{aligned} d(f \cdot g) &= f \cdot dg + g \cdot df \\ d(rf + sg) &= r \cdot df + s \cdot dg \end{aligned} \tag{4.3}$$

for all $f, g \in S$, and $r, s \in R$. In other words

$$\Omega^1_{S/R} := \left(\bigoplus_{f \in S} S \cdot df \right) / \langle N \rangle$$

where

$$N = \{d(fg) - fdg - gdf, d(rf + sg) - rdf - sdg : f, g \in S, r, s \in R\}.$$

Remark 4.7. If we consider the map $d : S \to \Omega^1_{S/R}$, the second relation in (4.3) says that d is an R-linear map. The first relation in (4.3) is called the **Leibniz' rule**, and d is a **derivation**. This map d is called the **universal R-linear derivation**, since every R-linear derivation $d' : S \to M$ to an S-module M factors as the composition of d with a unique S-linear map $\Omega^1_{S/R} \to M$. For a proof of this property and more about $\Omega^1_{S/R}$ see for instance Eisenbud (1995, Chapter 16). For us, the relevant constructions to keep in mind are summarized in the following examples:

Example 4.2. For a polynomial ring $S = R[x_1, \ldots, x_n]$,

$$\Omega^1_{S/R} = \bigoplus_{i=1}^{n} S \cdot dx_i.$$

Example 4.3. For $S = R[x_1, \ldots, x_n]/I$, $I = \langle f_1, \ldots, f_m \rangle$, then

$$\begin{aligned} \Omega^1_{S/R} &= \left(\bigoplus_{i=1}^{n} S \cdot dx_i \right) / \langle df_1, \ldots, df_m \rangle \\ &\cong \Omega^1_{R[x_1,\ldots,x_n]/R} / \langle I \cdot \Omega^1_{R[x_1,\ldots,x_n]/R}, df_1, \ldots, df_m \rangle. \end{aligned}$$

Example 4.4. If $U \subseteq S$ is a multiplicative subset, then

$$\Omega^1_{S[U^{-1}]/R} \cong S[U^{-1}] \otimes_R \Omega^1_{S/R},$$

in particular, localizing the module of Kähler differentials we obtain the module of Kähler differentials of the localization.

Definition 4.9. Let $X \subseteq \mathbb{A}^n_{\mathsf{k}}$ be an affine algebraic variety. We define

$$\Omega^1_{X/\mathsf{k}}(X) := \Omega^1_{\mathsf{k}[X]/\mathsf{k}}.$$

Definition 4.10. Let X be an algebraic variety over k. For every open affine set $U \subseteq X$ we have an $\mathcal{O}_X(U)$-module of Kähler differentials $\Omega^1_{U/\mathsf{k}}(U)$. For any pair $U \subseteq V$ of open affine subsets of X we have the natural restriction (morphism of k-algebras)

$$\mathcal{O}_X(V) \to \mathcal{O}_X(U),$$

which induces the restriction of Kähler differentials

$$\Omega^1_{V/\mathsf{k}}(V) \to \Omega^1_{U/\mathsf{k}}(U).$$

Using these restrictions, we can glue these modules to define the **sheaf of algebraic differential forms on X**, which we denote $\Omega^1_{X/\mathsf{k}}$. This construction gives us a coherent $\mathcal{O}_X$-module. By taking the exterior power of $\Omega^1_{X/\mathsf{k}}$ we obtain the coherent **sheaf of algebraic differential k-forms on X**

$$\Omega^k_{X/\mathsf{k}} := \bigwedge^k \Omega^1_{X/\mathsf{k}}.$$

When X is smooth of dimension n, these sheaves are locally free of rank $\binom{n}{k}$.

Definition 4.11. More generally, given any morphism of algebraic varieties $f : X \to Y$ defined over the field k, we can define the **sheaf of relative algebraic differential forms** $\Omega^1_{X/Y}$ as the $\mathcal{O}_X$-module obtained by pasting the $\mathcal{O}_X(V)$-modules $\Omega^1_{S/R}$, where $R = \mathsf{k}[U]$, $S = \mathsf{k}[V]$ and $f(V) \subseteq U$. We can also define

$$\Omega^k_{X/Y} := \bigwedge^k \Omega^1_{X/Y}.$$

Furthermore, the differential map $d : \mathcal{O}_X \to \Omega^1_{X/Y}$ induces a differential map $d : \Omega^k_{X/Y} \to \Omega^{k+1}_{X/Y}$ which in turn defines the **algebraic de Rham complex**

$$0 \to \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/Y} \xrightarrow{d} \Omega^2_{X/Y} \xrightarrow{d} \cdots$$

Remark 4.8. In what follows we will mostly consider morphisms $f : X \to Y$ between complex algebraic varieties. When X is a complex algebraic variety instead of denoting the sheaf of algebraic forms over $\mathbb{C}$ by $\Omega^k_{X/\mathbb{C}}$, we will simply write Ω^k_X. In this case the differential map d is nothing else than the usual holomorphic differential ∂. The difference between them is that d only applies to differential forms with rational function coefficients, while ∂ applies to any holomorphic differential form.

Example 4.5. Let $\pi : X \to Y$ be a morphism of complex algebraic varieties, then

$$\Omega^k_{X/Y} \cong \frac{\Omega^k_X}{\pi^*\Omega^1_Y \wedge \Omega^{k-1}_X}.$$

Remark 4.9. One should not confuse Ω^k_X with the sheaf of holomorphic k-forms, we denote the latter by $\Omega^k_{X^{\mathrm{an}}}$. The sheaf of algebraic forms is defined over the Zariski topology of X, while the sheaf of holomorphic (or analytic) forms is defined over the analytic topology of X.

4.5 Serre's GAGA principle

One of the most celebrated results in complex algebraic geometry is Serre (1955/1956) GAGA (*Géométrie algébrique et géométrie analytique*) principle, which roughly speaking states that for complex projective varieties every result about coherent sheaves holds in the algebraic category if and only if it holds in the analytic category. This principle was later extended by Grothendieck and Raynaud (2002, Chapter XII) to proper complex algebraic varieties. In this section we will recall only the first part of GAGA following Serre (1955/1956).

Let X be a complex algebraic variety. We can associate to X a natural structure of analytic variety X^{an}. In fact, for each affine open set $U \subseteq X$ we have an isomorphism

$$U \underset{\varphi}{\overset{\sim}{\to}} V = \{f_1 = \cdots = f_m = 0\} \subseteq \mathbb{C}^n,$$

with $f_1, \ldots, f_m \in \mathbb{C}[x_1, \ldots, x_n]$. Since $f_1, \ldots, f_m$ are polynomials, they are in particular holomorphic functions, and so V has a natural analytic structure (and topology) induced by $\mathbb{C}^n$. By means of φ we can transport this analytic structure to U. This analytic structure is independent of chosen isomorphism φ and is denoted U^{an}. Since these open sets cover X and their analytic structures are compatible, they induce the analytic structure (and topology) of X^{an}. Furthermore, every $\mathcal{O}_X$-module $\mathcal{S}$ has a natural **analytification** $\mathcal{S}^{\mathrm{an}}$ defined over X^{an} as

$$\mathcal{S}_x^{\mathrm{an}} := \mathcal{S}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X^{\mathrm{an}},x}, \qquad \forall x \in X.$$

The GAGA principle describes the properties of the analytification functor $\mathcal{S} \mapsto \mathcal{S}^{\mathrm{an}}$ over a projective variety.

Example 4.6. Let X be a complex algebraic variety, then $(\Omega_X^k)^{\mathrm{an}} = \Omega_{X^{\mathrm{an}}}^k$. If $Y \subseteq X$ is an algebraic subvariety then $(\mathcal{I}_Y)^{\mathrm{an}} = \mathcal{I}_{Y^{\mathrm{an}}}$. If $f : X \to Z$ is any morphism of complex algebraic varieties and $\mathcal{S}$ is an $\mathcal{O}_X$-module, then $(f_*\mathcal{S})^{\mathrm{an}} = f_*^{\mathrm{an}}\mathcal{S}^{\mathrm{an}}$, where $f^{\mathrm{an}} : X^{\mathrm{an}} \to Z^{\mathrm{an}}$ is the induced morphism of analytic varieties given by f.

One of the main facts we need to establish the GAGA principle is the following.

Theorem 4.9. *Let X be a complex algebraic variety, then for every $x \in X$, $\mathcal{O}_{X^{\mathrm{an}},x}$ is a flat $\mathcal{O}_{X,x}$-module.*

Proof. See Serre (ibid., Section 6, Corollary 1). □

Corollary 4.2. *The analytification functor $\mathcal{S} \mapsto \mathcal{S}^{\mathrm{an}}$ is exact and takes coherent algebraic sheaves to coherent analytic sheaves.*

Theorem 4.10 (1st GAGA principle)**.** *Let X be a projective algebraic variety over $\mathbb{C}$ and let $\mathcal{S}$ be an algebraic coherent sheaf over X. Then we have a natural isomorphism*

$$\epsilon : H^q(X, \mathcal{S}) \xrightarrow{\sim} H^q(X^{\mathrm{an}} \mathcal{S}^{\mathrm{an}}), \qquad \forall q \geqslant 0.$$

Where ϵ is the map induced in Čech cohomology by any affine open covering $\mathcal{U}$. In other words the natural inclusion of sheaves over X

$$i : \mathcal{S} \hookrightarrow \mathcal{S}^{\mathrm{an}}$$

induces the map

$$\epsilon : H^q(X, \mathcal{S}) \cong H^q(\mathcal{U}, \mathcal{S}) \xrightarrow{i} H^q(\mathcal{U}, \mathcal{S}^{\mathrm{an}}) \cong H^q(X^{\mathrm{an}}, \mathcal{S}^{\mathrm{an}}).$$

In order to prove this result, we will prove first some lemmas.

Lemma 4.1. *Theorem 4.10 holds for $X = \mathbb{P}^n$ and $\mathcal{S} = \mathcal{O}_X$.*

Proof. It is clear that $H^0(X, \mathcal{O}_X) = \mathbb{C} = H^0(X, \mathcal{O}_{X^{\mathrm{an}}})$ and by Proposition 4.4 we also have $H^q(X, \mathcal{O}_X) = 0$ for all $q > 0$. On the other hand by Dolbeault Theorem 2.11, $H^q(X, \mathcal{O}_X) \cong H^{0,q}(\mathbb{P}^n) = 0$ for all $q > 0$ by the Hodge decomposition. □

Lemma 4.2. *Theorem 4.10 holds for $X = \mathbb{P}^n$ and $\mathcal{S} = \mathcal{O}_X(m)$ for all $m \in \mathbb{Z}$.*

Proof. We proceed by induction on n. Let $\mathbb{P}^{n-1} := \{x_n = 0\} \subseteq \mathbb{P}^n$ and let $i : \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$ be the inclusion map. Consider the short exact sequence of $\mathcal{O}_{\mathbb{P}^n}$-modules

$$0 \to \mathcal{O}_{\mathbb{P}^n}(m-1) \xrightarrow{\cdot x_n} \mathcal{O}_{\mathbb{P}^n}(m) \to i_* \mathcal{O}_{\mathbb{P}^{n-1}}(m) \to 0.$$

It induces the following diagram with exact rows

$$\begin{tikzcd}
H^{q-1}(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(m)) \arrow[r] \arrow[d, "\wr"] & H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m-1)) \arrow[r] \arrow[d, "\epsilon_{m-1}"] & H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) \arrow[r] \arrow[d, "\epsilon_m"] & H^q(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(m)) \arrow[d, "\wr"] \\
H^{q-1}(\mathbb{P}^{n-1}, \mathcal{O}^{\mathrm{an}}_{\mathbb{P}^{n-1}}(m)) \arrow[r] & H^q(\mathbb{P}^n, \mathcal{O}^{\mathrm{an}}_{\mathbb{P}^n}(m-1)) \arrow[r] & H^q(\mathbb{P}^n, \mathcal{O}^{\mathrm{an}}_{\mathbb{P}^n}(m)) \arrow[r] & H^q(\mathbb{P}^{n-1}, \mathcal{O}^{\mathrm{an}}_{\mathbb{P}^{n-1}}(m))
\end{tikzcd}$$

and so ϵ_m is an isomorphism for all $q \geqslant 0$ if and only if ϵ_{m-1} is an isomorphism for all $q \geqslant 0$. Since ϵ_0 is an isomorphism for all $q \geqslant 0$ (by Lemma 4.1) we are done. □

Lemma 4.3. *If Theorem 4.10 holds for $X = \mathbb{P}^n$, then it holds for any X.*

Proof. Just note that if $i : X \hookrightarrow \mathbb{P}^n$ is the inclusion map, then $i_*\mathcal{S}$ is a coherent $\mathcal{O}_{\mathbb{P}^n}$-module (by Proposition 4.3) with $(i_*\mathcal{S})^{\mathrm{an}} = i_*\mathcal{S}^{\mathrm{an}}$. Thus

$$H^q(X,\mathcal{S}) = H^q(\mathbb{P}^n, i_*\mathcal{S}) \cong H^q(\mathbb{P}^n, i_*\mathcal{S}^{\mathrm{an}}) = H^q(X^{\mathrm{an}}, \mathcal{S}^{\mathrm{an}}).$$

□

of Theorem 4.10. After Lemma 4.3 we are reduced to $X = \mathbb{P}^n$. Since $\mathbb{P}^n$ has an affine open covering given by $n+1$ open sets, the result is trivially true for $q > n$. Let us proceed by decreasing induction on q. By Serre's Theorem 4.7 we can write

$$0 \to \mathcal{G} \to \mathcal{L} \to \mathcal{S} \to 0$$

with $\mathcal{G}$ a coherent $\mathcal{O}_{\mathbb{P}^n}$-module and $\mathcal{L}$ a finite direct sum of twisted sheaves. By Lemma 4.2, Theorem 4.10 holds for $\mathcal{L}$. We obtain the following commutative diagram with exact rows

$$\begin{tikzcd}
H^q(\mathbb{P}^n,\mathcal{G}) \arrow[r] \arrow[d, "\epsilon_1"] & H^q(\mathbb{P}^n,\mathcal{L}) \arrow[r] \arrow[d, "\wr"] & H^q(\mathbb{P}^n,\mathcal{S}) \arrow[r] \arrow[d, "\epsilon_2"] & H^{q+1}(\mathbb{P}^n,\mathcal{G}) \arrow[r] \arrow[d, "\epsilon_3"] & H^{q+1}(\mathbb{P}^n,\mathcal{L}) \arrow[d, "\wr"] \\
H^q(\mathbb{P}^n,\mathcal{G}^{\mathrm{an}}) \arrow[r] & H^q(\mathbb{P}^n,\mathcal{L}^{\mathrm{an}}) \arrow[r] & H^q(\mathbb{P}^n,\mathcal{S}^{\mathrm{an}}) \arrow[r] & H^{q+1}(\mathbb{P}^n,\mathcal{G}^{\mathrm{an}}) \arrow[r] & H^{q+1}(\mathbb{P}^n,\mathcal{L}^{\mathrm{an}})
\end{tikzcd}$$

Let us prove first by induction that ϵ is surjective for all $q \geqslant 0$. By induction hypothesis ϵ_3 is surjective, and so by diagram chasing ϵ_2 is also surjective, completing the induction. Now let us prove by induction that ϵ is bijective for all $q \geqslant 0$. y induction hypothesis ϵ_3 is bijective, and we already know that ϵ_1 and ϵ_2 are surjective. By five lemma ϵ_2 is bijective, finishing the proof. □

Corollary 4.3. *Let X be a complex projective variety and $\mathcal{S}$ be an algebraic coherent sheaf over X. Then $H^q(X^{\mathrm{an}},\mathcal{S}^{\mathrm{an}})$ is a finite dimensional $\mathbb{C}$-vector space and there exists some n_0 such that*

$$H^q(X^{\mathrm{an}},\mathcal{S}^{\mathrm{an}}(n)) = 0, \qquad \forall q > 0, \forall n \geqslant n_0.$$

Proof. It is a direct consequence of the 1st GAGA principle (Theorem 4.10) applied to Theorem 4.8. □

Remark 4.10. The previous corollary holds for any coherent analytic sheaf, since in fact the 3rd GAGA principle asserts that every analytic coherent sheaf over a projective variety is the analytification of an algebraic coherent sheaf Serre (1955/1956, Section 12, Theorem 3).

Corollary 4.4. *Let $\mathcal{F}^\bullet$ be any bounded complex of algebraic coherent sheaves over a complex projective variety X. Then we have a natural isomorphism*

$$\mathbb{H}^k(X, \mathcal{F}^\bullet) \cong \mathbb{H}^k(X^{an}, (\mathcal{F}^\bullet)^{an}).$$

Proof. Let $\mathcal{U}$ be an affine open covering of X. The natural inclusions $\mathcal{F}^\bullet \hookrightarrow (\mathcal{F}^\bullet)^{\text{an}}$ induce a natural map

$$\mathbb{H}^k(X, \mathcal{F}^\bullet) \cong \mathbb{H}^k(\mathcal{U}, \mathcal{F}^\bullet) \to \mathbb{H}^k(\mathcal{U}, (\mathcal{F}^\bullet)^{an}) \cong \mathbb{H}^k(X^{an}, (\mathcal{F}^\bullet)^{an})$$

compatible with the naive filtrations. Thus it is enough to show that

$$Gr_F^p \mathbb{H}^{p+q}(X, \mathcal{F}^\bullet) = (E_\infty^{p,q})^{alg} \cong (E_\infty^{p,q})^{an} = Gr_F^p \mathbb{H}^{p+q}(X^{an}, (\mathcal{F}^\bullet)^{an}).$$

Since the complex is bounded, both spectral sequences degenerate. And since $E_{r+1}^{p,q}$ is the cohomology of $(E_r^{p,q}, d_r)$, it is enough to show the isomorphism at some page. The result follows by the 1st GAGA principle at the page 1

$$(E_1^{p,q})^{alg} \cong H^q(X, \mathcal{F}^p) \xrightarrow[\text{GAGA}]{\sim} H^q(X^{an}, (\mathcal{F}^p)^{an}) \cong (E_1^{p,q})^{an}.$$

□

4.6 Algebraic de Rham cohomology on affine varieties

Let X be a smooth affine variety over $\mathbb{C}$. Consider the algebraic de Rham complex of X

$$0 \to \mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^n \xrightarrow{d} 0.$$

Serre's Theorem 4.6 implies this is an acyclic complex. On the other hand, considering the C^∞ de Rham cohomology complex of X

$$0 \to \mathcal{O}_{X^\infty} \xrightarrow{d} \Omega_{X^\infty}^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_{X^\infty}^{2n} \xrightarrow{d} 0.$$

We know this is an acyclic resolution of the constant sheaf $\mathbb{C}$, by Poincaré Lemma (Corollary 2.2) and the fact each sheaf $\Omega^q_{X^\infty}$ is fine. Considering now the analytic de Rham complex of X

$$0 \to \mathcal{O}_{X^{\mathrm{an}}} \xrightarrow{\partial} \Omega^1_{X^{\mathrm{an}}} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega^n_{X^{\mathrm{an}}} \xrightarrow{\partial} 0,$$

we see it is also resolution of the constant sheaf $\mathbb{C}$ over X^{an}, by Corollary 3.1. And furthermore, it is acyclic by the fact that every affine variety is Stein (Corollary 4.1) and Cartan B Theorem 4.5.

Applying Proposition 3.2 and Proposition 3.7, we obtain the following isomorphisms

$$H^q(\Gamma(\Omega^\bullet_X), d) \cong \mathbb{H}^q(X, \Omega^\bullet_X), \tag{4.4}$$

$$H^q_{\mathrm{dR}}(X^{\mathrm{an}}) := H^q(\Gamma(\Omega^\bullet_{X^{\mathrm{an}}}), \partial) \cong \mathbb{H}^q(X^{\mathrm{an}}, \Omega^\bullet_{X^{\mathrm{an}}}) \cong H^q(X^{\mathrm{an}}, \mathbb{C}), \tag{4.5}$$

$$H^q_{\mathrm{dR}}(X) = H^q(\Gamma(\Omega^\bullet_{X^\infty}), d) \cong \mathbb{H}^q(X^{\mathrm{an}}, \Omega^\bullet_{X^\infty}) \cong H^q(X^{\mathrm{an}}, \mathbb{C}). \tag{4.6}$$

So it is clear that (4.5) is isomorphic to (4.6). On the other hand, applying Proposition 3.2 to the acyclic complex $(\Omega^\bullet_{X^{\mathrm{an}}}, \partial)$ over the Zariski topology of X we obtain the isomorphism

$$H^q_{\mathrm{dR}}(X^{\mathrm{an}}) \cong \mathbb{H}^q(X, \Omega^\bullet_{X^{\mathrm{an}}}).$$

Definition 4.12. Let X be an affine algebraic variety over a field k. We define the **algebraic de Rham cohomology** of X as

$$H^k_{\mathrm{dR}}(X/\mathsf{k}) := H^k(\Gamma(\Omega^\bullet_{X/\mathsf{k}}), d).$$

In view of the previous isomorphisms and Proposition 3.6, it is enough to show that the natural inclusion (of sheaves over the Zariski topology)

$$(\Omega^\bullet_X, d) \hookrightarrow (\Omega^\bullet_{X^{\mathrm{an}}}, \partial) \tag{4.7}$$

is a quasi-isomorphism to conclude that the algebraic de Rham cohomology (4.4) is isomorphic to analytic de Rham cohomology (4.5) and the usual de Rham cohomology (4.6). In other words that global algebraic differential forms are enough to recover the de Rham cohomology groups. The difficulty in proving (4.7) is a quasi-isomorphism lies on working over the Zariski topology, because we do not understand the stalks of the cohomology sheaves associated to each complex. Despite this fact, we can prove the following result.

Theorem 4.11 (Atiyah and Hodge (1955)). *Let X be an affine smooth variety over the field $\mathbb{C}$ of complex numbers. Then the canonical map*

$$H^q(\Gamma(\Omega_X^\bullet), d) \to H^q_{\mathrm{dR}}(X)$$

is an isomorphism of $\mathbb{C}$-vector spaces.

4.7 Proof of Atiyah–Hodge theorem

The idea of the proof is to get rid of Zariski topology, and reduce everything to prove a quasi-isomorphism between complexes of sheaves over the analytic topology. For this we need to complete the variety X to a well behaved projective variety $\bar{X}$, transport the algebraic sheaf to it and use Serre's GAGA principle (Theorem 4.10) in order to work with sheaves over the analytic topology of $\bar{X}$. The following proof was essentially taken from Narasimhan (1968), except for Lemma 4.4 which was taken from Voisin (2003, page 160, Lemma 6.6). Let us recall first some notions about divisors.

Definition 4.13. Let X be a complex algebraic variety, a codimension 1 algebraic subvariety $Y \subseteq X$ is called a **divisor**. We define the sheaf $\mathcal{O}_X(Y)$ to be the $\mathcal{O}_X$-module of rational functions with pole of order at most one along Y, this sheaf is coherent since it is locally free of rank one. We say that a locally free sheaf of rank one is an **invertible sheaf**.

Definition 4.14. Let X be a smooth complex algebraic variety and $Y \subseteq X$ be a divisor. We say Y is a **normal crossing divisor** if for every $x \in X$ there exists a coordinate chart $(z_1, \ldots, z_n) : U \xrightarrow{\varphi} V \subseteq \mathbb{C}^n$ for U an open neighbourhood of $x \in X$ (in the analytic topology of X) and V an open neighbourhood of $0 \in \mathbb{C}^n$ such that $\varphi(Y \cap U) = \{(z_1, \ldots, z_n) \in V : z_1 \cdots z_r \neq 0\}$ for some $1 \leqslant r \leqslant n$.

Definition 4.15. Let X be a complex projective variety, and $Y \subseteq X$ be a divisor. We say that Y is a **very ample divisor** if there exists a projective embedding $i : X \hookrightarrow \mathbb{P}^N$ such that $i_*\mathcal{O}_X(Y) \cong \mathcal{O}_{\mathbb{P}^N}(1)$. In general we say that an invertible sheaf $\mathcal{L}$ is **very ample** if for some projective embedding $i_*\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^N}(1)$. We say that Y is an **ample divisor** if $\mathcal{O}_X(kY) := \mathcal{O}_X(Y)^{\otimes k}$ (i.e. the sheaf of rational functions with pole of order at most k along Y) is very ample for some $k \geqslant 1$. We say that an invertible sheaf $\mathcal{L}$ is **ample** if $\mathcal{L}^{\otimes k}$ is very ample for some $k \geqslant 1$.

Proposition 4.5. *Let X be a complex projective variety, $\mathcal{L}$ be an ample invertible sheaf and $\mathcal{S}$ be an algebraic coherent sheaf. Then there exists some $k_0 \in \mathbb{N}$ such* *that $\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes k}$ is acyclic for all $k \geqslant k_0$, and so by GAGA principle $\mathcal{S}^{\mathrm{an}} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \mathcal{L}^{\mathrm{an}\otimes k}$ is also acyclic over X^{an}.*

Proof. Let $m_0 \in \mathbb{N}$ such that $\mathcal{L}^{m_0} \cong i_*\mathcal{O}_{\mathbb{P}^N}(1)$. Applying Theorem 4.8 to $\mathcal{S}, \mathcal{S} \otimes \mathcal{L}, \dots, \mathcal{S} \otimes \mathcal{L}^{m_0-1}$ we find $n_0, n_1, \dots, n_{m_0-1} \in \mathbb{N}$ such that $\mathcal{S} \otimes \mathcal{L}^j(k)$ is acyclic for all $k \geqslant n_j$. Taking $k_0 := \max\{n_j \cdot m_0 : j = 0, \dots, m_0 - 1\}$ we get for every $k \geqslant k_0$ that $k = j + m_0 \cdot q_j$ for some $0 \leqslant j < m_0$ and $q_j \geqslant n_j$. Thus $\mathcal{S} \otimes \mathcal{L}^k = \mathcal{S} \otimes \mathcal{L}^j(q_j)$ is acyclic.

□

Remark 4.11. By Hironaka's resolution of singularities theorem, we can take $j : X \hookrightarrow \bar{X}$ where $\bar{X}$ is a smooth projective variety, j is an open immersion and $Y = \bar{X} \setminus X$ is an ample normal crossing divisor of $\bar{X}$, see Kollár (2007, Theorem 3.35).

Proof of Theorem 4.11. Transporting the sheaf of algebraic p-forms over X to $\bar{X}$, we know that

$$j_*\Omega^p_X = \lim_{k\to\infty} \Omega^p_{\bar{X}} \otimes \mathcal{O}_{\bar{X}}(kY).$$

Then, if we take the analytic sheaf

$$\Omega^p_{\bar{X}^{\mathrm{an}}}(*Y) := \lim_{k\to\infty} \Omega^p_{\bar{X}^{\mathrm{an}}} \otimes \mathcal{O}_{\bar{X}^{\mathrm{an}}}(Y)^{\otimes k},$$

we have (passing to the limit in Serre's GAGA correspondence) that

$$\Gamma(\Omega^p_{\bar{X}^{\mathrm{an}}}(*Y)) \cong \Gamma(j_*\Omega^p_X) = \Gamma(\Omega^p_X).$$

Hence, it is enough to show that the cohomology of global sections associated to the complex $(\Omega^\bullet_{\bar{X}^{\mathrm{an}}}(*Y), \partial)$ corresponds to the de Rham cohomology of X. Using Proposition 4.5, we see that $(\Omega^\bullet_{\bar{X}^{\mathrm{an}}}(*Y), \partial)$ is an acyclic complex, then by Proposition 3.2

$$H^q(\Gamma(\Omega^\bullet_{\bar{X}^{\mathrm{an}}}(*Y)), \partial) \cong \mathbb{H}^q(\bar{X}, \Omega^\bullet_{\bar{X}^{\mathrm{an}}}(*Y)).$$

On the other hand, considering the $\mathcal{C}^\infty$ de Rham complex of X transported to $\bar{X}$, we have another acyclic complex $(j_*\Omega^\bullet_{X^\infty}, d)$, and

$$H^q_{\mathrm{dR}}(X) = H^q(\Gamma(j_*\Omega^\bullet_{X^\infty}), d) \cong \mathbb{H}^q(\bar{X}, j_*\Omega^\bullet_{X^\infty}).$$

This reduces the proof to show that the natural inclusion of complexes of sheaves over $\bar{X}$ (over the analytic topology)

$$\iota : (\Omega^\bullet_{\bar{X}^{\mathrm{an}}}(*Y), \partial) \hookrightarrow (j_*\Omega^\bullet_{X^\infty}, d)$$

is a quasi-isomorphism.

Since polydiscs form a basis of the analytic topology, we can focus on them. We divide the analysis on whether the polydisc intersects Y or not. If Δ is a polydisc not intersecting Y, we have that

$$(\Omega^\bullet_{\bar{X}^{\mathrm{an}}}(*Y)|_\Delta, \partial) = (\Omega^\bullet_{\Delta^{\mathrm{an}}}, \partial) \quad \text{and} \quad (j_*\Omega^\bullet_{X^\infty}|_\Delta, d) = (\Omega^\bullet_{\Delta^\infty}, d)$$

Since both are resolutions of the constant sheaf $\mathbb{C}$ over Δ, we have the desired quasi-isomorphism. This means $H^q(\iota)_x$ is an isomorphism for all $q \geqslant 0$, $x \in X$, i.e. $H^q(\iota)|_X$ is an isomorphism.

For $x \in Y$, we can choose a local chart and a polydisc Δ such that $\Delta \setminus Y$ is biholomorphic to $\{(z_1, \ldots, z_n) \in \mathbb{D}^n | z_1 \cdots z_r \neq 0\} = (\mathbb{D}^\times)^r \times \mathbb{D}^{n-r}$. These polydiscs form a neighbourhoods system for x, so we can use them to compute the stalk at x of the cohomology sheaves associated to each complex.

We know that $\Delta \setminus Y$ is homotopically equivalent to $(\mathbb{D}^\times)^r$, and so we get the isomorphism

$$H^q(\Gamma(\Delta, j_*\Omega^\bullet_{X^\infty}), d) \cong H^q_{\mathrm{dR}}((\mathbb{D}^\times)^r) = \begin{cases} 0 & \text{if } q > r, \\ \bigoplus_{1 \leqslant i_1 < \cdots < i_q \leqslant r} \mathbb{C}\left[\frac{dz_{i_1} \wedge \cdots \wedge dz_{i_q}}{z_{i_1} \cdots z_{i_q}}\right] & \text{if } 1 \leqslant q \leqslant r, \\ \mathbb{C} & \text{if } q = 0. \end{cases}$$

On the other hand, it is clear that $H^0(\Gamma(\Delta, \Omega^\bullet_{\bar{X}^{\mathrm{an}}}(*Y)), \partial) = \mathbb{C}$, then $H^0(\iota)_x$ is an isomorphism.

For $q > 0$, we clearly see that $H^q(\iota)_x$ is surjective (since the representatives of $H^q(j_*\Omega^\bullet_{X^\infty}, d)_x$ belong to $H^q(\Omega^\bullet_{\bar{X}^{\mathrm{an}}}(*Y), \partial)_x$). So, it is enough to show it is injective in order to finish the proof. This is what we prove in the following lemma. □

Lemma 4.4. *Let ω be a closed meromorphic q-form on $\Delta = \mathbb{D}^n$, which is holomorphic in $\Delta \setminus Y$ with poles along $Y = \{(z_1, \ldots, z_n) \in \Delta \mid z_1 \cdots z_r = 0\}$. Further, assume that $\omega = 0$ in $H^q_{\mathrm{dR}}(\Delta \setminus Y)$. Then there exist a meromorphic $(q-1)$-form η (defined on a possibly smaller polydisc) also with poles along Y, such that $\omega = \partial\eta$.*

Proof. We claim there exists a meromorphic $(q-1)$-form ν in some (possibly smaller) polydisc Δ with poles along Y such that $\omega - \partial\nu$ does not depend on z_i

and dz_i for $i = r+1, \dots, n$. In fact, assuming the claim for $i > m \geqslant r+1$, in a possibly smaller polydisc we have

$$\omega - \partial \nu = \frac{\mu}{(z_1 \cdots z_r)^l},$$

for some μ holomorphic q-form not depending on z_i and dz_i for $i = m+1, \dots, n$. Writing

$$\mu = \mu_1 + dz_m \wedge \mu_2,$$

with μ_1 and μ_2 two holomorphic forms not depending on z_j for $j = m+1, \dots, n$, and on dz_i for $i = m, \dots, n$. We can formally integrate the power series expansion of μ_2 to obtain a holomorphic $(q-1)$-form δ such that $\mu_2 = \frac{\partial \delta}{\partial z_m}$ (here $\frac{\partial \delta}{\partial z_m}$ is obtained by applying $\frac{\partial}{\partial z_m}$ to every component of δ) and also does not depend on z_j for $j = m+1, \dots, n$, and on dz_i for $i = m, \dots, n$. Taking $\widetilde{\nu} = \nu + \frac{\delta}{(z_1 \cdots z_r)^l}$, the form $\omega - \partial \widetilde{\nu}$ does not depend on z_j for $j = m+1, \dots, n$, and on dz_i for $i = m, \dots, n$. Furthermore, since it is ∂-closed, it follows that it does not depend on z_m. This proves the claim. Now, we can assume

$$\omega = \frac{\mu}{(z_1 \cdots z_r)^l}$$

with μ holomorphic q-form not depending on z_i and dz_i for $i = r+1, \dots, n$.

We have reduced the lemma to the following assertion:

"For any closed meromorphic q-form ω in $\Delta = \mathbb{D}^r$ with poles along $Y = \{z_1 \cdots z_r = 0\}$ such that $\omega = 0$ in $H^q_{\mathrm{dR}}((\mathbb{D}^\times)^r)$, there exist a meromorphic $(q-1)$-form η (in a possibly smaller polydisc) with poles along Y such that $\omega = \partial \eta$."

We prove this assertion by induction on r, being the case $r = 1$ clear. For $r > 1$, we can write $\mu = \mu_1 + dz_r \wedge \mu_2$ with μ_1 and μ_2 not depending on dz_r. Taking the power series expansion of μ_2 we can write

$$\omega = \psi_1 + dz_r \wedge \psi_2,$$

where

$$\psi_2 = \sum_{j=-l}^{\infty} \phi_j(z_1, \dots, z_{r-1}) z_r^j,$$

with ϕ_j meromorphic form in $\mathbb{D}^{r-1}$ with poles along $\{z_1 \cdots z_{r-1} = 0\}$. Using formal integration we obtain a meromorphic $(q-1)$-form φ such that

$$\psi_2 - \frac{\phi_{-1}}{z_r} = \frac{\partial \varphi}{\partial z_r}.$$

Then

$$\omega - \partial\varphi = \xi_1 + \xi_2 \wedge \frac{dz_r}{z_r},$$

with ξ_1 meromorphic not depending on dz_r, and ξ_2 meromorphic not depending on z_r and dz_r. Since ω is closed, we have $\frac{\partial \xi_1}{\partial z_r} = \frac{\pm 1}{z_r}\partial\xi_2$. Then $z_r \frac{\partial \xi_1}{\partial z_r}$ does not depend on z_r, i.e. ξ_1 also does not depend on z_r. Consequently, ξ_1 and ξ_2 are closed meromorphic forms in $\mathbb{D}^{r-1}$.

Künneth's theorem (see Bott and Tu (1982, Chapter 1, Section 5)) and the exactness of ω imply that ξ_1 and ξ_2 are exact forms on $\mathbb{D}^{r-1}$. By induction hypothesis we conclude the assertion. This finishes the proof of the lemma and completes the proof of Atiyah–Hodge's theorem.

$\square$

5 Algebraic de Rham cohomology

I do feel however that while we wrote algebraic GEOMETRY they [Weil, Zariski, Grothendieck] made it ALGEBRAIC geometry with all that it implies, (Solomon Lefschetz (1968)).

5.1 Introduction

The main objective of this chapter is to define the algebraic de Rham cohomology $H^m_{\mathrm{dR}}(X/\mathsf{k})$ of a smooth algebraic variety X defined over a field k of characteristic zero. When $\mathsf{k} = \mathbb{C}$ we have the underlying C^∞ manifold X^∞ and we aim to construct explicitly the isomorphism between the algebraic de Rham cohomology $H^m_{\mathrm{dR}}(X/\mathsf{k})$ and the classical de Rham cohomology $H^m_{\mathrm{dR}}(X)$. In this way, we are able to write down explicit formulas for cup products, Gauss–Manin connection etc, in algebraic de Rham cohomology. A. Grothendieck is the main responsible for the definition of algebraic de Rham cohomology, however, it must be noted that he was largely inspired by the work of Atiyah and Hodge (1955). His paper, Grothendieck (1966), was originally written as a letter to Atiyah and Hodge and it would be fair to call this Atiyah–Hodge–Grothendieck algebraic de Rham cohomology. However in the literature, and mainly in Algebraic Geometry, we find the name Grothendieck's algebraic de Rham cohomology. Algebraic de Rham cohomology for singular schemes has been studied by several authors during the seventies, see for example Hartshorne (1975) and the references therein.

5.2 The definition

The following is the final format of the definition of an algebraic de Rham cohomology by Grothendieck (1966) after a long period of dealing with integrals with algebraic integrand.

Definition 5.1. Let X be a smooth variety over a field k. We consider the complex $(\Omega^\bullet_{X/\mathsf{k}}, d)$ of regular differential forms on X. The **algebraic de Rham cohomology** of X is defined to be the hypercohomology

$$H^q_{\mathrm{dR}}(X/\mathsf{k}) := \mathbb{H}^q(X, \Omega^\bullet_{X/\mathsf{k}}),\ q = 0, 1, 2, \ldots.$$

We take a covering $\mathcal{U}$ of X/k by open affine subsets. By Serre's vanishing theorem (Theorem 4.6) this covering is acyclic with respect to all sheaves $\Omega^i_{\mathsf{X}/\mathsf{k}}$. Hence by Theorem 3.1 the hypercohomology in the above definition can be taken relative to $\mathcal{U}$. Therefore we can write an element of $H^q_{\mathrm{dR}}(X/\mathsf{k})$ relative to $\mathcal{U}$ in the format described in Section 3.3.

Remark 5.1. It might be useful to consider a projective variety X over a ring R and define the algebraic de Rham cohomology $H^q_{\mathrm{dR}}(X/\mathsf{R})$ in a similar way. This might not be a free R-module of the correct rank (which are Betti numbers if R is a subring of $\mathbb{C}$). However, in the framework of tame polynomials and Brieskorn modules, one gets such free modules. For further details see Movasati (2021, Chapter 10).

5.3 The isomorphism

Let X be a smooth affine variety over $\mathbb{C}$. We have

$$H^i_{\mathrm{dR}}(X/\mathbb{C}) \cong H^i(\Gamma(X, \Omega^\bullet_{X/\mathbb{C}}), d) \cong H^i_{\mathrm{dR}}(X)$$

The first isomorphism follows from Serre's vanishing theorem (Theorem 4.6) and Proposition 3.2. The second isomorphism is the statement of the Atiyah–Hodge theorem (Theorem 4.11).

For an arbitrary algebraic variety X over the complex numbers, the Atiyah–Hodge theorem implies that the inclusion $\Omega^\bullet_{X/\mathbb{C}} \to \Omega^\bullet_{X^\infty}$ is a quasi-isomorphism over the Zariski topology. In this way, using Proposition 3.2 and Proposition 3.6 we get the isomorphism

$$H^m_{\mathrm{dR}}(X/\mathbb{C}) \cong H^m_{\mathrm{dR}}(X).$$

In Section 3.7 we have described this isomorphism and its inverse explicitly. Since this will be an important tool for later applications, we are going to describe again the explicit construction of the maps

$$A: \quad H^m_{\mathrm{dR}}(X) \to H^m_{\mathrm{dR}}(X/\mathbb{C}), \tag{5.1}$$
$$A^{-1}: \quad H^m_{\mathrm{dR}}(X/\mathbb{C}) \to H^m_{\mathrm{dR}}(X). \tag{5.2}$$

Let us describe first the map (5.1). Take an element in $H^m_{\mathrm{dR}}(X)$, represented by a closed differential m-form $\check{\omega}$. In what follows, differential forms with ˇ will represent C^∞ differential forms, and those without ˇ will be algebraic differential forms. We restrict $\check{\omega}$ to each open set U_i, say $\check{\omega}^m_i = \check{\omega}|_{U_i}$. Using Atiyah–Hodge theorem we find algebraic differential m-forms ω^m_i and C^∞ $(m-1)$-forms $\check{\eta}^{m-1}_i$ in U_i such that

$$\check{\omega}^m_i = \omega^m_i - d\check{\eta}^{m-1}_i, \text{ and } d\omega^m = 0.$$

Applying δ to ω^m we obtain

$$\omega^m_j - \omega^m_i = d(\check{\eta}^{m-1}_j - \check{\eta}^{m-1}_i) \text{ in } U_i \cap U_j.$$

Again by Atiyah–Hodge theorem, the right hand side of the above equality represents the zero element in the $(m-1)$-th de Rham cohomology of $U_i \cap U_j$. Therefore there are algebraic differential $(m-1)$-forms ω^{m-1}_{ij} in $U_i \cap U_j$ such that

$$\omega^m_j - \omega^m_i - d\omega^{m-1}_{ij} = 0,$$

and so

$$d(\check{\eta}^{m-1}_j - \check{\eta}^{m-1}_i - \omega^{m-1}_{ij}) = 0.$$

Using Atiyah–Hodge theorem in the intersections $U_i \cap U_j$, we can add a closed algebraic differential form to ω^{m-1}_{ij} and assume that

$$\check{\eta}^{m-1}_j - \check{\eta}^{m-1}_i - \omega^{m-1}_{ij} = d\check{\eta}^{m-2}_{ij},$$

where $\check{\eta}_{ij}^{m-2}$ are C^∞ $(m-2)$-forms in $U_i \cap U_j$. Applying δ to ω^{m-1} we get in $U_i \cap U_j \cap U_k$ the relation

$$\omega_{jk}^{m-1} - \omega_{ik}^{m-1} + \omega_{ij}^{m-1} = -d(\eta_{jk}^{m-1} - \eta_{ik}^{m-1} + \eta_{ij}^{m-1}).$$

Again we have an algebraic $(m-1)$-form which is exact using C^∞ differential forms and so it must be exact using algebraic differential forms. We repeat the process to produce ω^{m-2}. This process at the k-th step gives us ω^{m-k} such that

$$\delta\omega^{m-k} + (-1)^{k+1} d(\delta\check{\eta}^{m-k-1}) = 0.$$

Using Atiyah–Hodge theorem we can produce ω^{m-k-1} such that

$$\begin{aligned} \delta\omega^{m-k} + (-1)^{k+1} d\omega^{m-k-1} &= 0, \\ \delta\check{\eta}^{m-k-1} - \omega^{m-k-1} - (-1)^{k+2} d\check{\eta}^{m-k-2} &= 0. \end{aligned}$$

At the end of the process we get the element $A\check{\omega} = \omega = \omega^0 + \omega^1 + \cdots + \omega^m \in H_{\mathrm{dR}}^m(X/\mathbb{C})$. Note that the last equality for $k = m-1$ is just $\delta\check{\eta}^0 - \omega^0 = 0$.

Let us now describe the map (5.2). Take an element in $H_{\mathrm{dR}}^m(X/\mathbb{C})$, represented by $\omega = \sum_{j=0}^m \omega^j$ such that $D\omega = 0$. In particular, $\delta\omega^0 = 0$. Since the Čech cohomology of the sheaf of C^∞ differential forms is zero (except in dimension zero), we have

$$\omega^0 = \delta\check{\eta}^0,$$

for some C^∞ differential forms $\check{\eta}^0$. Replacing ω by $\omega - D\check{\eta}^0$, we can assume that $\omega^0 = 0$. This process continues with $\delta\omega^1 = 0$. In the final step we get a closed C^∞ m-form in X. The complete description can be done in terms of a partition of unity.

Proposition 5.1. *Let X be a smooth algebraic variety over $\mathbb{C}$. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a locally finite affine cover of X. Given $\omega = \omega^0 + \cdots + \omega^m \in H_{\mathrm{dR}}^m(X/\mathbb{C})$, and a partition of unity $\{a_i\}_{i\in I}$ subordinated to $\mathcal{U}$, we can compute*

$$(A^{-1}\omega)_i = \sum_{l=0}^{m} \sum_{k_1,\ldots,k_l \in I} da_{k_1} \wedge \cdots \wedge da_{k_l} \wedge \omega_{ik_1\cdots k_l}^{m-l} \in H^0(\mathcal{U}, \Omega_{X^\infty}^m).$$

Proof. We claim that for every $j = 0, \ldots, m$ we can represent ω as $\widetilde{\omega}^j + \omega^{j+1} + \cdots + \omega^m$, where

$$\widetilde{\omega}^j_{i_0\cdots i_{m-j}} := \sum_{l=0}^{j} \sum_{k_1,\dots,k_l \in I} da_{k_1} \wedge \cdots \wedge da_{k_l} \wedge \omega^{j-l}_{i_0\cdots i_{m-j} k_1 \cdots k_l} \in C^{m-j}(\mathcal{U}, \Omega^j_{X\infty}).$$

In fact, this is clear for $j = 0$. Assuming the claim for $j \geqslant 0$, define $\eta^j \in C^{m-j-1}(\mathcal{U}, \Omega^j_{X\infty})$ by

$$\eta^j_{i_0\cdots i_{m-j-1}} := \sum_{k \in I} a_k \widetilde{\omega}^j_{i_0\cdots i_{m-j-1} k}.$$

Then $\delta\eta^j = (-1)^{m-j}\widetilde{\omega}^j$, and so $D\eta^j = (-1)^{m-j}(\widetilde{\omega}^j - d\eta^j)$. Representing ω by $\omega + (-1)^{m-j-1} D\eta^j$ we obtain the claim for $j + 1$. □

5.4 Hodge filtration

For the complex of differential forms $\Omega^\bullet_{X/\mathsf{k}}$, we have the truncated complex of differential forms

$$\Omega^{\bullet \geqslant i}_{X/\mathsf{k}} : \quad \cdots \to 0 \to 0 \to \Omega^i_{X/\mathsf{k}} \to \Omega^{i+1}_{X/\mathsf{k}} \to \cdots$$

and a natural map

$$\Omega^{\bullet \geqslant i}_{X/\mathsf{k}} \to \Omega^\bullet_{X/\mathsf{k}}.$$

We define the **algebraic Hodge filtration**

$$0 = F^{m+1} \subset F^m \subset \cdots \subset F^1 \subset F^0 = H^m_{\mathrm{dR}}(X/\mathsf{k}),$$

as follows

$$F^q = F^q H^m_{\mathrm{dR}}(X/\mathsf{k}) := \mathrm{Im}\left(\mathbb{H}^m(X, \Omega^{\bullet \geqslant i}_{X/\mathsf{k}}) \to \mathbb{H}^m(X, \Omega^\bullet_{X/\mathsf{k}})\right).$$

For the special case of the complex of differential forms, Theorem 3.2 becomes:

Theorem 5.1. *Let* k *be an algebraically closed field of characteristic zero and* X *be a smooth projective variety over* k*. We have*

$$F^q / F^{q+1} \cong H^{m-q}(X, \Omega^q_{X/\mathsf{k}})$$

By Lefschetz principle we can assume that X is defined over complex numbers. We denote by X^{an} the underlying complex manifold of X. Let $\check{\Omega}^i_{X^{\mathrm{an}}}$ be the sheaf of closed holomorphic differential form on X^{an}.

Theorem 5.2. *(Dolbeault) Let X be a smooth projective variety over $\mathbb{C}$. The maps*

$$H^j(X^{\mathrm{an}}, \Omega^i_{X^{\mathrm{an}}}) \to H^j(X^{\mathrm{an}}, \check{\Omega}^{i+1}_{X^{\mathrm{an}}}),\ i, j \in \mathbb{N}_0,$$

induced by the differential map $d : \Omega^i_{X^{\mathrm{an}}} \to \check{\Omega}^{i+1}_{X^{\mathrm{an}}}$ are all zero.

This theorem was taken from Griffiths (1969a, p. II). He uses this property in order to prove that the Hodge filtration varies holomorphically. He associates this proposition to Dolbeault, and in the appendix of Griffiths (1969b), he gives a proof using Laplacian operators. The situation is quite similar to the case of Hard Lefschetz theorem, where the only available proof is done using harmonic forms. Theorem 5.2 is equivalent to the fact the Frölicher spectral sequence of X^{an} degenerates at E_1, see for instance Voisin (2002, Theorem 8.28). In Voisin's book the mentioned fact is basically derived from the Hodge decomposition.

Proof of Theorem 5.1. We need to check that the hypothesis of Theorem 3.2 are satisfied in our case. By GAGA principle the natural morphisms

$$H^j(X, \Omega^i_X) \to H^j(X^{\mathrm{an}}, \Omega^i_{X^{\mathrm{an}}})$$

are isomorphisms of $\mathbb{C}$-vector spaces and under this identifications the algebraic d_r coincides with the holomorphic d_r. Therefore it is enough to check that the holomorphic d_r satisfies the hypothesis of Theorem 3.2. In the holomorphic context, by Theorem 5.2 we have $H^j(X^{\mathrm{an}}, \check{\Omega}^i_{X^{\mathrm{an}}}) \cong H^j(X^{\mathrm{an}}, \Omega^i_{X^{\mathrm{an}}})$. For this we write the long exact sequence of the short exact sequence

$$0 \to \check{\Omega}^i_{X^{\mathrm{an}}} \to \Omega^i_{X^{\mathrm{an}}} \to \check{\Omega}^{i+1}_{X^{\mathrm{an}}} \to 0.$$

In the definition if d_1 we can choose a representative of $\omega^i \in H^j(X^{\mathrm{an}}, \Omega^i_{X^{\mathrm{an}}})$ such that $d\omega^i = 0$, and so by definition all d_r's are zero. $\square$

According to Voisin (ibid., page 207) the algebraic proof of degeneracy at E_1 of the complex of algebraic differential forms (and hence an algebraic proof of Theorem 5.1) was done by Deligne and Illusie (1987).

Corollary 5.1. *Let X be a smooth projective variety, then via the natural isomorphism*

$$H^k_{\mathrm{dR}}(X/\mathbb{C}) \cong H^k_{\mathrm{dR}}(X), \tag{5.3}$$

F^i corresponds with the classical Hodge filtration

$$F^i \cong F^i H^k_{\mathrm{dR}}(X) := \bigoplus_{p=i}^{k} H^{p,k-p}(X).$$

Proof. The isomorphism (5.3) is given by the quasi-isomorphisms

$$(\Omega^\bullet_X, d) \hookrightarrow (\Omega^\bullet_{X^{\mathrm{an}}}, \partial), \tag{5.4}$$

$$(\Omega^\bullet_{X^{\mathrm{an}}}, \partial) \hookrightarrow (\Omega^\bullet_{X^\infty}, d). \tag{5.5}$$

Where (5.4) is over the Zariski topology of X, while (5.5) is over the analytic topology of X. If we consider the truncated complexes, we have the corresponding quasi-isomorphisms

$$(\Omega^{\bullet \geqslant i}_X, d) \hookrightarrow (\Omega^{\bullet \geqslant i}_{X^{\mathrm{an}}}, \partial), \tag{5.6}$$

$$\left(\Omega^{\bullet \geqslant i}_{X^{\mathrm{an}}}, \partial\right) \hookrightarrow \left(\bigoplus_{p \geqslant i} \Omega^{p, \bullet - p}_{X^\infty}, d\right). \tag{5.7}$$

Note that both complexes of (5.7) are resolutions of the sheaf of holomorphic ∂-closed i-forms. Therefore the image of F^i inside $H^k_{\mathrm{dR}}(X)$ is exactly

$$\check{F}^i := \operatorname{Im}\left(H^k\left(\Gamma\left(\bigoplus_{p \geqslant i} \Omega^{p,\bullet-p}_{X^\infty}\right), d\right) \to H^k_{\mathrm{dR}}(X)\right).$$

In particular, $F^i H^k_{\mathrm{dR}}(X) \subseteq \check{F}^i$. Since

$$\begin{aligned} \check{F}^i / \check{F}^{i+1} \cong F^i / F^{i+1} \cong H^{k-i}(X, \Omega^i_X) \cong \\ \cong H^{i,k-i}(X) = F^i H^k_{\mathrm{dR}}(X) / F^{i+1} H^k_{\mathrm{dR}}(X), \end{aligned}$$

we conclude $\check{F}^i = F^i H^k_{\mathrm{dR}}(X)$. □

5.5 Cup product

In the usual de Rham cohomology we have the cup/wedge product

$$H^m_{\mathrm{dR}}(X) \times H^n_{\mathrm{dR}}(X) \to H^{n+m}_{\mathrm{dR}}(X),$$

$$(\check{\omega}, \check{\alpha}) \mapsto \check{\omega} \wedge \check{\alpha}$$

and it is natural to ask for the corresponding bilinear map in algebraic de Rham cohomology. For partial result in this direction see Carlson and Griffiths (1980).

Theorem 5.3. *The cup product of* $\omega \in H^m_{\mathrm{dR}}(X/\mathsf{k})$ *and* $\alpha \in H^n_{\mathrm{dR}}(X/\mathsf{k})$ *is given by* $\gamma = \omega \cup \alpha$*, where*

$$\begin{aligned}
\gamma^{n+m}_{i_0} &= \omega^m_{i_0} \wedge \alpha^n_{i_0} \\
\gamma^{n+m-1}_{i_0 i_1} &= (-1)^m \omega^m_{i_0} \wedge \alpha^{n-1}_{i_0 i_1} + \omega^{m-1}_{i_0 i_1} \wedge \alpha^n_{i_1} \\
&\vdots \\
\gamma^{n+m-j}_{i_0 i_1 \cdots i_j} &= \sum_{r=0}^{j} (-1)^{m(j-r)+r(j-1)} \omega^{m-r}_{i_0 \cdots i_r} \wedge \alpha^{n-j+r}_{i_r \cdots i_j} \\
\gamma^0_{i_0 \cdots i_{n+m}} &= \omega^0_{i_0 \cdots i_m} \cdot \alpha^0_{i_m \cdots i_{n+m}}
\end{aligned}$$

Proof. We have to prove that under the canonical isomorphism between the classical and algebraic de Rham cohomology, the wedge product is transformed in the product given in the theorem. Let us take $\check{\omega} \in H^m_{\mathrm{dR}}(X)$ and $\check{\alpha} \in H^n_{\mathrm{dR}}(X)$ and construct the algebraic counterpart of $\check{\omega}, \check{\alpha}, \check{\omega} \cup \check{\alpha}$. For this we will use the explicit construction of (5.1). Following this, it is possible to compute the first two lines in Theorem 5.3. Once the general formula is guessed the proof is as bellow: First we check that $D\gamma = 0$.

$$
\begin{aligned}
(\delta\gamma^{m+n-j})_{i_0\cdots i_{j+1}} &= \sum_{k=0}^{j+1}(-1)^k \gamma^{m+n-j}_{i_0\cdots\hat{i}_k\cdots i_{j+1}} \\
&= \sum_{k=0}^{j+1}\sum_{r=0}^{k-1}(-1)^{k+m(j-r)+r(j-1)}\omega^{m-r}_{i_0\cdots i_r}\wedge\alpha^{n-j+r}_{i_r\cdots\hat{i}_k\cdots i_{j+1}} \\
&+ \sum_{k=0}^{j+1}\sum_{r=k+1}^{j+1}(-1)^{k+m(j-r+1)+(r-1)(j-1)}\omega^{m-r+1}_{i_0\cdots\hat{i}_k\cdots i_r}\wedge\alpha^{n-j+r}_{i_r\cdots i_{j+1}} \\
&= \sum_{r=0}^{j}\sum_{k=r+1}^{j+1}(-1)^{k+m(j-r)+r(j-1)}\omega^{m-r}_{i_0\cdots i_r}\wedge\alpha^{n-j+r}_{i_r\cdots\hat{i}_k\cdots i_{j+1}} \\
&+ \sum_{r=1}^{j+1}\sum_{k=0}^{r-1}(-1)^{k+m(j-r+1)+(r-1)(j-1)}\omega^{m-r+1}_{i_0\cdots\hat{i}_k\cdots i_r}\wedge\alpha^{n-j+r}_{i_r\cdots i_{j+1}} \\
&= \sum_{r=0}^{j}(-1)^{r+m(j-r)+r(j-1)}\omega^{m-r}_{i_0\cdots i_r}\wedge\left[(\delta\alpha^{n-j+r})_{i_r\cdots i_{j+1}}-\alpha^{n-j+r}_{i_{r+1}\cdots i_{j+1}}\right] \\
&+ \sum_{r=1}^{j+1}(-1)^{m(j-r+1)+(r-1)(j-1)}\left[(\delta\omega^{m-r+1})_{i_0\cdots i_r}-(-1)^r\omega^{m-r+1}_{i_0\cdots i_{r-1}}\right]\wedge\alpha^{n-j+r}_{i_r\cdots i_{j+1}} \\
&= \sum_{r=0}^{j}(-1)^{j+m(j-r)+r(j-1)}\omega^{m-r}_{i_0\cdots i_r}\wedge d\alpha^{n-j+r-1}_{i_r\cdots i_{j+1}} \\
&+ \sum_{r=1}^{j+1}(-1)^{r-1+m(j-r+1)+(r-1)(j-1)}(d\omega^{m-r})_{i_0\cdots i_r}\wedge\alpha^{n-j+r}_{i_r\cdots i_{j+1}} \\
&= (-1)^j d\left(\sum_{r=0}^{j+1}(-1)^{m(j+1-r)+rj}\omega^{m-r}_{i_0\cdots i_r}\wedge\alpha^{n-j-1+r}_{i_r\cdots i_{j+1}}\right) \\
&= (-1)^j d\gamma^{m+n-j-1}_{i_0\cdots i_{j+1}}
\end{aligned}
$$

Second, it is easy to verify that the cup product formula announced in Theorem 5.3 satisfies

$$
\begin{aligned}
\omega\cup\alpha &= (-1)^{mn}\alpha\cup\omega, && (5.8)\\
D\eta\cup\alpha &= D(\eta\cup\alpha) \quad \text{for} \quad D\alpha = 0. && (5.9)
\end{aligned}
$$

Therefore, the cup is independent of the choice of representatives.

We have to prove that $\check{\omega}\wedge\check{\alpha}$ and $\check{\gamma}$ induce the same element in $\mathbb{H}^{n+m}(X^\infty, \Omega^\bullet_{X^\infty})$, for $\mathsf{k} = \mathbb{C}$. This follows from the following equalities in $\mathbb{H}^{n+m}(X^\infty, \Omega^\bullet_{X^\infty})$

$$[\check{\gamma}] = [\gamma] = [\omega \cup \alpha] = [\check{\omega} \cup \check{\alpha}] = [\check{\omega} \wedge \check{\alpha}].$$

where $[\cdot]$ means the induced class in hypercohomology. □

Let us assume that m and n are even numbers and ω and α have only the middle pieces $\omega^{\frac{m}{2}}_{i_0 i_1 \cdots i_{\frac{m}{2}}}$ and $\alpha^{\frac{n}{2}}_{i_0 i_1 \cdots i_{\frac{n}{2}}}$, respectively. In this particular case, $\omega \cup \alpha$ has also only the middle piece given by

$$(\omega \cup \alpha)^{\frac{n+m}{2}}_{i_0 i_1 \cdots i_{\frac{n+m}{2}}} = (-1)^{\frac{nm}{4}} \omega^{\frac{m}{2}}_{i_0 i_1 \cdots i_{\frac{m}{2}}} \wedge \alpha^{\frac{n}{2}}_{i_{\frac{m}{2}} i_{\frac{m}{2}+1} \cdots i_{\frac{n+m}{2}}}. \tag{5.10}$$

Remark 5.2. Let $\mathcal{U}$ be a convex locally finite open cover of X^{an}. Since each open convex set has trivial de Rham cohomology (since it is contractible), an element of $H^n_{\mathrm{dR}}(X/\mathbb{C})$ can be represented by $\omega^0 \in H^m(\mathcal{U}, \mathbb{C})$ and $\alpha^0 \in H^n(\mathcal{U}, \mathbb{C})$. Let $\{a_i\}_{i \in I}$ be a partition of unity subordinated to $\mathcal{U}$, then

$$\check{\omega} = \sum_{k_1, \ldots, k_m \in I} \omega^0_{i k_1 \cdots k_m} da_{k_1} \wedge \cdots \wedge da_{k_m},$$

and

$$\check{\alpha} = \sum_{l_1, \ldots, l_n \in I} \alpha^0_{j l_1 \cdots l_n} da_{l_1} \wedge \cdots \wedge da_{l_n},$$

for any $i, j \in I$. On the other hand $\gamma = \omega \cup \alpha$ will correspond to $\check{\gamma} \in H^{m+n}_{\mathrm{dR}}(X^\infty)$ given by

$$\check{\gamma} = \sum_{k_1, \ldots, k_m, l_1, \ldots, l_n \in I} da_{k_1} \wedge \cdots \wedge da_{k_m} \wedge da_{l_1} \wedge \cdots \wedge da_{l_n} \gamma^0_{i k_1 \cdots k_m l_1 \cdots l_n} = \check{\omega} \wedge \check{\alpha}.$$

For the description of cup product in Čech cohomology $H^m(X, \mathbb{Z})$ see Brylinski (2008). For simple applications of cup product in the case of elliptic curves see Movasati (2012).

5.6 Hodge filtration and cup product

Proposition 5.2. *The cup product and the Hodge filtration satisfy the relations*

$$F^i H^m_{\mathrm{dR}}(X) \cup F^j H^n_{\mathrm{dR}}(X) \subset F^{i+j} H^{m+n}_{\mathrm{dR}}(X). \tag{5.11}$$

Proof. This follows directly from the definition of cup product. □

5.7 Polarization

Let $\mathbb{P}^N$ be a projective space of dimension N, with projective coordinates $[x_0 : x_1 : \cdots : x_N]$ and $f_{ij} := \frac{x_i}{x_j}$. The rational function f_{ij} has a zero (resp. pole) of order 1 at $x_i = 0$ (resp. $x_j = 0$). For the algebraic de Rham cohomology of $\mathbb{P}^N$ we consider the standard covering and represent its elements in this covering.

Definition 5.2. The following

$$\theta = \{\theta_{ij}\}, \quad \theta_{ij} := \frac{df_{ij}}{f_{ij}} = \frac{d(\frac{x_j}{x_i})}{\frac{x_j}{x_i}} = \frac{dx_j}{x_j} - \frac{dx_i}{x_i},$$

induces an element in $H^2_{\mathrm{dR}}(\mathbb{P}^N/\mathsf{k})$ and it is called the **polarization**. For any subvariety X of $\mathbb{P}^N$ over k, its restriction to $H^2_{\mathrm{dR}}(X/\mathsf{k})$ is also called the polarization.

Note that $\delta\theta = 0$ and $d\theta = 0$. Therefore, $D\theta = 0$ and hence θ induces an element in $H^2_{\mathrm{dR}}(X/\mathsf{k})$. Note also that if we write $\theta = \theta^0 + \theta^1 + \theta^2$, then θ^0 and θ^2 are zero.

Proposition 5.3. *The polarization in the usual de Rham cohomology $H^2_{\mathrm{dR}}(X)$ is given by the* $(1,1)$*-form*

$$\check{\theta} := \bar{\partial}\partial \log\left(\sum_{i=0}^{N} |x_i|^2\right). \tag{5.12}$$

Moreover, for any linear $\mathbb{P}^1 \subset \mathbb{P}^N$ we have

$$\int_{\mathbb{P}^1} \theta = 2\pi\sqrt{-1}.$$

Proof. The proof of the first part is as follows. We have to follow the recipe in Section 5.3 and construct $A^{-1}(\theta)$. We define:

$$p_i : \mathbb{P}^N \backslash \{x_i = 0\} \to \mathbb{R}^+, \quad p_i(x) := \sum_{j=0}^{N} \left|\frac{x_j}{x_i}\right|^2$$

and

$$\eta_i^1 := \frac{\partial p_i}{p_i} = \partial \log p_i.$$

We have

$$p_j = \left|\frac{x_j}{x_i}\right|^{-2} \cdot p_i \text{ and so } \eta^1_j - \eta^1_i = -\frac{df_{ij}}{f_{ij}} = -\theta_{ij}.$$

Let $\eta^1 = \{\eta^1_i,\ i \in I\}$. Recall the explicit description of $\mathbb{H}^2$ in (3.9),(3.10) and (3.11). We substitute θ by $\check{\theta} := \theta + D\eta$. This satisfies $\check{\theta}^0 = 0$, $\check{\theta}^1 = 0$ and

$$\check{\theta}^2_i = d\eta^1_i = \bar{\partial}\partial \log p_i.$$

The $(1,1)$-forms $\check{\theta}^2_i$'s in the intersection of U_i's coincide and give us the element (5.12) in the usual de Rham cohomology.

The second statement follows from the fact that $\frac{1}{2\pi i}\check{\theta}$ is the first Chern class of the tautological line bundle of $\mathbb{P}^N$, see for instance Movasati (2017a). An elementary proof is as follows. Since all lines in $\mathbb{P}^N$ are homologous to each other, we assume that $\mathbb{P}^1$ is given by $x_N = x_{N-1} = \cdots = x_2 = 0$, and hence, we only need to prove the case $N = 1$. In this case, let us take the chart $x_1 = 1$ and use $z := x_0$. With this new notation, let x and y be the real and complex part of z, that is, $z := x + iy$. We have

$$\bar{\partial}\partial \log(1 + |z|^2) = 2\sqrt{-1}\frac{dx \wedge dy}{(1 + x^2 + y^2)^2}$$

The desired result follows from

$$\int_{\mathbb{R}^2} \frac{dx \wedge dy}{(1 + x^2 + y^2)^2} = \pi \tag{5.13}$$

This can be checked directly or by the following Mathematica command:

```
Integrate[1/(x^2+y^2+1)^2,{x,-Infinity,Infinity},{y,-Infinity,Infinity}]
```

□

Using Theorem 5.3 we know that $\theta^m \in H^{2m}_{\mathrm{dR}}(\mathbb{P}^N/\mathsf{k})$ has only the middle piece. This middle piece is given by

$$(\theta^m)_{i_0 i_1 \cdots i_m} = (-1)^{\binom{m+1}{2}} \left(\frac{dx_{i_0}}{x_{i_0}} - \frac{dx_{i_1}}{x_{i_1}}\right) \wedge \left(\frac{dx_{i_1}}{x_{i_1}} - \frac{dx_{i_2}}{x_{i_2}}\right) \wedge \cdots \wedge \left(\frac{dx_{i_{m-1}}}{x_{i_{m-1}}} - \frac{dx_{i_m}}{x_{i_m}}\right). \tag{5.14}$$

In particular, in the top cohomology $H^{2n}_{\mathrm{dR}}(\mathbb{P}^N/\mathsf{k})$ we have the element θ^N given by its middle piece

$$(\theta^N)_{012\cdots N} = (-1)^{\binom{N+1}{2}}\left(\frac{dx_0}{x_0}-\frac{dx_1}{x_1}\right)\wedge\left(\frac{dx_1}{x_1}-\frac{dx_2}{x_2}\right)\wedge\cdots\wedge\left(\frac{dx_{N-1}}{x_{N-1}}-\frac{dx_N}{x_N}\right)$$
$$= (-1)^{\binom{N}{2}}\frac{\sum_{i=0}^{N}(-1)^i x_i\widehat{dx_i}}{x_0x_1x_2\cdots x_N}.$$

The summary of our discussion is in the following theorem:

Theorem 5.4. *Let k be a field of characteristic zero and not necessarily algebraically closed. The algebraic de Rham cohomology ring $H^*_{\mathrm{dR}}(\mathbb{P}^N/\mathsf{k})$ is generated by θ.*

Proof. We can take k small enough to have the embedding $\mathsf{k}\hookrightarrow\mathbb{C}$, and since the statement of the theorem remain valid under field extensions, we can assume that $\mathsf{k}=\mathbb{C}$. By Proposition 5.3 $\frac{1}{2\pi i}\theta$ maps to $u\in H^2(\mathbb{P}^N,\mathbb{Z})$ under the canonical isomorphism $H^m_{\mathrm{dR}}(\mathbb{P}^N/\mathbb{C})\cong H^m_{\mathrm{dR}}(\mathbb{P}^N)$. Here, u is the cohomology class of a hyperplane $\mathbb{P}^{N-1}\subset\mathbb{P}^N$. Now the the theorem follows from the same theorem in the topological side. □

It would be more convenient to give a proof of Theorem 5.4 without referring to the same statement in topology.

5.8 Top cohomology

Let $X\subset\mathbb{P}^N$ be a smooth projective variety of dimension n over k.

Proposition 5.4. *The top cohomology $H^{2n}_{\mathrm{dR}}(X/\mathsf{k})$ is of dimension 1 and it is generated by the restriction of $\theta^n\in H^{2n}_{\mathrm{dR}}(\mathbb{P}^N/\mathsf{k})$. Moreover, for an embedding $\mathsf{k}\hookrightarrow\mathbb{C}$ we have*

$$\int_X\theta^n = \deg(X)\cdot(2\pi i)^n.$$

Proof. The proof follows from a similar statement in topology and the fact that $u=\frac{1}{2\pi i}\theta\in H^2(X^{\mathrm{an}},\mathbb{Z})$ is the topological polarization. □

It turns out that we have a canonical isomorphism

$$\mathrm{Tr}: H^{2n}_{\mathrm{dR}}(X/\mathsf{k})\cong\mathsf{k},\ \ \mathrm{Tr}(\omega):=\deg(X)\cdot\frac{\omega}{\theta^n}$$

over the field k. In the complex context $\mathsf{k} = \mathbb{C}$, it is given by

$$\omega \mapsto \frac{1}{(2\pi i)^n} \int_X \omega.$$

Definition 5.3. The element $\alpha := \frac{\theta^n}{\deg(X)}$ is called the **volume top form** of X. It is characterized by the fact that $\mathrm{Tr}(\alpha) = 1$.

Let $\mathbb{P}^{N-n-1} \subset \mathbb{P}^N$ be a projective space such that $\mathbb{P}^{N-n-1} \cap X = \emptyset$. We assume that it is given by linear equations $g_0 = g_1 = \cdots = g_n = 0$ and consider the corresponding covering $\mathcal{U} = \{U_i,\ i = 0, 1, \ldots, n\},\ \ U_i := \{x \in X | g_i(x) \neq 0\}$. In this covering $\theta^n \in H^{2n}_{\mathrm{dR}}(X/\mathsf{k})$ is given by

$$(\theta^n)_{012\cdots n} = (-1)^{\binom{n+1}{2}} \left(\frac{dg_0}{g_0} - \frac{dg_1}{g_1}\right) \wedge \left(\frac{dg_1}{g_1} - \frac{dg_2}{g_2}\right) \wedge \cdots \wedge \left(\frac{dg_{n-1}}{g_{n-1}} - \frac{dg_n}{g_n}\right). \tag{5.15}$$

Remark 5.3. The appearance of the sign $(-1)^{\binom{n+1}{2}}$ in our formulas is natural. In Deligne, Milne, et al. (1982) the pairing between the N dimensional homology and de Rham cohomology is modified with this sign and it has been remarked: "Our signs differ from the usual signs because the standard sign conventions

$$\int_\sigma d\omega = \int_{d\sigma} \omega, \quad \int_{X \times Y} pr_1^* \omega \wedge pr_2^* \eta = \int_X \omega \cdot \int_Y \eta, \quad \text{etc.}$$

violate the sign conventions for complexes. "

5.9 Bilinear maps in cohomology

The cup product composed with the trace map gives us the bilinear maps

$$\langle \cdot, \cdot \rangle : H^m_{\mathrm{dR}}(X) \times H^{2\mathrm{n}-m}_{\mathrm{dR}}(X) \to \mathsf{k}, \ \ (\alpha, \beta) \mapsto \mathrm{Tr}(\alpha \cup \beta) \tag{5.16}$$

and

$$\langle \cdot, \cdot \rangle : H^m_{\mathrm{dR}}(X) \times H^m_{\mathrm{dR}}(X) \to \mathsf{k}, \ \ (\alpha, \beta) \mapsto \mathrm{Tr}(\alpha \cup \beta \cup \theta^{\mathrm{n}-m}). \tag{5.17}$$

Note that the second bilinear map depends on the polarization.

Proposition 5.5. *The bilinear maps* (5.16) *and* (5.17) *are non-degenerate.*

Proof. The fact that the map (5.16) is non-degenerate follows from the same statement in the topological context. The same statement for (5.17) follows from the Hard Lefschetz theorem and the previous statement. □

Proposition 5.6. *The bilinear maps* (5.16) *and* (5.17) *satisfy*

$$\langle F^i, F^j \rangle = 0,$$

for $i + j > n$ and $i + j > m$ respectively.

Proof. The relation between cup product and Hodge filtration is described in Proposition 5.2. We use this for $m_1 = m$ and $m_2 = 2n - m$. Moreover, we know that $F^i H^{2n}_{\mathrm{dR}}(X) =$ for $i > n$. This finishes the proof for the bilinear map (5.16). For (5.17) note that the isomorphism of Hard Lefschetz theorem $H^m_{\mathrm{dR}}(X) \to H^{2n-m}_{\mathrm{dR}}(X)$, $\omega \mapsto \omega \cup \theta^{n-m}$ maps $F^i H^m_{\mathrm{dR}}(X)$ into $F^{i+n-m} H^{2n-m}_{\mathrm{dR}}(X)$. □

5.10 Volume top form in $\mathbb{P}^N$

All our methods to compute periods reduce at some point to compute the period of an element in the top cohomology of the projective space. Since $H^{2N}_{\mathrm{dR}}(\mathbb{P}^N) = H^{N,N}(\mathbb{P}^N) \cong H^N(\mathbb{P}^N, \Omega^N_{\mathbb{P}^N}) \cong \mathbb{C}$, we just need to know the period of one generator. This will give us the top form of $\mathbb{P}^N$ in the sense of Definition 5.3. We will fix the element

$$\frac{\Omega}{x_0 \cdots x_N} = \frac{\sum_{i=0}^N (-1)^i x_i \widehat{dx_i}}{x_0 \cdots x_N} = \left(\frac{dx_1}{x_1} - \frac{dx_0}{x_0}\right) \wedge \cdots \wedge \left(\frac{dx_N}{x_N} - \frac{dx_0}{x_0}\right) \in H^N(\mathcal{U}, \Omega^N_{\mathbb{P}^N}),$$

and compute its period. Here $\mathcal{U} = \{U_i\}_{i=0}^N$ is the standard open covering of $\mathbb{P}^N$, i.e. $U_i = \{x_i \neq 0\}$.

Proposition 5.7. *We have*

$$\int_{\mathbb{P}^N} \frac{\Omega}{x_0 \cdots x_N} = (-1)^{\binom{N}{2}} (2\pi\sqrt{-1})^N.$$

Since we have natural isomorphisms $H^N(\mathbb{P}^N, \Omega^N_{\mathbb{P}^N}) \cong \mathbb{H}^{2N}(\mathbb{P}^N, \Omega^\bullet_{(\mathbb{P}^N)^\infty}) \cong H^{2N}_{\mathrm{dR}}(\mathbb{P}^N)$. The element $\frac{\Omega}{x_0 \cdots x_N} \in H^N(\mathbb{P}^N, \Omega^N_{\mathbb{P}^N})$ corresponds to a top form

$\omega \in H_{\mathrm{dR}}^{2N}(\mathbb{P}^N)$. By abuse of notation we will denote

$$\int_{\mathbb{P}^N} \frac{\Omega}{x_0 \cdots x_N} := \int_{\mathbb{P}^N} \omega.$$

We will always use this identification when we talk about periods. From Proposition 5.7 we get:

Proposition 5.8. *The volume top form of $\mathbb{P}^N$ in the sense of Definition 5.3 and associated to the standard covering $\mathcal{U}$ of $\mathbb{P}^N$ is given by*

$$(-1)^{\binom{N}{2}} \frac{\sum_{i=0}^{N} (-1)^i x_i \widehat{dx_i}}{x_0 \cdots x_N}.$$

Note that this top form is equal to θ^N, where $\theta \in H_{\mathrm{dR}}^2(\mathbb{P}^N/\mathbb{C})$ is the polarization element defined in Section 5.7. Therefore, Proposition 5.7 follows from Proposition 5.3. We would like to give a different proof of this using a partition of unity subordinated to the covering $\mathcal{U}$. The main reason for this is that the isomorphism between algebraic and C^∞ de Rham cohomology groups described in Section 5.3 can be explicitly written down using a partition of unity. However, this resulting formula might be huge. Thus the case of $\mathbb{P}^N$ serves as a good exercise in case one needs a more concrete presentation of the material in Section 5.3.

Proof of Proposition 5.7. We have to determine the image of $\frac{\Omega}{x_0 \cdots x_N}$ via the isomorphisms

$$H^N\left(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^N\right) \cong H_{\mathrm{dR}}^{2N}\left(\mathbb{P}^N\right) \cong \mathbb{H}^{2N}\left(\mathbb{P}^N, \Omega^\bullet_{(\mathbb{P}^N)_\infty}\right).$$

Let $\{a_i\}_{i=0}^N$ be a partition of unity subordinated to $\{U_i\}_{i=0}^N$. In the affine chart $\mathbb{C}^N \subset \mathbb{P}^N$, consider the coordinate system z with $z_i := x_i/x_0$ for $i = 1, 2, \ldots, N$. Without loss of generality, we can take the partition of unity such that

$$a_i = \begin{cases} 0 & \text{if } |z_i| \leqslant 1, \\ 1 & \text{if } |z_i| \geqslant 2, |z_j| \leqslant 1 \, \forall j \in \{1, \cdots, N\} \setminus \{i\}, \end{cases}$$

and

$$a_1 + \cdots + a_N = 1 \text{ if } \exists j \in \{1, \cdots, N\} : |z_j| \geqslant 2.$$

Thus

$$\mathrm{Supp}(da_1 \wedge \cdots \wedge da_N) \subseteq \{z \in \mathbb{C}^N : 1 \leqslant |z_i| \leqslant 2, \forall i = 1, \ldots, N\} = (\overline{\mathbb{D}(0;2)} \setminus \mathbb{D}(0;1))^N.$$

Using Proposition 5.1, we know that the image of $\frac{\Omega}{x_0 \cdots x_N}$ in $H^{2N}_{\mathrm{dR}}(\mathbb{P}^N)$ is represented by the global closed $2N$-form ω^{2N} which in the chart U_i is given by

$$(\omega^{2N})_i = N! \cdot (-1)^i da_0 \wedge \cdots \widehat{da_i} \cdots \wedge da_N \wedge \frac{\Omega}{x_0 \cdots x_N} \in \Omega^{2N}_{(\mathbb{P}^N)^\infty}(U_i).$$

Since $Supp(\omega^{2N}) \subset U_{0,1,\ldots,N}$. In order to integrate ω^{2N} over $\mathbb{P}^N$, is enough to do it on any affine chart:

$$\begin{aligned}
\int_{\mathbb{P}^N} \frac{\Omega}{x_0 \cdots x_N} &= \int_{U_0} (\omega^{2N})_0 \\
&= N! \cdot \int_{\mathbb{C}^N} da_1 \wedge \cdots \wedge da_N \wedge \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_N}{z_N} \\
&= N! \cdot (-1)^{\binom{N}{2}} \int_{\mathbb{C}^N} d\left(a_1 \cdot \frac{dz_1}{z_1}\right) \wedge \cdots \wedge d\left(a_N \cdot \frac{dz_N}{z_N}\right) \\
&= (-1)^{\binom{N}{2}} (2\pi i)^N.
\end{aligned}$$

where in the last equality we have used Stokes formula and the Cauchy residue formula.

□

It is easy to see that an element of $Z^N(\mathcal{U}, \Omega^N_{\mathbb{P}^N})$ is of the form

$$\frac{P\Omega}{x_0^{\alpha_0} \cdots x_N^{\alpha_N}}$$

with $\alpha_0, .., \alpha_N \in \mathbb{N}$ such that $\alpha_0 + \cdots + \alpha_N = deg(P) + N + 1$. Then it is a $\mathbb{C}$-linear combination of terms of the form

$$x_0^{\beta_0} \cdots x_N^{\beta_N} \Omega$$

with $\beta_0, \ldots, \beta_N \in \mathbb{Z}$ such that $\beta_0 + \cdots + \beta_N = -N - 1$. The following proposition tells us how to compute the period of any such form.

Proposition 5.9. *The form*

$$x_0^{\beta_0} \cdots x_N^{\beta_N} \Omega \in H^N(\mathcal{U}, \Omega^N_{\mathbb{P}^N})$$

represents an exact top form of $\mathbb{P}^N$ *if and only if* $(\beta_0, \ldots, \beta_N) \neq (-1, \cdots, -1)$. *In particular,*

$$\int_{\mathbb{P}^N} x_0^{\beta_0} \cdots x_N^{\beta_N} \Omega = \begin{cases} 0 & \text{if } (\beta_0, \ldots, \beta_N) \neq (-1, \ldots, -1), \\ (-1)^{\binom{N}{2}} (2\pi\sqrt{-1})^N & \text{if } (\beta_0, \ldots, \beta_N) = (-1, \ldots, -1). \end{cases}$$

Proof. The only thing left to prove is that for $(\beta_0, \ldots, \beta_N) \neq (-1, \ldots, -1)$ the form is D-exact. Using the isomorphism $H^N(\mathbb{P}^N, \Omega^N_{\mathbb{P}^N}) \cong \mathbb{H}^{2N}(\mathbb{P}^N, \Omega^\bullet_{\mathbb{P}^N})$ it is enough to show that the element $\omega \in \mathbb{H}^{2N}(\mathbb{P}^N, \Omega^\bullet_{\mathbb{P}^N})$ given by

$$\omega^N = x_0^{\beta_0} \cdots x_N^{\beta_N} \Omega,$$

and $\omega^i = 0$ for $i \neq N$, is zero in hypercohomology. Note that if $(\beta_0, \ldots, \beta_N) \neq (-1, \ldots, -1)$, then there exist some $\beta_i \geqslant 0$. Therefore,

$$x_0^{\beta_0} \cdots x_N^{\beta_N} \Omega \in \Omega^N_{\mathbb{P}^N}(\cap_{j \neq i} U_j).$$

Define $\eta \in \bigoplus_{j=0}^{2N-1} C^{2N-1-j}(\mathcal{U}, \Omega^j_{\mathbb{P}^N})$ by

$$\eta^N_J := (-1)^i x_0^{\beta_0} \cdots x_N^{\beta_N} \Omega,$$

for $J = (0, \ldots, i-1, i+1, \ldots, N)$, $\eta^N_{J'} = 0$ for $J' \neq J$, and $\eta^j = 0$ for $j \neq N$. This form clearly satisfy $D\eta = \omega$ as desired. □

5.11 Chern class

Consider the short exact sequence

$$0 \to 2\pi i \mathbb{Z} \to \mathcal{O}_{X^{\mathrm{an}}} \to \mathcal{O}^*_{X^{\mathrm{an}}} \to 0 \tag{5.18}$$

over a complex manifold X^{an}. The map $\mathcal{O}_{X^{\mathrm{an}}} \to \mathcal{O}^*_{X^{\mathrm{an}}}$ is given by the exponential map

$$f \mapsto e^f.$$

We write the corresponding long exact sequence

$$\cdots \to H^1(X^{\mathrm{an}}, \mathcal{O}^*_{X^{\mathrm{an}}}) \xrightarrow{c} H^2(X^{\mathrm{an}}, 2\pi i \mathbb{Z}) \xrightarrow{p} H^2(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \to \cdots \tag{5.19}$$

and call c the **Chern class map**. For a line bundle $L \in H^1(X^{\mathrm{an}}, \mathcal{O}^*_{X^{\mathrm{an}}})$ on X^{an}, $c(L)$ is called the **Chern class** of L.

By Serre (1955/1956) GAGA, holomorphic line bundles in X^{an} are algebraic and the natural map $H^1(X, \mathcal{O}^*_{X/\mathbb{C}}) \to H^1(X^{\mathrm{an}}, \mathcal{O}^*_{X^{\mathrm{an}}})$ is an isomorphism. Now, let us consider X over the field k.

Proposition 5.10. *The algebraic Chern class map* $c : H^1(X, \mathcal{O}^*_{X/\mathsf{k}}) \to H^2_{\mathrm{dR}}(X/\mathsf{k})$ *is given by*

$$c(L) := \left\{ \frac{dh_{ij}}{h_{ij}} \right\} \in H^2_{\mathrm{dR}}(X/\mathsf{k}), \quad L := \{h_{ij}\} \in H^1(X, \mathcal{O}^*_{X/\mathsf{k}}).$$

Proof. We prove the proposition for $\mathsf{k} = \mathbb{C}$. Take an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ in the analytic/usual topology of X^{an}such that there exist $g_{ij} = \log h_{ij}$ for every $i, j \in I$. Note that this is not necessarily true for affine open sets. Then it is easy to see that the coboundary c is given by

$$c(L) = \delta g \in H^2(\mathcal{U}, 2\pi i \mathbb{Z}).$$

Now, we have to take this element into algebraic de Rham cohomology. Representing $c(L) \in \mathbb{H}^2(X, \Omega^\bullet_{X^{\mathrm{an}}}) = H^2_{\mathrm{dR}}(X/\mathsf{k})$ we see that $D(g) = c(L) - dg$. Therefore

$$c(L)_{ij} = \frac{dh_{ij}}{h_{ij}},$$

as claimed. □

Definition 5.4. We define

$$\mathrm{Pic}(X/\mathsf{k}) := H^1(X^{\mathrm{an}}, \mathcal{O}^*_{X/\mathsf{k}})$$

and call it the **Picard group** of X/k. We also define

$$\begin{aligned} \mathrm{NS}(X/\mathsf{k}) &:= \frac{H^1(X, \mathcal{O}^*_{X/\mathsf{k}})}{\mathrm{Im}(H^1(X, \mathcal{O}_{X/\mathsf{k}}) \to H^1(X, \mathcal{O}^*_{X/\mathsf{k}}))} \\ &\cong \mathrm{Im}(H^1(X, \mathcal{O}^*_{X/\mathsf{k}}) \xrightarrow{c} H^2_{\mathrm{dR}}(X/\mathsf{k})) \end{aligned}$$

and call it the **Néron–Severi group** of X/k.

Let us now consider an **algebraic cycle** $Z = \sum_{i=1}^{k} n_i Z_i,\ \ n_i \in \mathbb{Z}$ and Z_i's are irreducible subvarieties of X of codimension 1. These are also called **divisors** in X. In affine open sets U_i, Z is the divisor of a rational function f_i and it turns out that

$$L_Z := \{h_{ij}\} = \left\{\frac{f_i}{f_j}\right\} \in H^1(X, \mathcal{O}^*_{X/\mathsf{k}})$$

is a line bundle in X/k.

5.12 The cohomological class of an algebraic cycle

Let $Z \subset X$ be an algebraic cycle of codimension p in a smooth projective variety X, both of them are defined over a field k of characteristic 0. We may assume that k is small enough so that we have an embedding $\mathsf{k} \hookrightarrow \mathbb{C}$. Since any projective variety X over k uses a finite number of coefficients in k, the assumption on the embedding of k in $\mathbb{C}$ is not problematic.

In this section we would like to construct the cohomology class $[Z]^{\mathsf{pd}} \in H^p_{\mathrm{dR}}(X)$. Note that the notation $[Z]$ is reserved for the homology class of Z in $H_{2n-2p}(X^{\mathrm{an}}, \mathbb{Z})$ and our notation $[Z]^{\mathsf{pd}}$ is justified by Theorem 5.5. According to Deligne, Milne, et al. (1982) the construction of $cl(Z) = [Z]^{\mathsf{pd}}$ is as follows: "In each cohomology theory there is a canonical way of attaching a class $cl(Z)$ in $H^{2p}(X)(p)$ to an algebraic cycle Z on X of pure codimension p. Since our cohomology groups are without torsion, we can do this using Chern classes (Grothendieck 1958). Starting with a functorial isomorphism $c_1 : \mathrm{Pic}(X) \to H^2(X)(1)$, one uses the splitting principle to define the Chern polynomial

$$c_t(E) = \sum c_p(E)t^p,\ c_p(E) \in H^{2p}(X)(p),$$

of a vector bundle E on X. The map $E \mapsto c_t(E)$ is additive, and therefore factors through the Grothendieck group of the category of vector bundles on X. But, as X is smooth, this group is the same as the Grothendieck group of the category of coherent $\mathcal{O}_X$-modules, and we can therefore define

$$cl(Z) := \frac{1}{(p-1)!} c_p(\mathcal{O}_Z)."$$

see also Voisin (2007) regarding this definition. It is not clear to the authors how to use the above definition and compute $[Z]^{\mathsf{pd}}$. In other words, if Z is given by an explicit ideal and $H^p_{\mathrm{dR}}(X)$ has an explicit basis (for instance Griffiths' basis in the case of hypersurfaces) then we would like to compute $[Z]^{\mathsf{pd}}$ in terms of such a basis. There is an alternative way to construct $[Z]^{\mathsf{pd}}$ using the resolution of singularities in characteristic 0 which is also not practical from the computational point of view. However, for the main purposes of the present text it is enough.

Definition 5.5. We know that there is a smooth projective variety $\tilde{Z}$ and a morphism $\pi : \tilde{Z} \to Z$ defined over k. Now, we have the induced map in de Rham cohomology composed with the trace map

$$H^{2(n-p)}_{\mathrm{dR}}(X/\mathsf{k}) \xrightarrow{r} H^{2(n-p)}_{\mathrm{dR}}(\tilde{Z}/\mathsf{k}) \xrightarrow{\mathrm{Tr}} \mathsf{k} \tag{5.20}$$

Since the pairing $H^{2(n-p)}_{\mathrm{dR}}(X/\mathsf{k}) \times H^{2p}_{\mathrm{dR}}(X/\mathsf{k}) \to \mathsf{k}$ is non-degenerate we get the class $[Z]^{\mathsf{pd}} \in H^{2p}_{\mathrm{dR}}(X/\mathsf{k})$. It is uniquely determined by the property:

$$\mathrm{Tr}(\omega \cup [Z]^{\mathsf{pd}}) = \mathrm{Tr} \circ r(\omega), \quad \forall \omega \in H^{2(n-p)}_{\mathrm{dR}}(X/\mathsf{k}). \tag{5.21}$$

Theorem 5.5 (Poincaré duality for algebraic cycles)**.** *We have*

$$\int_{[Z]} \omega = \int_{X^{\mathrm{an}}} \omega \cup \frac{[Z]^{\mathsf{pd}}}{(2\pi i)^p}, \quad \forall \omega \in H^{2n-2p}_{\mathrm{dR}}(X). \tag{5.22}$$

Proof. This follows from (5.21) and the formula of the trace map using integrals in Section 5.8. Note that if $\pi : \tilde{Z} \to Z$ is the desingularization map and $Z_{\mathrm{smooth}} \subset Z$ is the open subset of smooth points of Z then π is a biholomorphism between $\pi^{-1}(Z_{\mathrm{smooth}})$ and Z_{smooth} and hence

$$(2\pi i)^{n-p}\mathrm{Tr} \circ r(\omega) = \int_{\tilde{Z}} r(\omega) = \int_{\pi^{-1}(Z_{\mathrm{smooth}})} r(\omega) = \int_{Z_{\mathrm{smooth}}} \omega = \int_{[Z]} \omega \tag{5.23}$$

for all $\forall \omega \in H^{2n-2p}_{\mathrm{dR}}(X)$.

□

In the algebraic context, it is not clear why $[Z]^{\mathsf{pd}}$ is independent of the resolution map. In the complex geometry context, this follows from the last equality in (5.23). Note that the topological Poincaré dual of Z is $\frac{[Z]^{\mathsf{pd}}}{(2\pi i)^p}$. For this reason we sometimes call $[Z]^{\mathsf{pd}}$ the **algebraic Poincaré dual of** $[Z]$.

Corollary 5.2. *We have*

$$\frac{1}{(2\pi i)^{\dim(Z)}} \int_{[Z]} \omega = \mathrm{Tr}(\omega \cup [Z]^{\mathsf{pd}}), \qquad \forall \omega \in H^{2n-2m}_{\mathrm{dR}}(X/\mathsf{k})$$

Proof. This is just a reformulation of Theorem 5.5. □

Corollary 5.2 tell us that the integration over algebraic cycles is a purely algebraic operation for which we do not need to embed the base field inside complex numbers.

Proposition 5.11. *(Deligne, Milne, et al. (1982, Proposition 1.5)) Let X be a projective smooth variety over an algebraically closed field $\mathsf{k} \subset \mathbb{C}$ and let Z be an irreducible subvariety in X of dimension $\frac{m}{2}$. For any element of the algebraic de Rham cohomology $H^m_{\mathrm{dR}}(X/\mathsf{k})$, we have*

$$\frac{1}{(2\pi i)^{\dim(Z)}} \int_{[Z]} \omega \quad \in \quad \mathsf{k} \tag{5.24}$$

where $[Z] \subset H_n(X^{\mathrm{an}}, \mathbb{Z})$ is the homology class of Z.

Proof. This is a direct consequence of Corollary 5.2. □

Proposition 5.12. *Let X be a smooth projective variety over an algebraically closed field $\mathsf{k} \subset \mathbb{C}$ and let Z be an irreducible subvariety in X of codimension p. We have $[Z]^{\mathsf{pd}} \in F^{p+1}H^{2p}(X/\mathsf{k})$ and*

$$\int_{[Z]} F^{n-p+1} H^{2n-2p}_{\mathrm{dR}}(X/\mathsf{k}) = 0, \tag{5.25}$$

where $[Z] \subset H_{2(n-p)}(X^{\mathrm{an}}, \mathbb{Z})$ is the homology class of Z.

Proof. The restriction map r in (5.20) restricted to $F^{n-p+1} H^{2n-2p}_{\mathrm{dR}}(X/\mathsf{k})$ is identically zero. □

Proposition 5.12 leads us to the notion of a Hodge cycle and ultimately to the Hodge conjecture.

5.13 Hodge cycles

We start this section with the classical definition of a Hodge class. In this chapter we consider integral cohomology groups up to torsion, and therefore, for a smooth projective variety Y we freely write $H^m(Y^{\mathrm{an}}, \mathbb{Z}) \subset H^m_{\mathrm{dR}}(Y)$.

Definition 5.6. Let Y be a smooth projective variety and $m \geqslant 0$ be an even number. A **Hodge class** is any element in the intersection of the integral cohomology $H^m(Y^{\mathrm{an}}, \mathbb{Z}) \subset H^m_{\mathrm{dR}}(Y)$ and $F^{\frac{m}{2}} \subset H^m_{\mathrm{dR}}(Y)$, where $F^{\frac{m}{2}} = F^{\frac{m}{2}} H^m_{\mathrm{dR}}(Y)$ is the $\frac{m}{2}$-th piece of the Hodge filtration of $H^m_{\mathrm{dR}}(Y)$.

Therefore, the $\mathbb{Z}$-vector space of Hodge classes is simply the intersection

$$H^m(Y^{\mathrm{an}}, \mathbb{Z}) \cap H^{\frac{m}{2},\frac{m}{2}}(Y) = H^m(Y^{\mathrm{an}}, \mathbb{Z}) \cap F^{\frac{m}{2}}.$$

The equality follows from the definition $H^{p,q}(Y) := F^p \cap \overline{F^q}$ and the fact that $H^m(Y^{\mathrm{an}}, \mathbb{Z})$ is invariant under complex conjugation. For our main purposes in this text, it is better to define Hodge cycles which live in homology.

Definition 5.7. A homological cycle $\delta \in H_m(Y^{\mathrm{an}}, \mathbb{Z})$ is called a **Hodge cycle** if

$$\int_\delta F^{\frac{m}{2}+1} H^m_{\mathrm{dR}}(Y) = 0. \tag{5.26}$$

The number of equalities to define a Hodge cycle is the dimension of $F^{\frac{m}{2}+1} H^m_{\mathrm{dR}}(Y)$ which is the sum of Hodge numbers $h^{m,0} + h^{m-1,1} + \cdots + h^{\frac{m}{2}+1,\frac{m}{2}-1}$. We will see that in general this is bigger than the dimension of the moduli of Y, and so, Hodge cycles are usually rare to find.

Proposition 5.13. *The Poincaré duality* $P : H_m(Y^{\mathrm{an}}, \mathbb{Z}) \cong H^{2n-m}(Y^{\mathrm{an}}, \mathbb{Z})$ *induces an isomorphism between Hodge cycles and Hodge classes.*

Proof. Let $\delta \in H_m(Y^{\mathrm{an}}, \mathbb{Z})$ and $\delta^{\mathsf{pd}} := P(\delta)$ be its topological Poincaré dual. By definition of Poincaré duality and the trace map Tr we have

$$\int_\delta \omega = \int_Y \omega \wedge \delta^{\mathsf{pd}} = (2\pi i)^n \mathrm{Tr}(\omega \cup \delta^{\mathsf{pd}}) = (2\pi i)^n \langle \omega, \delta^{\mathsf{pd}} \rangle, \ \omega \in H^m_{\mathrm{dR}}(Y)$$

If δ^{pd} is a Hodge class then it is in $F^{n-\frac{m}{2}}$ and by Proposition 5.6 we have $\langle F^i, F^{n-\frac{m}{2}} \rangle = 0$ for $i > \frac{m}{2}$ which means that δ is a Hodge cycle. Conversely, if δ is a Hodge cycle then $\langle F^{\frac{m}{2}+1}, \delta^{\mathsf{pd}} \rangle = 0$. We know that $\langle F^{\frac{m}{2}+1}, F^{n-\frac{m}{2}} \rangle = 0$ (Proposition 5.6), $\langle \cdot, \cdot \rangle$ is non-degenerate (Proposition 5.5) and $F^{\frac{m}{2}+1}$, $F^{n-\frac{m}{2}}$ have complimentary dimensions in $H^{2n-m}_{\mathrm{dR}}(Y)$. This implies that $(F^{\frac{m}{2}+1})^{\perp} = F^{n-\frac{m}{2}}$ and so $\delta^{\mathsf{pd}} \in F^{n-\frac{m}{2}}$.

□

Conjecture 5.1 (Hodge conjecture). *For any Hodge cycle $\delta \in H_m(X^{\mathrm{an}}, \mathbb{Z})$ there is a natural number $a \in \mathbb{N}$ such that $a \cdot \delta$ is an algebraic cycle, that is, there exist irreducible subvarieties $Z_i \subset X$, $i = 1, 2, \ldots, k$ of dimension $\frac{m}{2}$ and $n_i \in \mathbb{N}$ such that*

$$a\delta = \sum r_i [Z_i].$$

The conjecture reformulated with $a = 1$ is known as the **integral Hodge conjecture**. It is known that it is false. Note that if δ is a torsion cycle then there is $a \in \mathbb{N}$ such that $a\delta = 0$ and so in the way that we have introduced the Hodge conjecture, torsion cycles are Hodge cycles hence they do not violate the Hodge conjecture.

5.14 Lefschetz (1,1) theorem

Recall the construction of cohomology class of an algebraic cycle in Section 5.12 and the Chern class in Section 5.11. In this section we introduce Lefschetz (1,1) theorem. In Movasati (2021, Chapter 9) the reader can find another presentation of this which uses the classical definition of Hodge filtration using (p, q)-forms.

Proposition 5.14. *Let Z be a divisor in X. The Chern class of L_Z and the cohomology class of Z are the same, that is,*

$$c(L_Z) = [Z]^{\mathsf{pd}}. \tag{5.27}$$

Note that we have defined both sides of (5.27) for an arbitrary field k of characteristic 0, and in principle we might ask for a purely algebraic proof. But, once again, we assume that $\mathsf{k} = \mathbb{C}$ and use topological and analytic methods. First, we may try to prove this proposition in Čech cohomology. For this we have to know the generator of the top Čech cohomology, study the effect of pull-back of Čech cohomology elements by desingularization etc. An alternative way is to use C^∞ de Rham cohomology and integration. The latter strategy is exploited in Griffiths and Harris (1994, page 141, Proposition 1). The following theorem says that the integral Hodge conjecture is true for $(2n-2)$-dimensional cycles.

Theorem 5.6 (Lefschetz (1,1) theorem). *Let X be a smooth projective variety of dimension n over $\mathbb{C}$. Any Hodge cycle $\delta \in H_{2n-2}(X^{\mathrm{an}}, \mathbb{Z})$ (including torsion cycles) is algebraic, that is, there is a divisor $Z = \sum_{i=1}^{k} n_i Z_i$, $n_i \in \mathbb{Z}$, Z_i irreducible, such that $\delta = \sum_{i=1}^{k} n_i [Z_i]$.*

Proof. Let us consider the topological Poincaré dual $\delta^{\mathsf{pd}} \in H^2(X^{\mathrm{an}}, \mathbb{Z})$ and its image (which we denote it again by δ^{pd}) under the canonical map $H^2(X^{\mathrm{an}}, \mathbb{Z}) \to H^2_{\mathrm{dR}}(X/\mathbb{C})$ (which kills torsions). By Proposition 5.13 we have $\delta^{\mathsf{pd}} \in F^1 H^2_{\mathrm{dR}}(X/\mathbb{C})$. From another side we have a canonical map

$$H^2_{\mathrm{dR}}(X/\mathbb{C}) \to F^0/F^1 \to H^2(X, \mathcal{O}_{X/\mathbb{C}})$$

which maps δ^{pd} to zero. The latter by Theorem 5.1 is actually an isomorphism, however, we will not need this fact. We get the map

$$\check{p} : H^2(X^{\mathrm{an}}, \mathbb{Z}) \to H^2(X, \mathcal{O}_{X/\mathbb{C}})$$

and we claim that $p(2\pi i \cdot) = \check{p}(\cdot)$, where p is the map in (5.19). For this we first observe that by GAGA principle we have $H^2(X, \mathcal{O}_{X/\mathbb{C}}) \cong H^2(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$. Moreover, $\mathbb{H}^2(X, \Omega^\bullet_{X/\mathbb{C}}) \cong \mathbb{H}^2(X^{\mathrm{an}}, \Omega^\bullet_{X^{\mathrm{an}}})$ and under this isomorphism the algebraic Hodge filtration is mapped to the holomorphic one. Now, if we consider $\mathbb{C}$ as a complex $\mathbb{C} \to 0 \to 0 \to \cdots$ then the natural inclusion $\mathbb{C} \subset \Omega^\bullet_{X^{\mathrm{an}}}$ is a quasi-isomorphism and $p(2\pi i \cdot) = \check{p}(\cdot)$ follows. The exactness of the sequence (5.19) implies that δ^{pd} is the Chern class of some line bundle L in X^{an}. We know that any line bundle has a meromorphic section, that is, $L = L_Z$ for some divisor Z. The theorem follows from Proposition 5.14. □

Remark 5.4. The proof of Lefschetz $(1, 1)$ theorem is not at all constructive, that is, if we are given a topological cycle δ, it does not give any hint how to construct the defining ideal of Z. The most critical part of the proof is taking a meromorphic section of a line bundle L. In many concrete situations like a Fermat surfaces in Shioda (1981), we can write down line bundles on X explicitly, however, we do not know how to write down sections of such line bundles.

5.15 Pull-back in algebraic de Rham cohomology

Given a morphism $Y \to X$ of smooth algebraic varieties defined over a field of characteristic zero $\mathsf{k} \subset \mathbb{C}$. In the topological side we have an induced pull-back map in de Rham cohomology

$$H^k_{\mathrm{dR}}(X) \to H^k_{\mathrm{dR}}(Y).$$

In this section we describe these pull-back homomorphisms algebraically.

Definition 5.8. Let X and Y be smooth algebraic varieties defined over $\mathsf{k} \subset \mathbb{C}$, and $\mathcal{U}$ an affine open covering of X. Consider an affine morphism $\varphi : Y \to X$ (i.e. such that $\varphi^{-1}(U)$ is affine, for each open affine U of X), and $\omega \in H^k_{\mathrm{dR}}(X/\mathsf{k})$. We denote $\varphi^{-1}(\mathcal{U}) := \{\varphi^{-1}(U)\}_{U \in \mathcal{U}}$. The **pull-back map** in algebraic de Rham cohomology $\varphi^*\omega \in H^k_{\mathrm{dR}}(Y/\mathsf{k})$ is defined as

$$\varphi^*\omega = \sum_{i=0}^{k} \varphi^*\omega^i \in \bigoplus_{i=0}^{k} C^{k-i}(\varphi^{-1}(\mathcal{U}), \Omega^i_Y),$$

with

$$\omega = \sum_{i=0}^{k} \omega^i \in \bigoplus_{i=0}^{k} C^{k-i}(\mathcal{U}, \Omega^i_X),$$

where

$$(\varphi^*\omega^i)_{j_0 \cdots j_{k-i}} := \varphi^*(\omega^i_{j_0 \cdots j_{k-i}}) \in \Omega^i_Y(\varphi^{-1}(U_{j_0 \cdots j_{k-i}})),$$

and

$$\varphi^*\left(\sum_I a_I\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right) := \sum_I \varphi^* a_I\, d(\varphi^* x_{i_1}) \wedge \cdots \wedge d(\varphi^* x_{i_k}).$$

Remark 5.5. It follows from the above definition that φ^* commutes with d and δ, then it also commutes with D.

Proposition 5.15. *Under the same hypothesis of Definition 5.8 the pull-back map in algebraic de Rham cohomology corresponds to the usual pull-back map in the classical de Rham cohomology. In other words we have the following commutative diagram*

$$\begin{array}{ccc} H^k_{\mathrm{dR}}(X/\mathsf{k}) & \xrightarrow{\varphi^*} & H^k_{\mathrm{dR}}(Y/\mathsf{k}) \\ \downarrow & & \downarrow \\ H^k_{\mathrm{dR}}(X) & \xrightarrow{\varphi^*} & H^k_{\mathrm{dR}}(Y) \end{array}$$

Proof. It is easy to see that $D(\varphi^*\omega) = \varphi^*(D\omega) = 0$. Now, in order to show that $\varphi^*\omega$ corresponds to the pull-back of the form ω, we have to show there exist a representative of the hypercohomology class of ω of the form

$$\mu = \mu^0 + \cdots + \mu^k \in \bigoplus_{i=0}^{k} C^{k-i}(\mathcal{U}, \Omega^i_{X^\infty}),$$

with $\mu^i = 0$, $\forall i = 0, \ldots, k-1$. And a representative of the hypercohomology class of $\varphi^*\omega$ of the form

$$\widetilde{\mu} = \widetilde{\mu}^0 + \cdots + \widetilde{\mu}^k \in \bigoplus_{i=0}^{k} C^{k-i}(\varphi^{-1}(\mathcal{U}), \Omega^i_{Y\infty}),$$

with $\widetilde{\mu}^i = 0$, $\forall i = 0, \ldots, k-1$. Such that $\widetilde{\mu}^k \in \Omega^k_{Y\infty}(Y)$ is the pull-back of $\mu^k \in \Omega^k_{X\infty}(X)$ as C^∞ differential k-forms. We will in fact show more, we will show inductively that for every $l = 0, \ldots, k$ there exist a representative of the hypercohomology class of ω of the form

$$\mu_l = \mu_l^0 + \cdots + \mu_l^k \in \bigoplus_{i=0}^{k} C^{k-i}(\mathcal{U}, \Omega^i_{X\infty}),$$

with $\mu_l^i = 0$, $\forall i = 0, \ldots, l-1$. And a representative of the hypercohomology class of $\varphi^*\omega$ of the form

$$\widetilde{\mu}_l = \widetilde{\mu}_l^0 + \cdots + \widetilde{\mu}_l^k \in \bigoplus_{i=0}^{k} C^{k-i}(\varphi^{-1}(\mathcal{U}), \Omega^i_{Y\infty}),$$

with $\widetilde{\mu}_l^i = 0$, $\forall i = 0, \ldots, l-1$. Such that, for every $j = l, \ldots, k$, the form $(\widetilde{\mu}_l^j)_{i_0 \cdots i_{k-j}} \in \Omega^j_{Y\infty}(\varphi^{-1}(U_{i_0 \cdots i_{k-j}}))$ is the pull-back of the form $(\mu_l^j)_{i_0 \cdots i_{k-j}} \in \Omega^j_{X\infty}(U_{i_0 \cdots i_{k-j}})$. In fact, the claim follows for $l = 0$ by the definition of $\varphi^*\omega$. Assuming the claim for l we will show it for $l+1$. Consider $\{a_h\}_{h=0}^N$ a partition of unity subordinated to the covering $\mathcal{U}$. Define

$$\eta = \eta^0 + \cdots + \eta^{k-1} \in \bigoplus_{i=0}^{k-1} C^{k-1-i}(\mathcal{U}, \Omega^i_{X\infty}),$$

such that

$$\eta^i = \begin{cases} 0 & \text{if } i \neq l, \\ \lambda & \text{if } i = l, \end{cases}$$

where

$$\lambda_{i_0 \cdots i_{k-1-l}} := \sum_{h=0}^{N} a_h (\mu_l^l)_{i_0 \cdots i_{k-1-l} h} \in \Omega^l_{X\infty}(U_{i_0 \cdots i_{k-1-l}}).$$

And define

$$\widetilde{\eta} = \widetilde{\eta}^0 + \cdots + \widetilde{\eta}^{k-1} \in \bigoplus_{i=0}^{k-1} C^{k-1-i}(\varphi^{-1}(\mathcal{U}), \Omega^i_{Y\infty}),$$

such that

$$\widetilde{\eta}^i = \begin{cases} 0 & \text{if } i \neq l, \\ \widetilde{\lambda} & \text{if } i = l, \end{cases}$$

where

$$\widetilde{\lambda}_{i_0\cdots i_{k-1-l}} := \sum_{h=0}^{N} \varphi^* a_h (\widetilde{\mu}^l_l)_{i_0\cdots i_{k-1-l}h} \in \Omega^l_{Y\infty}(\varphi^{-1}(U_{i_0\cdots i_{k-1-l}})).$$

Then, using that $\delta(\mu^l_l) = 0$ we see that $\delta(\lambda) = (-1)^{k-l}\mu^l_l$. Also noticing that $\delta(\widetilde{\mu}^l_l) = 0$ (and using that $\{\varphi^* a_h\}_{h=0}^N$ is a partition of unity subordinated to $\varphi^{-1}(\mathcal{U})$) we get $\delta(\widetilde{\lambda}) = (-1)^{k-l}\widetilde{\mu}^l_l$. Thus, defining $\mu_{l+1} := \mu_l + (-1)^{k-l+1} D\eta$, and $\widetilde{\mu}_{l+1} := \widetilde{\mu}_l + (-1)^{k-l+1} D\widetilde{\eta}$, the claim follows for $l+1$ since $d\widetilde{\lambda} = \varphi^*(d\lambda)$ (because $\widetilde{\lambda} = \varphi^*(\lambda)$). □

A first application of Proposition 5.15 is the description of pull-backs in Čech cohomology for each piece of the Hodge structure.

Corollary 5.3. *Let X and Y be smooth complex algebraic varieties, and $\mathcal{U}$ an affine open covering of X. Consider an affine morphism $\varphi : Y \to X$, and $\omega \in H^q(\mathcal{U}, \Omega^p_X)$, then the pull-back*

$$\varphi^*\omega \in H^q(\varphi^{-1}(\mathcal{U}), \Omega^p_Y),$$

is given by

$$\varphi^*\omega_{j_0\cdots j_q} = \varphi^*(\omega_{j_0\cdots j_q}) \in \Omega^p_Y(\varphi^{-1}(U_{j_0\cdots j_q})).$$

Proof. The pull-back morphism in algebraic de Rham cohomology

$$\varphi^* : H^{p+q}_{\mathrm{dR}}(X/\mathbb{C}) \to H^{p+q}_{\mathrm{dR}}(Y/\mathbb{C})$$

is compatible with the Hodge filtration (i.e. $\varphi^*(F^k(X)) \subseteq F^k(Y)$), thus the pull-back in Čech cohomology is induced by

$$\varphi^* : H^q(X, \Omega^p_X) \cong F^q(X)/F^{q+1}(X) \to F^q(Y)/F^{q+1}(Y) \cong H^q(Y, \Omega^p_Y).$$

□

5.16 Hard Lefschetz theorem

For m even number we write

$$\theta^{\frac{m}{2}} := \underbrace{\theta \cup \theta \cup \cdots \cup \theta}_{\frac{m}{2}-\text{ times}} \in H^m_{\mathrm{dR}}(X/\mathsf{k}).$$

In particular, for $m = 2n$ we get the element θ^n in the top cohomology $H^{2n}_{\mathrm{dR}}(X/\mathsf{k})$. Recall that the trace map

$$\mathrm{Tr} : H^{2n}_{\mathrm{dR}}(X/\mathsf{k}) \to \mathsf{k}$$

is a k-linear map. Let

$$L^i : H^*_{\mathrm{dR}}(X/\mathsf{k}) \to H^*_{\mathrm{dR}}(X/\mathsf{k}), \ \ \alpha \mapsto \alpha \cup \theta^i.$$

The m**-th primitive cohomology** is defined to be

$$H^m_{\mathrm{dR}}(X/\mathsf{k})_0 := \ker\left(L^{n-m+1} : H^m_{\mathrm{dR}}(X/\mathsf{k}) \to H^{2n-m+2}_{\mathrm{dR}}(X/\mathsf{k})\right).$$

Theorem 5.7 (Hard Lefschetz theorem)**.** *The map*

$$L^{n-m} : H^m_{\mathrm{dR}}(X/\mathsf{k}) \to H^{2n-m}_{\mathrm{dR}}(X/\mathsf{k})$$

is an isomorphism of k*-vector spaces.*

We do not have a purely algebraic proof for the above theorem. We can assume that $\mathsf{k} \subset \mathbb{C}$ and then it is enough to prove the theorem for $\mathsf{k} = \mathbb{C}$. We may further use the isomorphism in Section 5.3 and consider the classical de Rham cohomology given by C^∞ forms. In this context, the proof uses harmonic forms, see for instance Voisin (2002, Theorem 6.25). For further comments on a proof with an arithmetic flavour see the paragraph after Movasati (2021, Theorem 5.4).

Theorem 5.8 (Lefschetz decomposition)**.** *We have*

$$\oplus_q H^{m-2q}_{\mathrm{dR}}(X/\mathsf{k})_0 \cong H^m_{\mathrm{dR}}(X/\mathsf{k})$$

which is given by $\oplus_q L^q$.

Proof. This is a direct consequence of Hard Lefschetz theorem, see for instance Movasati (ibid., Theorem 5.5) for the proof in homology. The proof in cohomology is similar. □

In the case $X \subseteq \mathbb{P}^{n+1}$ is a smooth projective hypersurface, we know by Lefschetz hyperplane section theorem (see Movasati (ibid., Theorem 5.2)) that

$$H^m_{\mathrm{dR}}(X/\mathsf{k}) = \begin{cases} \mathsf{k}\cdot\theta^{\frac{m}{2}} & \text{if } m \neq n \text{ is even,} \\ 0 & \text{if } m \neq n \text{ is odd.} \end{cases}$$

In consequence

$$H^m_{\mathrm{dR}}(X/\mathsf{k})_0 = 0 \qquad \forall\, m \neq n,$$

and for $m = n$ we have

$$H^n_{\mathrm{dR}}(X/\mathsf{k}) = \begin{cases} H^n_{\mathrm{dR}}(X/\mathsf{k})_0 \oplus \mathsf{k}\cdot\theta^{\frac{n}{2}} & \text{if } n \text{ is even,} \\ H^n_{\mathrm{dR}}(X/\mathsf{k})_0 & \text{if } n \text{ is odd.} \end{cases}$$

5.17 Relative algebraic de Rham cohomology

Relative algebraic de Rham cohomology is important in itself, as the Eilenberg–Steenrod axioms of cohomology theories are built upon the relative case. From another side our need for relative algebraic de Rham cohomology comes from the geometrization of Jacobi forms and open string amplitudes, see Movasati (2020a, Chapter 11). After a web search of the title of the present section one lands on a link on MathOverflow which offers two definitions of relative de Rham cohomology in the C^∞ context. In the following we present the hypercohomology (and hence algebraic) version of this. These are inspired by the corresponding definitions in the C^∞ context.

Let X be a variety and $Y \subset X$ be a subvariety both defined over a field k. We define $\Omega^m_{X,Y}$ the sheaf of differential m-forms in X whose restriction to Y is zero. These sheaves form a complex $(\Omega^\bullet_{X,Y}, d)$. We define the **relative algebraic de Rham cohomology groups**

$$H^m_{\mathrm{dR}}(X,Y) := \mathbb{H}^m(\Omega^\bullet_{X,Y}, d) \tag{5.28}$$

This definition in the C^∞ context is in Godbillon (1998, Chapter XIII). We call this **Godbillon relative algebraic de Rham cohomology**. We may also define

$$\Omega^m_{X,Y} := \Omega^m_X \times \Omega^{m-1}_Y,$$

$$d : \Omega^m_{X,Y} \to \Omega^{m+1}_{X,Y},\ d(\omega,\alpha) := (d\omega, \omega|_Y - d\alpha)$$

and define $H^m_{\mathrm{dR}}(X,Y)$ to be the hypercohomology of this complex. We call this **Bott–Tu relative algebraic de Rham cohomology**.

Theorem 5.9. *For varieties X and Y (possibly singular) with Y closed in X, the map $\omega \mapsto (\omega, 0)$ from the Godbillon complex to Bott–Tu complex is a quasi-isomorphism and hence it induces an isomorphism between Godbillon and Bott–Tu relative algebraic de Rham cohomologies.*

Proof. This basically follows from the fact that if X is affine and Y is a closed sub affine variety of X then by definition any differential form in Y is a restriction of a differential form in X. We first check the surjectivity. Let (ω, α) be a closed element in the Bott–Tu sense. This means that $d\omega = 0, \ \omega|_Y = d\alpha$. We extend α to X and call it $\tilde{\alpha}$, that is, $\tilde{\alpha}|_Y = \alpha$. Now, (ω, α) is equivalent to $(\omega - d\tilde{\alpha}, 0)$ modulo exact elements:

$$(\omega, \alpha) - (\omega - d\tilde{\alpha}, 0) = (d\tilde{\alpha}, \alpha) = d(\tilde{\alpha}, 0).$$

Next we check the injectivity. Assume that $(\omega, 0)$ is zero in the Bott–Tu case. This means that $(\omega, 0) = (d\tilde{\omega}, \tilde{\omega}|_Y - d\tilde{\alpha})$. We extend $\tilde{\alpha}$ to X and denote it by the same letter $\tilde{\alpha}$. Now, $(\omega, 0)$ is the the differential of $(\tilde{\omega} - d\tilde{\alpha}, 0)$ and $\tilde{\omega} - d\tilde{\alpha}$ restricted to Y is zero.

□

Similar to the case of algebraic de Rham cohomology of projective varieties, it is natural to define the **Hodge filtration in the relative case** by truncating the underlying complex

$$F^q = F^q H^m_{\mathrm{dR}}(X, Y) = \mathrm{Im}\left(\mathbb{H}^m(X, \Omega^{\bullet \geqslant q}_{X,Y}) \to \mathbb{H}^m(X, \Omega^{\bullet}_{X,Y})\right).$$

and get the filtration which we call it again the Hodge filtration.

$$0 = F^{m+1} \subset F^m \subset \cdots \subset F^1 \subset F^0 = H^m_{\mathrm{dR}}(X),$$

It is worth to note that the relative de Rham cohomology $H^m_{\mathrm{dR}}(X, Y)$ carries a mixed Hodge structure with the weight filtration:

$$0 = W_{m-2} \subset W_{m-1} = \ker\left(H^m_{\mathrm{dR}}(X, Y) \to H^m_{\mathrm{dR}}(X)\right) \subset W_m = H^m_{\mathrm{dR}}(X, Y).$$

In this way the long exact sequence of the pair (X, Y)

$$\cdots \to H^{m-1}_{\mathrm{dR}}(Y) \to H^m_{\mathrm{dR}}(X, Y) \to H^m_{\mathrm{dR}}(X) \to H^m_{\mathrm{dR}}(Y) \to \cdots$$

is a morphism of mixed Hodge structures, for further details see also Peters and Steenbrink (2008).

Exercise 5.1. Let X be a smooth curve and Y be a finite set of points in X (a zero dimensional subvariety of X). Let also assume that X is covered by two open sets U_0, U_1. Show that

$$H^1_{\mathrm{dR}}(X,Y) = \frac{\left\{(\omega^0_{i_1i_2}, (\omega^1_{i_1}, \alpha^0_{i_1}), (\omega^1_{i_2}, \alpha^0_{i_2})\,)\,\middle|\, d\omega^0_{i_1i_2} = \omega^1_{i_2} - \omega^1_{i_1},\ \ \omega^0_{i_1i_2}|_Y = \alpha^0_{i_2} - \alpha^0_{i_1}\right\}}{\left\{(\omega^0_{i_2} - \omega^0_{i_1}, (d\omega^0_{i_1}, \omega^0_{i_1}|_Y), (d\omega^0_{i_2}, \omega^0_{i_2}|_Y)\right\}}$$

where ω^i (resp. α^i) stands for a differential form in an open set in X (resp. Y). Such an open set is determined by the sub indices, for instance, for $i_1 i_2$ it is $U_{i_1} \cap U_{i_2}$ etc.

The following theorem can be considered as the fundamental theorem in the topology of algebraic varieties. Its original proof uses Picard–Lefschetz theory and vanishing cycles, see Lamotke (1981) and Movasati (2021, Section 5.2), and it is highly desirable to have an algebraic proof for it.

Theorem 5.10. *Let X be a smooth projective variety of dimension n. Let also Y, Z be two codimension one transversal hyperplane sections of X which intersect each other transversely and set $X' := Y \cap Z$. We have*

$$H^q_{\mathrm{dR}}(X\backslash Z, Y\backslash X') = \begin{cases} 0 & \text{if } q \neq n \\ \text{free } \mathbb{Z}\text{-module of finite rank} & \text{if } q = n \end{cases}, \quad n := \dim(X).$$

.

The above theorem is a stronger version of the so called Lefschetz hyperplane section theorem, see Movasati (ibid., Theorem 5.1 and Theorem 5.2).

6 Logarithmic differential forms

Writing a paper takes a lot of time. Writing it is very useful, to have everything put together in a correct way, and one learns a lot doing so, but it's also somewhat painful. So in the beginning of forming ideas, I find it very convenient to write a letter. I send it, but often it is really a letter to myself. Because I don't have to dwell on things the recipient knows about, some short-cuts will be all right. Sometimes the letter, or a copy of it, will stay in a drawer for some years, but it preserves ideas, and when I eventually write a paper, it serves as a blueprint. (P. Deligne in Raussen and Skau (2014) page 183).

6.1 Introduction

Let X be an affine variety. It follows from Theorem 4.6 and Theorem 5.1 that the Hodge filtration of X defined in (5.4) is trivial, that is, all the pieces of the Hodge filtration are equal. Hence the truncated complexes $\Omega_X^{\bullet \geqslant i}$ give us the correct Hodge filtration for projective varieties but beyond this we might be in trouble to define the "correct" Hodge filtrations. One has to clarify what the adjective correct means. Deligne (1971) uses logarithmic differential forms and defines the mixed Hodge structure of affine varieties, and in particular its basic ingredient, namely the Hodge filtration. In this section we give an exposition of this topic avoiding entering into mixed Hodge structures. Even if one is interested in projective varieties, computations are usually done in affine varieties, and this topic is indispensable for further study of projective varieties. The reader may also consult Voisin (2002, Section 8.2.3) and Voisin (2003, Section 6.1) .

6.2 Logarithmic differential forms

For simplicity, we will restrict ourselves to the following context: Let X be a smooth projective variety, and $Y \subseteq X$ be a smooth hyperplane section. We will consider affine varieties of the form

$$U := X \setminus Y.$$

In the general case, one can take Y to be a normal crossing divisor in X.

Definition 6.1. Let X be a smooth projective variety and $Y \subseteq X$ be a smooth hyperplane section. We define the **sheaf of rational p-forms with logarithmic poles along Y** as

$$\Omega_X^p(\log Y) := \operatorname{Ker}(\Omega_X^p(Y) \xrightarrow{d} \Omega_X^{p+1}(2Y)/\Omega_X^{p+1}(Y)).$$

Analogously, we define the **sheaf of meromorphic p-forms with logarithmic poles along Y** as

$$\Omega_{X^{\mathrm{an}}}^p(\log Y) := \operatorname{Ker}(\Omega_{X^{\mathrm{an}}}^p(Y) \xrightarrow{\partial} \Omega_{X^{\mathrm{an}}}^{p+1}(2Y)/\Omega_{X^{\mathrm{an}}}^{p+1}(Y)).$$

By Serre's GAGA principle we know $\Omega_{X^{\mathrm{an}}}^p(\log Y)$ is the analytification of $\Omega_X^p(\log Y)$.

The following proposition describes how logarithmic forms look like in local systems of coordinates.

Proposition 6.1. *Let X be a smooth projective variety of dimension n and $Y \subseteq X$ be a smooth hyperplane section. Let $z_1, \dots, z_n$ be local coordinates on an open set V of X, such that $V \cap Y = \{z_1 = 0\}$. Then $\Omega_{X^{\mathrm{an}}}^p(\log Y)|_V$ is a free $\mathcal{O}_{V^{\mathrm{an}}}$-module, for which $dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ and $\frac{dz_1}{z_1} \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_{p-1}}$, $i_k, j_l \in \{2, \dots, n\}$, form a basis. In particular $\Omega_{X^{\mathrm{an}}}^p(\log Y)$ is locally free.*

Proof. Let $\alpha \in \Gamma(V, \Omega_{X^{\mathrm{an}}}^p(\log Y))$. Since it has pole of order 1 there exist $\beta \in \Gamma(V, \Omega_{X^{\mathrm{an}}}^p)$ such that $\alpha = \frac{\beta}{z_1}$. Furthermore, since the same happens to $d\alpha$, we conclude $dz_1 \wedge \beta = 0$, i.e. $\beta = dz_1 \wedge \gamma$, where γ is a holomorphic $(p-1)$-form, just depending on $dz_2, \dots, dz_n$. $\square$

6.3 Deligne's theorem

The following Theorem due to Deligne allows us to define another filtration on algebraic de Rham cohomology of affine varieties.

Theorem 6.1 (Deligne (1970)). *In the context of the previous definition, let $i : U = X \setminus Y \hookrightarrow X$ be the inclusion. Then, the natural map*

$$\Omega_X^\bullet(\log Y) \hookrightarrow i_*\Omega_U^\bullet =: \Omega_X^\bullet(*Y),$$

induces the isomorphism

$$\mathbb{H}^k(X, \Omega_X^\bullet(\log Y)) \cong H_{\mathrm{dR}}^k(U/\mathbb{C}).$$

We will provide two different proofs of this fact. The first one being more conceptual (following the argument of Atiyah–Hodge's theorem), while the second one is more computational, and relies on a lemma due to Carlson and Griffiths.

First proof of Theorem 6.1 Consider the composition

$$\iota : \Omega_{X^{\mathrm{an}}}^\bullet(\log Y) \hookrightarrow \Omega_{X^{\mathrm{an}}}^\bullet(*Y) \hookrightarrow i_*\Omega_{U^\infty}^\bullet,$$

of complexes of sheaves over the analytic topology. If we prove that ι is a quasi-isomorphism, we obtain the result by Serre's GAGA correspondence (since $\Omega_X^p(\log Y)$ is coherent).

In order to show the quasi-isomorphism we proceed in the same way we did for Atiyah–Hodge's theorem. It is clear that for $x \in U$, $H^k(\iota)_x$ is an isomorphism for all $k \geqslant 0$. And for $x \in Y$, is clear that $H^k(\iota)_x$ is surjective. Then it remains to show the injectivity. For this, we use the following logarithmic counterpart of Lemma 4.4 whose proof is analogous:

Lemma 6.1. *If ω is a closed meromorphic k-form on $\Delta = \mathbb{D}^n$, which is holomorphic in $\Delta \setminus Y$ with logarithmic poles along $Y = \{(z_1, \ldots, z_n) \in \Delta | z_1 \cdots z_r = 0\}$. And $\omega = 0$ in $H_{\mathrm{dR}}^k(\Delta \setminus Y)$. Then there exists a meromorphic $(k-1)$-form η (on a possibly smaller polydisc) also with logarithmic poles along Y, such that $\omega = \partial\eta$.*

Remark 6.1. Deligne's Theorem 6.1 remains valid for $U = X \setminus Y$, and Y an ample normal crossing divisor of X. In fact, the proof above applies in this context without modifications.

6.4 Carlson–Griffiths Lemma

For the second proof of Theorem 6.1, we will use the following Lemma due to Carlson and Griffiths.

Definition 6.2. Suppose X is embedded in a projective space $\mathbb{P}^N$, and $Y = X \cap \{F = 0\}$ for some homogeneous polynomial $F \in \mathbb{C}[x_0, \ldots, x_N]$. Consider $\mathcal{U} = \{U_i\}_{i=0}^N$ the Jacobian covering of X given by $U_i = X \cap \{F_i \neq 0\}$, where $F_i := \frac{\partial F}{\partial x_i}$. For every $l \geqslant 2$ define

$$H_l : C^q(\mathcal{U}, \Omega_X^p(lY)) \to C^q(\mathcal{U}, \Omega_X^{p-1}((l-1)Y)),$$

$$(H_l\omega)_{j_0 \cdots j_q} := \frac{(-1)^q}{1-l}\frac{F}{F_{j_0}}\iota_{\frac{\partial}{\partial x_{j_0}}}(\omega_{j_0 \cdots j_q}).$$

Theorem 6.2 (Carlson and Griffiths (1980))**.** *For every $l \geqslant 2$, letting $D = \delta + (-1)^q d$, then*

$$DH + HD : \bigoplus_{p+q=k} \frac{C^q(\mathcal{U}, \Omega_X^p(lY))}{C^q(\mathcal{U}, \Omega_X^p((l-1)Y))} \to \bigoplus_{p+q=k} \frac{C^q(\mathcal{U}, \Omega_X^p(lY))}{C^q(\mathcal{U}, \Omega_X^p((l-1)Y))}$$

is the identity map. Note that D is the differential map used in the hypercohomology groups.

Proof. We claim

$$(-1)^q \cdot (dH_l + H_{l+1}d) : \frac{C^q(\mathcal{U}, \Omega_X^p(lY))}{C^q(\mathcal{U}, \Omega_X^p((l-1)Y))} \to \frac{C^q(\mathcal{U}, \Omega_X^p(lY))}{C^q(\mathcal{U}, \Omega_X^p((l-1)Y))}$$

is the identity map. In fact, take $\omega_J = \frac{\alpha_J}{F^l} \in C^q(\mathcal{U}, \Omega_X^p(lY))$, then

$$\begin{aligned}
(-1)^q \cdot (dH_l + H_{l+1}d)\omega_J &= d\left(\frac{1}{1-l}\frac{F}{F_{j_0}}\iota_{\frac{\partial}{\partial x_{j_0}}}\left(\frac{\alpha_J}{F^l}\right)\right) - \frac{1}{l}\frac{F}{F_{j_0}}\iota_{\frac{\partial}{\partial x_{j_0}}}\left(\frac{d\alpha_J}{F^l} - l\frac{dF \wedge \alpha_J}{F^{l+1}}\right) \\
&\equiv \frac{1}{1-l}\frac{dF}{F_{j_0}} \wedge \frac{\iota_{\frac{\partial}{\partial x_{j_0}}}(\alpha_J)}{F^l} - \frac{l}{1-l}\frac{F}{F_{j_0}}\frac{dF \wedge \iota_{\frac{\partial}{\partial x_{j_0}}}(\alpha_J)}{F^{l+1}} \\
&\qquad + \frac{F}{F_{j_0}}\frac{(F_{j_0}\alpha_J - dF \wedge \iota_{\frac{\partial}{\partial x_{j_0}}}(\alpha_J))}{F^{l+1}} \\
&= \frac{\alpha_J}{F^l} = \omega_J,
\end{aligned}$$

where the congruence is taken mod $C^q(\mathcal{U}, \Omega^p_X((l-1)Y))$. Using this, take $\omega = \sum_{p=0}^k \omega^p \in \bigoplus_{p+q=k} C^q(\mathcal{U}, \Omega^p_X(lY))$ then

$$\begin{aligned}\omega - DH\omega &= \sum_{p+q=k} (\omega^p - (-1)^q dH\omega^p - \delta H\omega^p) \\ &\equiv \sum_{p+q=k} ((-1)^q H d\omega^p - \delta H\omega^p) \\ &\equiv \sum_{p+q=k} H((-1)^q d\omega^p + \delta\omega^p) = HD\omega.\end{aligned}$$

Where in the last congruence we used $\delta H\omega^p \equiv -H\delta\omega^p$. In fact

$$\begin{aligned}(\delta H_l\omega^p)_{j_0\cdots j_{q+1}} &= \sum_{m=0}^{q+1} (-1)^m (H_l\omega^p)_{j_0\cdots\widehat{j_m}\cdots j_{q+1}} \\ &= \frac{(-1)^q}{1-l}\left(\frac{F}{F_{j_1}}\iota_{\frac{\partial}{\partial x_{j_1}}}(\omega^p_{j_1\cdots j_{q+1}}) + \frac{F}{F_{j_0}}\iota_{\frac{\partial}{\partial x_{j_0}}}\left(\sum_{m=1}^{q+1}(-1)^m\omega^p_{j_0\cdots\widehat{j_m}\cdots j_{q+1}}\right)\right) \\ &= \frac{(-1)^q F}{1-l}\left(\frac{\iota_{\frac{\partial}{\partial x_{j_1}}}(\omega^p_{j_1\cdots j_{q+1}})}{F_{j_1}} - \frac{\iota_{\frac{\partial}{\partial x_{j_0}}}(\omega^p_{j_1\cdots j_{q+1}})}{F_{j_0}}\right) - (H_l\delta\omega^p)_{j_0\cdots j_{q+1}}.\end{aligned}$$

□

6.5 Second proof of Deligne's Theorem

The second proof of Deligne's Theorem 6.1 follows from Carlson–Griffiths Lemma (Theorem 6.2). The idea is to use the operator H defined in Definition 6.2 to reduce the pole order of a given algebraic form with poles along Y until transforming it into a logarithmic form representing the same hypercohomology class.

In fact, consider the inclusion $\iota : \Omega^\bullet_X(\log Y) \hookrightarrow \Omega^\bullet_X(*Y)$. It induces the morphism

$$\varphi : \mathbb{H}^k(\mathcal{U}, \Omega^\bullet_X(\log Y)) \to \mathbb{H}^k(\mathcal{U}, \Omega^\bullet_X(*Y)).$$

Let us prove φ is an isomorphism. Given

$$\omega = \sum_{p=0}^k \omega^p \in \bigoplus_{p+q=k} C^q(\mathcal{U}, \Omega^p_X(*Y))$$

such that $D\omega = 0$, there exists $l \geqslant 1$ such that

$$\omega \in \bigoplus_{p+q=k} C^q(\mathcal{U}, \Omega_X^p(lY)).$$

If $l = 1$, we claim $\omega \in \bigoplus_{p+q=k} C^q(\mathcal{U}, \Omega_Y^p(\log Y))$. In fact, since $D\omega = 0$, we get

$$(-1)^q d\omega^p = -\delta\omega^{p+1} \in C^q(\mathcal{U}, \Omega_X^{p+1}(Y)),$$

as desired. For $l \geqslant 2$, it follows from Theorem 6.2 that ω is cohomologous (in hypercohomology) to the D-closed element

$$\nu := (1 - DH)^{l-1}\omega \in \bigoplus_{p+q=k} C^q(\mathcal{U}, \Omega_X^p(Y)).$$

And it follows from the case $l = 1$ that

$$\nu \in \bigoplus_{p+q=k} C^q(\mathcal{U}, \Omega_Y^p(\log Y)),$$

i.e. φ is surjective. Now, for the injectivity, consider

$$\omega \in \bigoplus_{p+q=k} C^q(\mathcal{U}, \Omega_X^p(\log Y)),$$

such that there exist

$$\eta \in \bigoplus_{p+q=k-1} C^q(\mathcal{U}, \Omega_X^p(lY))$$

for some $l \geqslant 1$, with

$$D\eta = \omega.$$

If $l = 1$, we claim $\eta \in \bigoplus_{p+q=k-1} C^q(\mathcal{U}, \Omega_Y^p(\log Y))$. In fact, since $D\eta = \omega$, we get

$$d\eta^{k-1} = \omega^k \in C^0(\mathcal{U}, \Omega_X^k(Y)),$$

then $\eta^{k-1} \in C^0(\mathcal{U}, \Omega_X^{k-1}(\log Y))$. Inductively, if we assume

$$\eta^{p+1} \in C^{q-1}(\mathcal{U}, \Omega_X^{p+1}(\log Y))$$

then

$$(-1)^q d\eta^p = \omega^{p+1} - \delta\eta^{p+1} \in C^q(\mathcal{U}, \Omega_X^{p+1}(\log Y)),$$

in consequence $\eta^p \in C^q(\mathcal{U}, \Omega_X^p(\log Y))$. Finally, for $l \geqslant 2$, if we take

$$\phi := (1 - DH)^{l-1}\eta.$$

Since $(1 - DH)\eta = H\omega + \mu \in \bigoplus_{p+q=k} C^q(\mathcal{U}, \Omega_X^p((l-1)Y))$, and $D((1 - DH)\eta) = D\eta = \omega$, it is clear that $D\phi = \omega$ and

$$\phi \in \bigoplus_{p+q=k-1} C^q(\mathcal{U}, \Omega_X^p(Y)).$$

By the case $l = 1$, we actually see that

$$\phi \in \bigoplus_{p+q=k-1} C^q(\mathcal{U}, \Omega_X^p(\log Y))$$

as desired.

6.6 Residue map

Let X be a smooth projective variety, $Y \subseteq X$ be a smooth hyperplane section, and $U := X \setminus Y$. Taking the long exact sequence in cohomology of the pair (X, U), and using Leray–Thom–Gysin isomorphism (see Movasati (2021, Chapter 4, Section 6)) we obtain the exact sequence

$$\cdots \to H_{\mathrm{dR}}^{k+1}(X) \to H_{\mathrm{dR}}^{k+1}(U) \xrightarrow{\text{res}} H_{\mathrm{dR}}^{k}(Y) \xrightarrow{\tau} H_{\mathrm{dR}}^{k+2}(X) \to \cdots \tag{6.1}$$

where res is the residue map, and τ corresponds to the wedge product with the polarization θ. In particular, we obtain a surjective map

$$\text{res} : H_{\mathrm{dR}}^{k+1}(U) \to H_{\mathrm{dR}}^{k}(Y)_0 = \operatorname{Ker} \tau,$$

where $H_{\mathrm{dR}}^{k}(Y)_0$ is the primitive part of de Rham cohomology, i.e. is the complementary space to $\theta^{\frac{k}{2}}$ inside $H_{\mathrm{dR}}^{k}(Y)$ (see Movasati (ibid., Chapter 5, Section 7)). We want to determine the algebraic counterpart of this map (together with its long exact sequence).

Definition 6.3. Let X be a smooth complex projective variety of dimension n and $Y \subseteq X$ be a smooth hyperplane section. Let $z_1, \ldots, z_n$ be local coordinates on an open set V of X, such that $V \cap Y = \{z_1 = 0\}$. For $\alpha \in \Gamma(V, \Omega^p_{X^{\mathrm{an}}}(\log Y))$ we define its **residue at** Y to be

$$\operatorname{res}(\alpha) := \beta|_{Y \cap V} \in \Gamma(V, j_* \Omega^{p-1}_{Y^{\mathrm{an}}}),$$

where $\alpha = \beta \wedge \frac{dz_1}{z_1} + \gamma$, and β, γ are holomorphic forms. This definition does not depend on the choice of the coordinates, and defines the **residue map**

$$\operatorname{res} : \Omega^p_{X^{\mathrm{an}}}(\log Y) \to j_* \Omega^{p-1}_{Y^{\mathrm{an}}}.$$

Which is part of the following exact sequence

$$0 \to \Omega^p_{X^{\mathrm{an}}} \to \Omega^p_{X^{\mathrm{an}}}(\log Y) \xrightarrow{\operatorname{res}} j_* \Omega^{p-1}_{Y^{\mathrm{an}}} \to 0,$$

called **Poincaré residue sequence**. Using Serre's GAGA principle we have its algebraic counterpart

$$0 \to \Omega^p_X \to \Omega^p_X(\log Y) \xrightarrow{\operatorname{res}} j_* \Omega^{p-1}_Y \to 0,$$

which we also call Poincaré residue sequence.

This sequence gives rise to a short exact sequence of complexes

$$0 \to \Omega^\bullet_X \to \Omega^\bullet_X(\log Y) \xrightarrow{\operatorname{res}} j_* \Omega^{\bullet-1}_Y \to 0. \tag{6.2}$$

Taking the long exact sequence in hypercohomology, and applying Deligne's Theorem 6.1 we get the exact sequence

$$\cdots \to H^{k+1}_{\mathrm{dR}}(X/\mathbb{C}) \to H^{k+1}_{\mathrm{dR}}(U/\mathbb{C}) \xrightarrow{\operatorname{res}} H^k_{\mathrm{dR}}(Y/\mathbb{C}) \xrightarrow{\tau} H^{k+2}_{\mathrm{dR}}(X/\mathbb{C}) \to \cdots, \tag{6.3}$$

Which turns out to be the algebraic counterpart of the sequence (6.1).

6.7 Hodge filtration for affine varieties

Since for every $i \geqslant 0$ we have the short exact sequence of complexes

$$0 \to \Omega^{\bullet \geqslant i}_X \to \Omega^{\bullet \geqslant i}_X(\log Y) \xrightarrow{\operatorname{res}} j_* \Omega^{\bullet \geqslant i-1}_Y \to 0,$$

we have the following commutative diagram with exact rows

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{H}^{k+1}(X,\Omega_X^{\bullet\geqslant i}) \arrow[r] \arrow[d, "f"'] & \mathbb{H}^{k+1}(X,\Omega_X^{\bullet\geqslant i}(\log Y)) \arrow[r] \arrow[d, "g"'] & \mathbb{H}^{k}(Y,\Omega_Y^{\bullet\geqslant i-1}) \arrow[r] \arrow[d, "h"'] & \cdots \\
\cdots \arrow[r] & H_{\mathrm{dR}}^{k+1}(X/\mathbb{C}) \arrow[r] & H_{\mathrm{dR}}^{k+1}(U/\mathbb{C}) \arrow[r, "\mathrm{res}"] & H_{\mathrm{dR}}^{k}(Y/\mathbb{C}) \arrow[r, "\tau"] & \cdots
\end{tikzcd}$$

The vertical arrows of this diagram are all injective. In fact, the maps f and h are injective by Proposition 3.9. The injectivity of g is more delicate and is consequence of a theorem due to Deligne on the degeneration of the spectral sequence associated to the naive filtration of $\Omega_X^{\bullet}(\log Y)$ (see Voisin (2002, Theorem 8.35) or Deligne (1971, Corollary 3.2.13)). In consequence, the exact sequence (6.3) is compatible with Hodge filtrations, i.e. we have the exact sequence

$$\cdots \to F^i H_{\mathrm{dR}}^{k+1}(X/\mathbb{C}) \to F^i H_{\mathrm{dR}}^{k+1}(U/\mathbb{C}) \xrightarrow{\mathrm{res}} F^{i-1} H_{\mathrm{dR}}^{k}(Y/\mathbb{C}) \xrightarrow{\tau} F^i H_{\mathrm{dR}}^{k+2}(X/\mathbb{C}) \to \cdots .$$

Definition 6.4. For $U = X \setminus Y$, where X is smooth projective, and Y is a smooth hyperplane section. We define the **(algebraic) Hodge filtration of** U as

$$\begin{aligned} F^i H_{\mathrm{dR}}^k(U/\mathbb{C}) &:= \mathrm{Im}(\mathbb{H}^k(X,\Omega_X^{\bullet\geqslant i}(\log Y)) \to \mathbb{H}^k(X,\Omega_X^{\bullet}(\log Y)) \\ &\cong H_{\mathrm{dR}}^k(U/\mathbb{C})). \end{aligned}$$

7 Cohomology of hypersurfaces

When I was a graduate student at Princeton, it was frequently said that Lefschetz never stated a false theorem nor gave a correct proof, Griffiths, Spencer, and Whitehead (1992, page 289).

7.1 Introduction

In this chapter we will present Griffiths' results on the cohomology of hypersurfaces Griffiths (1969a). This work culminates with an explicit basis for the primitive cohomology of a hypersurface, compatible with the Hodge filtration.

Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d, and $U = \mathbb{P}^{n+1} \setminus X$. In order to construct the basis for $H^k_{\mathrm{dR}}(X/\mathbb{C})_0$ we will give generators for $H^{k+1}_{\mathrm{dR}}(U/\mathbb{C})$ compatible with the Hodge filtration $F^{i+1}H^{k+1}_{\mathrm{dR}}(U/\mathbb{C})$. Hence we will obtain the desired basis by applying the algebraic residue map to the generators, and reduce the set of generators to a basis. Furthermore, we will prove a theorem due to Carlson and Griffiths which describes this basis in each graded part induced by the Hodge filtration

$$Gr^i_F H^k_{\mathrm{dR}}(X/\mathbb{C}) = F^i H^k_{\mathrm{dR}}(X/\mathbb{C})/F^{i+1}H^k_{\mathrm{dR}}(X/\mathbb{C}) \cong H^{k-i}(X, \Omega^i_X).$$

With a little more effort we will also describe this basis in all $H^k_{\mathrm{dR}}(X/\mathbb{C})$.

7.2 Bott's formula

In this section we will prove a theorem due to Bott that determines the Čech cohomology groups of $\mathbb{P}^N$ with coefficients in $\Omega^p_{\mathbb{P}^N}(k)$. This theorem contains the cases of Bott vanishing theorem. In order to prove this theorem we need to introduce Euler's exact sequence.

Definition 7.1. Consider over $\mathbb{C}^{N+1}$ the global vector field

$$E := \sum_{i=0}^{N} x_i \frac{\partial}{\partial x_i}.$$

This field is called **Euler's vector field**. We define **Euler's sequence**

$$0 \to \Omega^{p+1}_{\mathbb{P}^N}(p+1) \to \mathcal{O}_{\mathbb{P}^N}^{\oplus\binom{N+1}{p+1}} \to \Omega^p_{\mathbb{P}^N}(p+1) \to 0, \tag{7.1}$$

as follows: For every U_i in the standard covering of $\mathbb{P}^N$, and every

$$\omega = \sum_{|J|=p} a_J dx_{j_0} \wedge \cdots \wedge dx_{j_p} \in \Omega^{p+1}_{\mathbb{P}^N}(p+1)(U_i)$$

the first map sends ω to $(a_J)_{|J|=p}$. While the second one is sending $(b_J)_{|J|=p}$, where each $b_J \in \mathcal{O}_{\mathbb{P}^N}(U_i)$, to the form $\eta = \iota_E(\sum_{|J|=p} b_J dx_{j_0} \wedge \cdots \wedge dx_{j_p})$.

Proposition 7.1. *Euler's sequence* (7.1) *is an exact sequence.*

Proof. To show the exactness, we will show it remains exact in every U_i of the standard covering. Let us do it for $U_0 = \mathbb{P}^N \setminus V(x_0)$. The injectivity of the left morphism is clear. In order to show that Euler's sequence is a complex, it is enough to notice that $\iota_E(d(\frac{x_i}{x_0})) = 0$. In order to show that the right morphism is well defined, i.e. that $\iota_E(\sum_{|J|=p} b_J dx_{j_0} \wedge \cdots \wedge dx_{j_p}) \in \Omega^p_{\mathbb{P}^N}(p+1)(U_0)$, it is enough to show that any $\eta = \iota_E(dx_{j_0} \wedge \cdots \wedge dx_{j_p}) \in \Omega^p_{\mathbb{P}^N}(p)(U_0)$. This follows from the equality

$$\eta = x_0^p x_{j_0} \cdots x_{j_p} \left(d\left(\frac{x_{j_1}}{x_0}\right)\frac{1}{x_{j_1}} - d\left(\frac{x_{j_0}}{x_0}\right)\frac{1}{x_{j_0}}\right) \wedge \cdots \wedge \left(d\left(\frac{x_{j_p}}{x_0}\right)\frac{1}{x_{j_p}} - d\left(\frac{x_{j_0}}{x_0}\right)\frac{1}{x_{j_0}}\right).$$

The surjectivity of the right morphism follows from taking any $\eta \in \Omega^p_{\mathbb{P}^N}(p+1)(U_0)$ and write it as $\eta = \iota_E(x_0^{-1} dx_0 \wedge \eta)$ (notice that $\iota_E(\eta) = 0$). Finally, the middle exactness follows in the same way. If $\omega = \sum_{|J|=p} b_J dx_{j_0} \wedge \cdots \wedge dx_{j_p}$ is such that $\iota_E(\omega) = 0$, then we write $\omega = \iota_E(x_0^{-1} dx_0 \wedge \omega)$, and we know the image of ι_E determines a form in the projective space, so $\omega \in \Omega^{p+1}_{\mathbb{P}^N}(p+1)(U_0)$. $\square$

Theorem 7.1 (Bott's formula, Bott (1957))**.**

$$\dim_{\mathbb{C}} H^q(\mathbb{P}^N, \Omega^p_{\mathbb{P}^N}(k)) = \begin{cases} \binom{k+N-p}{k}\binom{k-1}{p} & \text{if } q = 0, 0 \leqslant p \leqslant N, k > p, \\ 1 & \text{if } k = 0, 0 \leqslant p = q \leqslant N, \\ \binom{-k+p}{-k}\binom{-k-1}{N-p} & \text{if } q = N, 0 \leqslant p \leqslant N, k < p - N, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The result follows by induction on p. For $p = 0$ we know by Proposition 4.4 that

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k)) = \mathbb{C}[x_0, \dots, x_N]_k,$$

and

$$H^N(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k)) \cong \mathbb{C}[x_0, \dots, x_N]_{-k-N-1}, \tag{7.2}$$

for $k < -N$. Where the isomorphism (7.2) takes each monomial

$$x^I \in \mathbb{C}[x_0, \dots, x_N]_{-k-N-1} \qquad \text{to} \qquad \frac{1}{x^I x_0 \cdots x_N} \in H^N(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k)).$$

And the rest of groups are zero. For the induction step we use the long exact sequence in cohomology associated to the twist of Euler's sequence

$$0 \to \Omega^{p+1}_{\mathbb{P}^N}(k) \to \mathcal{O}_{\mathbb{P}^N}^{\oplus\binom{N+1}{p+1}}(k-p-1) \to \Omega^p_{\mathbb{P}^N}(k) \to 0. \tag{7.3}$$

The proof of the induction step is tedious but straightforward if we separate in the cases given by the formula. First of all, (7.3) remains exact in global sections. In fact, since each global section of $\Omega^p_{\mathbb{P}^N}(k)$ lift to a global section of $\Omega^p_{\mathbb{C}^{N+1}}$ of the form

$$\eta = \sum_{|J|=p-1} c_J dx_{j_0} \wedge \cdots \wedge dx_{j_{p-1}}$$

with $c_J \in \mathbb{C}[x_0, \dots, x_N]_{k-p}$. We can use Euler's identity to write

$$\eta = x_0\eta_0 + \cdots + x_N\eta_N.$$

Then $\eta = \iota_E(dx_0 \wedge \eta_0 + \cdots + dx_N \wedge \eta_N)$. This gives us the short exact sequence

$$0 \to H^0(\mathbb{P}^N, \Omega^{p+1}_{\mathbb{P}^N}(k)) \to H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k-p-1))^{\binom{N+1}{p+1}} \to H^0(\mathbb{P}^N, \Omega^p_{\mathbb{P}^N}(k)) \to 0,$$

and we can deduce the inductive step for $q = 0$. On the other hand, by induction hypothesis we have the short exact sequence

$$0 \to H^N(\mathbb{P}^N, \Omega^{p+1}_{\mathbb{P}^N}(k)) \to H^N(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k-p-1))^{\binom{N+1}{p+1}} \to H^N(\mathbb{P}^N, \Omega^p_{\mathbb{P}^N}(k)) \to 0,$$

from which we deduce the inductive step for $q = N$. For the remaining cases, we have by the base case $p = 0$ that $H^q(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k-p-1)) = 0$ for $0 < q \leqslant N-1$. In consequence

$$H^q(\mathbb{P}^N, \Omega^p_{\mathbb{P}^N}(k)) \cong H^{q+1}(\mathbb{P}^N, \Omega^{p+1}_{\mathbb{P}^N}(k)),$$

for $0 < q < N-1$, and $H^1(\mathbb{P}^N, \Omega^{p+1}_{\mathbb{P}^N}(k)) = 0$. These two facts are what we need to finish the induction for the remaining values of q. $\square$

Corollary 7.1. *Every $\omega \in H^0(\mathbb{P}^N, \Omega^{N-1}_{\mathbb{P}^N}(k))$ is of the form*

$$\omega = \sum_{i=0}^{N} T_i \iota_{\frac{\partial}{\partial x_i}}(\Omega),$$

with $T_i \in \mathbb{C}[x_0, \ldots, x_N]_{k-N}$. Where

$$\Omega := \iota_E(dx_0 \wedge \cdots \wedge dx_N) = \sum_{i=0}^{N} (-1)^i x_i \widehat{dx_i},$$

and $\widehat{dx_i} := dx_0 \wedge dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_N$.

Proof. Using (7.3) for $p = N-1$ and Bott's formula we see that $H^0(\mathbb{P}^N, \Omega^{N-1}_{\mathbb{P}^N}(k))$ is generated by

$$\begin{aligned}
\iota_E\left(\sum_{i=0}^{N} P_i \widehat{dx_i}\right) &= \iota_E\left(\sum_{i=0}^{N} (-1)^i P_i \iota_{\frac{\partial}{\partial x_i}}(dx_0 \wedge \cdots \wedge dx_N)\right), \\
&= \iota_E\left(\iota_{\sum_{i=0}^{N}(-1)^i P_i \frac{\partial}{\partial x_i}}(dx_0 \wedge \cdots \wedge dx_N)\right), \\
&= -\iota_{\sum_{i=0}^{N}(-1)^i P_i \frac{\partial}{\partial x_i}}(\iota_E(dx_0 \wedge \cdots \wedge dx_N)), \\
&= \sum_{i=0}^{N} (-1)^{i+1} P_i \iota_{\frac{\partial}{\partial x_i}}(\Omega),
\end{aligned}$$

where $P_i \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k-N))$. Take $T_i := (-1)^{i+1} P_i$. $\square$

7.3 Griffiths' Theorem I

Since the residue map

$$H^{k+1}_{\mathrm{dR}}(U/\mathbb{C}) \xrightarrow{\mathrm{res}} H^{k}_{\mathrm{dR}}(X/\mathbb{C})_0$$

is an isomorphism (because $H^{k+1}_{\mathrm{dR}}(\mathbb{P}^{n+1})_0 = 0$). We conclude that $H^k_{\mathrm{dR}}(X/\mathbb{C})_0 = 0$ for all $k \neq n$ (because $H^{k+1}_{\mathrm{dR}}(U/\mathbb{C}) = 0$ for $k+1 \neq n+1$, see Movasati (2021, Chapter 5, Section 5)). Thus, we are just interested in determining a basis for the non-trivial primitive cohomology group of X

$$H^n_{\mathrm{dR}}(X/\mathbb{C})_0.$$

We start by giving a set of generators.

Theorem 7.2 (Griffiths (1969a)). *For every $q = 0, \dots, n$, the natural map*

$$H^0(\mathbb{P}^{n+1}, \Omega^{n+1}_{\mathbb{P}^{n+1}}((q+1)X)) \to H^{n+1}_{\mathrm{dR}}(U/\mathbb{C})$$

has image equal to $F^{n+1-q}H^{n+1}_{\mathrm{dR}}(U/\mathbb{C})$. Consequently, every piece of the Hodge filtration

$$F^{n-q}H^n_{\mathrm{dR}}(X/\mathbb{C})_0,$$

is generated by the residues of global forms with pole of order at most $q+1$ along X.

Proof. Consider $\omega^{n+1} \in H^0(\mathbb{P}^{n+1}, \Omega^{n+1}_{\mathbb{P}^{n+1}}((q+1)X))$. The natural map sends it to

$$\omega \in H^{n+1}_{\mathrm{dR}}(U/\mathbb{C}) = \mathbb{H}^{n+1}(\mathbb{P}^{n+1}, \Omega^{\bullet}_{\mathbb{P}^{n+1}}(*X)),$$

by letting $\omega^k = 0$ for $k = 0, \dots, n$. To see in which part of Hodge filtration ω is, we need to write it as an element of $\mathbb{H}^{n+1}(\mathbb{P}^{n+1}, \Omega_{\mathbb{P}^{n+1}}(\log X))$, i.e. we need to reduce the order of the pole of ω^{n+1} up to order 1. Thanks to Carlson–Griffiths Theorem 6.2, we know how to do this applying the operator $(1 - DH)$. In order to obtain a form with poles of order 1 we need to apply it q times, i.e. the image of ω in $\mathbb{H}^{n+1}(\mathbb{P}^{n+1}, \Omega^{\bullet}_{\mathbb{P}^{n+1}}(\log X))$ is represented by $(1-DH)^q\omega$. It is clear by the definition of H and D that

$$(1-DH)^l\omega \in \mathrm{Im}(\mathbb{H}^{n+1}(\mathbb{P}^{n+1}, \Omega^{\bullet \geqslant n+1-l}_{\mathbb{P}^{n+1}}((q+1-l)X)) \to \mathbb{H}^{n+1}(\mathbb{P}^{n+1}, \Omega^{\bullet}_{\mathbb{P}^{n+1}}((q+1-l)X))).$$

In consequence, for $l = q$ we see that

$$(1-DH)^q\omega \in F^{n+1-q}H^{n+1}_{\mathrm{dR}}(U/\mathbb{C}).$$

Conversely, let $\omega \in F^{n+1-q} H^{n+1}_{\mathrm{dR}}(U/\mathbb{C})$. Then we can represent $\omega = \omega^{n+1-q} + \cdots + \omega^{n+1}$, where each

$$\omega^k \in C^{n+1-k}(\mathcal{U}, \Omega^k_{\mathbb{P}^{n+1}}(\log X)),$$

and $\omega^k = 0$ for $k = 0, \ldots, n-q$. We claim that for every $l = 0, \ldots, q$ we can represent $\omega \in \mathbb{H}^{n+1}(\mathbb{P}^{n+1}, \Omega^{\bullet}_{\mathbb{P}^{n+1}}(*X))$ as $\omega = \eta_l^{n+1-q+l} + \omega^{n+2-q+l} + \cdots + \omega^{n+1}$ with

$$\eta_l^{n+1-q+l} \in C^{q-l}(\mathcal{U}, \Omega^{n+1-q+l}_{\mathbb{P}^{n+1}}((l+1)X)).$$

We prove this claim by induction on l. The case $l = 0$ is clear taking $\eta_0^{n+1-q} = \omega^{n+1-q}$. Now, for $l > 0$ suppose $\omega = \eta_{l-1}^{n-q+l} + \omega^{n+1-q+l} + \cdots + \omega^{n+1}$. Since $D\omega = 0$, we know $\delta\eta_{l-1}^{n-q+l} = 0$. By Bott's formula (Theorem 7.1)

$$H^{q-l+1}(\mathbb{P}^{n+1}, \Omega^{n-q+l}_{\mathbb{P}^{n+1}}(lX)) = 0.$$

Then, there exist $\mu \in C^{q-l}(\mathcal{U}, \Omega^{n-q+l}_{\mathbb{P}^{n+1}}(lX))$ such that $\delta\mu = \eta_{l-1}^{n-q+l}$. Subtracting from ω the exact form in hypercohomology $D\mu$, we get the claim for l, and we finish the induction. Finally, applying the claim for $l = q$ we can write $\omega = \eta_q^{n+1}$ with

$$\eta_q^{n+1} \in H^0(\mathcal{U}, \Omega^{n+1}_{\mathbb{P}^{n+1}}((q+1)X)),$$

as desired. □

7.4 Griffiths' Theorem II

Griffiths' Theorem 7.2 tells us that the elements of the form

$$\mathrm{res}\left(\frac{P\Omega}{F^{q+1}}\right) \in F^{n-q} H^n_{\mathrm{dR}}(X/\mathbb{C})_0,$$

where $P \in H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d(q+1)-n-2))$ and $q = 0, \ldots, n$, generate all $H^n_{\mathrm{dR}}(X/\mathbb{C})_0$.

The following theorem tells us how we can choose a basis from these generators.

Theorem 7.3 (Griffiths (1969a)). *For every $q = 0, \dots, n$ the kernel of the map*

$$\begin{aligned} \varphi : H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d(q+1)-n-2)) &\twoheadrightarrow F^{n-q} H^n_{\mathrm{dR}}(X/\mathbb{C})_0 / F^{n+1-q} H^n_{\mathrm{dR}}(X/\mathbb{C})_0 \\ P &\mapsto \operatorname{res}\left(\frac{P\Omega}{F^{q+1}}\right) \end{aligned}$$

is the degree $N = d(q+1) - n - 2$ part of the Jacobian ideal of F, $J^F_N \subseteq \mathbb{C}[x_0, \dots, x_{n+1}]_N$.

Definition 7.2. Recall that the **Jacobian ideal of F** is the homogeneous ideal

$$J^F := \langle F_0, \dots, F_{n+1} \rangle \subseteq \mathbb{C}[x_0, \dots, x_{n+1}],$$

where, from now on, we denote

$$F_i := \frac{\partial F}{\partial x_i},$$

for $i = 0, \dots, n+1$. The **Jacobian ring of F** is

$$R^F := \mathbb{C}[x_0, \dots, x_{n+1}]/J^F.$$

Remark 7.1. Theorem 7.3 implies that to choose a basis for

$$F^{n-q} H^n_{\mathrm{dR}}(X/\mathbb{C})_0 / F^{n+1-q} H^n_{\mathrm{dR}}(X/\mathbb{C})_0 \cong H^{n-q,q}(X)_0$$

it is enough to take the elements of the form $\operatorname{res}\left(\frac{P\Omega}{F^{q+1}}\right)$, for $P \in \mathbb{C}[x_0, \dots, x_{n+1}]_N$ forming a basis of R^F_N. In particular

$$h^{n-q,q}(X)_0 = \dim_{\mathbb{C}} R^F_N.$$

Proof of Theorem 7.3. By Theorem 7.2, it is clear that P is in the kernel of φ, if and only if, there exist $Q \in H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(dq-n-2))$ such that

$$\operatorname{res}\left(\frac{P\Omega}{F^{q+1}}\right) = \operatorname{res}\left(\frac{Q\Omega}{F^q}\right).$$

Since the residue map is an isomorphism between $H^{n+1}_{\mathrm{dR}}(U/\mathbb{C}) \cong H^n_{\mathrm{dR}}(X/\mathbb{C})_0$. This is equivalent to say

$$\frac{(P - FQ)\Omega}{F^{q+1}} = 0 \in H^{n+1}_{\mathrm{dR}}(U/\mathbb{C}). \tag{7.4}$$

Since $H^{n+1}_{\mathrm{dR}}(U/\mathbb{C}) \cong H^{n+1}(\Gamma(\Omega^\bullet_U), d)$, (7.4) is equivalent to

$$\frac{(P - QF)\Omega}{F^{q+1}} = d\gamma,$$

for some $\gamma \in H^0(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}(qX))$. Recall from Corollary 7.1 that every $\gamma \in H^0(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}(qX))$ is of the form

$$\gamma = \frac{\sum_{i=0}^{n+1} T_i \iota_{\frac{\partial}{\partial x_i}}(\Omega)}{F^q},$$

for some $T_i \in \mathbb{C}[x_0, \dots, x_{n+1}]_{dq-n-1}$. In consequence, P is in the kernel of φ, if and only if,

$$\frac{P\Omega}{F^{q+1}} \equiv \frac{-q\sum_{i=0}^{n+1} T_i F_i \Omega}{F^{q+1}} \ (\mathrm{mod}\ H^0(\mathbb{P}^{n+1}, \Omega^n_{\mathbb{P}^{n+1}}(qX))),$$

in other words

$$P \equiv -q \sum_{i=0}^{n+1} T_i F_i \ (\mathrm{mod}\ F).$$

Since $F \in J^F$ (by Euler's identity), this is equivalent to $P \in J^F$. $\square$

Corollary 7.2. *Let $X \subseteq \mathbb{P}^{n+1}$ be a smooth degree d hypersurface. The Hodge numbers increase up to the middle*

$$h^{n,0}(X)_0 \leqslant h^{n-1,1}(X)_0 \leqslant \cdots \leqslant h^{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}(X)_0.$$

Proof. Since the Hodge numbers are the same for all smooth degree d hypersurfaces of $\mathbb{P}^{n+1}$, it is enough to show the corollary for $X = \{x_0^d + \cdots + x_{n+1}^d = 0\}$ the Fermat variety. By Griffiths' basis theorem we know

$$\begin{aligned} h^{n-k,k}(X)_0 &= \#R^F_{d(k+1)-n-2} = \\ &= \#\left\{ x_0^{i_0} \cdots x_{n+1}^{i_{n+1}} : \sum_{j=0}^{n+1} i_j = d(k+1) - n - 2\ ,\ 0 \leqslant i_j \leqslant d-2\ ,\ \forall j = 0, \dots, n+1 \right\} \\ &= \#\left\{ (i_0, \dots, i_{n+1}) \in \{0, \dots, d-2\}^{n+2} : i_0 + \cdots + i_{n+1} = d(k+1) - n - 2 \right\}. \end{aligned}$$

Let us define

$$I^n_N := \{(i_0, \dots, i_{n+1}) \in \{0, \dots, d-2\}^{n+2} : i_0 + \cdots + i_{n+1} = N\}$$

Thus, it is enough to show that

$$I_k^n \leqslant I_{k+1}^n, \qquad \forall\, N < (d-2)(\frac{n}{2}+1). \tag{7.5}$$

Let us prove (7.5) by induction on n. Using the map

$$(i_0, i_1, \ldots, i_{n+1}) \mapsto (i_0 + 1, i_1, \ldots, i_{n+1})$$

we have the correspondence

$$\#\{i_0 + \cdots + i_{n+1} = k\ ,\ i_0 \neq d-2\} = \#\{i_0 + \cdots + i_{n+1} = k+1\ ,\ i_0 \neq 0\}.$$

Therefore, (7.5) is reduced to show

$$I_{k-(d-2)}^{n-1} \leqslant I_{k+1}^{n-1}. \tag{7.6}$$

If $k+1 \leqslant (d-2)(\frac{n-1}{2}+1)$, then (7.6) follows by induction hypothesis. If $k+1 > (d-2)(\frac{n-1}{2}+1)$, since $k+1 \leqslant (d-2)(\frac{n}{2}+1)$, then

$$k-(d-2) < (d-2)(n+1)-(k+1) < (d-2)(\frac{n-1}{2}+1).$$

It follows from induction hypothesis that

$$I_{k-(d-2)}^{n-1} \leqslant I_{(d-2)(n+1)-(k+1)}^{n-1} = I_{k+1}^{n-1},$$

as desired. Note that the last equality is given by the correspondence

$$(i_1, \ldots, i_{n+1}) \longleftrightarrow (d-2-i_1, \ldots, d-2-i_{n+1}).$$

$\square$

Remark 7.2. The above property holds more generally for smooth complete intersection projective varieties. For some references about this phenomenon see Movasati (2021, p. 223).

7.5 Carlson–Griffiths Theorem

In this section we give an explicit description in Čech cohomology of the residue map for the generators given by Griffiths' theorem. This was done by Carlson and Griffiths (1980) as a consequence of Carlson–Griffiths Lemma (Theorem 6.2).

Let $X \subseteq \mathbb{P}^{n+1}$ be a smooth degree d hypersurface given by $X = \{F = 0\}$. Recall that $H^0(\mathbb{P}^{n+1}, \Omega^{n+1}_{\mathbb{P}^{n+1}}(n+2)) \cong \mathbb{C}$ is generated by

$$\Omega = \sum_{i=0}^{n+1} (-1)^i x_i \, dx_0 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_{n+1}.$$

Theorem 7.4 (Carlson and Griffiths (ibid.)). *Let $q \in \{0, 1, \ldots, n\}$,*

$$P \in H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d(q+1) - n - 2)).$$

Then

$$\operatorname{res}\left(\frac{P\Omega}{F^{q+1}}\right) = \frac{(-1)^n}{q!} \left\{\frac{P\Omega_J}{F_J}\right\}_{|J|=q} \in H^q(\mathcal{U}, \Omega_X^{n-q}). \tag{7.7}$$

Where $\Omega_J := \iota_{\frac{\partial}{\partial x_{j_q}}}(\cdots \iota_{\frac{\partial}{\partial x_{j_0}}}(\Omega)\cdots)$, $F_J := F_{j_0} \cdots F_{j_q}$ and $\mathcal{U} = \{U_i\}_{i=0}^{n+1}$ is the Jacobian covering restricted to X, given by $U_i = \{F_i \neq 0\} \cap X$.

Proof. Let $U := \mathbb{P}^{n+1} \setminus X$. For $l = 0, \ldots, q$ define

$$^{(l)}\omega := (1 - DH)^l \left(\frac{P\Omega}{F^{q+1}}\right) \in H^{n+1}_{\mathrm{dR}}(U/\mathbb{C}),$$

where H is the operator defined in Definition 6.2. We claim

$$^{(l)}\omega^{n+1-l} = \left\{\frac{(q-l)!(-1)^n P}{q! \cdot F^{q-l}} \left(\frac{\Omega_J}{F_J} \wedge \frac{dF}{F} + d \cdot (-1)^n \frac{V_J}{F_J}\right)\right\}_{|J|=l} \in$$
$$\in C^l(\mathcal{U}, \Omega^{n+1-l}_{\mathbb{P}^{n+1}}((q-l+1)X)),$$

where $V := dx_0 \wedge \cdots \wedge dx_{n+1}$, and $V_J := \iota_{\frac{\partial}{\partial x_{j_m}}}(\cdots \iota_{\frac{\partial}{\partial x_{j_0}}}(V)\cdots)$. In fact, for $l = 0$, the claim follows from the identity

$$\frac{\Omega}{F} = \frac{dF}{F} \wedge \frac{\Omega_{(i)}}{F_i} + d \cdot \frac{V_{(i)}}{F_i}.$$

(Which is obtained by contracting $\iota_{\frac{\partial}{\partial x_i}}$ to the equality $dF \wedge \Omega = d \cdot F \cdot V$.) Assuming the claim for $l \geqslant 0$, then

$$H_{q-l+1}({}^{(l)}\omega^{n-l+1})_J = \frac{-(q-l-1)!P}{q! \cdot F^{q-l}} \frac{\Omega_J}{F_J}.$$

In consequence,

$${}^{(l+1)}\omega_J^{n-l} = -\delta H_{q-l+1}({}^{(l)}\omega^{n+1-l})_J = \frac{(q-l-1)!}{q!} \sum_{m=0}^{l+1} (-1)^m \frac{P\Omega_{J\setminus\{j_m\}}}{F^{q-l} F_{J\setminus\{j_m\}}}.$$

Using the following identity

$$\Omega_J \wedge dF + (-1)^n d \cdot F \cdot V_J = (-1)^n \sum_{m=0}^{l+1} (-1)^m F_{j_m} \Omega_{J\setminus\{j_m\}}, \tag{7.8}$$

(this identity is obtained by successive contraction of the identity $dF \wedge \Omega = d \cdot F \cdot V$ by the $\iota_{\frac{\partial}{\partial x_{j_m}}}$, for $m = 0, \dots, l+1$) we obtain the claim for $l+1$. In conclusion

$$\begin{aligned} {}^{(q)}\omega^{n+1-q} = \left\{ \frac{(-1)^n P}{q!} \left(\frac{\Omega_J}{F_J} \wedge \frac{dF}{F} + d \cdot (-1)^n \frac{V_J}{F_J} \right) \right\}_{|J|=q} &\in \\ \in C^q(\mathcal{U}, \Omega^{n+1-q}_{\mathbb{P}^{n+1}}(\log X)), \end{aligned}$$

the rest is just to apply the residue map. □

7.6 Computing the residue map in algebraic de Rham cohomology

Using Carlson–Griffiths' Lemma, it is possible to describe explicitly the residue map in all algebraic de Rham cohomology, i.e. as an element of

$$\operatorname{res}\left(\frac{P\Omega}{F^{q+1}}\right) \in F^{n-q} H^n_{\mathrm{dR}}(X/\mathbb{C}).$$

This is resumed in the following theorem.

Theorem 7.5. *Let $q \in \{0, 1, \dots, n\}$, $P \in H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d(q+1) - n - 2))$. For $l = 0, \dots, q$ define*

$$^{(l)}\omega := (1 - DH)^l \left(\frac{P\Omega}{F^{q+1}} \right).$$

Then, for each $m = 0, \dots, l$

$$^{(l)}\omega^{n+1-m} = \left\{ \frac{^{(l)}\alpha_J^{n+1-m}}{F^{q-l}} \left(\frac{\Omega_J}{F_J} \wedge \frac{dF}{F} + d \cdot (-1)^n \frac{V_J}{F_J} \right) \right\}_{|J|=m} \in$$
$$\in C^m(\mathcal{U}, \Omega^{n+1-m}_{\mathbb{P}^{n+1}}((q+1-l)X)).$$

Where $V := dx_0 \wedge \cdots \wedge dx_{n+1}$, $V_J := \iota_{\frac{\partial}{\partial x_{j_m}}}(\cdots \iota_{\frac{\partial}{\partial x_{j_0}}}(V)\cdots)$, $\mathcal{U}$ is the Jacobian covering of $\mathbb{P}^{n+1}$. And $^{(l)}\alpha^{n+1-m} \in C^m(\mathcal{U}, \mathcal{O}_{\mathbb{P}^{n+1}}(d(q-l+m+1) - n - 2))$ are such that

$$^{(l)}\alpha_J^{n+1-m} = \frac{1}{q-l+1} \left({}^{(l-1)}\alpha^{n+2-m}_{J\setminus\{j_k\}} + F_{J\setminus\{j_k\}} \frac{\partial}{\partial x_{j_k}} \left(\frac{^{(l-1)}\alpha_J^{n+1-m}}{F_J} \right) \right), \tag{7.9}$$

for any $k \in \{0, \dots, m\}$, and $^{(0)}\alpha_j^{n+1} = (-1)^n P$ for every $j = 0, \dots, n+1$.

Proof. Let us proceed by induction in $l = 0, \dots, q$. The base case follows directly from the identity

$$\frac{\Omega}{F} = \frac{dF}{F} \wedge \frac{\Omega_{(i)}}{F_i} + d \cdot \frac{V_{(i)}}{F_i}.$$

(To obtain this identity just apply $\iota_{\frac{\partial}{\partial x_i}}$ to the identity $dF \wedge \Omega = d \cdot FV$.) Now, assume the theorem is true for some $l \geqslant 0$. Then

$$^{(l+1)}\omega = (1 - DH)^{(l)}\omega.$$

For any $m = 0, \dots, l+1$ we have that

$$H_{q-l+1}({}^{(l)}\omega_J^{n+1-m}) = \frac{(-1)^{n+1}}{(q-l)} \frac{^{(l)}\alpha_J^{n+1-m}\Omega_J}{F^{q-l}F_J}.$$

Consider the following identity

$$\Omega_J \wedge dQ + (-1)^n g Q V_J = (-1)^n \sum_{k=0}^m (-1)^k Q_{j_k} \Omega_{J\setminus\{j_k\}}, \tag{7.10}$$

for $Q \in \mathbb{C}(x_0,\dots,x_{n+1})$ homogeneous of degree g (this identity is obtained by successive contraction of the identity $dQ \wedge \Omega = gQV$ by the $\iota_{\frac{\partial}{\partial x_{j_k}}}$, for $k = 0,\dots,m$). Noting that

$$d\Omega_J = (-1)^m(m-n-1)V_J,$$

and applying (7.10) to $Q = \frac{(-1)^{n+1-m}.{}^{(l)}\alpha_J^{n+1-m}}{(q-l)F^{q-l}F_J}$ we obtain

$$dH_{q-l+1}({}^{(l)}\omega_J^{n+1-m}) = \sum_{k=0}^{m}(-1)^k\frac{\partial}{\partial x_{j_k}}\left(\frac{(-1)^{n+1-m}}{(q-l)}\frac{{}^{(l)}\alpha_J^{n+1-m}}{F^{q-l}F_J}\right)\Omega_{J\setminus\{j_k\}}.$$

By the other hand, applying (7.10) for $Q = F$ we get

$${}^{(l)}\omega_J^{n+1-m} = (-1)^n\sum_{k=0}^{m}(-1)^k\left(\frac{{}^{(l)}\alpha_J^{n+1-m}F_{j_k}}{F^{q+1-l}F_J}\right)\Omega_{J\setminus\{j_k\}}.$$

Using these relations we conclude

$${}^{(l+1)}\omega_J^{n+1-m} =$$

$$= \sum_{k=0}^{m}\left(\frac{\frac{(-1)^n}{q-l}\left({}^{(l)}\alpha_{J\setminus\{j_k\}}^{n+2-m} + F_{J\setminus\{j_k\}}\frac{\partial}{\partial x_{j_k}}\left(\frac{{}^{(l)}\alpha_J^{n+1-m}}{F_J}\right)\right)}{F^{q-l}F_J}\right)(-1)^kF_{j_k}\Omega_{J\setminus\{j_k\}},$$

and the result follows if we prove that the expression

$${}^{(l)}E_k^{n+1-m} := {}^{(l)}\alpha_{J\setminus\{j_k\}}^{n+2-m} + F_{J\setminus\{j_k\}}\frac{\partial}{\partial x_{j_k}}\left(\frac{{}^{(l)}\alpha_J^{n+1-m}}{F_J}\right) \tag{7.11}$$

is independent of k (and we conclude that ${}^{(l+1)}\alpha_J^{n+1-m} = {}^{(l)}E_k^{n+1-m}/(q-l)$). If $l = 0$ this is clear, and for $l > 0$ we know by induction hypothesis this is true for ${}^{(l-1)}E_h^{n+2-m} = {}^{(l)}\alpha_{J\setminus\{j_k\}}^{n+2-m}(q-l+1)$ and ${}^{(l-1)}E_h^{n+1-m} = {}^{(l)}\alpha_J^{n+1-m}(q-l+1)$, i.e. we can write

$${}^{(l)}\alpha_{J\setminus\{j_k\}}^{n+2-m}(q-l+1) = {}^{(l-1)}\alpha_{J\setminus\{j_k,j_h\}}^{n+3-m} + F_{J\setminus\{j_k,j_h\}}\frac{\partial}{\partial x_{j_h}}\left(\frac{{}^{(l-1)}\alpha_{J\setminus\{j_k\}}^{n+2-m}}{F_{J\setminus\{j_k\}}}\right),$$

and

$$ {}^{(l)}\alpha_J^{n+1-m}(q-l+1) = {}^{(l-1)}\alpha_{J\setminus\{j_h\}}^{n+2-m} + F_{J\setminus\{j_h\}} \frac{\partial}{\partial x_{j_h}} \left(\frac{{}^{(l-1)}\alpha_J^{n+1-m}}{F_J} \right), $$

for $h \in \{0, \dots, m\} \setminus \{k\}$. Replacing this in (7.11) we obtain

$$ {}^{(l)}E_k^{n+1-m} = \frac{S(k,h)}{q-l+1} - \frac{\frac{\partial^2 F}{\partial x_{j_k} \partial x_{j_h}}}{F_{j_k} F_{j_h}} {}^{(l-1)}E_h^{n+1-m}, $$

where

$$ \begin{aligned} S(k,h) \quad &:= {}^{(l-1)}\alpha_{J\setminus\{j_k,j_h\}}^{n+3-m} + F_{J\setminus\{j_k,j_h\}} \left(\frac{\partial^2}{\partial x_{j_k} \partial x_{j_h}} \left(\frac{{}^{(l-1)}\alpha_J^{n+1-m}}{F_J} \right) \right. \\ &\quad \left. + \frac{\partial}{\partial x_{j_k}} \left(\frac{{}^{(l-1)}\alpha_{J\setminus\{j_h\}}^{n+2-m}}{F_{J\setminus\{j_h\}}} \right) + \frac{\partial}{\partial x_{j_h}} \left(\frac{{}^{(l-1)}\alpha_{J\setminus\{j_k\}}^{n+2-m}}{F_{J\setminus\{j_k\}}} \right) \right) = S(h,k), \end{aligned} $$

is a symmetric expression in terms of k and h. Since

$$ {}^{(l-1)}E_h^{n+1-m} = {}^{(l-1)}E_k^{n+1-m} $$

we conclude

$$ {}^{(l)}E_k^{n+1-m} = {}^{(l)}E_h^{n+1-m}, $$

as desired. □

Remark 7.3. Theorem 7.5 is giving us an explicit expression of $P\Omega/F^{q+1}$ in hypercohomology with meromorphic forms with logarithmic poles along X, namely ${}^{(q)}\omega$. By (7.9) we know that

$$ {}^{(l)}\alpha_J^{n+1-l} = \frac{1}{(q-l+1)} {}^{(l-1)}\alpha_{J\setminus\{j_k\}}^{n+2-l}, $$

for $l = 0, \dots, q$. Recursively, we get the expression

$$ {}^{(q)}\alpha_J^{n+1-q} = \frac{1}{q!} {}^{(0)}\alpha_j^{n+1} = \frac{(-1)^n P}{q!}. $$

In consequence,

$$ \begin{aligned} {}^{(q)}\omega^{n+1-q} = \left\{ \frac{(-1)^n P}{q!} \left(\frac{\Omega_J}{F_J} \wedge \frac{dF}{F} + d \cdot (-1)^n \frac{V_J}{F_J} \right) \right\}_{|J|=q} &\in \\ \in C^m(\mathcal{U}, \Omega_{\mathbb{P}^{n+1}}^{n+1-q}(\log X)). \end{aligned} $$

Applying residue to ${}^{(q)}\omega$, we obtain Carlson–Griffiths' Theorem 7.4.

7.7 Cup product for hypersurfaces

Now we have all the necessary tools to describe the cup product for hypersurfaces. We content ourselves with the following result due to Carlson and Griffiths (1980, Theorem 2).

Theorem 7.6. *Let $p+q=n$. For every $P,Q\in\mathbb{C}[x_0,\dots,x_{n+1}]$ with* $\deg P = d(q+1)-n-2$ *and* $\deg Q = d(p+1)-n-2$ *we have*

$$\operatorname{res}\left(\frac{P\Omega}{F^{q+1}}\right)\cup\operatorname{res}\left(\frac{Q\Omega}{F^{p+1}}\right)=0 \quad \textit{if and only if} \quad PQ\in J^F.$$

The proof of the above theorem will be given in the next chapter as a consequence of our computations of periods, see Proposition 8.4. The following is the explicit computation of the cup in the Čech cohomology group $H^n(X,\Omega_X^n)$ relative to the Jacobian covering.

Proposition 7.2. *Let $p+q=n$. Consider $P,Q\in\mathbb{C}[x_0,\dots,x_{n+1}]$ homogeneous polynomials with* $\deg P = d(q+1)-n-2$ *and* $\deg Q = d(p+1)-n-2$. *Then for every $m=0,\dots,n+1$*

$$\left(\operatorname{res}\left(\frac{P\Omega}{F^{q+1}}\right)\cup\operatorname{res}\left(\frac{Q\Omega}{F^{p+1}}\right)\right)^n_{0\cdots\widehat{m}\cdots n+1}=\begin{cases}\dfrac{(-1)^{p+m}PQx_m\Omega_{(q+1)}F_m}{p!\cdot q!F_0\cdots F_{n+1}\cdot F_{q+1}} & \textit{if } m\leqslant q,\\[2ex] \dfrac{(-1)^{p+m}PQx_m\Omega_{(q)}F_m}{p!\cdot q!F_0\cdots F_{n+1}\cdot F_q} & \textit{if } m>q.\end{cases}$$

Proof. Since $\operatorname{res}\left(\frac{P\Omega}{F^{q+1}}\right)\in F^pH^n_{\mathrm{dR}}(X/\mathbb{C})$ and $\operatorname{res}\left(\frac{Q\Omega}{F^{p+1}}\right)\in F^qH^n_{\mathrm{dR}}(X/\mathbb{C})$ it follows from the cup product formula (Theorem 5.3) that

$$\begin{aligned}
&\left(\operatorname{res}\left(\frac{P\Omega}{F^{q+1}}\right)\cup\operatorname{res}\left(\frac{Q\Omega}{F^{p+1}}\right)\right)^n_{0\cdots\widehat{m}\cdots n+1}=\\
&=\sum_{r=0}^{m-1}(-1)^{n+r}\left(\operatorname{res}\left(\frac{P\Omega}{F^{q+1}}\right)\right)^{n-r}_{0\cdots r}\wedge\left(\operatorname{res}\left(\frac{Q\Omega}{F^{p+1}}\right)\right)^{r}_{r\cdots\widehat{m}\cdots n+1}+\\
&+\sum_{r=m+1}^{n+1}(-1)^{n+r+1}\left(\operatorname{res}\left(\frac{P\Omega}{F^{q+1}}\right)\right)^{n-r+1}_{0\cdots\widehat{m}\cdots r}\wedge\left(\operatorname{res}\left(\frac{Q\Omega}{F^{p+1}}\right)\right)^{r-1}_{r\cdots n+1}=\\
&=\begin{cases}(-1)^p\left(\operatorname{res}\left(\frac{P\Omega}{F^{q+1}}\right)\right)^p_{0\cdots q}\wedge\left(\operatorname{res}\left(\frac{Q\Omega}{F^{p+1}}\right)\right)^q_{q\cdots\widehat{m}\cdots n+1} & \text{if } q<m,\\ (-1)^p\left(\operatorname{res}\left(\frac{P\Omega}{F^{q+1}}\right)\right)^p_{0\cdots\widehat{m}\cdots q+1}\wedge\left(\operatorname{res}\left(\frac{Q\Omega}{F^{p+1}}\right)\right)^q_{q+1\cdots n+1} & \text{if } q\geqslant m.\end{cases}
\end{aligned}$$

By Carlson–Griffiths theorem (Theorem 7.4) we can compute this explicitly as

$$\left(\operatorname{res}\left(\frac{P\Omega}{F^{q+1}}\right)\cup\operatorname{res}\left(\frac{Q\Omega}{F^{p+1}}\right)\right)^n_{0\cdots\widehat{m}\cdots n+1}=$$
$$=\begin{cases}\frac{(-1)^p PQ}{p!\dot{q}!}\frac{\Omega_{(0\cdots q)}}{F_{0\cdots q}}\wedge\frac{\Omega_{(q\cdots\widehat{m}\cdots n+1)}}{F_{q\cdots\widehat{m}\cdots n+1}} & \text{if } q<m,\\ \frac{(-1)^p PQ}{p!\dot{q}!}\frac{\Omega_{(0\cdots\widehat{m}\cdots q+1)}}{F_{0\cdots\widehat{m}\cdots q+1}}\wedge\frac{\Omega_{(q+1\cdots n+1)}}{F_{q+1\cdots n+1}} & \text{if } q\geqslant m.\end{cases}$$

The result follows from the next identities

$$\begin{aligned}\Omega_{(0\cdots q)}\wedge\Omega_{(q\cdots\widehat{m}\cdots n+1)} &= (-1)^{q+1}\iota_E\iota_{\frac{\partial}{\partial x_q}}\left(V_{(0\cdots q-1)}\right)\wedge\Omega_{(q\cdots\widehat{m}\cdots n+1)}\\ &= (-1)^{q+1}\iota_E\iota_{\frac{\partial}{\partial x_q}}\left(V_{(0\cdots q-1)}\wedge\Omega_{(q\cdots\widehat{m}\cdots n+1)}\right)\\ &= (-1)^{q+1}\iota_E\iota_{\frac{\partial}{\partial x_q}}\Big(V_{(0\cdots q-1)}\wedge\\ &\qquad\wedge(-1)^{(n+1)q-m}((-1)^q x_m dx_0\wedge\cdots\wedge dx_{q-1}+\\ &\qquad+\sum_{j=0}^{q-1}(-1)^j x_j dx_0\wedge\cdots\widehat{dx_j}\cdots\wedge dx_{q-1}\wedge dx_m)\Big)\\ &= (-1)^{(n+1)q+m+1}x_m\iota_E\iota_{\frac{\partial}{\partial x_q}}(dx_q\wedge\cdots\wedge dx_{n+1}\wedge\\ &\qquad\qquad\wedge dx_0\wedge\cdots\wedge dx_{q-1})\\ &= (-1)^{m+1}x_m\iota_E\iota_{\frac{\partial}{\partial x_q}}(V)\\ &= (-1)^m x_m\Omega_{(q)},\end{aligned}$$

and similarly

$$\begin{aligned}
\Omega_{(0\cdots\widehat{m}\cdots q+1)} \wedge \Omega_{(q+1\cdots n+1)} &= (-1)^{q+1}\iota_E \iota_{\frac{\partial}{\partial x_{q+1}}}\left(V_{(0\cdots\widehat{m}\cdots q)}\right) \wedge \Omega_{(q+1\cdots n+1)} \\
&= (-1)^{q+1}\iota_E \iota_{\frac{\partial}{\partial x_{q+1}}}\left(V_{(0\cdots\widehat{m}\cdots q)} \wedge \Omega_{(q+1\cdots n+1)}\right) \\
&= (-1)^{q+1}\iota_E \iota_{\frac{\partial}{\partial x_{q+1}}}\left(V_{(0\cdots\widehat{m}\cdots q)}\wedge\right. \\
&\qquad \left.\wedge\ (-1)^{nq}\sum_{j=0}^{q}(-1)^j x_j\, dx_0 \wedge \cdots \widehat{dx_j} \cdots \wedge dx_q\right) \\
&= (-1)^{nq+1} x_m \iota_E \iota_{\frac{\partial}{\partial x_{q+1}}}(dx_m \wedge dx_{q+1} \wedge \cdots \wedge \\
&\qquad \wedge dx_{n+1} \wedge dx_0 \wedge \cdots \widehat{dx_m} \cdots \wedge dx_q) \\
&= (-1)^{m+1} x_m \iota_E \iota_{\frac{\partial}{\partial x_{q+1}}}(V) = (-1)^m x_m \Omega_{(q+1)}.
\end{aligned}$$

$\square$

8 Periods of algebraic cycles

In its early phase (Abel, Riemann, Weierstrass), algebraic geometry was just a chapter in analytic function theory, Solomon Lefschetz (1968).

8.1 Introduction

Periods of algebraic cycles play a fundamental role when we look at Hodge conjecture in families. In fact, they determine the cohomology class of the algebraic cycle which together with the infinitesimal variation of Hodge structure gives us the tangent space of the underlying Hodge locus. This is our main motivation to compute those periods, and it is the central topic of this chapter. Since non-zero dimensional algebraic cycles cannot lie inside affine varieties, their periods do not fit well into the multiple integral context of Picard and Simart (1897, 1906). This has actually produced Picard's ρ_0 puzzle, for details see Movasati (2021, Chapter 3). Historically, the first use of periods of algebraic cycles in the literature is by Deligne, Milne, et al. (1982, Proposition 1.5) which leads him to define the notion of an absolute Hodge cycle. This has been reproduced in Proposition 5.11. When algebraic cycles are known exactly, which is the case in this chapter, computation of such periods can be done by theoretical Čech cohomology manipulations such as in Carlson and Griffiths (1980), and this eventually leads us to the verification of the alternative Hodge conjecture in explicit examples. This method was suggested by Movasati (2017b, Section 3.5) and carried out by Movasati and Villaflor (2018) for linear algebraic cycles. The full computation for complete intersection algebraic cycles was done by the second author in his thesis, see Villaflor (2019, n.d.[a]), and this is the main content of this chapter. The applications are left to Chapters 11 and 12 in which we discuss the variational and alternative Hodge conjecture. For Fermat varieties the periods are known exactly and the problem reduces to finding algebraic relations between values of the Gamma function on rational numbers, see Deligne, Milne, et al. (1982, Section 7) and Movasati (2021, Chapter 16).

In Proposition 5.11 we have already seen that integration over algebraic cycles is a purely algebraic operation. In this chapter we consider the case of an even dimensional smooth hypersurface $X = \{F = 0\} \subseteq \mathbb{P}^{n+1}$ given by a homogeneous polynomial F of degree d. Every $\frac{n}{2}$-dimensional subvariety Z of X determines an algebraic cycle $[Z] \in H_n(X, \mathbb{Z})$. Recall that by Griffiths' Theorem 7.2, each piece of the Hodge filtration is generated by the differential forms

$$\omega_P := \operatorname{res}\left(\frac{P\Omega}{F^{q+1}}\right) \in F^{n-q} H^n_{\mathrm{dR}}(X/\mathbb{C})_0,$$

for $P \in \mathbb{C}[x_0, \ldots, x_{n+1}]_{d(q+1)-n-2}$. Note that, since Z is a projective variety of positive dimension, it intersects every divisor of X. This means that it is impossible to find an affine chart of X containing Z, and hence, the computation of periods of Z by calculus methods would be impossible. Our strategy to compute the periods is to reduce the computation to a period of some projective space $\mathbb{P}^N$.

As mentioned before, the content of this chapter was mainly taken from Villaflor (n.d.[a]). Nevertheless we alert the reader that if one compares those results with the corresponding ones in this book, one encounters several sign discrepancies. These differences are due to our choice of the differential $D = \delta + (-1)^q d$ in algebraic de Rham cohomology (in Villaflor (ibid.) the chosen differential in algebraic de Rham cohomology is $D' = d + (-1)^p \delta$).

8.2 Periods of top forms

In Section 5.10 we computed the period of the standard top form relative to the standard covering of $\mathbb{P}^N$. Using Corollary 5.3 we can compute the periods of top forms in $\mathbb{P}^N$ described with other open coverings (not just the standard one), for instance the Jacobian covering associated to a smooth hypersurface.

Proposition 8.1. *Let $f_0, \dots, f_N \in \mathbb{C}[x_0, \dots, x_N]_d$ homogeneous polynomials of the same degree $d > 0$, such that*

$$\{f_0 = \cdots = f_N = 0\} = \varnothing.$$

They define the finite morphism $F : \mathbb{P}^N \to \mathbb{P}^N$ given by

$$F(x_0 : \cdots : x_N) := (f_0 : \cdots : f_N).$$

Let $\mathcal{U}_F = \{V_i\}_{i=0}^N$ be the open covering associated, i.e. $V_i = \{f_i \neq 0\}$. Then the top form

$$\frac{\Omega_F}{f_0 \cdots f_N} := \frac{\sum_{i=0}^N (-1)^i f_i \widehat{df_i}}{f_0 \cdots f_N} \in H^N(\mathcal{U}_F, \Omega^N_{\mathbb{P}^N}),$$

has period

$$\int_{\mathbb{P}^N} \frac{\Omega_F}{f_0 \cdots f_N} = d^N \cdot (-1)^{\binom{N}{2}} (2\pi\sqrt{-1})^N.$$

Proof. If Ω is the standard top form associated to the standard covering, applying Corollary 5.3 we get $F^{-1}(\mathcal{U}) = \mathcal{U}_F$ and $F^*\Omega = \Omega_F$. Then it follows from topological degree theory that

$$\int_{\mathbb{P}^N} \frac{\Omega_F}{f_0 \cdots f_N} = \deg(F) \cdot \int_{\mathbb{P}^N} \frac{\Omega}{x_0 \cdots x_N} = \deg(F) \cdot (-1)^{\binom{N}{2}} (2\pi\sqrt{-1})^N.$$

Since F is defined by a base point free linear system, the fiber of F is generically reduced by Bertini's theorem (see Hartshorne (1977, p. 179)), and corresponds to d^N points by Bézout's theorem, i.e. $\deg(F) = d^N$. □

Before going further we will recall the following theorem due to Macaulay (for a proof see Voisin (2003, Theorem 6.19)):

Theorem 8.1 (Macaulay (1916))**.** *Given $f_0, \dots, f_N \in \mathbb{C}[x_0, \dots, x_N]$ homogeneous polynomials with* $\deg(f_i) = d_i$ *and*

$$\{f_0 = \cdots = f_N = 0\} = \varnothing.$$

Letting

$$R := \frac{\mathbb{C}[x_0, \dots, x_N]}{\langle f_0, \dots, f_N \rangle},$$

then for $\sigma := \sum_{i=0}^N (d_i - 1)$, we have that

(i) *For every* $0 \leqslant i \leqslant \sigma$ *the product* $R_i \times R_{\sigma-i} \to R_\sigma$ *is a perfect pairing.*

(ii) $\dim_{\mathbb{C}} R_\sigma = 1$.

(iii) $R_e = 0$ *for* $e > \sigma$.

Definition 8.1. A ring of the form $R = \mathbb{C}[x_0, \ldots, x_N]/I$ for some homogeneous ideal I is called an **Artinian Gorenstein algebra of socle degree** σ, if there exist $\sigma \in \mathbb{N}$ such that R satisfies properties (i), (ii) and (iii) of Macaulay's Theorem 8.1. We also say that I is an **Artinian Gorenstein ideal of socle** σ.

Remark 8.1. It is easy to see (using Euler's identity) that

$$\Omega_F = d^{-1} \det(\mathrm{Jac}(F))\Omega$$

where $\mathrm{Jac}(F) = \left[\frac{\partial f_i}{\partial x_j}\right]_{0 \leqslant i,j \leqslant N}$ is the Jacobian matrix of F. Any element of $H^N(\mathcal{U}_F, \Omega^N_{\mathbb{P}^N})$ is of the form

$$\omega = \frac{P\Omega}{f_0^{\alpha_0} \cdots f_N^{\alpha_N}}$$

where $\alpha_0, \ldots, \alpha_N \in \mathbb{Z}_{>0}$ with $d \cdot (\alpha_0 + \cdots + \alpha_N) = \deg(P) + N + 1$. Using Macaulay's Theorem 8.1, we see that

$$P = \sum_{d(\beta_0 + \cdots + \beta_N) = \deg(P) - d(N+1)} f_0^{\beta_0} \cdots f_N^{\beta_N} P_\beta$$

with $\deg(P_\beta) = (d-1)(N+1) = \sigma$. This reduces the problem of computation of periods, to forms of the form

$$\frac{P_\beta \Omega}{f_0^{\alpha_0} \cdots f_N^{\alpha_N}}, \tag{8.1}$$

with $\alpha_0, \ldots, \alpha_N \in \mathbb{Z}$ such that $\alpha_0 + \cdots + \alpha_N = N + 1$ and $\deg(P_\beta) = (d-1)(N+1)$. It is clear that such a form represents an exact top form of $\mathbb{P}^N$ if some α_i is non-positive (in fact, it is equivalent to show that it is zero in hypercohomology and this is clear because the form extends to a d-closed form on $V_{0\cdots\hat{i}\cdots N}$, as in the proof of Proposition 5.9) then we reduce the computation to forms of the form

$$\frac{Q\Omega}{f_0 \cdots f_N},$$

where $\deg(Q) = \sigma$.

Corollary 8.1. *If $Q \in \mathbb{C}[x_0, \ldots, x_N]_\sigma$, then*

$$\int_{\mathbb{P}^N} \frac{Q\Omega}{f_0 \cdots f_N} = c \cdot d^{N+1} \cdot (-1)^{\binom{N}{2}} (2\pi\sqrt{-1})^N,$$

where $c \in \mathbb{C}$ is the unique number such that

$$Q \equiv c \cdot \det(\mathrm{Jac}(F)) \ (mod\ \langle f_0, \ldots, f_N \rangle).$$

Proof. To show the existence and uniqueness of $c \in \mathbb{C}$ we use item (ii) of Macaulay's Theorem 8.1. So, it is enough to show

$$\det(\mathrm{Jac}(F)) \notin \langle f_0, \ldots, f_N \rangle.$$

This is direct from the previous considerations and the fact

$$\frac{\Omega_F}{f_0 \cdots f_N} \in H^N(\mathbb{P}^N, \Omega^N_{\mathbb{P}^N})$$

does not represent an exact top form by Proposition 8.1. □

Remark 8.2. In summary, the computation of the period reduces to the computation of such constant c that relates Q with $\det(\mathrm{Jac}(F))$ in R_σ.

8.3 Coboundary map

In order to compute periods of complete intersection algebraic cycles, we need to compute periods of smooth hyperplane sections Y of a given projective smooth variety X. In other words, for $Y \hookrightarrow X$ a smooth hypersurface given by $\{F = 0\}$, we need an explicit description of the isomorphism

$$\begin{array}{ccc} H^n(Y, \Omega^n_Y) & \overset{\tau}{\simeq} & H^{n+1}(X, \Omega^{n+1}_X), \\ \omega & \mapsto & \widetilde{\omega} \end{array} \tag{8.2}$$

together with the relation of periods, that is, the number $a \in \mathbb{C}$ such that

$$\int_X \widetilde{\omega} = a \int_Y \omega.$$

For this purpose recall the exact sequence (6.3)

$$\cdots \to H^{k+1}_{\mathrm{dR}}(X/\mathbb{C}) \to H^{k+1}_{\mathrm{dR}}(U/\mathbb{C}) \xrightarrow{\mathrm{res}} H^k_{\mathrm{dR}}(Y/\mathbb{C}) \xrightarrow{\tau} H^{k+2}_{\mathrm{dR}}(X/\mathbb{C}) \to \cdots,$$

induced by the short exact sequence of complexes

$$0 \to \Omega_X^\bullet \to \Omega_X^\bullet(\log Y) \xrightarrow{\text{res}} j_*\Omega_Y^{\bullet-1} \to 0.$$

Since $H^{2n+1}_{\text{dR}}(U) = H^{2n+2}_{\text{dR}}(U) = 0$, the coboundary map is an isomorphism

$$H^{2n}_{\text{dR}}(Y/\mathbb{C}) \overset{\tau}{\simeq} H^{2n+2}_{\text{dR}}(X/\mathbb{C}).$$

These vector spaces are one dimensional, and τ preserves the Hodge filtration. Therefore, it induces the desired isomorphism in (8.2).

Proposition 8.2. *Let $X \subseteq \mathbb{P}^N$ be an smooth complete intersection of dimension $n+1$, and $Y \subseteq X$ be a smooth hyperplane section given by $\{F = 0\} \cap X$, for some homogeneous $F \in \mathbb{C}[x_0, \dots, x_N]_d$. Let $\omega \in C^n(X, \Omega_X^n)$ such that $\omega|_Y \in \ker(C^n(Y, \Omega_Y^n) \xrightarrow{\delta} C^{n+1}(Y, \Omega_Y^{n+1}))$. For any $\overline{\omega} \in C^n(X, \Omega_X^{n+1}(\log Y))$ such that*

$$\overline{\omega} \equiv \omega \wedge \frac{dF}{F} \ (\textit{mod}\ C^n(X, \Omega_X^{n+1})),$$

we have

$$\tau(\omega) = \widetilde{\omega} := \delta(\overline{\omega}) \in \ker(C^{n+1}(X, \Omega_X^{n+1}) \xrightarrow{\delta} C^{n+2}(X, \Omega_X^{n+1})).$$

Furthermore, $\widetilde{\omega} \in H^{n+1}(X, \Omega_X^{n+1})$ is uniquely determined by $\omega|_Y \in H^n(Y, \Omega_Y^n)$ and

$$\int_X \widetilde{\omega} = -2\pi\sqrt{-1} \int_Y \omega.$$

Remark 8.3. Since any hypersurface section $\{F = 0\} \cap X$ as above is a hyperplane section after using a Veronese embedding (see for instance Movasati (2021, Section 5.2)) we will only use the latter.

Proof. The map described in the proposition is the coboundary map τ, i.e. $\tau(\omega) = \widetilde{\omega}$. Therefore, it is left to prove the period relation. By the fact that τ is an isomorphism of one dimensional spaces we have a constant $a_{X,Y} \in \mathbb{C}^\times$ such that

$$\int_X \tau(\omega) = a_{X,Y} \int_Y \omega,$$

for every $\omega \in H^n(Y, \Omega_Y^n)$. Since X is a complete intersection, Lefschetz hyperplane section theorem (see for instance Movasati (ibid., Chapter 4)) implies

$[Y] = d \cdot [X \cap \mathbb{P}^{N-1}]$ for some general hyperplane $\mathbb{P}^{N-1} \subseteq \mathbb{P}^N$, and so we can assume $F = x_N^d$. Let us evaluate the above period relation at $\omega = \theta^n$. By (5.14) we have for the standard open covering of $\mathbb{P}^N$ that

$$\begin{aligned}\theta^n_{i_0\cdots i_n} &= (-1)^{\binom{n+1}{2}} \left(\frac{dx_{i_0}}{x_{i_0}} - \frac{dx_{i_1}}{x_{i_1}}\right) \wedge \left(\frac{dx_{i_1}}{x_{i_1}} - \frac{dx_{i_2}}{x_{i_2}}\right) \wedge \cdots \wedge \left(\frac{dx_{i_{n-1}}}{x_{i_{n-1}}} - \frac{dx_{i_n}}{x_{i_n}}\right) \\ &= (-1)^{\binom{n}{2}} \left(\frac{dx_{i_1}}{x_{i_1}} - \frac{dx_{i_0}}{x_{i_0}}\right) \wedge \left(\frac{dx_{i_2}}{x_{i_2}} - \frac{dx_{i_0}}{x_{i_0}}\right) \wedge \cdots \wedge \left(\frac{dx_{i_n}}{x_{i_n}} - \frac{dx_{i_0}}{x_{i_0}}\right).\end{aligned}$$

Thus

$$\begin{aligned}&\overline{\omega}_{i_0\cdots i_n} = \\ &d\cdot(-1)^{\binom{n}{2}} \left(\frac{dx_{i_1}}{x_{i_1}} - \frac{dx_{i_0}}{x_{i_0}}\right)\wedge\left(\frac{dx_{i_2}}{x_{i_2}} - \frac{dx_{i_0}}{x_{i_0}}\right)\wedge\cdots\wedge\left(\frac{dx_{i_n}}{x_{i_n}} - \frac{dx_{i_0}}{x_{i_0}}\right)\wedge\left(\frac{dx_N}{x_N} - \frac{dx_{i_0}}{x_{i_0}}\right),\end{aligned}$$

hence

$$\begin{aligned}&\tau(\theta^n)_{i_0\cdots i_{n+1}} = \\ &= d \cdot (-1)^{\binom{n-1}{2}} \left(\frac{dx_{i_1}}{x_{i_1}} - \frac{dx_{i_0}}{x_{i_0}}\right) \wedge \left(\frac{dx_{i_2}}{x_{i_2}} - \frac{dx_{i_0}}{x_{i_0}}\right) \wedge \cdots \wedge \left(\frac{dx_{i_{n+1}}}{x_{i_{n+1}}} - \frac{dx_{i_0}}{x_{i_0}}\right) = \\ &\qquad = -d \cdot \theta^{n+1}_{i_0\cdots i_{n+1}}.\end{aligned}$$

By Proposition 5.4 it follows that

$$\begin{aligned}-d \cdot \deg(X) \left(2\pi\sqrt{-1}\right)^{n+1} &= \int_X \tau(\theta^n) = \\ &= a_{X,Y} \int_Y \theta^n = a_{X,Y} \cdot \deg(Y) \left(2\pi\sqrt{-1}\right)^n\end{aligned}$$

and so $a_{X,Y} = -2\pi\sqrt{-1}$. □

8.4 Periods of complete intersection algebraic cycles

In this section we compute periods of complete intersection algebraic cycles inside a smooth hypersurface $X \subseteq \mathbb{P}^{n+1}$, of even dimension n. After Carlson–Griffiths' theorem we know that the integrands in these periods are of the form

$$\omega_P = \operatorname{res}\left(\frac{P\Omega}{F^{\frac{n}{2}+1}}\right)^{\frac{n}{2},\frac{n}{2}} = \frac{1}{\frac{n}{2}!}\left\{\frac{P\Omega_J}{F_J}\right\}_{|J|=\frac{n}{2}} \in H^{\frac{n}{2}}(X, \Omega_X^{\frac{n}{2}}), \qquad (8.3)$$

where $P \in \mathbb{C}[x_0, \ldots, x_{n+1}]_\sigma$, and $\sigma = (d-2)(\frac{n}{2}+1)$. In order to compute these periods over a complete intersection subvariety Z of $\mathbb{P}^{n+1}$ (contained in X), the main ingredient is the explicit description of the coboundary map. For a complete intersection $Z \subseteq X$ of dimension $\frac{n}{2}$, we construct a chain of subvarieties

$$Z = Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \cdots \subseteq Z_{\frac{n}{2}+1} = \mathbb{P}^{n+1},$$

where each Z_i is a hypersurface section of Z_{i+1}, and apply inductively the coboundary map, to reduce the computation of the period of Z to the computation of the integral of a top form in $\mathbb{P}^{n+1}$. Recall that for functions $F : \mathbb{C}^{n+1} \to \mathbb{C}$ and $H = (H_0, H_1, \ldots, H_{n+1}) : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ the Hessian and Jacobian matrices are defined in the following way:

$$\mathrm{Hess}(F) := \left[\frac{\partial^2 F}{\partial x_i \partial x_j}\right]_{(n+1)\times(n+1)}, \quad \mathrm{Jac}(H) := \left[\frac{\partial H_i}{\partial x_j}\right]_{(n+1)\times(n+1)} \tag{8.4}$$

Theorem 8.2 (Villaflor (n.d.[a])). *Let $X \subseteq \mathbb{P}^{n+1}$ be a smooth hypersurface given by $X = \{F = 0\}$. Suppose*

$$F = f_1 g_1 + \cdots + f_{\frac{n}{2}+1} g_{\frac{n}{2}+1},$$

such that $Z := \{f_1 = \cdots = f_{\frac{n}{2}+1} = 0\} \subseteq X$ is a complete intersection (i.e. $\dim(Z) = \frac{n}{2}$*). Define*

$$H = (h_0, \ldots, h_{n+1}) := (f_1, g_1, f_2, g_2, \ldots, f_{\frac{n}{2}+1}, g_{\frac{n}{2}+1}).$$

Then

$$\int_Z \omega_P = \frac{(-2\pi\sqrt{-1})^{\frac{n}{2}}}{\frac{n}{2}!} c \cdot (d-1)^{n+2}, \tag{8.5}$$

where ω_P is given by (8.3)*, and $c \in \mathbb{C}$ is the unique number such that*

$$P \cdot \det(\mathrm{Jac}(H)) \equiv c \cdot \det(\mathrm{Hess}(F)) \ (\mathit{mod}\ J^F).$$

Remark 8.4. To understand the statement of Theorem 8.2 recall that Macaulay's Theorem 8.1 implies that the Jacobian ring

$$R^F = \frac{\mathbb{C}[x_0, \ldots, x_{n+1}]}{J^F},$$

where $J^F = \langle F_0, \ldots, F_{n+1}\rangle$ is the Jacobian ideal, is an Artinian Gorenstein algebra of socle 2σ. In particular $\dim_{\mathbb{C}} R^F_{2\sigma} = 1$. Furthermore, by Proposition 8.1 and Remark 8.1, $R^F_{2\sigma}$ is generated by $\det(\mathrm{Hess}(F))$. In consequence, for any pair of polynomials $P, Q \in \mathbb{C}[x_0, \ldots, x_{n+1}]_\sigma$ there exists a unique $c \in \mathbb{C}$ such that

$$P \cdot Q \equiv c \cdot \det(\mathrm{Hess}(F)) \pmod{J^F}.$$

Definition 8.2. We say that a algebraic cycle $\delta \in H_n(X, \mathbb{Z})$ is of **complete intersection type** if

$$\delta = \sum_{i=1}^{k} n_i [Z_i],$$

for $Z_1, \ldots, Z_k \subseteq X$ a set of $\frac{n}{2}$-dimensional subvarieties that are complete intersection inside $\mathbb{P}^{n+1}$, given by

$$Z_i = \{f_{i,1} = \cdots = f_{i,\frac{n}{2}+1} = 0\}, \quad i = 1, \ldots, k,$$

such that there exist $g_{i,1}, \ldots, g_{i,k} \in \mathbb{C}[x_0, \ldots, x_{n+1}]$ with

$$F = \sum_{j=1}^{\frac{n}{2}+1} f_{i,j} g_{i,j}.$$

For such an algebraic cycle, we define its **associated polynomial**

$$P_\delta := \sum_{i=1}^{k} n_i \cdot \det(\mathrm{Jac}(H_i)) \in R^F_\sigma,$$

where $\sigma := (d-1)(\frac{n}{2}+1)$ and $H_i := (f_{i,1}, g_{i,1}, \ldots, f_{i,\frac{n}{2}+1}, g_{i,\frac{n}{2}+1})$.

Remark 8.5. Theorem 8.2 tells us that in order to compute the periods of a complete intersection type cycle δ it is enough to know its associated polynomial P_δ. In fact, we are determining the Poincaré dual of the cycle δ in primitive cohomology

$$\delta_0^{\mathsf{pd}} = \mathrm{res}\left(\frac{P_\delta \Omega}{F^{\frac{n}{2}+1}}\right)^{\frac{n}{2},\frac{n}{2}} \in H^{\frac{n}{2},\frac{n}{2}}(X)_0,$$

that is, it satisfies (up to some constant non-zero factor)

$$\int_{\delta} \omega = \int_{X} \omega \cup \operatorname{res}\left(\frac{P_{\delta}\Omega}{F^{\frac{n}{2}+1}}\right), \qquad \forall \omega \in H^{n}_{\mathrm{dR}}(X)_0.$$

Let $\omega = \omega_P$ be as in (8.3). In order to prove Theorem 8.2, we will use Proposition 8.2 to construct inductively

$$\begin{aligned}
\omega^{(0)} &:= \omega|_Z \in H^{\frac{n}{2}}(Z, \Omega_Z^{\frac{n}{2}}),\\
\omega^{(l)} &:= \widetilde{\omega^{(l-1)}} \in H^{\frac{n}{2}+l}(Z_l, \Omega_{Z_l}^{\frac{n}{2}+l}),\ l = 1, \dots, \frac{n}{2}+1
\end{aligned}$$

with $Z_l := \{f_{l+1} = \cdots = f_{\frac{n}{2}+1} = 0\} \subseteq \mathbb{P}^{n+1}$ and $Z_0 = Z$ and $Z_{\frac{n}{2}+1} = \mathbb{P}^{n+1}$.

Proposition 8.3. *Both sides of the periods equation* (8.5) *depend continuously on the parameters*

$$(f_1, g_1, \dots, f_{\frac{n}{2}+1}, g_{\frac{n}{2}+1}) \in \bigoplus_{i=1}^{\frac{n}{2}+1} \mathbb{C}[x_0, \dots, x_{n+1}]_{d_i} \oplus \mathbb{C}[x_0, \dots, x_{n+1}]_{d-d_i},$$

such that $F := f_1 g_1 + \cdots + f_{\frac{n}{2}+1} g_{\frac{n}{2}+1}$.

Proof. Consider

$$U := \left\{ (f_1, g_1, \dots, f_{\frac{n}{2}+1}, g_{\frac{n}{2}+1}) \in \bigoplus_{i=1}^{\frac{n}{2}+1} \mathbb{C}[x]_{d_i} \oplus \mathbb{C}[x]_{d-d_i} : \right.$$
$$X := \{f_1 g_1 + \cdots + f_{\frac{n}{2}+1} g_{\frac{n}{2}+1} = 0\} \text{ is smooth and}$$
$$\left. Z := \{f_1 = f_2 = \cdots = f_{\frac{n}{2}+1} = 0\} \text{ is a complete intersection} \right\}.$$

Let $\sigma := (d-2)(\frac{n}{2}+1)$ and fix any $P \in \mathbb{C}[x]_\sigma$. For $(f_1, g_1, \dots, f_{\frac{n}{2}+1}, g_{\frac{n}{2}+1}) \in U$, we know that the Jacobian ideal $J^F := \langle \frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_{n+1}} \rangle$ (where $F := f_1 g_1 + \cdots + f_{\frac{n}{2}+1} g_{\frac{n}{2}+1} \in \mathbb{C}[x]_d$) is Artinian Gorenstein of $soc(J^F) = 2\sigma$, and that $\det(\mathrm{Hess}(F)) \in \mathbb{C}[x]_{2\sigma} \setminus J^F_{2\sigma}$ (by Corollary 8.1). Therefore there exists a unique number $c \in \mathbb{C}$ such that

$$P \cdot \det(\mathrm{Jac}(f_1, g_1, \dots, f_{\frac{n}{2}+1}, g_{\frac{n}{2}+1})) \equiv c \cdot \det(\mathrm{Hess}(F)) \pmod{J^F}.$$

We claim that this number c depends continuously on

$$\lambda := (f_1, g_1, \dots, f_{\frac{n}{2}+1}, g_{\frac{n}{2}+1}) \in U.$$

In fact, consider the $\mathbb{C}$-vector space $V := \mathbb{C}[x]_{2\sigma}$. For every $\lambda \in U$ define the hyperplane $V_\lambda := J^F_{2\sigma} \subseteq V$, we claim that V_λ varies continuously with respect to λ in the space of hyperplanes of V, in fact, each V_λ is generated as $\mathbb{C}$-vector space by the vectors

$$V_\lambda = \left\langle \frac{\partial F_\lambda}{\partial x_i} x^I : i = 0, \dots, n+1, x^I \text{ monomials with } |I| = 2\sigma - d + 1 \right\rangle,$$

where $F_\lambda := F = f_1 g_1 + \cdots + f_{\frac{n}{2}+1} g_{\frac{n}{2}+1}$, and each of these vectors depend continuously on $\lambda \in U$ (here we are using the non-trivial fact that we know a priori that the generated spaces are hyperplanes). In consequence, there exists a continuous map

$$\varphi : U \to \mathbb{P}(V^*)$$

such that $V_\lambda = \operatorname{Ker} \varphi_\lambda$. Now we can compute c in terms of continuous functions depending on $\lambda \in U$ as

$$c = \frac{\varphi_\lambda(P \cdot \det(\operatorname{Jac}(\lambda)))}{\varphi_\lambda(\det(\operatorname{Hess}(F_\lambda)))}.$$

□

Remark 8.6. In order to use the coboundary map, we will assume that every Z_l is a smooth hyperplane section of Z_{l-1}. We can reduce ourselves to this situation by noticing this will hold for a general pair (Z, X) satisfying the hypothesis of the theorem, then by continuity the result will extend to every such pair.

Lemma 8.1. *For each* $l = 0, \dots, \frac{n}{2}+1$ *and* $J = \{j_0, \dots, j_{\frac{n}{2}+l}\} \subseteq \{0, \dots, n+1\}$

let $K = \{k_0, \ldots, k_{\frac{n}{2}-l}\} = \{0, 1, \ldots, n+1\} \setminus J$. *We have*

$$(\omega^{(l)})_J = \frac{(-1)^{\binom{\frac{n}{2}+2}{2}+j_0+\cdots+j_{\frac{n}{2}}+l} P \cdot d^l \cdot d_1 \cdot d_2 \cdots d_l}{\frac{n}{2}! \cdot F_J} \cdot$$

$$\left[\sum_{m=1}^{l} (-1)^{m-1} g_m \frac{\widehat{dg_m}}{d} \bigwedge_{r=0}^{\frac{n}{2}-l} dx_{k_r} \bigwedge_{t=1}^{l} \frac{df_t}{d_t} + \right.$$

$$+(-1)^l \sum_{p=0}^{\frac{n}{2}-l} (-1)^p x_{k_p} \bigwedge_{s=1}^{l} \frac{dg_s}{d} \wedge \widehat{dx_{k_p}} \wedge \bigwedge_{t=1}^{l} \frac{df_t}{d_t}$$

$$\left. +(-1)^{\frac{n}{2}+l} \sum_{q=1}^{l} \frac{\widehat{dg_q}}{d} \wedge \frac{dF}{d} \bigwedge_{r=0}^{\frac{n}{2}-l} dx_{k_r} \wedge \frac{\widehat{df_q}}{d_q} \right],$$

Proof. We proceed by induction on l:

<u>$l = 0$</u>: We have

$$\Omega_J := \iota_{\frac{\partial}{\partial x_{j_{\frac{n}{2}}}}} (\cdots \iota_{\frac{\partial}{\partial x_{j_0}}} (\Omega) \cdots) = (-1)^{j_0+\cdots+j_{\frac{n}{2}}+\binom{\frac{n}{2}+2}{2}} \sum_{l=0}^{\frac{n}{2}} (-1)^l x_{k_l} \widehat{dx_{k_l}}, \tag{8.6}$$

This gives us

$$(\omega^{(0)})_{j_0 \cdots j_{\frac{n}{2}}} = (\omega)_{j_0 \cdots j_{\frac{n}{2}}} = \frac{(-1)^{\binom{\frac{n}{2}+2}{2}+j_0+\cdots+j_{\frac{n}{2}}} P}{\frac{n}{2}! \cdot F_J} \left[\sum_{p=0}^{\frac{n}{2}} (-1)^p x_{k_p} \widehat{dx_{k_p}}\right].$$

$\underline{l \Rightarrow l+1}$:

$$
\begin{aligned}
(\omega^{(l)})_J \wedge \frac{df_{l+1}}{f_{l+1}} \quad &\equiv \frac{(-1)^{\binom{\frac{n}{2}+2}{2}+j_0+\cdots+j_{\frac{n}{2}+l}} P d^l d_1 \cdots d_{l+1}}{\frac{n}{2}! \cdot F_J \cdot f_{l+1}} \\
&\cdot \Bigg[\sum_{m=1}^{l} (-1)^{m-1} g_m \frac{d\widehat{g_m}}{d} \bigwedge_{r=0}^{\frac{n}{2}-l} dx_{k_r} \bigwedge_{t=1}^{l+1} \frac{df_t}{d_t} \\
&+(-1)^l \sum_{p=0}^{\frac{n}{2}-l} (-1)^p x_{k_p} \bigwedge_{s=1}^{l} \frac{dg_s}{d} \wedge d\widehat{x_{k_p}} \wedge \bigwedge_{t=1}^{l+1} \frac{df_t}{d_t} \\
&+(-1)^{\frac{n}{2}+l} \sum_{q=1}^{l} \frac{d\widehat{g_q}}{d} \wedge \frac{dF}{d} \bigwedge_{r=0}^{\frac{n}{2}-l} dx_{k_r} \wedge \frac{d\widehat{f_q}}{d_q} \wedge \frac{df_{l+1}}{d_{l+1}} \\
&+(-1)^{\frac{n}{2}+l+1} f_{l+1} \sum_{u=1}^{l+1} \frac{d\widehat{g_u}}{d_u} \bigwedge_{r=0}^{\frac{n}{2}-l} dx_{k_r} \wedge \frac{d\widehat{f_u}}{d_u} \Bigg].
\end{aligned}
$$

Applying δ of the Čech cohomology we get

$$
\begin{aligned}
\omega_J^{(l+1)} \quad &= \frac{(-1)^{\binom{\frac{n}{2}+2}{2}+j_0+\cdots+j_{\frac{n}{2}+l+1}} d^l d_1 \cdots d_{l+1}}{\frac{n}{2}! \cdot F_J \cdot f_{l+1}} \\
&\cdot \Bigg[\sum_{m=1}^{l} (-1)^{m-1} g_m \frac{d\widehat{g_m}}{d} \wedge \left(\sum_{p=0}^{\frac{n}{2}+l+1} F_{j_p} dx_{j_p} \right) \bigwedge_{r=0}^{\frac{n}{2}-l-1} dx_{k_r} \bigwedge_{t=1}^{l+1} \frac{df_t}{d_t} \\
&+(-1)^l \left(\sum_{p=0}^{\frac{n}{2}+l+1} F_{j_p} x_{j_p} \right) \bigwedge_{s=1}^{l} \frac{dg_s}{d} \bigwedge_{q=0}^{\frac{n}{2}-l-1} dx_{k_q} \bigwedge_{t=1}^{l+1} \frac{df_t}{d_t} \\
&+(-1)^{l+1} \sum_{p=0}^{\frac{n}{2}-l-1} (-1)^p x_{k_p} \bigwedge_{s=1}^{l} \frac{dg_s}{d} \wedge \left(\sum_{r=0}^{\frac{n}{2}+l+1} F_{j_r} dx_{j_r} \right) \wedge d\widehat{x_{k_p}} \bigwedge_{t=1}^{l+1} \frac{df_t}{d_t} \\
&+(-1)^{\frac{n}{2}+l} \sum_{q=1}^{l} \frac{d\widehat{g_q}}{d} \wedge \frac{dF}{d} \wedge \left(\sum_{p=0}^{\frac{n}{2}+l+1} F_{j_p} dx_{j_p} \right) \bigwedge_{r=0}^{\frac{n}{2}-l-1} dx_{k_r} \wedge \frac{d\widehat{f_q}}{d_q} \wedge \frac{df_{l+1}}{d_{l+1}} \\
&+(-1)^{\frac{n}{2}+l+1} f_{l+1} \sum_{u=1}^{l+1} \frac{d\widehat{g_u}}{d_u} \wedge \left(\sum_{p=0}^{\frac{n}{2}+l+1} F_{j_p} dx_{j_p} \right) \bigwedge_{r=0}^{\frac{n}{2}-l} dx_{k_r} \wedge \frac{d\widehat{f_u}}{d_u} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(-1)^{\binom{\frac{n}{2}+2}{2}+j_0+\cdots+j_{\frac{n}{2}+l+1}} d^l d_1 \cdots d_{l+1}}{\frac{n}{2}! \cdot F_J \cdot f_{l+1}} \Bigg[\sum_{m=1}^{l} (-1)^{m-1} g_m \frac{d\widehat{g_m}}{d} \wedge \frac{dF}{d} \bigwedge_{r=0}^{\frac{n}{2}-l-1} dx_{k_r} \bigwedge_{t=1}^{l+1} \frac{df_t}{d_t} \\
&+(-1)^l F \bigwedge_{s=1}^{l} \frac{dg_s}{d} \bigwedge_{q=0}^{\frac{n}{2}-l-1} dx_{k_q} \bigwedge_{t=1}^{l+1} \frac{df_t}{d_t} + (-1)^{l+1} \sum_{p=0}^{\frac{n}{2}-l-1} (-1)^p x_{k_p} \bigwedge_{s=1}^{l} \frac{dg_s}{d} \wedge \frac{dF}{d} \wedge \widehat{dx_{k_p}} \bigwedge_{t=1}^{l+1} \frac{df_t}{d_t} \\
&+(-1)^{\frac{n}{2}+l} \sum_{q=1}^{l} \frac{d\widehat{g_q}}{d} \wedge \frac{dF}{d} \wedge \frac{dF}{d} \bigwedge_{r=0}^{\frac{n}{2}-l-1} dx_{k_r} \wedge \frac{d\widehat{f_q}}{d_q} \wedge \frac{df_{l+1}}{d_{l+1}} \\
&+(-1)^{\frac{n}{2}+l+1} f_{l+1} \sum_{u=1}^{l+1} \frac{d\widehat{g_u}}{d_u} \wedge \frac{dF}{d} \bigwedge_{r=0}^{\frac{n}{2}-l} dx_{k_r} \wedge \frac{d\widehat{f_u}}{d_u} \Bigg].
\end{aligned}
$$

Replacing $F = f_1 g_1 + \cdots + f_{\frac{n}{2}+1} g_{\frac{n}{2}+1}$ in the first three expressions we finish the induction. □

Proof of Theorem 8.2. Using Lemma 8.1 for $l = \frac{n}{2} + 1$ we get

$$
\begin{aligned}
(\omega^{(\frac{n}{2}+1)})_{0\cdots n+1} &= \frac{(-1)^{\binom{\frac{n}{2}+1}{2}} P d^{\frac{n}{2}+1} d_1 \cdots d_{\frac{n}{2}+1}}{\frac{n}{2}! \cdot F_0 \cdots F_{n+1}} \Bigg[\sum_{m=1}^{\frac{n}{2}+1} (-1)^{m-1} g_m \frac{d\widehat{g_m}}{d} \bigwedge_{t=1}^{\frac{n}{2}+1} \frac{df_t}{d_t} \\
&\quad +(-1)^{n+1} \sum_{q=1}^{\frac{n}{2}+1} \frac{d\widehat{g_q}}{d} \wedge \frac{dF}{d} \wedge \frac{d\widehat{f_q}}{d_q} \Bigg].
\end{aligned}
$$

Replacing $F = f_1 g_1 + \cdots + f_{\frac{n}{2}+1} g_{\frac{n}{2}+1}$ we obtain

$$
\begin{aligned}
(\omega^{(\frac{n}{2}+1)})_{0\cdots n+1} &= \frac{(-1)^{\binom{\frac{n}{2}+1}{2}} P d^{\frac{n}{2}+1} d_1 \cdots d_{\frac{n}{2}+1}}{\frac{n}{2}! \cdot F_0 \cdots F_{n+1}} \Bigg[\sum_{m=1}^{\frac{n}{2}+1} (-1)^{m-1} \left(\frac{d-d_m}{d}\right) g_m \frac{d\widehat{g_m}}{d} \bigwedge_{t=1}^{\frac{n}{2}+1} \frac{df_t}{d_t} \\
&\quad +(-1)^{\frac{n}{2}} \sum_{q=1}^{\frac{n}{2}+1} (-1)^q f_q \bigwedge_{s=1}^{\frac{n}{2}+1} \frac{dg_s}{d} \wedge \frac{d\widehat{f_q}}{d_q} \Bigg],
\end{aligned}
$$

in other words

$$
(\omega^{(\frac{n}{2}+1)})_{0\cdots n+1} = \frac{(-1)^{\frac{n}{2}+1} P e_0 \cdots e_{n+1}}{\frac{n}{2}! \cdot F_0 \cdots F_{n+1}} \sum_{k=0}^{n+1} (-1)^k h_k \frac{d\widehat{h_k}}{e_k},
$$

where $e_k = \deg(h_k)$. Replacing $e_i \cdot h_i = \sum_{j=0}^{n+1} \frac{\partial h_i}{\partial x_j} \cdot x_j$ and $dh_i = \sum_{j=0}^{n+1} \frac{\partial h_i}{\partial x_j} dx_j$ we obtain

$$
(\widetilde{\omega}_i^{(\frac{n}{2}+1)})_{0\cdots n+1} = \frac{(-1)^{\frac{n}{2}+1} P \cdot \det(\mathrm{Jac}(H))}{\frac{n}{2}! \cdot F_0 \cdots F_{n+1}} \sum_{k=0}^{n+1} (-1)^k x_k \widehat{dx_k}.
$$

The theorem follows from Proposition 8.2, Proposition 5.7, and Corollary 8.1. □

8.5 Cohomology class of complete intersection algebraic cycles

In this section we rewrite Theorem 8.2 in the language of cohomology class of algebraic cycles.

Theorem 8.3. *Let $X \subseteq \mathbb{P}^{n+1}$ be a smooth degree d hypersurface of even dimension n given by $X = \{F = 0\}$. Suppose that $Z := \{f_1 = \cdots = f_{\frac{n}{2}+1} = 0\} \subseteq X$ is a complete intersection inside $\mathbb{P}^{n+1}$ and*

$$I(Z) = \langle f_1, \ldots, f_{\frac{n}{2}+1} \rangle \subseteq \mathbb{C}[x_0, \ldots, x_{n+1}].$$

Write

$$F = f_1 g_1 + \cdots + f_{\frac{n}{2}+1} g_{\frac{n}{2}+1},$$

and define

$$H = (h_0, \ldots, h_{n+1}) := (f_1, g_1, \ldots, f_{\frac{n}{2}+1}, g_{\frac{n}{2}+1}).$$

Then

$$[Z] = \frac{\deg(Z)}{\deg(X)} \theta^{\frac{n}{2}} - \frac{\frac{n}{2}!(-1)^{\frac{n}{2}}}{\deg(X)} \operatorname{res} \left(\frac{\det(\operatorname{Jac}(H)) \Omega}{F^{\frac{n}{2}+1}} \right)^{\frac{n}{2},\frac{n}{2}} \in H^{\frac{n}{2},\frac{n}{2}}(X),$$

where

$$\Omega = \sum_{i=0}^{n+1} (-1)^i x_i \, dx_0 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_{n+1}$$

is the generator of $H^0(\mathbb{P}^{n+1}, \Omega^{n+1}_{\mathbb{P}^{n+1}}(n+2))$ and $\theta \in H^{1,1}(X)$ is the polarization.

After Griffiths basis theorem we know that

$$[Z] = (\omega_{P_Z})^{\frac{n}{2},\frac{n}{2}} + \alpha \theta^{\frac{n}{2}} \in H^{\frac{n}{2},\frac{n}{2}}(X) \tag{8.7}$$

for some $\alpha \in \mathbb{C}$ and some $P_Z \in \mathbb{C}[x_0, \ldots, x_{n+1}]_{(d-2)(\frac{n}{2}+1)}$. In order to compute α let us integrate the polarization $\theta^{\frac{n}{2}}$ over Z

$$\deg(Z) = \frac{1}{(2\pi\sqrt{-1})^{\frac{n}{2}}} \int_Z \theta^{\frac{n}{2}} = \frac{1}{(2\pi\sqrt{-1})^n} \int_X \theta^{\frac{n}{2}} \cup \alpha \theta^{\frac{n}{2}} = \alpha \cdot \deg(X),$$

and so $\alpha = \frac{\deg(Z)}{\deg(X)}$. We will need the following fact whose proof was essentially done in the proof of Carlson and Griffiths (1980, Theorem 2).

Proposition 8.4. *Let $X \subseteq \mathbb{P}^{n+1}$ be a smooth degree d hypersurface of dimension n (not necessarily even). For every pair of homogeneous polynomials $P, Q \in \mathbb{C}[x_0, \dots, x_{n+1}]$ with $\deg P = (q+1)d - n - 2$ and $\deg Q = (p+1)d - n - 2$ for $p + q = n$, then*

$$\int_X \omega_P \cup \omega_Q = \frac{(-1)^{\binom{n+1}{2}+q+1}(2\pi\sqrt{-1})^n}{p!q!} c \cdot (d-1)^{n+2} d,$$

where $c \in \mathbb{C}$ is the unique number such that

$$PQ \equiv c \cdot \det(\mathrm{Hess}(F)) \quad (mod\ J^F).$$

Proof. Let $\mathcal{U}$ be the Jacobian covering of $\mathbb{P}^{n+1}$. By (8.3) we know explicitly how $(\omega_P)^{p,p}$ and $(\omega_Q)^{q,q}$ look like in Čech cohomology. Then we can also compute $(\omega_P \cup \omega_Q)^{n,n} \in H^n(\mathcal{U}, \Omega_X^n)$ by performing the twisted product (see Proposition 7.2)

$$((\omega_P)^{p,p} \cup (\omega_Q)^{q,q})_{0\cdots\widehat{m}\cdots n+1} = \begin{cases} \frac{(-1)^{p+m} PQ x_m \Omega_{(q+1)} F_m}{p!q! F_0 \cdots F_{n+1} \cdot F_{q+1}} & \text{if } m \leqslant q \\ \frac{(-1)^{p+m} PQ x_m \Omega_{(q)} F_m}{p!q! F_0 \cdots F_{n+1} \cdot F_q} & \text{if } m > q \end{cases}$$

where $\Omega_{(i)} = \iota_{\frac{\partial}{\partial x_i}}(\Omega)$ for $i = q, q+1$. A direct application of Proposition 8.2 gives us

$$\int_{\mathbb{P}^{n+1}} \widetilde{\omega} = -2\pi\sqrt{-1} \int_X \omega_P \cup \omega_Q,$$

for

$$\widetilde{\omega} = \frac{d \cdot (-1)^q PQ\Omega}{p!q! F_0 \cdots F_{n+1}} \in C^{n+1}(\mathcal{U}, \Omega^{n+1}_{\mathbb{P}^{n+1}}).$$

The result follows from Corollary 8.1. □

Proof of Theorem 8.3. Let

$$R_Z := \frac{(-1)^{\frac{n}{2}+1} \frac{n}{2}!}{\deg(X)} \det(\mathrm{Jac}(H)) \in \mathbb{C}[x_0, \dots, x_{n+1}]_{(d-2)(\frac{n}{2}+1)}$$

we claim that $P_Z = R_Z$ (where P_Z is given by (8.7)). In fact, since the wedge product in $H^n_{\mathrm{dR}}(X)_{prim}$ is not degenerated it is enough to check that

$$\frac{1}{(2\pi\sqrt{-1})^{\frac{n}{2}}}\int_Z \omega_P = \frac{1}{(2\pi\sqrt{-1})^n}\int_X \omega_{P_Z}\cup\omega_P = \frac{1}{(2\pi\sqrt{-1})^n}\int_X \omega_{R_Z}\cup\omega_P,$$

$\forall P \in \mathbb{C}[x_0,\dots,x_{n+1}]_{(d-2)(\frac{n}{2}+1)}$, which follows from Theorem 8.2 and Proposition 8.4. □

Proof of Theorem 7.6. Since $H^n(X,\Omega_X^{\frac{n}{2}}) = \mathbb{C}\cdot\theta^n$ and $\int_X \theta^n \neq 0$, it follows that

$$\omega_P\cup\omega_Q = 0 \iff \int_X \omega_P\cup\omega_Q = 0 \iff c=0 \iff PQ\in J^F.$$

□

8.6 Applications

We denote by $CH^n(X)_{cit}$ the space of algebraic cycles of complete intersection type. We define the degree of $\delta\in CH^n(X)_{cit}$ as its degree as an element of $H_n(\mathbb{P}^{n+1},\mathbb{Z})$, i.e. $\deg(\delta) := \sum_{i=1}^k n_i\cdot\deg(Z_i)$. It follows from Theorem 8.3 and the linearity of the cycle class map that

$$[\delta] = \frac{\deg(\delta)}{\deg(X)}\theta^{\frac{n}{2}} - \frac{\frac{n}{2}!(-1)^{\frac{n}{2}}}{\deg(X)}(\omega_{P_\delta})^{\frac{n}{2},\frac{n}{2}}. \tag{8.8}$$

Corollary 8.2. *Let $X\subseteq\mathbb{P}^{n+1}$ be a smooth hypersurface given by*

$$X=\{F=0\}.$$

If $\delta,\mu\in CH^n(X)_{cit}$ are complete intersection type algebraic cycles, then

(i) $P_\delta\in J^F$ if and only if $[\delta]=\alpha\cdot[X\cap\mathbb{P}^{\frac{n}{2}+1}]$, for $\alpha=\deg(\delta)/\deg(X)$.

(ii) Let $c\in\mathbb{C}$ be the unique number such that $P_\delta\cdot P_\mu\equiv c\cdot\det(\mathrm{Hess}(F))$ (mod J^F), then

$$\delta\cdot\mu = \frac{\deg(\delta)\cdot\deg(\mu)}{\deg(X)} - c\cdot\frac{(\deg(X)-1)^{n+2}}{\deg(X)}. \tag{8.9}$$

Proof. The first part is a direct application of Griffiths basis theorem and (8.8). The second part is a direct application of the fact

$$\delta \cdot \mu = \frac{1}{(2\pi\sqrt{-1})^n} \int_X [\delta] \cup [\mu],$$

together with equation (8.8), Corollary 8.1 and Proposition 8.4. □

Remark 8.7. It follows from (8.9) that for every pair of algebraic cycles $\delta, \mu \in CH^{\frac{n}{2}}(X)$, the unique number $c \in \mathbb{C}$ such that $P_\delta \cdot P_\mu \equiv c \cdot \det(\mathrm{Hess}(F)) \pmod{J^F}$ is in fact a rational number such that

$$c \cdot (d-1)^{n+2} \in \mathbb{Z} \text{ and } c \cdot (d-1)^{n+2} \equiv \delta \cdot \mu \pmod{d}.$$

In general, it is not known how to determine when a given element of Griffiths basis $(\omega_P)^{\frac{n}{2},\frac{n}{2}} \in H^{\frac{n}{2},\frac{n}{2}}(X)$ is an integral or rational class in terms of the polynomial $P \in \mathbb{C}[x_0, \ldots, x_{n+1}]_{(d-2)(\frac{n}{2}+1)}$. Equation (8.9) gives us a (computable) necessary condition: If $(\omega_P)^{\frac{n}{2},\frac{n}{2}} \in H^{\frac{n}{2},\frac{n}{2}}(X) \cap H^n(X,\mathbb{Z})$ then for every complete intersection type algebraic cycle $\delta \in CH^{\frac{n}{2}}(X)_{cit}$

$$P \cdot P_\delta \equiv c \cdot \det(\mathrm{Hess}(F)) \pmod{J^F}, \tag{8.10}$$

for some $c \in \mathbb{Q}$ such that $c \cdot (d-1)^{n+2} \in \mathbb{Z}$. Other necessary condition that follows from Proposition 8.4 is

$$P^2 \equiv c_P \cdot \det(\mathrm{Hess}(F)) \pmod{J^F}, \tag{8.11}$$

for some $c_P \in \mathbb{Q}$ such that $c_P(d-1)^{n+2}d \in (\frac{n}{2}!)^2\mathbb{Z}$.

Remark 8.8. Another observation we can derive from Theorem 8.2 is that each period is of the form $(2\pi\sqrt{-1})^{\frac{n}{2}}$ times a number in a number field k, where k is the smallest number field such that $f_1, g_1, \ldots, f_{\frac{n}{2}+1}, g_{\frac{n}{2}+1} \in k[x_0, \ldots, x_{n+1}]$, i.e. the periods belong to the same field where we can decompose F as $f_1g_1 + \cdots + f_{\frac{n}{2}+1}g_{\frac{n}{2}+1}$. This was already mentioned in Deligne's work about absolute Hodge cycles, see Deligne, Milne, et al. (1982, Proposition 7.1).

Corollary 8.3. *Let*

$$X = \{x_0^d + \cdots + x_{n+1}^d = 0\}$$

be the Fermat variety. For $\alpha_0, \alpha_2, \ldots, \alpha_n \in \{1, 3, \ldots, 2d-1\}$ consider

$$\mathbb{P}_\alpha^{\frac{n}{2}} := \{x_0 - \zeta_{2d}^{\alpha_0}x_1 = \cdots = x_n - \zeta_{2d}^{\alpha_n}x_{n+1} = 0\}, \tag{8.12}$$

and $\delta := \mathbb{P}^{\frac{n}{2}}_{\alpha}$. *Its associated polynomial is*

$$P_\delta = d^{\frac{n}{2}+1}\zeta_{2d}^{\alpha_0+\cdots+\alpha_n}\prod_{j=1}^{\frac{n}{2}+1}\left(\sum_{l=0}^{d-2} x_{2j-2}^{d-2-l}\zeta_{2d}^{\alpha_{2j-2}l}x_{2j-1}^{l}\right). \tag{8.13}$$

In particular

$$\mathbb{P}^{\frac{n}{2}}_{\alpha}\cdot\mathbb{P}^{\frac{n}{2}}_{\beta} = \frac{1-(1-d)^{m+1}}{d}$$

where $m = \dim \mathbb{P}^{\frac{n}{2}}_{\alpha}\cap\mathbb{P}^{\frac{n}{2}}_{\beta}$.

Proof. Computing the Jacobian matrix of H as in Theorem 8.2, we see it is diagonal by 2×2 blocks, and each block has determinant

$$\frac{d(\zeta_{2d}^{\alpha_{2j-2}}x_{2j-2}^{d-1}+x_{2j-1}^{d-1})}{x_{2j-2}-\zeta_{2d}^{\alpha_{2j-2}}x_{2j-1}},$$

and so (8.13) follows. In order to compute the intersection product apply Corollary 8.2, part (ii). We just need to compute $c\in\mathbb{C}$ such that $P_\delta\cdot P_\mu \equiv c\cdot d^{n+2}(d-1)^{n+2}(x_0\cdots x_{n+1})^{d-2} \pmod{\langle x_0^{d-1},\ldots,x_{n+1}^{d-1}\rangle}$, where $\delta = \mathbb{P}^{\frac{n}{2}}_{\alpha}$ and $\mu = \mathbb{P}^{\frac{n}{2}}_{\beta}$. It follows from (8.13) that

$$\begin{aligned} c &= \frac{\zeta_{2d}^{(\alpha_0+\beta_0)+\cdots+(\alpha_n+\beta_n)}}{(d-1)^{n+2}}\prod_{j=1}^{\frac{n}{2}+1}\left(\sum_{l=0}^{d-2}\zeta_{2d}^{\alpha_{2j-2}l+\beta_{2j-2}(d-2-l)}\right) \\ &= \frac{\prod_{j=1}^{\frac{n}{2}+1}\left(\sum_{l=0}^{d-2}\zeta_{2d}^{\alpha_{2j-2}(l+1)+\beta_{2j-2}(d-1-l)}\right)}{(d-1)^{n+2}}. \end{aligned}$$

For every $j = 1,\ldots,\frac{n}{2}+1$

$$\sum_{l=0}^{d-2}\zeta_{2d}^{\alpha_{2j-2}(l+1)+\beta_{2j-2}(d-1-l)} =$$

$$= -\sum_{l=1}^{d-1}\zeta_{2d}^{(\alpha_{2j-2}-\beta_{2j-2})l} = \begin{cases} 1-d & \text{if } \alpha_{2j-2}=\beta_{2j-2}, \\ 1 & \text{if } \alpha_{2j-2}\neq\beta_{2j-2}. \end{cases}$$

Therefore $c(d-1)^{n+2} = (1-d)^{m+1}$ and so by (8.9) the result follows. $\square$

We close this section by computing the periods of linear cycles inside Fermat varieties. Consider the following set

$$I_{(d-2)(\frac{n}{2}+1)}:=$$
$$:=\{(i_0,\dots,i_{n+1})\in\{0,\dots,d-2\}^{n+2}: i_0+\cdots+i_{n+1}=(d-2)(\frac{n}{2}+1)\},$$

we define for every $i\in I_{(d-2)(\frac{n}{2}+1)}$

$$\omega_i:=\operatorname{res}\left(\frac{x^i\Omega}{F^{\frac{n}{2}+1}}\right)=\frac{1}{\frac{n}{2}!}\left\{\frac{x^i\Omega_J}{F_J}\right\}_{|J|=\frac{n}{2}}\in H^{\frac{n}{2}}(X,\Omega_X^{\frac{n}{2}}).$$

We know these forms are a Griffiths basis for $H^{\frac{n}{2},\frac{n}{2}}(X)_{prim}$.

Corollary 8.4 (Movasati and Villaflor (2018)). *Let $X\subseteq\mathbb{P}^{n+1}$ be the degree d even dimensional Fermat variety, let $\mathbb{P}^{\frac{n}{2}}\subseteq X$ as in (8.12) for $\alpha_0=\cdots=\alpha_n=1$, and let $i\in I_{(d-2)(\frac{n}{2}+1)}$. Then*

$$\int_{\mathbb{P}^{\frac{n}{2}}}\omega_i=\begin{cases}\frac{(-2\pi\sqrt{-1})^{\frac{n}{2}}}{d^{\frac{n}{2}+1}\cdot\frac{n}{2}!}\zeta_{2d}^{\frac{n}{2}+1+i_0+i_2+\cdots+i_n} & \text{if } i_{2l-2}+i_{2l-1}=d-2,\ \forall l=1,\dots,\frac{n}{2}+1,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. By Theorem 8.2 we just need to compute $c\in\mathbb{C}$ such that

$$x^iP_\delta\equiv c\cdot d^{n+2}(d-1)^{n+2}(x_0\cdots x_{n+1})^{d-2}\ (\mathrm{mod}\ \langle x_0^{d-1},\dots,x_{n+1}^{d-1}\rangle).$$

By Corollary 8.3

$$x^iP_\delta=d^{\frac{n}{2}+1}\zeta_{2d}^{\frac{n}{2}+1}x^i\prod_{j=1}^{\frac{n}{2}+1}\left(\sum_{l=0}^{d-2}x_{2j-2}^{d-2-l}\zeta_{2d}^{l}x_{2j-1}^{l}\right),$$
$$\equiv c_i\cdot(x_0\cdots x_{n+1})^{d-2}\ (\mathrm{mod}\ \langle x_0^{d-1},\dots,x_{n+1}^{d-1}\rangle).$$

If for every $j=1,\dots,\frac{n}{2}+1$ there exist $l_j\in\{0,\dots,d-2\}$ such that $l_j+i_{2j-1}=d-2$ and $d-2-l_j+i_{2j-2}=d-2$, then $c_i=\zeta_{2d}^{\frac{n}{2}+1+l_1+\cdots+l_{\frac{n}{2}}}$. This condition is equivalent to $l_j=i_{2j-2}$ and $i_{2j-2}+i_{2j-1}=d-2$. Otherwise $c_i=0$, and the result follows. □

9 Gauss–Manin connection

Grâce à l'article de Manin (1964), les équations de Picard–Fuchs redeviennent à la mode, sous le nouveau nom de "connexion de Gauss–Manin" (dû à Grothendieck, je crois)-pourquoi "connexion"?-sans doute parce que "équations différentielles" sonne moins bien, aux oreilles d'un géomètre, que "connexion sur un fibré vectoriel", (F. Pham (1979) page 18).

9.1 Introduction

The Gauss–Manin connection of families of projective varieties in a language closer to Gauss' original works has been fairly explained in Movasati (2021, Chapter 12). This includes a variety of examples such as the Weierstrass family of elliptic curves. In that book one captures the cohomology of a projective variety by looking in a fixed affine chart of it. This was a common procedure in the 19th century's works on abelian and multiple integrals such as Picard and Simart (1897, 1906). Soon after the development of hypercohomology and algebraic de Rham cohomology by A. Grothendieck among many others (see Chapters 3 and 5), the Gauss–Manin connection was reformulated in this new framework by Katz and Oda (1968), see also Deligne (1969). In this chapter we aim to present this formulation. Our computational approach to algebraic de Rham cohomology in Chapter 5 is the ground for the more explicit and computable presentation of Gauss–Manin connection in this chapter. In particular, the reader will see that the so called infinitesimal variation of Hodge structures developed by Griffiths and his coauthors in Carlson, Green, et al. (1983), Griffiths (1983), and Griffiths and Harris (1983) is just a small package derived from Gauss–Manin connection. The computation of Gauss–Manin connection usually produces huge polynomials and the available algorithms work only for families of lower dimensional varieties with few parameters. For more on this topic see Movasati (2011, 2012, 2021).

9.2 De Rham cohomology of families of projective varieties

Let $\pi : \mathsf{X} \to \mathsf{T}$ be a family of smooth projective varieties over k. Recall from Section 4.4 that the sheaf of relative differential forms in X is

$$\Omega^m_{\mathsf{X}/\mathsf{T}} := \frac{\Omega^m_{\mathsf{X}}}{\pi^*\Omega^1_{\mathsf{T}} \wedge \Omega^{m-1}_{\mathsf{X}}},$$

where π^* is the pull-back of differential forms in Ω^1_{T} to X. Later, we will need the sheaf of holomorphic and C^∞ differential forms in X. For these we will use the notations $\Omega^m_{\mathsf{X}^{\mathrm{an}}/\mathsf{T}^{\mathrm{an}}}$ and $\Omega^m_{\mathsf{X}^\infty/\mathsf{T}^\infty}$, respectively.

The differential operator d for differential forms in X induces a differential operator $d : \Omega^m_{\mathsf{X}/\mathsf{T}} \to \Omega^{m+1}_{\mathsf{X}/\mathsf{T}}$ for which we use the same letter d. We denote by $(\Omega^\bullet_{\mathsf{X}/\mathsf{T}}, d)$ the resulting complex.

Definition 9.1. We define the m-th **relative de Rham cohomology** of X/T as the m-th hypercohomology of the complex $(\Omega^\bullet_{\mathsf{X}/\mathsf{T}}, d)$, that is

$$H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}) := \mathbb{H}^m(\Omega^\bullet_{\mathsf{X}/\mathsf{T}}, d). \tag{9.1}$$

This is a sheaf of $\mathcal{O}_{\mathsf{T}}$-modules in a natural way. Following Chapter 5, we explain how the elements of the hypercohomology (9.1) look like and how to calculate them. This will provide us with a natural definition of the Gauss–Manin connection.

Remark 9.1. From a computational point of view it is convenient to describe the Gauss–Manin connection for a projective variety X defined over an affine variety $\mathsf{T} := \mathrm{Spec}(\mathsf{R})$ with a ring R described in Movasati (2021, §10.2). In other words we consider a projective variety X over the ring R. The hypercohomology $\mathbb{H}^m(\Omega^\bullet_{\mathsf{X}}, d)$ is now an R-module, but not necessarily free. In the context of moduli of enhanced projective varieties, see Movasati (2020b), it is natural to detect when $\mathbb{H}^m(\Omega^\bullet_{\mathsf{X}}, d)$ is a free R-module.

Let $\mathcal{U} = \{U_i\}_{i\in I}$ be any open covering of X by open affine subsets, where I is a totally ordered finite set. This gives us a covering of each fiber $X_t,\ t \in \mathsf{T}$ which we denote it by the same notation. We have the following double complex

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
(\Omega_{\mathsf{X}/\mathsf{T}})^0_2 & \to & (\Omega_{\mathsf{X}/\mathsf{T}})^1_2 & \to & (\Omega_{\mathsf{X}/\mathsf{T}})^2_2 & \to & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
(\Omega_{\mathsf{X}/\mathsf{T}})^0_1 & \to & (\Omega_{\mathsf{X}/\mathsf{T}})^1_1 & \to & (\Omega_{\mathsf{X}/\mathsf{T}})^2_1 & \to & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
(\Omega_{\mathsf{X}/\mathsf{T}})^0_0 & \to & (\Omega_{\mathsf{X}/\mathsf{T}})^1_0 & \to & (\Omega_{\mathsf{X}/\mathsf{T}})^2_0 & \to & \cdots
\end{array}
\quad . \tag{9.2}
$$

Here $(\Omega_{\mathsf{X}/\mathsf{T}})^i_j$ is the product over $I_1 \subset I,\ \#I_1 = j+1$ of the set of global sections ω_σ of $\Omega^i_{\mathsf{X}/\mathsf{T}}$ in the open set $\sigma = \cap_{i\in I_1} U_i$. The horizontal arrows are usual differential operator d of $(\Omega_{\mathsf{X}/\mathsf{T}})^i_{\mathsf{X}/\mathsf{T}}$'s and vertical arrows are differential operators δ in the sense of Čech cohomology, that is,

$$
\delta : (\Omega_{\mathsf{X}/\mathsf{T}})^i_j \to (\Omega_{\mathsf{X}/\mathsf{T}})^i_{j+1},\ \ \{\omega_\sigma\}_\sigma \mapsto \left\{ \sum_{k=0}^{j+1} (-1)^k \omega_{\widetilde{\sigma}_k}\,|_{\widetilde{\sigma}} \right\}_{\widetilde{\sigma}} . \tag{9.3}
$$

Here $\widetilde{\sigma}_k$ is obtained from $\widetilde{\sigma}$, neglecting the k-th open set in the definition of $\widetilde{\sigma}$. The k-th piece of the total chain of (9.2) is

$$
\mathcal{L}^k := \oplus_{i=0}^k (\Omega_{\mathsf{X}/\mathsf{T}})^i_{k-i}
$$

with the differential operator

$$
D = \delta + (-1)^{k-i} d : \mathcal{L}^k \to \mathcal{L}^{k+1}. \tag{9.4}
$$

Using Theorem 4.6 and Theorem 3.1, we know that the relative de Rham cohomology $H^k_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$ is the total cohomology of the double complex (9.2), that is,

$$
H^k_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}) := \mathbb{H}^k(\mathsf{X}, \Omega^\bullet_{\mathsf{X}/\mathsf{T}}) = \frac{\ker(\mathcal{L}^k \overset{d}{\to} \mathcal{L}^{k+1})}{\mathrm{Im}(\mathcal{L}^{k-1} \overset{d}{\to} \mathcal{L}^k)}.
$$

Theorem 9.1. *Let* $\mathsf{X} \to \mathsf{T}$ *be a family of smooth projective varieties over* k*. The relative de Rham cohomology* $H^i_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$ *is a free sheaf of finite rank of* $\mathcal{O}_{\mathsf{T}}$*-module.*

Topological proof. We first remark that the natural map

$$\pi : H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}) \to H^m_{\mathrm{dR}}(\mathsf{X}^\infty/\mathsf{T}^\infty) \tag{9.5}$$

is an inclusion. Here, $H^m_{\mathrm{dR}}(\mathsf{X}^\infty/\mathsf{T}^\infty)$ is the sheaf of sections of the C^∞ cohomology bundle over T^∞. For a sheaf $\mathcal{S}$ in T and t a point in the ambient space T, let $\check{\mathcal{S}}_t := \mathcal{S}/\mathcal{M}_t\mathcal{S}$ be the evaluation vector space, where $\mathcal{M}_t \subset \mathcal{O}_{\mathsf{T},t}$ is the maximal ideal. The evaluation vector spaces in both sides of (9.5) gives us the morphism $\pi_t : H^m_{\mathrm{dR}}(X_t/\mathsf{k}) \to H^m_{\mathrm{dR}}(X_t)$. By the comparison theorem between C^∞ and algebraic de Rham cohomology, see Section 5.3, we know that π_t is an isomorphism. This is enough to conclude that π is an inclusion.

Now, we would like to prove that $H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$ is free. We know that $H^m_{\mathrm{dR}}(\mathsf{X}^\infty/\mathsf{T}^\infty)$ is a free $\mathcal{C}^\infty_{\mathsf{T}}$-module. Since π_t is an isomorphism, we can take local sections of $H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$ at $t \in \mathsf{T}$ which generate $H^m_{\mathrm{dR}}(\mathsf{X}^\infty/\mathsf{T}^\infty)$ freely, and hence, for any other section α of $H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$ we can write $\alpha = \sum_i f_i\alpha_i$, where f_i's are C^∞ functions. We have to argue that f_i's are regular algebraic functions in a neighborhood of t. For this we may use Hard Lefschetz theorem to consider only the case $m <$ dimension of the fibers of X/T. Then we use the fact that the cup produce induces a non-degenerate map in $H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$. □

Algebraic proof. Let us show first that $H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$ is a coherent $\mathcal{O}_{\mathsf{T}}$-module. In other words we have to show that for $U \subseteq \mathsf{T}$ affine open subset, $H^m_{\mathrm{dR}}(\mathsf{X}_U/U)$ is a finitely generated $\mathcal{O}_{\mathsf{T}}(U)$-module. In order to do this, consider the Hodge filtration $F^iH^m_{\mathrm{dR}}(\mathsf{X}_U/U)$. It is enough to show that $F^iH^m_{\mathrm{dR}}(\mathsf{X}_U/U)/F^{i+1}H^m_{\mathrm{dR}}(\mathsf{X}_U/U)$ is a finitely generated $\mathcal{O}_{\mathsf{T}}(U)$-module for every $i = 0,\dots,m$. By Hartshorne (1977, Theorem III.8.8 (b)) $H^{m-i}(\mathsf{X}_U, \Omega^i_{\mathsf{X}_U/U})$ is a finitely generated $\mathcal{O}_{\mathsf{T}}(U)$-module. Consider the map

$$\sum_{j=i}^m \omega^j \in \mathbb{H}^m(\mathsf{X}_U, \Omega^{\bullet\geqslant i}_{\mathsf{X}_U/U}) \xrightarrow{\phi} \omega^i \in H^{m-i}(\mathsf{X}_U, \Omega^i_{\mathsf{X}_U/U}).$$

This map is well-defined and has kernel

$$\widetilde{F}^{i+1} := \mathrm{Im}(\mathbb{H}^m(\mathsf{X}_U, \Omega^{\bullet\geqslant i+1}_{\mathsf{X}_U/U}) \to \mathbb{H}^m(\mathsf{X}_U, \Omega^{\bullet\geqslant i}_{\mathsf{X}_U/U})).$$

Therefore, $\mathbb{H}^m(\mathsf{X}_U, \Omega^{\bullet\geq i}_{\mathsf{X}_U/U})/\widetilde{F}^{i+1}$ is a finitely generated $\mathcal{O}_\mathsf{T}(U)$-module. On the other hand we have a natural epimorphism

$$\mathbb{H}^m(\mathsf{X}_U, \Omega^{\bullet\geq i}_{\mathsf{X}_U/U})/\widetilde{F}^{i+1} \twoheadrightarrow F^i H^m_{\mathrm{dR}}(\mathsf{X}_U/U)/F^{i+1} H^m_{\mathrm{dR}}(\mathsf{X}_U/U)$$

and so $F^i H^m_{\mathrm{dR}}(\mathsf{X}_U/U)/F^{i+1} H^m_{\mathrm{dR}}(\mathsf{X}_U/U)$ is finitely generated. Finally, by Hartshorne (1977, Exercise II.5.8 (c)) $H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$ is locally free since it is coherent and for every $t \in \mathsf{T}$

$$H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})/\mathfrak{m}_t H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}) \simeq H^m_{\mathrm{dR}}(X_t/\mathsf{k}) \simeq H^m_{\mathrm{dR}}(X_0/\mathsf{k})$$

where $\mathfrak{m}_t$ is the maximal ideal of $\mathcal{O}_{\mathsf{T},t}$ and in the last isomorphism we have used Ehresmann's fibration theorem. □

It would be essential for computational purposes to give an algorithmic proof to Theorem 9.1 without referring to C^∞ context which uses statements such as Ehresmann's fibration theorem. Let $0 \in \mathsf{T}$ and consider the canonical projection

$$\pi : H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}) \to H^m_{\mathrm{dR}}(X_0/\mathsf{k}),$$

which is surjective. We choose a basis $\beta_1, \beta_2, \ldots, \beta_b$ of $H^m_{\mathrm{dR}}(X_0/\mathsf{k})$ and consider $\alpha_1, \alpha_2, \ldots, \alpha_b$ in the stalk $\mathcal{S}_0$ of $\mathcal{S} := H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$ at 0 which are mapped to β_i's, that is, $\pi(\alpha_i) = \beta_i$. It would be interesting to give an algorithmic proof to the fact that α_i's freely generate $\mathcal{S}_0$, that is, the proof gives an algorithm for computing the underlying coefficients.

9.3 Algebraic Gauss–Manin connection

Let X/T be as in the previous section. In this section we construct a connection

$$\nabla : H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}) \to \Omega^1_\mathsf{T} \otimes_{\mathcal{O}_\mathsf{T}} H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}),$$

where Ω^1_T is by definition the sheaf of differential 1-forms in T. By definition of a connection, ∇ is k-linear and satisfies the Leibniz rule

$$\nabla(r\omega) = dr \otimes \omega + r\nabla\omega, \; r \in \mathcal{O}_\mathsf{T}, \;\; \omega \in H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}). \tag{9.6}$$

For simplicity, we assume that $\mathsf{T} := \mathrm{Spec}(\mathsf{R})$ is an affine variety and X is a projective variety over R. The general context can be easily recovered replacing R with $\mathcal{O}_\mathsf{T}(U)$ for open subsets U of T. Therefore, in the following we have

$\mathcal{O}_\mathsf{T} = \mathsf{R}, \Omega^1_\mathsf{T} = \Omega_\mathsf{R}$ etc. We need to distinguish between the differential operator relative to $\mathcal{O}_\mathsf{T}$

$$d_\mathsf{R} = d_{\mathsf{X}/\mathsf{T}} : \Omega^m_{\mathsf{X}/\mathsf{T}} \to \Omega^{m+1}_{\mathsf{X}/\mathsf{T}}$$

and the differential operator d_X relative to k:

$$d_\mathsf{k} = d_\mathsf{X} : \Omega^m_\mathsf{X} \to \Omega^{m+1}_\mathsf{X}.$$

Let us take a covering $\mathcal{U} = \{U_i\}_{i\in I}$ of X/T by affine open sets. We need to consider the double complex similar to (9.2) relative to k:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
(\Omega_\mathsf{X})^0_2 & \to & (\Omega_\mathsf{X})^1_2 & \to & (\Omega_\mathsf{X})^2_2 & \to & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
(\Omega_\mathsf{X})^0_1 & \to & (\Omega_\mathsf{X})^1_1 & \to & (\Omega_\mathsf{X})^2_1 & \to & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
(\Omega_\mathsf{X})^0_0 & \to & (\Omega_\mathsf{X})^1_0 & \to & (\Omega_\mathsf{X})^2_0 & \to & \cdots
\end{array} \tag{9.7}$$

The differential operator of the double complexes (9.2) and (9.7) is respectively denoted by $D_{\mathsf{X}/\mathsf{T}}$ and D_X. There is a natural projection map from the double complex (9.7) to (9.2). Let $\omega \in H^k_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$. By our definition, ω is represented by

$$\oplus_{i=0}^m \omega^i, \ \omega^i \in (\Omega_{\mathsf{X}/\mathsf{T}})^i_{m-i}$$

and ω^i is a collection of i-forms $\{\omega^i_\sigma\}_\sigma$. By definition we have $D_{\mathsf{X}/\mathsf{T}}\omega = 0$. The differential map $d_{\mathsf{X}/\mathsf{T}}$ used in the definition of $D_{\mathsf{X}/\mathsf{T}}$ is relative to R, that is, by definition $dr = 0, r \in \mathsf{R}$. Now, let us take any element $\check{\omega}$ in the double complex (9.7) which is mapped into ω under the canonical projection. We apply D_X on $\check{\omega}$ and the result is not necessarily zero. However, by our choice of ω we have $D_{\mathsf{X}/\mathsf{T}}\omega = 0$, and so, $D_\mathsf{X}\check{\omega}$ maps to zero in the double complex (9.2). This is equivalent to say that

$$D_\mathsf{X}\check{\omega} = \check{\eta} = \oplus_{i=0}^{m+1} \check{\eta}^i,$$

$$\check{\eta}^i = \sum_a \beta_{i,a} \wedge \alpha_a \in (\Omega_\mathsf{X})^i_{m+1-i} \wedge \Omega^1_\mathsf{T}. \tag{9.8}$$

where α_a's generate Ω^1_T as $\mathcal{O}_\mathsf{T}$-module. We map $\check{\eta}$ into

$$\Omega^1_\mathsf{T} \otimes_{\mathcal{O}_\mathsf{T}} \left(\oplus_{i=0}^{m+1} (\Omega_{\mathsf{X}/\mathsf{T}})^i_{m+1-i} \right)$$

which is the tensor product over $\mathcal{O}_\mathsf{T}$ of Ω^1_T with the double complex (9.2) and we get an element

$$\check{\eta}^i = \sum_a \alpha_a \otimes \beta_{i,a} \in \Omega^1_\mathsf{T} \otimes_{\mathcal{O}_\mathsf{T}} (\Omega_{\mathsf{X}/\mathsf{T}})^i_{k+1-i},$$

which for simplicity we have used the same notation. This new double complex has the differential operator $\mathrm{id} \otimes D_{\mathsf{X}/\mathsf{T}}$, that is in Ω^1_T it acts as identity and in $\oplus_{i=0}^k (\Omega_{\mathsf{X}/\mathsf{T}})^i_{k-i}$ as $D_{\mathsf{X}/\mathsf{T}}$. Since $D_{\mathsf{X}/\mathsf{T}}$ is $\mathcal{O}_\mathsf{T}$-linear, $\mathrm{id} \otimes D_{\mathsf{X}/\mathsf{T}}$ is well-defined. Note that we replace the notation $\wedge$ with the tensor product $\otimes$ and we first write the elements of Ω^1_T.

Proposition 9.1. *We have*

$$(\mathrm{id} \otimes D_{\mathsf{X}/\mathsf{T}})(\check{\eta}) = 0. \tag{9.9}$$

Proof. This is a direct consequence of $D_\mathsf{X} \circ D_\mathsf{X}(\check{\omega}) = 0$. We know that $\Omega_\mathsf{T} = \Omega_\mathsf{R}$ as $\mathcal{O}_\mathsf{T}$-module is generated by exact 1-forms. This together with the fact that $D_{\mathsf{X}/\mathsf{T}}$ is $\mathcal{O}_\mathsf{T}$-linear, implies that in (9.8) we can assume that α_a are exact, and hence, closed. In particular,

$$0 = D_\mathsf{X} \circ D_\mathsf{X}(\check{\omega}) = \sum_a D_\mathsf{X}\beta_{i,a} \wedge \alpha_a.$$

By definition we have $(\mathrm{id} \otimes D_{\mathsf{X}/\mathsf{T}})(\check{\eta}) = \sum_a \alpha_{i,a} \otimes D_\mathsf{X}\beta_{i,a}$, and hence, we get the desired result. □

We use (9.9) and get an element in $\Omega^1_T \otimes_{\mathcal{O}_\mathsf{T}} H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$ which is the definition of the **algebraic Gauss–Manin connection**

$$\nabla\omega := (-1)^m \check{\eta}.$$

Remark 9.2. We have inserted $(-1)^m$ in the definition of Gauss–Manin connection because in this format it satisfies the fundamental property Theorem 9.3.

Exercise 9.1. Show that ∇ defined as above is well-defined, that is, it does not depend on the chosen representative $\check{\omega}$, and that it satisfies (9.6).

A vector field v in T is an R-linear map $\Omega^1_{\mathsf{T}} \to \mathsf{R}$. We define

$$\nabla_v : H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}) \to H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$$

to be ∇ composed with

$$v \otimes \mathrm{Id} : \Omega^1_{\mathsf{T}} \otimes_{\mathsf{R}} H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}) \to \mathsf{R} \otimes_{\mathsf{R}} H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}) = H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}).$$

If R is a polynomial ring $\mathbb{Q}[t_1, t_2, \ldots]$ then we have vector fields $\frac{\partial}{\partial t_i}$ which are defined by the rule

$$\frac{\partial}{\partial t_i}(dt_j) = 1 \text{ if } i = j \text{ and } = 0 \text{ if } i \neq j.$$

In this case we simply write $\frac{\partial}{\partial t_i}$ instead of $\nabla_{\frac{\partial}{\partial t_i}}$. Sometimes it is useful to choose a basis $\omega_1, \omega_2, \ldots, \omega_h$ of the R-module $H^m(\mathsf{X}/\mathsf{T})$ and write the Gauss–Manin connection in this basis

$$\nabla \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_h \end{pmatrix} = \mathsf{A} \otimes \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_h \end{pmatrix} \tag{9.10}$$

where A is a $h \times h$ matrix with entries in Ω^1_{T}. For simplicity, we take X to be a variety over the ring R and $\mathsf{T} := \mathrm{Spec}(\mathsf{R})$.

9.4 Integrability

The Gauss–Manin connection (and in general any connection) induces maps

$$\nabla_i : \Omega^i_{\mathsf{T}} \otimes_{\mathcal{O}_{\mathsf{T}}} H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}) \to \Omega^{i+1}_{\mathsf{T}} \otimes_{\mathcal{O}_{\mathsf{T}}} H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}). \tag{9.11}$$

It is uniquely defined by the fact that it is k-linear and satisfies the equality

$$\nabla_i(\alpha \otimes \omega) = d\alpha \otimes \omega + (-1)^i \alpha \wedge \nabla\omega, \ \alpha \in \Omega^i_{\mathsf{T}}, \ \omega \in H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}),$$

for $i = 0, 1, 2, \ldots$.

Proposition 9.2. *The Gauss–Manin connection is integrable, that is,*

$$\nabla_{i+1} \circ \nabla_i = 0, \ \ i = 0, 1, 2, \ldots. \tag{9.12}$$

Proof. The proof is a direct consequence of the definition of ∇_i. Note that if we take representative $\check{\omega} \in \Omega_{\mathsf{X}}^m$ for ω then $\nabla_i(\alpha \otimes \omega)$ is represented by $\check{\omega} \wedge \alpha$ and ∇_i is just D_{X}. In this way the affirmation follows from $D_{\mathsf{X}} \circ D_{\mathsf{X}} = 0$. □

Let A be the Gauss–Manin connection matrix as in (9.10). The integrability of the Gauss–Manin connection is translated into the following identity

$$d\mathsf{A} = \mathsf{A} \wedge \mathsf{A}. \tag{9.13}$$

This follows from

$$\begin{aligned} 0 &= \nabla_1 \circ \nabla_0(\omega) \\ &= \nabla_1(\mathsf{A} \otimes \omega) \\ &= d\mathsf{A} \otimes \omega - \mathsf{A} \wedge \nabla(\omega) \\ &= (d\mathsf{A} - \mathsf{A} \wedge \mathsf{A}) \otimes \omega, \end{aligned}$$

where $\omega = [\omega_1, \omega_2, \cdots, \omega_{\mathsf{b}}]^{\mathsf{tr}}$.

9.5 Griffiths transversality

Let X/T be as before. The relative algebraic de Rham cohomology $H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$ carries a natural filtration which is called the Hodge filtration:

$$0 = F^{m+1} \subset F^m \subset \cdots \subset F^1 \subset F^0 = H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$$

Its ingredients are defined by

$$F^k = F^k H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}) = \mathrm{Im}\left(\mathbb{H}^m(\mathsf{X}, \Omega^{\bullet \geqslant k}_{\mathsf{X}/\mathsf{T}}) \to \mathbb{H}^m(\mathsf{X}, \Omega^{\bullet}_{\mathsf{X}/\mathsf{T}})\right)$$

Theorem 9.2 (Griffiths transversality)**.** *The Gauss–Manin connection maps F^k to $\Omega^1_{\mathsf{T}} \otimes_{\mathsf{R}} F^{k-1}$, that is,*

$$\nabla(F^k) \subset \Omega^1_{\mathsf{T}} \otimes_{\mathsf{R}} F^{k-1}, \;\; k = 1, 2, \ldots, m.$$

Proof. This follows from the definition of the Gauss–Manin connection. Let $\omega \in F^k$ and so ω is represented by $\check{\omega} := \oplus_{i=k}^m \check{\omega}^i, \; \check{\omega}^i \in (\Omega_{\mathsf{X}})^i_{m-i}$

$$
\begin{matrix}
0 & \check{\eta}^k & & & & & \\
 & \check{\omega}^k & \check{\eta}^{k+1} & & & & \\
 & & \ddots & \ddots & & & \\
 & & & \check{\omega}^{m-1} & \check{\eta}^m & & \\
 & & & & \check{\omega}^m & \check{\eta}^{m+1} &
\end{matrix}
\tag{9.14}
$$

We have $D_{\mathsf{X}}(\check{\omega}) = \check{\eta} = \oplus_{i=k}^{m+1} \check{\eta}^i, \check{\eta}^i \in (\Omega_{\mathsf{X}})^i_{m+1-i} \wedge \Omega^1_{\mathsf{T}}$ which implies the result. $\square$

For a $\mathsf{b}_m \times \mathsf{b}_m$ matrix M we denote by M^{ij}, $i, j = 0, 1, 2, \ldots, m$ the $\mathsf{h}^{m-i,i} \times \mathsf{h}^{m-j,j}$ sub matrix of M corresponding to the decomposition $\mathsf{b}_m := \mathsf{h}^{m,0} + \mathsf{h}^{m-1,1} + \cdots + \mathsf{h}^{0,m}$.

$$
M = [M^{ij}] = \begin{pmatrix}
M^{00} & M^{01} & M^{02} & \cdots & M^{0m} \\
M^{10} & M^{11} & M^{12} & \cdots & M^{1m} \\
M^{20} & M^{21} & M^{22} & \cdots & M^{2m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
M^{m0} & M^{m1} & M^{m2} & \cdots & M^{mm}
\end{pmatrix}.
$$

We call M^{ij}, $i, j = 0, 1, 2 \ldots, m$ the (i, j)**-th Hodge block of** M.

Let A be the Gauss–Manin connection matrix as in (9.10). We assume that ω_i's are compatible with the Hodge filtration of $H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$. The Griffiths transversality implies that

$$
\mathsf{A}_m^{i,j} = 0, \quad j \geqslant i + 2 \tag{9.15}
$$

where we have used Hodge blocks notation for a matrix. For instance for $m = 4$ we have

$$
\mathsf{A}_m = \begin{pmatrix}
* & * & 0 & 0 & 0 \\
* & * & * & 0 & 0 \\
* & * & * & * & 0 \\
* & * & * & * & * \\
* & * & * & * & *
\end{pmatrix}
$$

9.6 Polarization, cup product and Gauss–Manin connection

The following assertions are well-known in the topological context and we verify them in the algebraic context.

Proposition 9.3. *The Gauss–Manin connection satisfies the following properties:*

1. *The polarization $\theta \in H^2_{\rm dR}(\mathsf{X}/\mathsf{T})$ is a flat section, that is,*
$$\nabla(\theta) = 0, \tag{9.16}$$
2. *For $\alpha \in H^{m_1}_{\rm dR}(\mathsf{X}/\mathsf{T})$ and $\beta \in H^{m_2}_{\rm dR}(\mathsf{X}/\mathsf{T})$ we have*
$$\nabla(\alpha \cup \beta) = \nabla(\alpha) \cup \beta + (-1)^{m_1} \alpha \cup \nabla(\beta). \tag{9.17}$$
3. *∇ sends the primitive cohomology $H^m_{\rm dR}(\mathsf{X}/\mathsf{T})_0$ to itself, and hence, it respects the Lefschetz decomposition.*

Proof. For the first item one has to recall the algebraic definition of the polarization in Section 5.7. We note that the canonical representative of θ in X satisfies $D_{\mathsf{X}}(\theta) = 0$, and hence, by definition of ∇ we have $\nabla(\theta) = 0$.

For the second item we have to remind the definition of $\cup$ in Section 5.5. We also note that for differential m_1-form α and m_2-form β in X we have $d_{\mathsf{X}}(\alpha \wedge \beta) = d_{\mathsf{X}}\alpha \wedge \beta + (-1)^{m_1}\alpha \wedge d_{\mathsf{X}}\beta$. This together with the fact that the Gauss–Manin connection in the level of representatives in X is just D_{X} we get the desired equality.

Item 1 and 2 imply that
$$\nabla(\alpha \cup \theta^m) = \nabla(\alpha) \cup \theta^m$$
and the last item follows from this and the definition of primitive cohomology. □

9.7 Analytic Gauss–Manin connection

The Gauss–Manin connection from a topological point of view is simple to describe, however, it becomes computationally complicated from an algebraic point of view. The topological description is as follows.

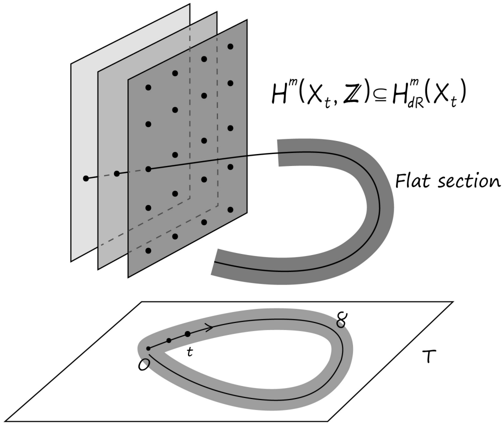

Figure 9.1: Gauss–Manin connection

Let $\mathsf{X} \to \mathsf{T}$ be a family of smooth projective varieties over $\mathbb{C}$. By Ehresmann's fibration theorem, this is a locally trivial C^∞ bundle over T, and hence, it gives us the cohomology bundle

$$H := \cup_{t \in \mathsf{T}} H^m(\mathsf{X}_t, \mathbb{C})$$

over T whose fiber at $t \in \mathsf{T}$ is the m-th cohomology of the fiber X_t, for more details see Movasati (2021, Chapter 6). This bundle has special holomorphic sections s such that for all $t \in \mathsf{T}$ we have $s(t) \in H^m(\mathsf{X}_t, \mathbb{Q})$. These are called flat sections. In a small neighborhood U of $t \in \mathsf{T}$ we can find flat sections $s_1, s_2, \ldots, s_{\mathsf{b}}$ such that any other holomorphic section in U can be written as $s = \sum_{i=1}^{\mathsf{b}} f_i s_i$, where f_i's are holomorphic functions in U. The Gauss–Manin connection on H is the unique connection on H with the prescribed flat sections:

$$\nabla : H \to \Omega^1_{\mathsf{T}} \otimes_{\mathcal{O}_{\mathsf{T}}} H, \;\; \nabla\left(\sum_{i=1}^{\mathsf{b}} f_i s_i\right) = \sum_{i=1}^{\mathsf{b}} df_i \otimes s_i,$$

see Figure 9.1 for a pictorial description of Gauss–Manin connection.

9.8 Algebraic vs. Analytic Gauss–Manin connection

Let us now assume that $\mathsf{k} = \mathbb{C}$. The main motivation, which is also the historical one, for defining the Gauss–Manin connection is the following:

Theorem 9.3. *For any $\omega \in H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$ and a continuous family of cycles $\delta_t \in H_m(X_t, \mathbb{Z})$ we have*

$$d\left(\int_{\delta_t} \omega\right) = \int_{\delta_t} \nabla\omega. \tag{9.18}$$

Note that in the right hand side of (9.18) the integration takes place only in the cohomology $H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$, that is,

$$\int_{\delta_t} \alpha \otimes \beta := \alpha \int_{\delta_t} \beta, \ \ \alpha \in \Omega^1_{\mathsf{T}}, \ \beta \in H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}).$$

Theorem 9.4. *The algebraic and analytic Gauss–Manin connections are the same under the canonical isomorphism $H^m_{\mathrm{dR}}(X_t/\mathbb{C}) \cong H^m_{\mathrm{dR}}(X_t)$ discussed in Section 5.3.*

Proof. The definition of the algebraic Gauss–Manin connection ∇^{alg} is done using algebraic differential forms. In a similar way we define ∇^∞ which is defined using C^∞ differential forms. We have to show that ∇^∞ coincides with the analytic Gauss–Manin connection defined in (9.7). We first find a refinement of the covering $\mathcal{U}$ in analytic topology such that the intersection of all open sets U_i are simply connected. This implies that the horizontal maps in the double complex $(\Omega_{\mathsf{X}^\infty/\mathsf{T}^\infty})^{\bullet}_{\bullet}$ are exact, and hence, $\omega = \omega^0 + \omega^1 + \cdots + \omega^m \in H^m_{\mathrm{dR}}(\mathsf{X}^\infty/\mathsf{T}^\infty)$ is equivalent to another one whose only non-zero piece is ω^0. The equality $D_{\mathsf{X}^\infty/\mathsf{T}^\infty}(\omega) = 0$ implies that $d_{\mathsf{X}^\infty/\mathsf{T}^\infty}\omega^0 = 0$ which in turn is equivalent to the fact that ω^0 is a C^∞ function in T. Moreover, $\delta(\omega^0) = 0$. Now, let us assume that $\nabla^\infty(\omega) = 0$. This means that $\omega^0_{i_0 i_1 \cdots i_m}$ are constants independent of t. Therefore, flat sections of ∇^∞ are generated by sections of the cohomology bundle with images in the cohomology with integral coefficients. $\square$

Proof of Theorem 9.3. The proof is the continuation of the proof of Theorem 9.4. The vertical arrows in $(\Omega_{\mathsf{X}^\infty/\mathsf{T}^\infty})^{\bullet}_{\bullet}$ are exact and hence $\omega = \omega^0 + \omega^1 + \cdots + \omega^m \in H^m_{\mathrm{dR}}(\mathsf{X}^\infty/\mathsf{T}^\infty)$ is equivalent to another one whose only non-zero piece is ω^m with $\delta(\omega) = 0$. Therefore, ω gives a global m-form in X. It may not be closed but restricted to each fiber X_t is closed. The rest of the proof is similar to

Movasati (2021, Proposition 13.1). Note that this proposition is valid without any sign ambiguity if we write the differential of parameters (α_i's in the definition of algebraic Gauss–Manin connection) first and then the differential forms involved in hypercohomology. This is the main reason for the appearance of $(-1)^m$ in our definition of algebraic Gauss–Manin connection. □

Proposition 9.4. *We have*

$$d\left(\left[\int_{\delta_j} \omega_i\right]\right) = \left[\int_{\delta_j} \omega_i\right] \cdot \mathsf{A}. \tag{9.19}$$

where A *is the Gauss–Manin connection matrix in* (9.10).

Proof. We just integrate both side of the equality (9.10) over a basis $\delta_1, \delta_2, \ldots, \delta_h \in H_m(X_t, \mathbb{Q})$. □

Remark 9.3. Gauss–Manin connection near the degeneracy locus of families of projective varieties is regular. The main references for this are Griffiths (1970, p. 237), Arnold, Gusein-Zade, and Varchenko (1988, Chapter 13) and Deligne (1970).

Remark 9.4. For a smooth hypersurface $X \subset \mathbb{P}^{n+1}$, Griffiths theorem gives us a basis of $H^n_{\mathrm{dR}}(U/\mathbb{C})$ for $U := \mathbb{P}^{n+1} - X$. Computation of the Gauss–Manin connection in this case is essentially a pole order reduction. This is mainly known as Griffiths–Dwork method. For more details see Movasati (2021, Chapter 12).

9.9 Gauss–Manin connection for hypersurfaces

The Gauss–Manin connection for hypersurfaces is simple. Recall that the cohomology of the complement of a smooth hypersurface $X : F = 0$ is given by algebraic differential forms $\frac{P\Omega}{F^k}$, where $\deg(P) + n + 2 = d \cdot k$. If the polynomial F depends on a parameter t then

$$\nabla_{\frac{\partial}{\partial t}}\left(\frac{P\Omega}{F^k}\right) = \left(-k\frac{\frac{\partial F}{\partial t} P\Omega}{F^{k+1}}\right) \otimes dt \tag{9.20}$$

In order to do pole order reduction we use

$$\left(\sum_{i=1}^{n+1} A_i \frac{\partial F}{\partial x_i}\right) \frac{\Omega}{F^{k+1}} = \frac{1}{k}\left(\sum_{i=1}^{n+1} \frac{\partial A_i}{\partial x_i}\right) \frac{\Omega}{F^k} + \text{exact terms}. \tag{9.21}$$

10 Infinitesimal variation of Hodge structures

The theory of the great Italian geometers was essentially, like part regarding the work of Picard. This was natural since in his time Poincaré's creation of algebraic topology was in its infancy. Indeed when I arrived on the scene (1915) it was hardly further along. [...] I cannot refrain, however, from mention of [....] the systematic algebraic attack on algebraic geometry by Oscar Zariski and his school, and beyond that of André Weil and Grothendieck, (S. Lefschetz in his mathematical autobiography Lefschetz 1968, page 855).

10.1 Introduction

Gauss–Manin connection carries many information of the underlying family of algebraic varieties, however, in general it is hard to compute it and verify some of its properties. For this reason it is sometimes convenient to break it into smaller pieces and study these by their own. Infinitesimal variation of Hodge structures, IVHS for short, is one of these pieces of the Gauss–Manin connection, and we explain it in this chapter. It was originated by the articles of Griffiths around sixties

Griffiths (1968a,b, 1969a), and was introduced by him and his collaborators in the subsequent articles Carlson, Green, et al. (1983), Carlson and Griffiths (1980), Griffiths (1983), and Griffiths and Harris (1983).

10.2 IVHS and Gauss–Manin connection

Recall that for a family of projective varieties $\mathsf{X} \to \mathsf{T}$ we have the Gauss–Manin connection

$$\nabla : H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}) \to \Omega^1_{\mathsf{T}} \otimes_{\mathcal{O}_{\mathsf{T}}} H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}).$$

We have also the Hodge filtration

$$0 = F^{m+1} \subset F^m \subset \cdots F^1 \subset F^0 = H^m_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$$

and the Gauss–Manin connection satisfies the so called Griffiths transversality:

$$\nabla(F^i) \subset \Omega^1_{\mathsf{T}} \otimes_{\mathcal{O}_{\mathsf{T}}} F^{i-1}, \ \ i = 1, 2, \ldots, m.$$

Therefore, the Gauss–Manin connection induces well-defined maps

$$\nabla_i : \frac{F^i}{F^{i+1}} \to \Omega^1_{\mathsf{T}} \otimes_{\mathcal{O}_{\mathsf{T}}} \frac{F^{i-1}}{F^i} \tag{10.1}$$

In Theorem 5.1 we have learned that

$$\frac{F^i}{F^{i+1}} \cong H^{m-i}(\mathsf{X}, \ \Omega^i_{\mathsf{X}/\mathsf{T}}).$$

Therefore, we get the following $\mathcal{O}_{\mathsf{T}}$-linear map

$$\nabla_i : H^{m-i}(\mathsf{X}, \ \Omega^i_{\mathsf{X}/\mathsf{T}}) \to \Omega^1_{\mathsf{T}} \otimes_{\mathcal{O}_{\mathsf{T}}} H^{m-i+1}(\mathsf{X}, \ \Omega^{i-1}_{\mathsf{X}/\mathsf{T}}). \tag{10.2}$$

After analysing the definition of the Gauss–Manin connection we get the following description of ∇_i. Let $\omega \in H^{m-i}(\mathsf{X}, \ \Omega^i_{\mathsf{X}/\mathsf{T}})$. It is given by a cocycle in

$$(\Omega_{\mathsf{X}/\mathsf{T}})^i_{m-i} = \frac{(\Omega_{\mathsf{X}})^i_{m-i}}{\pi^*\Omega^1_{\mathsf{T}} \wedge (\Omega_{\mathsf{X}})^{i-1}_{m-i}}.$$

Let $\check{\omega} \in (\Omega_{\mathsf{X}})^i_{m-i}$ such that it maps to ω under the canonical projection. We have

$$\delta(\check{\omega}) \in \pi^*\Omega^1_{\mathsf{T}} \wedge (\Omega_{\mathsf{X}})^{i-1}_{m-i+1}.$$

Lifting it to $\Omega^1_{\mathsf{T}} \otimes_{\mathcal{O}_{\mathsf{T}}} (\Omega_{\mathsf{X}})^{i-1}_{m-i+1}$ and then projecting it to $\Omega^1_{\mathsf{T}} \otimes_{\mathcal{O}_{\mathsf{T}}} (\Omega_{\mathsf{X}/\mathsf{T}})^{i-1}_{m-i+1}$ we get an element independent of the chosen lifting, corresponding to $\nabla_i(\omega)$. Sometimes it is useful to write (10.2) in the following way:

$$\nabla_i : \Theta_{\mathsf{T}} \to \hom\left(H^{m-i}(\mathsf{X},\ \Omega^i_{\mathsf{X}/\mathsf{T}}), H^{m-i+1}(\mathsf{X},\ \Omega^{i-1}_{\mathsf{X}/\mathsf{T}})\right). \tag{10.3}$$

We will also need the following map obtained by specialization:

$$\nabla_i : \mathbf{T}_0\mathsf{T} \to \hom\left(H^{m-i}(X,\ \Omega^i_X), H^{m-i+1}(X,\ \Omega^{i-1}_X)\right). \tag{10.4}$$

This is the so called **infinitesimal variation of Hodge structures (IVHS)**.

10.3 Kodaira–Spencer map I

Let $\pi : \mathsf{X} \to \mathsf{T}$ be a proper surjective morphism of varieties over a field k and let $\mathcal{S}$ be a sheaf over T. Our main example for $\mathcal{S}$ is the sheaf of vector fields over T. It follows that π is an open map, that is, for U open subset of X, $\pi(U)$ is an open subset of T, and this is what we need in order to define the inverse image of $\mathcal{S}$. The inverse image $\pi^{-1}\mathcal{S}$ of $\mathcal{S}$ by π is the sheaf over X which for every open subset U of X we have $\pi^{-1}\mathcal{S}(U) := \mathcal{S}(\pi(U))$.

Proposition 10.1. *We have*

$$H^m(\mathsf{X}, \pi^{-1}\Theta_{\mathsf{T}}) \quad = \quad H^m(\mathsf{T}, \Theta_{\mathsf{T}}), \;\; m = 0, 1, 2, \ldots.$$

Proof. This is a direct consequence of the definition of the inverse image of a sheaf and the definition of Čech cohomology. □

Definition 10.1. Let $\pi : \mathsf{X} \to \mathsf{T}$ be a family of smooth projective varieties over a field k as before. The sheaf $\Theta_{\mathsf{X}/\mathsf{T}}$ of **relative vector fields** is by definition the dual of the sheaf of $\Omega^1_{\mathsf{X}/\mathsf{T}}$ as $\mathcal{O}_{\mathsf{X}}$-module. In a small open set $U \subset \mathsf{T}$, an element $\mathsf{v} \in \Theta_{\mathsf{X}/\mathsf{T}}(U)$ is induced by a $\mathcal{O}_{\mathsf{X}}(U)$-linear map $\Omega^1_{\mathsf{X}/\mathsf{T}}(U) \to \mathcal{O}_X(U)$.

By definition it is clear that we have the inclusion

$$\Theta_{\mathsf{X}/\mathsf{T}} \subset \Theta_{\mathsf{X}}.$$

In geometric terms, $\Theta_{\mathsf{X}/\mathsf{T}}$ is the sheaf of vector fields tangent to the fibers of $\mathsf{X} \to \mathsf{T}$.

Definition 10.2. Let us define the **Kodaira–Spencer map**

$$\mathsf{K}: H^0(\mathsf{T}, \Theta_{\mathsf{T}}) \to H^1(\mathsf{X}, \Theta_{\mathsf{X}/\mathsf{T}}). \tag{10.5}$$

For a global vector field v in T we choose an acyclic covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X/T and vector fields v_i in U_i such that v_i is mapped to v under $\mathsf{X} \to \mathsf{T}$. The vector fields $v_i - v_j$ are tangent to the fibers of $\mathsf{X} \to \mathsf{T}$. We define $\mathsf{K}(v)$ as the cohomology class of the cocycle $\{v_i|_{U_i \cap U_j} - v_j|_{U_i \cap U_j}\}_{i,j \in I}$.

The Kodaira–Spencer map K corresponds to the connecting homomorphism of the long exact sequence associated to the following short exact sequence:

$$0 \to \Theta_{\mathsf{X}/\mathsf{T}} \to \check{\Theta}_{\mathsf{X}/\mathsf{T}} \to \pi^{-1}\Theta_{\mathsf{T}} \to 0 \tag{10.6}$$

where $\check{\Theta}_{\mathsf{X}/\mathsf{T}} \subset \Theta_{\mathsf{X}}$ is by definition the sheaf of vector fields in X which are mapped to vector fields in T, and $\pi^{-1}\Theta_{\mathsf{T}}$ the pull-back of the sheaf Θ_{T}. Note that $\check{\Theta}_{\mathsf{X}/\mathsf{T}}$ and $\pi^{-1}\Theta_T$ are not sheaves of $\mathcal{O}_{\mathsf{X}}$-modules, but sheaves of $\pi^{-1}\mathcal{O}_{\mathsf{T}}$-modules. The long exact sequence of (10.6) turns out to be

$$\cdots \to H^0(\mathsf{X}, \check{\Theta}_{\mathsf{X}/\mathsf{T}}) \to H^0(\mathsf{T}, \Theta_{\mathsf{T}}) \xrightarrow{\mathsf{K}} H^1(\mathsf{X}, \Theta_{\mathsf{X}/\mathsf{T}}) \to H^1(\mathsf{X}, \check{\Theta}_{\mathsf{X}/\mathsf{T}}) \to H^1(\mathsf{T}, \Theta_{\mathsf{T}}) \to \cdots \tag{10.7}$$

Proposition 10.2. *Let T be an affine variety. The Kodaira–Spencer map is surjective if and only if $H^1(\mathsf{X}, \check{\Theta}_{\mathsf{X}/\mathsf{T}}) = 0$.*

Proof. We know that Θ_{T} is a coherent sheaf and so by Serre's theorem $H^1(\mathsf{T}, \Theta_{\mathsf{T}}) = 0$. The statement follows from the long exact sequence (10.7). □

For the purposes of the present book, it would be essential to compute $H^i(\mathsf{X}, \check{\Theta}_{\mathsf{X}/\mathsf{T}})$, for $i = 0, 1$.

Let us describe the Kodaira–Spencer map for the family of all smooth degree d hypersurfaces of $\mathbb{P}^{n+1}$. In this case T is the affine space $\mathbb{A}_k^a$ minus a discriminant locus $\{\Delta = 0\}$. It has the coordinate system $(t_\alpha, \alpha \in \check{I})$, where

$$\check{I} := \left\{(\alpha_0, \alpha_1, \ldots, \alpha_{m+1}) \mid 0 \leqslant \alpha_e \leqslant d, \ \sum \alpha_e = d\right\}.$$

The polynomial expressions of Δ in terms of the variables t_α is in general huge. The variety $\mathsf{X} \subset \mathbb{P}^{m+1} \times \mathsf{T}$ is given by

$$\mathsf{X}: \ g = 0$$

$$\text{where } g := \sum_{\alpha \in \check{I}} t_\alpha x^\alpha$$

and $\mathsf{X} \to \mathsf{T}$ is the projection to T.

Definition 10.3. We call $\mathsf{X} \to \mathsf{T}$ as above, **the full family of smooth hypersurfaces**.

Let $\frac{\partial}{\partial t_\alpha}$, $\alpha \in \check{I}$ be the canonical vector fields over T. Since the fibers of $\mathsf{X} \to \mathsf{T}$ are smooth, we can take the Jacobian covering $\mathcal{U} = \{U_j\}_{j=0,1,2,\ldots,n+1}$ of X, given by

$$U_j : \frac{\partial g}{\partial x_j} \neq 0, \quad j = 0, 1, \ldots, n+1.$$

The vector field

$$\frac{\partial}{\partial t_\alpha} - \frac{\frac{\partial g}{\partial t_\alpha}}{\frac{\partial g}{\partial x_j}} \frac{\partial}{\partial x_j}$$

is defined in U_j and under the projection it is mapped to $\frac{\partial}{\partial t_\alpha}$. Therefore, the Kodaira–Spencer map is given by

$$\mathsf{K}(\frac{\partial}{\partial t_\alpha}) := \left\{ \frac{\partial g}{\partial t} \left((\frac{\partial g}{\partial x_j})^{-1} \frac{\partial}{\partial x_j} - (\frac{\partial g}{\partial x_i})^{-1} \frac{\partial}{\partial x_i} \right) \right\}_{i,j=0,1,\ldots,n+1}.$$

Conjecture 10.1. *For the full family of hypersurfaces $\mathsf{X} \to \mathsf{T}$, we have that $H^1(\mathsf{X}, \check{\Theta}_{\mathsf{X}/\mathsf{T}}) = 0$ except for hypersurface of dimension two and degree four. In this exceptional case it is a one dimensional $\mathbb{Q}$-vector space.*

The evidence to this conjecture comes from the classical presentation of Kodaira–Spencer map that we are going to discuss next.

10.4 Kodaira–Spencer map II

The content of Section 10.3 is not exactly the classical presentation of the Kodaira–Spencer map, and that is why, we landed on Conjecture 10.1 in a natural way. In the literature, see for instance the book of Voisin (2003, Lemma 6.15), one mainly take a point $0 \in \mathsf{T}$, set $X := \mathsf{X}_0$ and specialize (10.5) at a point $0 \in \mathsf{T}$:

$$\mathsf{K} : \mathbf{T}_0\mathsf{T} \to H^1(X, \Theta_X). \tag{10.8}$$

We start with

$$\begin{array}{ccccccccc} 0 & \to & \mathbf{T}X & \to & \mathbf{T}\mathsf{X}|_X & \to & \mathbf{N}_{X\subset\mathsf{X}} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbf{T}X & \to & \mathbf{T}\mathbb{P}^{n+1}\big|_X & \to & \mathbf{N}_{X\subset\mathbb{P}^{n+1}} & \to & 0. \end{array} \tag{10.9}$$

Here, $\mathbf{N}_{A\subset B}$ is the normal bundle of A inside B, the first down arrow map is the identity, the second is the derivation of the projection $\mathsf{X} \to \mathbb{P}^{n+1}$ and then restricted to X, and the third is the map induced in the quotient.

Proposition 10.3. *The normal bundle $\mathbf{N}_{X\subset\mathsf{X}}$ is trivial. More precisely, a trivialization of this bundle is given by the restriction of the derivation of the map $\mathsf{X} \to \mathsf{T}$ to the points of X. In particular, we have canonical identifications*

$$H^m(X, \mathbf{N}_{X\subset\mathsf{X}}) \;\cong\; \mathbf{T}_0\mathsf{T} \otimes_{\mathsf{k}} H^m(X, \mathcal{O}_X), \quad m = 0, 1, \ldots$$

Proof. For $x \in X$ the derivation $\mathbf{T}_x\mathsf{X} \to \mathbf{T}_{\pi(x)}\mathsf{T}$ of the map $\pi : \mathsf{X} \to \mathsf{T}$ restricted to to the tangent space of X inside X is zero as π sends X to the point 0 (recall that X is the fiber of π over 0). Taking the quotient we get the natural isomorphism $\mathbf{N}_{X\subset\mathsf{X}} \cong X \times \mathbf{T}_0\mathsf{T}$. Note that X is compact and hence global sections of $\mathcal{O}_X$ are only the constant functions. Therefore,

$$H^0(X, \mathbf{N}_{X\subset\mathsf{X}}) \cong \mathbf{T}_0\mathsf{T}.$$

□

We write the long exact sequence of (10.9) and use Proposition 10.3. We arrive at

$$\begin{array}{ccccccccc} H^0(X, \mathbf{T}\mathsf{X}|_X) & \to & \mathbf{T}_0\mathsf{T} & \xrightarrow{\mathsf{K}} & H^1(X, \mathbf{T}X) & \to & H^1(X, \mathbf{T}\mathsf{X}|_X) & \to & \mathbf{T}_0\mathsf{T} \otimes_{\mathsf{k}} H^1(X, \mathcal{O}_X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^0(X, \mathbf{T}\mathbb{P}^{n+1}|_X) & \xrightarrow{b} & H^0(X, \mathbf{N}_{X\subset\mathbb{P}^{n+1}}) & \xrightarrow{c} & H^1(X, \mathbf{T}X) & \to & H^1(X, \mathbf{T}\mathbb{P}^{n+1}|_X) & \to & H^1(X, \mathbf{N}_{X\subset\mathbb{P}^{n+1}}) \\ \uparrow a & & & & & & & & \\ H^0(\mathbb{P}^{n+1}, \mathbf{T}\mathbb{P}^{n+1}) & & & & & & & & \end{array} \tag{10.10}$$

The up arrow map is just the restriction map from $\mathbb{P}^{n+1}$ to X. We will need it later.

10.5 Kodaira–Spencer map for hypersurfaces

Let us now consider the parameter space T of smooth hypersurfaces $X = \{F = 0\}$ of degree d in the projective space $\mathbb{P}^{n+1}$ over a field k. For any point $t \in \mathsf{T}$ the tangent space $\mathbf{T}_t\mathsf{T}$ is canonically identified with $\mathsf{k}[x]_d$ for $x = (x_0, x_1, \ldots, x_{n+1})$ as follows. For $P \in \mathsf{k}[x]_d$ we consider the curve $F + tP$, $t \in \mathsf{k}$ in T whose derivation at $t = 0$ gives the corresponding vector in $\mathbf{T}_0\mathsf{T}$. Our main aim in this section is to prove the following theorem.

Theorem 10.1. *For a fixed hypersurface $X \subset \mathbb{P}^{n+1}, n > 1$ given by the homogeneous polynomial F of degree d and parameterized with $0 \in \mathsf{T}$ the Kodaira–Spencer map*

$$\mathsf{k}[x] \xrightarrow{\mathsf{K}} H^1(X, \Theta_X)$$

1. *is given by*

$$\mathsf{K}(P) := \left\{ P\left(\left(\frac{\partial F}{\partial x_j}\right)^{-1} \frac{\partial}{\partial x_j} - \left(\frac{\partial F}{\partial x_i}\right)^{-1} \frac{\partial}{\partial x_i} \right) \right\}_{i,j=0,1,\ldots,n+1}$$

   for $P \in \mathsf{k}[x]_d$.

2. *Its kernel is $J^F_d = \langle \frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_{n+1}} \rangle_d$.*

3. *For $(n,d) \neq (2,4)$ it is surjective and for $(n,d) = (2,4)$ its image is of codimension one and dimension 19 in $H^1(X, \Theta_X)$.*

The immediate consequence of the above theorem is that for $(n,d) \neq (2,4)$ the Kodaira–Spencer map induces an isomorphism:

$$\left(\frac{\mathsf{k}[x]}{J^F}\right)_d \overset{\mathsf{K}}{\cong} H^1(X, \Theta_X) \tag{10.11}$$

In order to prove Theorem 10.1 we need the following statements.

Proposition 10.4. *Let X be a smooth hypersurface of degree d in $\mathbb{P}^{n+1}$.*

1. *For $(n,d) \neq (1,3)$ we have*

$$H^1(\mathbb{P}^{n+1}, \mathbf{T}\mathbb{P}^{n+1}(X)) = 0,$$

   where $\mathbf{T}\mathbb{P}^{n+1}(X)$ is the sheaf of vector fields in $\mathbb{P}^{n+1}$ vanishing along X. In the exceptional case $(n,d) = (1,3)$ we have $\dim_{\mathsf{k}} H^1(\mathbb{P}^2, \mathbf{T}\mathbb{P}^2(X)) = 1$.

2. *For* $(n, d) \neq (2, 4)$ *we have*

$$H^1(X, \mathbf{T}\mathbb{P}^{n+1}|_X) = 0,$$

where $\mathbf{T}\mathbb{P}^{n+1}|_X$ *is the sheaf of sections of the restriction of the tangent space of* $\mathbb{P}^{n+1}$ *to* X. *In the exceptional case,* $(n, d) = (2, 4)$ *we have* $\dim_{\mathsf{k}} H^1(X, \mathbf{T}\mathbb{P}^3|_X) = 1$.

Proof. This follows from Bott's theorem, see Theorem 7.1, as follows. In this theorem we first put $p = 1, N = n + 1$, use Serre duality and and arrive at the formula

$$\dim_{\mathsf{k}}(H^{n+1-q}(\mathbb{P}^{n+1}, \mathbf{T}\mathbb{P}^{n+1}(-k-n-2))) = \begin{cases} 1 & \text{if } k = 0,\ q = 1, \\ \binom{-k+1}{-k}\binom{-k-1}{n} & \text{if } q = n+1,\ k < -n, \\ 0 & \text{otherwise.} \end{cases}$$

We have used the fact that the canonical bundle of $\mathbb{P}^{n+1}$ is $\mathcal{O}_{\mathbb{P}^{n+1}}(-n-2)$.

Proof of item 1. We must take $q = n$. This falls into the third case in Bott's formula as above except for $q = n = 1, k = 0$ which falls in the first case. Note that

$$\mathbf{T}\mathbb{P}^{n+1}(X) \cong \mathbf{T}\mathbb{P}^{n+1}(-d). \tag{10.12}$$

Proof of item 2. We write the long exact sequence of the restriction of $\mathbf{T}\mathbb{P}^{n+1}$ to X and arrive at

$$\cdots \to H^1(\mathbb{P}^{n+1}, \mathbf{T}\mathbb{P}^{n+1}) \to H^1(X, \mathbf{T}\mathbb{P}^{n+1}|_X) \to H^2(\mathbb{P}^{n+1}, \mathbf{T}\mathbb{P}^{n+1}(X)) \to \cdots$$

Bott's formula as above implies that $H^1(\mathbb{P}^{n+1}, \mathbf{T}\mathbb{P}^{n+1}) = 0$. Moreover $H^2(\mathbb{P}^{n+1}, \mathbf{T}\mathbb{P}^{n+1}(X)) = 0$ for $(n, d) \neq (2, 4)$, and for $(n, d) = (2, 4)$ we have

$$\dim_{\mathsf{k}} H^2(\mathbb{P}^{n+1}, \mathbf{T}\mathbb{P}^{n+1}(X)) = 1.$$

□

Proposition 10.5. *We have canonical identifications*

$$\begin{aligned} H^0(X, \mathbf{N}_{X\subset\mathsf{X}}) &\cong \mathbf{T}_0\mathsf{T} \cong \mathsf{k}[x]_d & (10.13)\\ H^0(X, \mathbf{N}_{X\subset\mathbb{P}^{n+1}}) &\cong \mathsf{k}[x]_d/\mathsf{k}F, \quad \textit{for } n > 1 & (10.14) \end{aligned}$$

and the map $H^0(X, \mathbf{N}_{X\subset\mathsf{X}}) \to H^0(X, \mathbf{N}_{X\subset\mathbb{P}^{n+1}})$ *after these identifications turns out to be the natural projection.*

Proof. The first isomorphism in (10.13) was proved in Proposition 10.3. We have the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(d)(X) \to \mathcal{O}_{\mathbb{P}^{n+1}}(d) \to \mathcal{O}_X(d) \to 0 \tag{10.15}$$

where $\mathcal{O}_{\mathbb{P}^{n+1}}(d)(X)$ is the the sheaf of section of $\mathcal{O}_{\mathbb{P}^{n+1}}(d)$ vanishing along X. We have also a canonical isomorphism $\mathcal{O}_X \cong \mathcal{O}_{\mathbb{P}^{n+1}}(d)(X)$, $s \mapsto sF$. Knowing that $H^1(X, \mathcal{O}_X) = 0$ for $n > 1$, the long exact sequence of the above short exact sequence gives us

$$0 \to \mathsf{k}F \to \mathsf{k}[x]_d \to H^0(X, \mathbf{N}_{X \subset \mathbb{P}^{n+1}}) \to 0$$

where we have used the isomorphism:

$$\mathbf{N}_{X \subset \mathbb{P}^{n+1}} \cong \mathcal{O}_X(d).$$

This finishes the proof. □

Remark 10.1. The long exact sequence of (10.15) also implies

$$H^i(X, \mathbf{N}_{X \subset \mathbb{P}^{n+1}}) = 0, \quad i \geqslant 1.$$

Proposition 10.6. *The* k*-vector space $H^0(X, \mathbf{T}\mathbb{P}^{n+1})$ is generated by $x_j \frac{\partial}{\partial x_i}$ and the composition $b \circ a$*

$$H^0(\mathbb{P}^{n+1}, \mathbf{T}\mathbb{P}^{n+1}) \xrightarrow{a} H^0(X, \mathbf{T}\mathbb{P}^{n+1}|_X) \xrightarrow{b} H^0(X, \mathbf{N}_{X \subset \mathbb{P}^{n+1}}) \cong \mathsf{k}[x]_d$$

is given by

$$x_j \frac{\partial}{\partial x_i} \to x_j \frac{\partial F}{\partial x_i}. \tag{10.16}$$

Proof. The maps a and b are just natural projections, and so the result follows from noting that the isomorphism $\mathbf{N}_{X \subset \mathbb{P}^{n+1}} \cong \mathcal{O}_X(d)$ is given locally by (10.16). □

Proof of Theorem 10.1. We have many canonical identifications and vanishings so that (10.10) becomes

$$
\begin{tikzcd}
H^0(X, \mathsf{T}X|_X) \arrow[r] \arrow[d] & \mathsf{k}[x]_d \arrow[r, "\mathsf{K}"] \arrow[d] & H^1(X, \mathbf{T}X) \arrow[r] \arrow[d] & H^1(X, \mathsf{T}X|_X) \arrow[r] \arrow[d] & 0 \arrow[d] \\
H^0(X, \mathbf{T}\mathbb{P}^{n+1}|_X) \arrow[r, "b"] & \mathsf{k}[x]_d/\mathsf{k}\cdot F \arrow[r, "c"] & H^1(X, \mathbf{T}X) \arrow[r] & H^1(X, \mathbf{T}\mathbb{P}^{n+1}|_X) \arrow[r] & H^1(X, \mathbf{N}_{X\subset\mathbb{P}^{n+1}}) \\
\langle x_i \frac{\partial}{\partial x_j} \rangle_{i,j=1,2,\ldots,n+1} \arrow[u, "a"] & & & &
\end{tikzcd}
\tag{10.17}
$$

The last zero in the above diagram is due to the vanishing $H^1(X, \mathcal{O}_X) = 0$. For this we assume that $n \neq 1$, that is, X is not a curve.

Proof of item 1. The proof is in Section 10.3.

Proof of item 2. The second and third down arrows in the above diagram are respectively surjective and isomorphism and so, the kernel of K projected into $\mathsf{k}[x]/\mathsf{k} \cdot F$ is the same as the kernel of c, and hence, the image of b. Since a is a surjective map (Proposition 10.4, Item 1), by Proposition 10.6 we conclude the first statement.

Proof of item 3. By the second part of Proposition 10.4 we know that for $(n, d) \neq (2, 4)$ we have $H^1(X, \mathbf{T}\mathbb{P}^{n+1}|_X) = 0$. Moreover, the Kodaira–Spencer map and c have the same image. This implies the result. In particular, this argument implies that $H^1(X, \mathsf{T}X|_X) = 0$. For $(n, d) = (2, 4)$ we have $\dim_{\mathsf{k}} H^1(X, \mathbf{T}\mathbb{P}^{n+1}|_X) = 1$ which finishes the proof. $\square$

10.6 A theorem of Griffiths

Recall the definition of the Kodaira–Spencer map in both Sections 10.3 and 10.4.

Theorem 10.2 (Griffiths)**.** *There is a canonical map*

$$\bar{\nabla}_i : H^1(X, \Theta_X) \to \hom\left(H^{m-i}(X, \Omega_X^i), H^{m-i+1}(X, \Omega_X^{i-1})\right) \tag{10.18}$$

such that the IVHS map (10.4) *factors through the Kodaira–Spencer map* (10.8)*, that is,*

$$\nabla_i = \bar{\nabla}_i \circ \mathsf{K}$$

Moreover, if $H^1(\mathsf{X}, \Theta_{\mathsf{X}}) = 0$ *there is also a canonical map*

$$\bar{\nabla}_i : H^1(\mathsf{X}, \Theta_{\mathsf{X}/\mathsf{T}}) \to \hom\left(H^{m-i}(\mathsf{X},\ \Omega^i_{\mathsf{X}/\mathsf{T}}), H^{m-i+1}(\mathsf{X},\ \Omega^{i-1}_{\mathsf{X}/\mathsf{T}})\right) \quad (10.19)$$

such that the IVHS map (10.3) *factors through the Kodaira–Spencer map* (10.5).

Proof. The proof of the first part must follow from the second part but we do not know how to do it. In Voisin (2002, Section 5.1.2) it is said that (10.18) is the cup product map, but we do not see the relation. One might look for Griffiths' original proof. This is also reproduced in Voisin (2003, Section 10.2.3). Below, we reproduce the proof only for the second part. It might indicate what kind of changes must be done, in order to have a proof of first part.

Let $v = \{v_{ij}\} \in H^1(\mathsf{X}, \Theta_{\mathsf{X}/\mathsf{T}})$ and $\omega \in H^{m-i}(\mathsf{X},\ \Omega^i_{\mathsf{X}/\mathsf{T}})$. We want to define $\bar{\nabla}_i(v)(\omega) \in H^{m-i+1}(\mathsf{X},\ \Omega^{i-1}_{\mathsf{X}/\mathsf{T}})$. We take an acyclic covering $\mathcal{U} := \{U_i\}_{i \in I}$ of X and a cocycle $\check{\omega} \in (\Omega_{\mathsf{X}})^i_{m-i}$ which maps to ω under the canonical projection $\Omega^i_{\mathsf{X}} \to \Omega^i_{\mathsf{X}/\mathsf{T}}$. Since ω is δ-closed, we have

$$\delta\check{\omega} \in \Omega^1_{\mathsf{T}} \wedge (\Omega^{i-1}_{\mathsf{X}})_{m-i+1}. \quad (10.20)$$

Since $\Theta_{\mathsf{X}/\mathsf{T}} \subset \Theta_{\mathsf{X}}$, we can consider v as an element in $H^1(\mathsf{X}, \Theta_{\mathsf{X}})$. However, we have assumed that this cohomology is zero. Therefore, we have vector fields v_i in U_i such that $v_{ij} = v_j - v_i$. In any intersections $U_0 \cap U_1 \cap \cdots \cap U_{i_m} \cap U_{m-i+1}$ we have

$$0 = i_{v_{i_a i_b}}(\delta\check{\omega}) = i_{v_{i_b}}(\delta\check{\omega}) - i_{v_{i_a}}(\delta\check{\omega}).$$

The first equality is in $\Omega^{i-1}_{\mathsf{X}/\mathsf{T}}$ and it follows from the following. We write $\delta\check{\omega} = \alpha \wedge \beta$, where α is a section of Ω^1_{T} and β is a section of $\Omega^{i-1}_{\mathsf{X}}$, see (10.20). We have

$$i_{v_{ij}}(\alpha \wedge \beta) = i_{v_{ij}}\alpha \wedge \beta + (-1)^1 \alpha \wedge i_{v_{ij}}\beta = -\alpha \wedge i_{v_{ij}}\beta.$$

where we have used $i_{v_{ij}}\alpha = 0$ because v_{ij} is tangent to the fibers of X/T. Therefore, $\{i_{v_{i_a}}(\delta\check{\omega})\}$ does not depend on the choice of $a = 0, 1, , \ldots, m - i + 1$. This gives us the desired element in $H^{m-i+1}(\mathsf{X},\ \Omega^{i-1}_{\mathsf{X}/\mathsf{T}})$. □

10.7 IVHS for hypersurfaces

Here is the main theorem of this section.

Theorem 10.3. *The infinitesimal variation of Hodge structures*

$$H^1(X,\Theta_X)\times H^{n-k}(X,\Omega_X^k)_0\to H^{n-k+1}(X,\Omega_X^{k-1})_0$$

for a smooth hypersurface $X=\{F=0\}$ of degree d and dimension n is given (up to some non-zero constant factor) by the multiplication of polynomials

$$\begin{aligned}(\mathbb{C}[X]/J^F)_d\times(\mathbb{C}[X]/J^F)_{(k+1)d-n-2}&\to(\mathbb{C}[X]/J^F)_{(k+2)d-n-2}\\ (P,Q)&\mapsto PQ,\end{aligned}\tag{10.21}$$

provided that $(n,d)\neq(2,4)$. *In this exceptional case, the same statement is true if we replace $H_1(X,\Theta_X)$ with the image of the Kodaira–Spencer map.*

Proof. The identifications of involved cohomologies with the homogeneous pieces of the ring $\mathbb{C}[x]/J^F$ is done in Theorem 7.3 and Theorem 10.1. Note that for n even and $k=\frac{n}{2}$, we have $H^k(X,\Omega_X^{n-k})=H^k(X,\Omega_X^{n-k})_0+\theta^{\frac{n}{2}}$. For all other n and k, $H^k(X,\Omega_X^{n-k})_0=H^k(X,\Omega_X^{n-k})$. It is now enough to argue that the IVHS map become multiplication of polynomials. This follows from the definition of the Gauss–Manin connection for hypersurfaces in (9.20). □

Remark 10.2. Let us consider the following IVHS

$$\bar{\nabla}:H^1(X,\Theta_X)\times H^1(X,\Omega_X^1)\to H^2(X,\Omega_X^0).$$

The polarization θ is a flat section of the Gauss–Manin connection. Therefore, it is natural to define

$$H^1(X,\Theta_X)_0:=\left\{\,a\in H^1(X,\Theta_X)\ \mid\ \bar{\nabla}(a,\theta)=0\,\right\}\tag{10.22}$$

In the case of a hypersurface $X\subset\mathbb{P}^{n+1}$, for $n\geqslant 3$ by Lefschetz theorems, and for $n=1$ by the Hodge decomposition of the top cohomology, we know that we have $H^2(X,\Omega_X^0)=0$. Therefore, in this case we have

$$H^1(X,\Theta_X)_0=H^1(X,\Theta_X).$$

Remark 10.3. A hypersurface $X\subset\mathbb{P}^3$ of degree 4 is called a K3 surface. Using Serre duality we have

$$H^1(X,\Theta_X)\cong H^1(X,\Omega_X^1)$$

Note that Ω_X^1 is dual to Θ_X and Ω_X^2 is the trivial line bundle. We find that the dimension of $H^1(X,\Theta_X)$ is the Hodge number $h^{1,1}$ of X. This is $h^{1,1}=20$. From another side $\dim(\mathbb{C}[x]/J^F)_4=19$. We conclude that the complex moduli space of a K3 surface is of dimension 20. Algebraic deformations correspond to a 19 dimensional subspace of this space.

11 Hodge cycles and Artinian Gorenstein algebras

Although intersection theory in manifolds is mainly of historical interest today, it is still of some value in aiding our geometric intuition about cocycles and cup products, at least in the case of manifolds. (W. S. Massey (1991, page 392)).

11.1 Introduction

In this chapter to any Hodge cycle of a smooth hypersurface we attach an Artinian Gorenstein algebra, and in this way many problems related to Hodge cycles, boil down to problems in commutative algebra. Some of the material of the present text are inspired from Dan (2017), Movasati and Villaflor (2018), Otwinowska (2002, 2003), Villaflor (n.d.[a]), and Voisin (2003). In this chapter, k is a field of characteristic zero, but not necessarily algebraically closed. Once we talk about Hodge cycles, it is assumed that $\mathsf{k} \subset \mathbb{C}$.

11.2 Artinian Gorenstein algebras

We have already introduced Artinian Gorenstein algebras and ideals in Section 8.2. We start by recalling their definitions.

Definition 11.1. Let $n \in \mathbb{N}$, and $I \subseteq \mathsf{k}[x_0, \ldots, x_{n+1}]$ be a homogeneous ideal. We say I is **Artinian Gorenstein** if $R := \mathsf{k}[x_0, \ldots, x_{n+1}]/I$ for some $\sigma \in \mathbb{N}$ satisfies

1. $\dim_{\mathsf{k}} R_\sigma = 1$.
2. For every $0 \leqslant i \leqslant \sigma$ the multiplication map
$$R_i \times R_{\sigma-i} \to R_\sigma$$
is a perfect pairing.
3. $R_e = 0$ for $e > \sigma$.

We also say that R is an **Artinian Gorenstein algebra**. The number σ is called the **socle** of I and R. In the literature, R_σ is also called the socle of R.

The above definition is mainly inspired from the following example, which is Macaulay's Theorem 8.1.

Example 11.1. Let $f_0, \ldots, f_{n+1} \in \mathsf{k}[x_0, \ldots, x_{n+1}]$ be homogeneous polynomials with $\deg(f_i) = d_i$ and

$$\{f_0 = \cdots = f_{n+1} = 0\} = \varnothing \subseteq \mathbb{P}^{n+1}.$$

By Theorem 8.1, the following

$$R := \frac{\mathsf{k}[x_0, \ldots, x_{n+1}]}{\langle f_0, \ldots, f_{n+1} \rangle},$$

is an Artinian Gorenstein algebra of socle $\sigma := \sum_{i=0}^{n+1}(d_i - 1)$.

For a proof see Voisin (2003, Theorem 6.19). Note that for the proof we can assume that $\mathsf{k} = \mathbb{C}$. For an arbitrary k, we first note that Theorem 8.1 is independent of the field extension, and we can take k small enough such that k can be embedded in $\mathbb{C}$.

Example 11.2. Let $F \in \mathsf{k}[x_0, x_1, \ldots, x_{n+1}]$ be a homogeneous polynomial of degree d such that its Jacobian ideal $J^F := \langle \frac{\partial F}{\partial x_0}, \frac{\partial F}{\partial x_1}, \cdots \frac{\partial F}{\partial x_{n+1}} \rangle$ has an isolated singularity at the origin $0 \in \mathbb{C}^{n+2}$. This is our main example of an Artinian Gorenstein ideal which is of socle $\sigma := (n+2)(d-2)$. For $F := x_0^d + x_1^d + \cdots + x_{n+1}^d$ this is

$$J^F = \langle x_0^{d-1}, x_1^{d-1}, \ldots, x_{n+1}^{d-1} \rangle.$$

In this case R_σ is generated by $x_1^{d-2} x_2^{d-2} \cdots x_{n+1}^{d-2}$.

Definition 11.2. For an ideal I of a ring R and some $P \in \mathsf{R}$, the **quotient ideal** is defined as follows:

$$(I : P) := \{Q \in \mathsf{R} : PQ \in I\}.$$

The following elementary proposition will be used frequently.

Proposition 11.1. *We have*

1. *If I is Artinian Gorenstein with socle σ, and $P \in \mathsf{k}[x_0, \ldots, x_{n+1}]_\mu \setminus I_\mu$, then $(I : P)$ is Artinian Gorenstein of socle $\sigma - \mu$.*

2. *If $I_1, I_2 \subset \mathsf{k}[x_0, x_1, \ldots, x_{n+1}]$ are two Artinian Gorenstein ideals with the same socle σ and $(I_1)_\sigma = (I_2)_\sigma$ then $I_1 = I_2$.*

Proof. The proof of the first statement is as follows. Let $\mathsf{R} = \mathsf{k}[x_0, \ldots, x_{n+1}]$. The first item in the definition of an Artinian Gorenstein algebra for $\mathsf{R}/(I : P)$ follows from the fact that $(\mathsf{R}/I)_\mu \times (\mathsf{R}/I)_{\sigma-\mu}$ is a perfect pairing. This also implies that $(\mathsf{R}/(I : P))_a \times (\mathsf{R}/(I : P))_{\sigma-\mu-a} \to (\mathsf{R}/(I : P))_{\sigma-\mu}$ is a perfect pairing.

For the second item we first observe that $I_1 \cap I_2$ is also Artinian Gorenstein. We then prove the same statement with the additional hypothesis $I_1 \subset I_2$. We only need to prove that the pairing $(\mathsf{k}[x]/I_1 \cap I_2)_a \times (\mathsf{k}[x]/I_1 \cap I_2)_{\sigma-a} \to (\mathsf{k}[x]/I_1 \cap I_2)_\sigma$ is perfect. If it not then we have $P \in (\mathsf{k}[x]_a$ with $P \notin (I_1)_a$ or $P \notin (I_2)_a$ whose product with all $Q \in \mathsf{k}[x]_{\sigma-a}$ is in $(I_1)_\sigma = (I_2)_\sigma$. If for instance, $P \notin (I_1)_a$ then this contradicts the same property for I_1. Now let us assume that $I_1 \subset I_2$. If $Q \in I_2$ and $Q \notin I_1$ then Q gives a non-zero element in R/I_1 and so there is an element $P \in \mathsf{R}/I_1$ such that PQ is a non-zero element of the socle $(\mathsf{R}/I_1)_\sigma$. This is equal to $(\mathsf{R}/I_2)_\sigma$ which implies that PQ is zero, and hence, we get a contradiction. □

Remark 11.1. The intersection of two Artinian Gorenstein ideal I_1, I_2 is not necessarily Artinian Gorenstein. For instance, consider the case in which I_1 and I_2 have the same socle and $(I_1)_\sigma \neq (I_2)_\sigma$ which implies that $\dim_{\mathbb{C}}(\mathbb{C}[x]/I_1 \cap I_2)_\sigma = 2$. Note also that for two polynomials $P, Q \in \mathbb{C}[x]_\mu$ and an Artinian Gorenstein ideal I we have the inclusion

$$(I : P) \cap (I : Q) \subset (I : P + Q)$$

which is a strict inclusion if $(I : P)_{\sigma-\mu} \neq (I : Q)_{\sigma-\mu}$.

11.3 Artinian Gorenstein algebra attached to a Hodge cycle

Let $X = \{F = 0\} \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d and even dimension n define over a field $\mathsf{k} \subset \mathbb{C}$, and

$$\sigma := (\frac{n}{2} + 1)(d - 2).$$

Let also Z_∞ be the intersection of a linear $\mathbb{P}^{\frac{n}{2}+1}$ with X and $[Z_\infty] \in H_n(X, \mathbb{Z})$ be the induced element in homology (the polarization). It is also well-known that $H_n(X, \mathbb{Z})$ has no torsion, see for instance Movasati (2021, Section 5.5)

Definition 11.3. For every Hodge cycle $\delta \in H_n(X, \mathbb{Z})/\mathbb{Z}[Z_\infty]$ we define its **associated Artinian Gorenstein ideal** as the homogeneous ideal

$$I(\delta)_a := \left\{ Q \in \mathbb{C}[x]_a \,\middle|\, \int_\delta \mathrm{res}\left(\frac{QP\Omega}{F^{\frac{n}{2}+1}}\right) = 0, \ \ \forall P \in \mathbb{C}[x]_{\sigma-a} \right\}.$$

We define the **Artinian Gorenstein algebra of** δ as $R(\delta) := \mathbb{C}[x]/I(\delta)$. By definition $I(\delta)_m = \mathbb{C}[x]_m$ for all $m \geqslant \sigma + 1$ and so $R(\delta)_a = 0$.

Definition 11.4. For a non-zero Hodge cycle $\delta \in H_n(X, \mathbb{Z})/\mathbb{Z}[Z_\infty]$, its **canonical associated polynomial** $\widetilde{P}_\delta \in \mathbb{C}[x]_{(\frac{n}{2}+1)(d-2)}$ is the one such that

$$\delta^{\mathsf{pd}} = \mathrm{res}\left(\frac{\widetilde{P}_\delta \Omega}{F^{\frac{n}{2}+1}}\right) \in F^{\frac{n}{2}} H^n_{\mathrm{dR}}(X). \tag{11.1}$$

Proposition 11.2. *We have*

$$I(\delta) = (J^F : \widetilde{P}_\delta)$$

and hence $I(\delta)$ is Artinian Gorenstein of socle $\sigma := (\frac{n}{2}+1)(d-2)$.

Proof. We first show that $J^F \subset I(\delta)$. This follows from

$$d\left(\frac{\omega}{F^{i-1}}\right) = \frac{d\omega}{F^{i-1}} - (i-1)\frac{dF \wedge \omega}{F^i} \tag{11.2}$$

$$= \frac{d\omega + d\cdot(i-1)\cdot P \cdot \iota_{\frac{\partial}{\partial x_j}} dx}{F^{i-1}} - (i-1)\frac{\frac{\partial F}{\partial x_j} P\Omega}{F^i} \tag{11.3}$$

for $i = \frac{n}{2}+1$ and $\omega = P\iota_{\frac{\partial}{\partial x_j}}\Omega,\ P \in \mathbb{C}[x]_{(i-1)\cdot d-n-1}$. Since δ is a Hodge cycle, after taking the residue and then integrating the above equality over δ we conclude that

$$\int_\delta \mathrm{res}\left(\frac{\frac{\partial F}{\partial x_j} P\Omega}{F^{\frac{n}{2}+1}}\right) = 0,$$

which implies that $\frac{\partial F}{\partial x_j} \in I(\delta),\quad j = 0,1,2,\dots,n+1$ (another way to see the above equality is noticing that by Griffiths' Theorem 7.3 the differential forms lies in $F^{\frac{n}{2}+1}$). Note that we cannot use directly the equality (11.2) as ι_E of its ingredients are not zero (and of the ingredients of the second equality are zero), and hence, they do not give us differential forms in the projective space $\mathbb{P}^{n+1}$. By Poincaré duality we have

$$\int_\delta \mathrm{res}\left(\frac{QP\Omega}{F^{\frac{n}{2}+1}}\right) = \int_X \frac{QP\Omega}{F^{\frac{n}{2}+1}} \cup \frac{\widetilde{P}_\delta\Omega}{F^{\frac{n}{2}+1}} \tag{11.4}$$

and by Theorem 7.6 the right hand side of the above equality is zero if and only if $QP\widetilde{P}_\delta \in J^F$. By Theorem 8.1, the Jacobian ideal J^F is Artinian Gorenstein, and in particular the multiplication is a perfect pairing. It follows that (11.4) is zero for all $P \in \mathbb{C}[x]_{\sigma-a}$ if and only if $Q\widetilde{P}_\delta \in J^F$. □

Remark 11.2. When δ is of complete intersection type, the relation between the canonical polynomial of δ and its associated polynomial defined in Definition 8.2 is

$$\widetilde{P}_\delta \equiv \frac{\frac{n}{2}!(-1)^{\frac{n}{2}+1}}{\deg(X)} P_\delta \pmod{J^F}. \tag{11.5}$$

In particular

$$(J^F : \widetilde{P}_\delta) = (J^F : P_\delta). \tag{11.6}$$

Proposition 11.3. *If $\delta = [Z]$ and Z is given by the ideal $\mathcal{I}_Z$ then*

$$\mathcal{I}_Z \subset I(\delta). \tag{11.7}$$

In particular, if the primitive part of the cycles $[Z_1], [Z_2], \ldots, [Z_k]$ form a one dimensional subspace of $H_n(X, \mathbb{Q})_0$ then

$$\sum_{i=1}^{k} \mathcal{I}_{Z_i} \subset I(\delta). \tag{11.8}$$

Proof. For (11.7) we use Carlson–Griffiths Theorem 7.4. In the equality (7.7) if $P = Q\widetilde{P}$ and $Q \in \mathcal{I}_Z$ then its right hand side restricted to Z is identically zero and so $Q \in I(\delta)$. The second part follows from the first part and the fact that $I(\delta)$ depends only the class of δ in $\mathbb{P}\left(H_n(X, \mathbb{Q})/\mathbb{Q}[Z_\infty]\right)$. □

Definition 11.5. An algebraic cycle Z is called **perfect** (resp. perfect at level $m \in \mathbb{N}$) if there are other algebraic cycles Z_i, $i = 2, \ldots, k$, $Z_1 = Z$ as in the above definition such that (11.8) is an equality (resp. equality for degree $\leqslant m$ pieces).

Proposition 11.4. *Complete intersection algebraic cycles inside hypersurfaces are perfect.*

Proof. We first recall the definition and related notations of a complete intersection algebraic cycle. Assume that $n \geqslant 2$ is even and $F \in \mathbb{C}[x]_d$ is of the following format:

$$F = f_1 f_{\frac{n}{2}+2} + f_2 f_{\frac{n}{2}+3} + \cdots + f_{\frac{n}{2}+1} f_{n+2}, \ \ f_i \in \mathbb{C}[x]_{d_i}, \ \ f_{\frac{n}{2}+1+i} \in \mathbb{C}[x]_{d-d_i}, \tag{11.9}$$

where $1 \leqslant d_i < d$, $i = 1, 2, \ldots, \frac{n}{2} + 1$ is a sequence of natural numbers. A complete intersection algebraic cycle Z is given by the ideal $\langle f_1, f_2, \ldots, f_{\frac{n}{2}+1} \rangle$. In $H_n(X, \mathbb{Z})$ the homology classes of all cycles

$$\check{Z} : g_1 = g_2 = \cdots = g_{\frac{n}{2}+1} = 0, \ g_i \in \{f_i, f_{\frac{n}{2}+1+i}\} \tag{11.10}$$

are equal up to sign and up to $\mathbb{Z}[Z_\infty]$. This with Proposition 11.3 imply that

$$J := \sum_{\check{Z}} \mathcal{I}_{\check{Z}} = \langle f_1, f_2, \ldots, f_{n+2} \rangle \subset (I_\delta).$$

where $\check{Z}$ runs through all algebraic cycles in (11.10). Now Macaulay's Theorem 8.1 implies that the ideal J is also Artinian Gorenstein of socle degree $\sigma := d - d_1 + d_1 + \cdots + d - d_{\frac{n}{2}+1} - d_{\frac{n}{2}+1} = (\frac{n}{2}+1)d$ and so $I(\delta) = J$. □

Remark 11.3. Meantime the present text was being written, the first author together with E. Sertöz analyzed the perfectness of curves inside surfaces. For instance, they prove the following. Let Z be a twisted cubic in a smooth quartic surface $X \subset \mathbb{P}^3$. The class $[Z]$ is perfect at level 2 if and only if $I([Z])_1 = 0$ and $\dim I([Z])_2 = 3$. In Cifani, Pirola, and Schlesinger (2021) the authors investigate the perfectness further and prove that arithmetically Cohen–Macaulay curves inside surfaces in $\mathbb{P}^3$ are perfect. They also prove that a smooth rational curve of degree 4 contained in a smooth surface $X \subset \mathbb{P}^3$ of degree 4 is not perfect at level 3 (and hence not perfect). This rises the question of geometric meaning of polynomials in $I(\delta)$ which are not in any ideal $\mathcal{I}_{Z_i}$.

11.4 Field of definition of Hodge cycles

In this section we define the field of definition of Hodge cycles. For this we assume that the projective variety X is defined over a subfield k of $\mathbb{C}$. In principle, all the notions that we are going to introduce depend on the embedding $\mathsf{k} \subset \mathbb{C}$.

Definition 11.6. Let X be a smooth hypersurface defined over the field $\mathsf{k} \subset \mathbb{C}$. For a Hodge cycle $\delta \in H_n(X,\mathbb{Z})/\mathbb{Z}[Z_\infty]$ let $\widetilde{P}_\delta \in \mathbb{C}[x]_{(\frac{n}{2}+1)(d-1)}$ be the canonical polynomial defined in (11.1). The subfield of $\mathbb{C}$ generated over k by the coefficients of $\widetilde{P}_\delta$ is called the **field of definition of** δ. We denote it by k_δ. By definition, k_δ is the smallest field such that $\widetilde{P}_\delta \in \mathsf{k}_\delta[x]$.

We have the following natural isomorphism of one dimensional vector spaces:

$$H(\delta) : R_\sigma \to \mathbb{C},\ P \mapsto \frac{1}{(2\pi i)^{\frac{n}{2}}} \int_\delta \frac{P\Omega}{F^{\frac{n}{2}+1}} \tag{11.11}$$

and hence we get:

$$H(\delta) : R_a \times R_{\sigma-a} \to R_\sigma \cong \mathbb{C}.$$

Proposition 11.5. *If $\delta = [Z]$ is an algebraic cycle, and both Z and X are defined over a field $\mathsf{k} \subset \mathbb{C}$ then $\mathsf{k}_\delta = \mathsf{k}$ and $I(\delta)$, $R(\delta)$ and $H(\delta)$ are defined over k.*

Proof. It follows from Proposition 5.11 that $P_\delta \in \mathsf{k}[x]$. The Jacobian ideal is defined over k and so $I(\delta)$, $R(\delta)$ are defined over k. In a similar way, because of Proposition 5.11 $H(\delta)$ is defined over k. □

Knowing Proposition 11.5, the following is a consequence of the Hodge conjecture.

Conjecture 11.1. *If F is defined over a field $\mathsf{k} \subset \mathbb{C}$ then $I(\delta)$, $R(\delta)$ and $H(\delta)$ are defined over a finite algebraic extension of k.*

It seems that the algebraic extension in Section 3.3 highly depends on the underlying variety X. The following particular case might indicate some general phenomena.

Theorem 11.1 (Deligne, Milne, et al. (1982))**.** *If F is the Fermat polynomial, and hence defined over $\mathbb{Q}$, then $I(\delta)$, $R(\delta)$ and $H(\delta)$ are defined over an abelian extension of $\mathbb{Q}$.*

A major problem in our way is that for a generic F there is no non-zero primitive Hodge cycle, and we might be interested to translate this into non-existence of certain Artinian Gorenstein rings of socle degree σ for such polynomials. Note that Conjecture 11.1 and Theorem 11.1 are the only manifestation of the fact that δ has coefficients in $\mathbb{Z}$.

11.5 A quotient ideal over a sum of two polynomials

We close this chapter with a proposition we will use in the proof of Theorem 12.4.

Proposition 11.6. *Consider the ideal $I := \langle x_0^{d-1}, \dots, x_{2r-1}^{d-1} \rangle \subseteq \mathbb{C}[x_0, \dots, x_{2r-1}]$. Let $d \geqslant 3$, and $\beta_1, \beta_2, c_1, c_2 \in \mathbb{C}^\times$ with $\beta_1 \neq \beta_2$. For $i = 1, 2$, define*

$$R_i := c_i \cdot \prod_{j=1}^{r} \frac{(x_{2j-2}^{d-1} - (\beta_i x_{2j-1})^{d-1})}{(x_{2j-2} - \beta_i x_{2j-1})}.$$

Then

$$(I : R_1)_e \cap (I : R_2)_e = (I : R_1 + R_2)_e, \tag{11.12}$$

if and only if $e \neq (d-2) \cdot r$.

Proof. First of all, note that $(I : R_1)$, $(I : R_2)$ and $(I : R_1 + R_2)$ are Artinian Gorenstein ideals of socle $(d-2)\cdot r$. In consequence,

$$(I : R_1) \cap (I : R_2) \neq (I : R_1 + R_2).$$

Otherwise, we would have $(I : R_1 + R_2) \subseteq (I : R_1)$, which implies

$$(I : R_1) = (I : R_1 + R_2) = (I : R_2),$$

a contradiction. Therefore, in order to prove the proposition, it is enough to prove (11.12) for $e \neq (d-2)\cdot r$. If $e > (d-2)\cdot r$, the equality (11.12) is trivial since $(d-2)\cdot r$ is the socle of the three ideals. If $e < (d-2)\cdot r$, we claim (11.12) reduces to the case $e = (d-2)\cdot r - 1$. In fact, if we assume (11.12) fails for some $e < (d-2)\cdot r$, we can choose

$$p \in (I : R_1 + R_2)_e \setminus (I : R_1)_e. \tag{11.13}$$

Since $(I : R_1)$ is Artinian Gorenstein of socle $(d-2)\cdot r$, the perfect pairing property implies that we can find a degree $(d-2)\cdot r - e$ monomial $x^i = x_0^{i_0}\cdots x_{2r-1}^{i_{2r-1}}$ such that

$$x^i \cdot p \in (I : R_1 + R_2)_{(d-2)\cdot r} \setminus (I : R_1)_{(d-2)\cdot r}. \tag{11.14}$$

Since $\deg(x^i) > 0$, there exist some $i_j > 0$, then (11.13) and (11.14) imply that

$$\frac{x^i}{x_j}\cdot p \in (I : R_1 + R_2)_{(d-2)\cdot r-1} \setminus (I : R_1)_{(d-2)\cdot r-1},$$

and so (11.12) would fail for $e = (d-2)\cdot r - 1$, as claimed. Therefore, we just consider the case $e = (d-2)\cdot r - 1$. It is enough to show that $(I : R_1 + R_2)_e \subseteq (I : R_1)_e \cap (I : R_2)_e$. Take $p \in (I : R_1 + R_2)_e$. Without loss of generality we may assume it can be written as

$$p = \sum_{k \text{ even}} \sum_{l=0}^{d-3} x_k^l x_{k+1}^{d-3-l} p_{k,l},$$

where each $p_{k,l}$ does not depend on x_k and x_{k+1}, and is a $\mathbb{C}$-linear combination of monomials of the form $x_0^{i_0}\cdots x_{k-1}^{i_{k-1}} x_{k+2}^{i_{k+2}}\cdots x_{2r-1}^{i_{2r-1}}$ with $i_{2j-2}+i_{2j-1} = d-2$, for all $j \in \{1,\dots,r\}\setminus\{\frac{k}{2}+1\}$. For every k and l, and $i = 1, 2$, there exist a constant $a_{k,l,i} \in \mathbb{C}$ such that

$$p_{k,l}\frac{R_i}{(x_k^{d-2} + x_k^{d-3}(\beta_i x_{k+1}) + \cdots + (\beta_i x_{k+1})^{d-2})} \equiv a_{k,l,i}\frac{(x_0\cdots x_{2r-1})^{d-2}}{(x_k x_{k+1})^{d-2}},$$

modulo $\langle x_0^{d-1}, \ldots, x_{k-1}^{d-1}, x_{k+2}^{d-1}, \ldots, x_{2r-1}^{d-1} \rangle$. Then

$$pR_i \equiv (x_0 \cdots x_{2r-1})^{d-2} \sum_{k \text{ even}} \left(\frac{1}{x_k} \sum_{l=0}^{d-3} a_{k,l,i} \beta_i^{l+1} + \frac{1}{x_{k+1}} \sum_{l=0}^{d-3} a_{k,l,i} \beta_i^{l} \right),$$

modulo I. Since $p \cdot (R_1 + R_2) \in I$ we conclude that

$$\sum_{l=0}^{d-3} a_{k,l,1} \beta_1^{l+1} + \sum_{l=0}^{d-3} a_{k,l,2} \beta_2^{l+1} = \sum_{l=0}^{d-3} a_{k,l,1} \beta_1^{l} + \sum_{l=0}^{d-3} a_{k,l,2} \beta_2^{l} = 0.$$

Since $\beta_1 \neq \beta_2$, this implies

$$\sum_{l=0}^{d-3} a_{k,l,1} \beta_1^{l} = \sum_{l=0}^{d-3} a_{k,l,2} \beta_2^{l} = 0,$$

and so $pR_i \in I$ for $i = 1, 2$. $\square$

12 Some components of Hodge loci

One may ask whether imposing a certain Hodge class upon a generic member of an algebraic family of polarized algebraic varieties amounts to an algebraic condition upon the parameters. A. Weil (1977, page 429).

12.1 Introduction

For a family of algebraic varieties $X \to V$ one may ask for the description of sub locus $\check{W} \subset V$ such that the varieties $X_t,\ t \in W$ enjoy certain algebraic properties which is not satisfied by a generic X_t, for instance, one can impose the existence of an algebraic cycle $Z_t \subset X_t$. Instead of the parameter space V, one can consider a moduli space for which the algebraic family $X \to V$ might not exist. Examples of such special loci W are abundant. Noether–Lefschetz loci parametrize hypersurfaces in $\mathbb{P}^3$ with Picard number > 1 (see for instance Griffiths and Harris (1985)), modular curves parameterize pairs of isogeneous elliptic curves (see for instance Galbraith (1996) or Movasati (2021, Exercise 16.8)) and Humbert surfaces parameterize abelian varieties with certain endomorphism structure (see for instance van der Geer (1988, Chapter IX) or Gruenewald (2008)). The last two examples can be reformulated in terms of algebraic cycles and if the algebraic cycle is replaced with a Hodge class/cycle then such special loci are called Hodge loci. In this chapter we aim to gather all well-known components of the Hodge locus in the case of hypersurfaces. Some of these components are conjectural and for some we have proofs.

12.2 Hodge locus

Hodge loci arise naturally when one wants to study Hodge cycles in families. Let $Y \to V$ be a family of smooth complex projective varieties ($Y \subset \mathbb{P}^N \times V$ and $Y \to V$ is obtained by projection on the second coordinate). Let $F^i H^m_{\mathrm{dR}}(Y/V)$ be the vector bundle of F^i pieces of the Hodge filtration of $H^m_{\mathrm{dR}}(Y_t),\ t \in V$. This notation is mainly used for the free sheaf of sections, however, in the definition below we use it as the total space of the bundle.

Definition 12.1. The **locus of Hodge classes** is the subset of $F^{\frac{m}{2}} H^m_{\mathrm{dR}}(Y/V)$ containing all Hodge classes.

Note that $F^{\frac{m}{2}} H^m_{\mathrm{dR}}(Y/V)$ is an algebraic bundle, however, the locus of Hodge classes is a union of local analytic varieties. This will be clear once we define the Hodge locus in terms of integrals. Now, we define the Hodge locus in V itself.

Definition 12.2. The projection of the locus of Hodge classes under $F^{\frac{m}{2}} H^m_{\mathrm{dR}}(Y/V) \to V$ is called the **Hodge locus** in V. An irreducible component H of the Hodge locus in a (usual) neighborhood of a point $t_0 \in V$ is characterized in the following way. It is an irreducible closed analytic subvariety of (V, t_0) with a continuous family of Hodge classes $\delta_t \in H^m(Y_t, \mathbb{Z}) \cap F^{\frac{m}{2}}$ in varieties $Y_t, t \in H$ such that for points t in a dense open subset of H, the monodromy of δ_t to a point in a neighborhood (in the usual topology of V) of t and outside H is no more a Hodge class.

Remark 12.1. Even though for all known examples of irreducible components H of the Hodge locus, the dense open subset of H in the above definition can be replaced by H itself, we do not expect this to be true in general. In Definition 12.3, we introduce the Hodge locus δ_0 attached to δ_0 which might have many irreducible components and H is just one of them. Here, the dense open subset of H refers to H minus all other components of V_{δ_0}.

The following consequence of the Hodge conjecture is well-known:

Conjecture 12.1. *Let $Y \to V$ be a family of smooth projective varieties defined over a field $\mathsf{k} \subset \mathbb{C}$. All the components of the locus of Hodge classes are algebraic subsets of $F^{\frac{m}{2}} H^m_{\mathrm{dR}}(Y/V)$ defined over the algebraic closure of k.*

In particular, the components of the Hodge locus in V are also algebraic.

Proposition 12.1. *Hodge conjecture implies Conjecture 12.1.*

Proof. We follow the argument in Cattani, Deligne, and Kaplan (1995, page 483). Let C_d be the Chow variety parameterizing projective sub schemes Z of $\mathbb{P}^N$ of degree d and dimension $n - \frac{m}{2}$, where n is the dimension of Y_t. Here, we have to use the language of schemes, as we would like to consider Z as an effective algebraic cycle $Z = \sum_{i=1}^{k} n_i Z_i$, $n_i \in \mathbb{N}$ and Z_i's are degree d subvarieties of $\mathbb{P}^N$ of degree d and dimension $n - \frac{m}{2}$. Moreover, by definition $\deg(Z) = \sum_{i=1}^{k} n_i \deg(Z_i)$. The multiplicities n_i's are encoded in the defining ideal of Z. Instead of Chow varieties we may also use Hilbert schemes. We consider the subvariety A_d of $C_d \times V$ containing the points (Z, t) with $Z \subset Y_t$. We only look at it as a local analytic variety near the point $p_0 := (Z_{t_0}, t_0)$ and then consider its projection $(B_d, t_0) \subset (V, t_0)$ in the second coordinate. Note that B_d is a branch of an algebraic variety in V near t_0, and this variety might have other branches due to different choices of the algebraic cycle Z_{t_0} with fixed t_0. Moreover, A_d and B_d might be singular at p_0 and t_0, respectively, and the map $(A_d, p_0) \to (B_d, t_0)$ might have non-zero dimensional fibers. After changing the base point t_0, if necessary, we can find a smooth subvariety $\widetilde{A}_d$ of A_d and of the same dimension as B_d such that the projection $(\widetilde{A}_d, p_0) \to (B_d, t_0)$ is a biholomorphism. This implies that we have a family of topological cycle $[Z_t] \in H^m(Y_t, \mathbb{Z})$, $t \in B_d$ which is obtained by the monodromy of $[Z_{t_0}] \in H^m(Y_{t_0}, \mathbb{Z})$. One might try to give a more convincing proof of the mentioned fact by applying a version of Ehresmann's theorem. Note that Z_t's might be singular. One can also try to construct $[Z_t]$ in the algebraic de Rham cohomology and prove that it is a flat section of the Gauss–Manin connection.

Let H be an irreducible component of the Hodge locus in V passing through t_0. For $t \in H$ we have the Hodge cycles δ_t which is also algebraic $\delta_t = \frac{1}{a}([Z_{1,t}] - [Z_{t,2}])$, where $Z_{1,t}$ and $Z_{2,t}$ are two sub schemes of Y_t of codimension $\frac{m}{2}$ and $a \in \mathbb{N}$ is a number depending on t. So far we do not know, how $Z_{i,t}$, $i = 1, 2$ varies with t. Since the degrees of $Z_{1,t}$ and $Z_{2,t}$ are discrete functions in t, we conclude that there is dense subset $\widetilde{H}$ (outside an enumerable proper analytic subvarieties of H) such that $\deg Z_{1,t} = d_1$ and $\deg(Z_{2,t}) = d_2$ is constant in $\widetilde{H}$ (independent of t). Without loss of generality, we can assume that our base point $t_0 \in \widetilde{H}$. We have $H \subset B_{d_1} \cap B_{d_2}$ and claim that H is an irreducible component of $B_{d_1} \cap B_{d_2}$. If not, it lies properly inside a component of $B_{d_1} \cap B_{d_2}$. The triple $(Y_{t_0}, Z_{1,t_0}, Z_{2,t_0})$ deforms to a point $t \notin H$. However, this deformation gives us the monodromy $\delta_t := \frac{1}{a}([Z_{1,t}] - [Z_{2,t}])$ which is Hodge. This is in contradiction with the definition of H.

□

The algebraicity statement in Conjecture 12.1 has been proved by Cattani, Deligne and Kaplan.

Theorem 12.1. *(Cattani, Deligne, and Kaplan (ibid.)) The irreducible components of the locus of Hodge classes in $F^{\frac{m}{2}} H^m_{\mathrm{dR}}(Y/V)$ are algebraic sets.*

The main ingredient of their proof is Schmid's nilpotent orbit theorem in Schmid (1973) together with some results in Cattani, Kaplan, and Schmid (1986). All these are purely transcendental methods in algebraic geometry, and hence, their proof does not give any light into the second part of Conjecture 12.1, that is, any component of the locus of Hodge classes is defined over the algebraic closure of the base field k.

The algebraicity statement for the locus of Hodge classes is slightly stronger than the same statement for the Hodge locus. Let us explain this. We take an irreducible component H of the Hodge locus. Above each point $t \in H$ we have a Hodge class β and the above theorem implies that the action of the monodromy representation $\pi_1(H, t) \to \mathrm{Aut}(H^m(Y_t, \mathbb{Q}))$ on β produces a finite number of cohomological classes (which are again Hodge classes). This topological fact does not follow just from the algebraicity of H.

We will work with Hodge cycles which live in homology in comparison with Hodge classes which live in cohomology. Both notions are related to each other by Poincaré duality. Let $Y \to V$ be a family of smooth projective varieties as before. Let also $\delta_t \in H_n(Y_t, \mathbb{Z})$ be a continuous family of cycles. We consider sections $\omega_1, \omega_2, \ldots, \omega_a$ of the cohomology bundle $H^m_{\mathrm{dR}}(Y_t)$, $t \in (V, 0)$ such that for any $t \in (V, 0)$ they form a basis of the $F^{\frac{m}{2}+1}$-piece of the Hodge filtration of $H^m_{\mathrm{dR}}(Y_t)$. Therefore, $a = h^{\frac{n}{2}+1}$. Let $\mathcal{O}_{V,0}$ be the ring of holomorphic functions in a neighborhood of 0 in V. We have the elements

$$\int_{\delta_t} \omega_i \in \mathcal{O}_{V,0}, \quad i = 1, 2, \ldots, a.$$

Definition 12.3. The **(analytic) Hodge locus** passing through 0 and corresponding to δ is the analytic variety

$$V_\delta := \left\{ t \in (V, 0) \;\middle|\; \int_{\delta_t} \omega_1 = \int_{\delta_t} \omega_2 = \cdots = \int_{\delta_t} \omega_a = 0 \right\}. \tag{12.1}$$

We consider it as an analytic scheme with

$$\mathcal{O}_{V_\delta} := \mathcal{O}_{V,0} \Big/ \left\langle \int_{\delta_t} \omega_1, \int_{\delta_t} \omega_2, \cdots, \int_{\delta_t} \omega_a \right\rangle. \tag{12.2}$$

In the two dimensional case, that is $\dim(Y_t) = 2$, the Hodge locus is usually called **Noether–Lefschetz locus**. In this case the Hodge conjecture is known as Lefschetz $(1,1)$ theorem. The Hodge locus is given by the vanishing of $a = h^{\frac{n}{2}+1}$ holomorphic functions in t. By definition of a Hodge cycle, we already know that 0 is a point of this variety. This is a local analytic, not necessarily irreducible, subset of V.

The fact that a Hodge locus is given by holomorphic functions is immediate in our context. However, this is not easily seen by the classical definition, "It seems to be a known fact (cf. e.g. P. Griffiths, *passim*) that to impose a Hodge class upon a manifold with complex structure imposes upon its local moduli a holomorphic condition", (Weil (1999, page 428)).

12.3 Hodge locus for hypersurfaces

Let $\mathsf{T} \subset \mathbb{C}[x]_d$ be the parameter space of smooth hypersurface of degree d and even dimension n in $\mathbb{P}^{n+1}$. For $t \in \mathsf{T}$ we have the hypersurface X_t given by $F_t \in \mathbb{C}[x]_d$. We fix $0 \in \mathsf{T}$. In the case of hypersurfaces, we have the Griffiths' description of the de Rham cohomology of a hypersurface, together with its Hodge filtration, and so a Hodge locus can be presented only with the knowledge of integrals. This is actually the way it is done in Movasati (2021, Section 16.5). For the convenience of the reader we repeat it here. For a Hodge cycle $\delta = \delta_0 \in H_m(X, \mathbb{Z})$, let $\delta_t \in H_n(X_t, \mathbb{Z})$ be the monodromy of δ to the hypersurface X_t. Let $\mathcal{O}_{\mathsf{T},0}$ be the ring of holomorphic functions in a neighborhood of 0 in T. We have the elements

$$f_{P,i}(t) := \int_{\delta_t} \mathrm{res}\left(\frac{P\Omega}{F^i}\right) \in \mathcal{O}_{\mathsf{T},0}, \quad P \in \mathbb{C}[x]_{\sigma-(\frac{n}{2}-i+1)d}, \; i = 1, 2, \ldots, \frac{n}{2}.$$

where $\sigma = (d-2)(\frac{n}{2}+1)$.

Definition 12.4. The **(analytic) Hodge locus** passing through 0 and corresponding to δ is the analytic variety

$$V_\delta := \left\{ t \in (\mathsf{T}, 0) \;\middle|\; f_{P,i}(t) = 0,\ \forall P \in \mathbb{C}[x]_{\sigma-(\frac{n}{2}-i+1)d},\ i = 1, 2, \dots, \frac{n}{2} \right\}. \tag{12.3}$$

We consider it as an analytic scheme with

$$\mathcal{O}_{V_\delta} := \mathcal{O}_{\mathsf{T},0} \Big/ \left\langle f_{P,i} \;\middle|\; P \in \mathbb{C}[x]_{\sigma-(\frac{n}{2}-i+1)d},\ i = 1, 2, \dots, \frac{n}{2} \right\rangle. \tag{12.4}$$

The main reason for the introduction of Artinian Gorenstein algebras in Chapter 11 is the following theorem.

Theorem 12.2. *Let $V_\delta \subset (\mathsf{T}, 0)$ be the Hodge locus passing through* 0 *and corresponding to $\delta = \delta_0$. The Zariski tangent space of V_δ at* 0 *is canonically identified with $I(\delta)_d$.*

Proof. This follows by our definition of Hodge locus above. Note that (up to some non-zero constant factor)

$$\frac{\partial}{\partial t_\alpha} \int_{\delta_t} \operatorname{res}\left(\frac{P\Omega}{F^i}\right) = \int_{\delta_t} \operatorname{res}\left(\frac{QP\Omega}{F^{i+1}}\right), \quad Q := -\frac{\partial F}{\partial t_\alpha}. \tag{12.5}$$

This implies that the linear part of $f_{P,i}$ for $i < \frac{n}{2}$ is zero and only the linear parts of $f_{P,\frac{n}{2}}$ contribute to the tangent space of V_δ at $0 \in \mathsf{T}$. □

In the terminology of Voisin (2002), ${}^t\nabla(\delta^{\mathsf{pd}}) = I(\delta)_d$ is the Zariski tangent space of V_δ at 0.

Definition 12.5. Note that the Zariski tangent space $\mathbf{T}_0 V_\delta$ is the the tangent space of V_δ as scheme. If V_δ is the zero set of $f_1, f_2, \cdots, f_k \in \mathcal{O}_{\mathsf{T},0}$ then $\mathbf{T}_0 V_\delta$ is the zero set of the linear part of f_i's. We say that V_δ is **smooth** if it is smooth in the scheme theoretical context, that is, the ideal defining V_δ is generated by $f_1, f_2, \dots, f_k$ and the linear parts of f_i's are linearly independent. This also implies that V_δ is **reduced**. Note that reducedness is a weaker property than smoothness.

Corollary 12.1. *Let* T *be the parameter space of smooth degree* d *hypersurfaces of* $\mathbb{P}^{n+1}$*, of even dimension* n*. For* $t \in \mathsf{T}$*, let* $X_t = \{F = 0\} \subseteq \mathbb{P}^{n+1}$ *be the corresponding hypersurface. If* $\delta \in CH^n(X_t)_{cit}$ *is a complete intersection type algebraic cycle, then*

$$T_t V_{[\delta]} = (J^F : P_\delta)_d.$$

Proof. By Theorem 12.2 and Theorem 8.2 we have

$$T_t V_{[\delta]} = \{P \in \mathbb{C}[x_0, \ldots, x_{n+1}]_d : P \cdot Q \cdot P_\delta \in J^F,$$

for all $Q \in \mathbb{C}[x_0, \ldots, x_{n+1}]_{d\frac{n}{2}-n-2}\}$.

By item (ii) of Macaulay's Theorem 8.1 applied to the Jacobian ring R^F, we conclude

$$T_t V_{[\delta]} = \{P \in \mathbb{C}[x_0, \ldots, x_{n+1}]_d : P \cdot P_\delta \in J^F\} = (J^F : P_\delta)_d.$$

$\square$

In order to prove Theorem 12.4 we will use Corollary 12.1 for $t = 0 \in \mathsf{T}$ corresponding to the Fermat variety, and $\delta = \mathbb{P}^{\frac{n}{2}} \in CH^n(X_0)$ a linear cycle inside it.

12.4 Alternative Hodge conjecture

In this section we are going to present the alternative Hodge conjecture (AHC) which is a stronger version of Grothendieck's variational Hodge conjecture (VHC). This appears first in Movasati (2021, Section 18.2) and it is false in general. However, its validity in many concrete examples is still a conjecture, and that is the main reason for letting the name conjecture in its title. The concept of a deformation of a pair of a variety X together with an algebraic cycle $Z \subset X$ is not rigorously established in Movasati (ibid.) and we fill this gap here.

Let $X \subset \mathbb{P}^N$ be a smooth projective variety and Z be an algebraic cycle of dimension $\frac{m}{2}$, where m is an even number between 0 and $2\dim(X)$. We assume that $X = X_0$ is the fiber over $0 \in \mathsf{T}$ of a family $\mathsf{X} \to \mathsf{T}$ of smooth projective varieties. We would like give a precise definition for deformations $Z_t \subset X_t$ of $Z_0 \subset X_0$ within this family.

Definition 12.6. The **deformation space** $V_Z \subset (\mathsf{T}, 0)$ of Z within T is defined in the following way. First, we write

$$Z = \sum_{i=1}^{k} n_i Z_i, \; n_i \in \mathbb{Z} \tag{12.6}$$

where Z_i's are effective algebraic cycles, that is, they can be written as sum of irreducible subvarieties of X of dimension $\frac{m}{2}$ and with positive coefficients. Let $\mathrm{Hilb}(X)$ and $\mathrm{Hilb}(X, Z_i)$, $i = 1, 2, \dots, k$ be the Hilbert scheme parametrizing deformations of X and the pair (X, Z_i), respectively. We have the canonical maps

$$\kappa_i : \mathrm{Hilb}(X, Z_i) \to \mathrm{Hilb}(X), \;\; i = 1, 2, \dots, k$$

We denote by $0 \in \mathrm{Hilb}(X)$ and $0 \in \mathrm{Hilb}(X, Z_i)$ the points corresponding to X and (X, Z_i), respectively. We define

$$V_Z := \cap_{i=1}^{k} \mathrm{Image}(\kappa_i : (\mathrm{Hilb}(X, Z_i)^{\mathrm{an}}, 0) \to (\mathrm{Hilb}(X)^{\mathrm{an}}, 0)).$$

For an arbitrary T we define V_Z by taking pull-back through $\mathsf{T} \to \mathrm{Hilb}(X)$. This definition depends on the decomposition (12.6). By definition we have a family $(Z_t, X_t), \; t \in V_Z$.

Remark 12.2. For a scheme V we denote by V^{an} the analytic scheme (and not the analytic variety as we have used in earlier chapters) attached to V and we denote by $(V^{\mathrm{an}}, 0)$ a small neighborhood of V^{an} in the analytic/usual topology. By definition V_Z is an analytic scheme whose scheme structure comes from the algebraic scheme structure of $\mathrm{Hilb}(X, Z_i)$.

Remark 12.3. Let $\mathsf{X} \to \mathsf{T}$ be a family of smooth projective varieties. If the Hodge conjecture is true then the components of the Hodge locus in T enjoy two different scheme theoretical structures, one is analytic coming from Hodge theory and the other is algebraic coming from Hilbert scheme arguments. It is not clear whether the former is the analytification of the latter.

Remark 12.4. For an effective algebraic cycle $Z = \sum_{i=1}^{k} n_i Z_i, \; n_i \in \mathbb{N}$ we consider it as scheme in the following way. First, we consider the ideal I_i of all regular functions vanishing on Z_i. Therefore, by definition I_i is radical. The defining ideal of Z is $I_1^{n_1} I_2^{n_2} \cdots I_k^{n_k}$. Note that for two ideals I and J, the ideal IJ is defined by the products $ab, \;\; a \in I, \;\; b \in J$. It is not clear for an arbitrary ideal I inducing an irreducible variety Z what kind of coefficient one must attach to Z, and this indicates that Definition 12.6 might need some improvements.

Conjecture 12.2 (Alternative Hodge Conjecture)**.** Let $\{X_t\}_{t\in\mathsf{T}}$ be a family of complex smooth projective varieties, and let Z_0 be an algebraic cycle of dimension $\frac{m}{2}$ in X_0 for $0 \in \mathsf{T}$. We say that the **weak alternative Hodge conjecture (WAHC)** holds if there is a deformation space V_Z such that the underlying analytic varieties of V_Z and $V_{[Z]}$ are the same. In other words, there is an open neighborhood U of 0 in T (in the usual topology) such that for all $t \in U$ if the monodromy $\delta_t \in H_n(X_t, \mathbb{Z})$ of $\delta_0 = [Z_0]$ is a Hodge cycle, then there is an algebraic deformation $Z_t \subset X_t$ of $Z_0 \subset X_0$ such that $\delta_t = [Z_t]$. In other words, deformations of Z_0 as a Hodge cycle and as an algebraic cycle are the same. We say that **strong alternative Hodge conjecture (SAHC)** holds if $V_{[Z]} = V_Z$ as analytic schemes. By AHC we mean WAHC.

We would like to get some information about the tangent space of V_Z at 0. Since V_Z is given as the image of another variety, we will be able to get some information about the image of the derivation of κ at 0. This might be strictly smaller than $\mathbf{T}_0 V_Z$.

12.5 Complete intersection algebraic cycle

Assume that $n \geqslant 2$ is even and $F \in \mathbb{C}[x]_d$ is of the following format:

$$F = f_1 f_{\frac{n}{2}+2} + f_2 f_{\frac{n}{2}+3} + \cdots + f_{\frac{n}{2}+1} f_{n+2}, \quad f_i \in \mathbb{C}[x]_{d_i}, \quad f_{\frac{n}{2}+1+i} \in \mathbb{C}[x]_{d-d_i}, \tag{12.7}$$

where $1 \leqslant d_i < d$, $i = 1, 2, \ldots, \frac{n}{2}+1$ is a sequence of natural numbers. We denote by $\mathsf{T}_{\underline{d}}$ the subvariety of T containing hypersurfaces X given by such an F. We say that the f_i and $f_{\frac{n}{2}+1+i}$ are companion of each other. Let $X \subset \mathbb{P}^{n+1}$ be the hypersurface given by $F = 0$ and $Z \subset X$ be the algebraic cycle given by

$$Z: \quad f_1 = f_2 = \cdots = f_{\frac{n}{2}+1} = 0.$$

In this section we aim to prove the following.

Theorem 12.3. *Let V_Z be the analytic branch of $\mathsf{T}_{\underline{d}}$ corresponding to deformations of (X, Z). We have*

$$V_Z = V_{[Z]},$$

that is SAHC holds for (X, Z). This implies that the subvariety $\mathsf{T}_{\underline{d}}$ of T is a component of the Hodge locus. In other words, there is a Zariski open (and hence dense) subset U of $\mathsf{T}_{\underline{d}}$ such that for all $t \in U$ and a complete intersection algebraic cycle $Z \subset X := X_t$ as above, deformations of Z as an algebraic cycle and Hodge cycle are the same.

This theorem is proved in Dan (2017, Theorem 1.1) in which the author assumes $d > deg(Z)$ which is not necessary, see also Kloosterman (2020) and Villaflor (n.d.[b]). The computational proof for particular examples of n and d is done in Movasati and Villaflor (2018). It has the advantage that it works for other algebraic cycles which are not complete intersections. The main result in Otwinowska (2003) implies Theorem 12.3 for very large degrees, however, the lower bound in this article is not explicitly computed.

Proof. In $H_n(X,\mathbb{Z})$ the homology classes of all cycles

$$g_1 = g_2 = \cdots = g_{\frac{n}{2}+1} = 0,\ g_i \in \{f_i, f_{\frac{n}{2}+1+i}\}$$

are equal up to sign and up to $\mathbb{Z}[Z_\infty]$. Let us denote it by δ. This with Proposition 11.3 imply that Z is perfect and

$$J_d := \langle f_1, f_2, \ldots, f_{n+2}\rangle_d \subset I(\delta)_d.$$

Now Macaulay's Theorem 8.1 implies that J is also Artinian Gorenstein of socle degree $\sigma := d - d_1 + d_1 + \cdots + d - d_{\frac{n}{2}+1} - d_{\frac{n}{2}+1} = (\frac{n}{2}+1)d$ and so $I(\delta) = J$, and in particular, $I(\delta)_d = J_d$. Note that J_d is the tangent space of $\mathsf{T}_{\underline{d}}$ at X and this proves the theorem. □

Remark 12.5. The Zariski open subset in Theorem 12.3 is the set of homogeneous polynomials F in (12.7) such that the zero set of the ideal $\langle f_1, f_2, \ldots, f_{n+1}\rangle$ in $\mathbb{P}^{n+1}$ is empty. This is needed in Macaulay's theorem used in the proof.

Exercise 12.1. In the case of Fermat variety X_n^d and

$$x_{2i-2}^d - x_{2i-1}^d = f_i f_{\frac{n}{2}+1+i},\ f_i \in \mathbb{C}[x_{2i-2}, x_{2i-1}]_{d_i},$$

$$f_{\frac{n}{2}+1+i} \in \mathbb{C}[x_{2i-2}, x_{2i-1}]_{d-d_i}, i = 1, \ldots, \frac{n}{2}+1$$

the Macaulay's theorem is an easy exercise in commutative algebra. Prove it.

12.6 Sum of two algebraic cycles

In this section we discuss a theoretical approach in order to generalize Theorem 12.3. For a complete algebraic cycle Z as in the previous section, let V_Z be the local analytic branch of $\mathsf{T}_{\underline{d}}$ passing through 0. It corresponds to deformations of the pair $Z \subset X$. We follow the notations in Movasati (2021, Section 17.9). In Theorem 12.3 we have actually proved that the tangent space of both V_Z and $V_{[Z]}$ are the same, and hence, these two analytic schemes are the same.

Let Z and $\check{Z}$ be two complete intersection algebraic cycles given by the ideals $\mathcal{I}_Z = \langle f_1, f_2, \ldots, f_{\frac{n}{2}+1}\rangle$ and $\mathcal{I}_{\check{Z}} = \langle \check{f}_1, \check{f}_2, \ldots, \check{f}_{\frac{n}{2}+1}\rangle$, respectively. Let also $\delta := r[Z] + \check{r}[\check{Z}] \in H_n(X, \mathbb{Z}),\ r, \check{r} \in \mathbb{Z} - \{0\}$ be the corresponding Hodge cycle. We define

$$I_{Z+\check{Z}} := \text{Radical}\left(\bigoplus \left(\underline{\mathcal{I}_Z} \cdot \underline{\mathcal{I}_{\check{Z}}}\right)\right),$$

where the sum is over all possible replacements of companions of the generators of $\mathcal{I}_Z$ and $\mathcal{I}_{\check{Z}}$. By Proposition 11.3 we know that

$$I_{Z+\check{Z}} \subset I(\delta).$$

By definition $I(\delta)$ depends on the numbers r and $\check{r}$, however, for many interesting cases it turns out that it is independent of these coefficients. If the the ring $\mathbb{C}[x]/I_{Z+\check{Z}}$ is Artinian Gorenstein of socle degree $\sigma := (\frac{n}{2} + 1)(d - 2)$ then we have the equality $I_{Z+\check{Z}} = I(\delta)$ and we expect that

$$V_{r[Z]+\check{r}[\check{Z}]} = V_{rZ+\check{r}\check{Z}} := V_Z \cap V_{\check{Z}},$$

and hence the alternative Hodge conjecture (see Movasati (ibid., Conjecture 18.2)) is true for $(X, rZ + \check{r}\check{Z})$. We only need to check that $(I_{Z+\check{Z}})_d$ is the tangent space of $V_Z \cap V_{\check{Z}}$. We will analyze this situation in the following particular case.

Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface given by the homogeneous polynomial

$$F = f_1 f_{s+1} + f_2 f_{s+1} + \cdots + f_m f_{s+m} + f_{m+1} f_{s+m+1} g_1 + \cdots + f_s f_{2s} g_{s-m},$$

with $s = \frac{n}{2} + 1$. We have the algebraic cycles:

$$\begin{aligned} Z &: \quad f_1 = f_2 = \cdots = f_m = f_{m+1} = \cdots = f_s = 0, \\ \check{Z} &: \quad f_1 = f_2 = \cdots = f_m = f_{s+m+1} = \cdots = f_{2s} = 0. \end{aligned}$$

Proposition 12.2. *The ideal $I_{Z+\check{Z}}$ in $\mathbb{C}[x]$ contains*

$$f_i,\ 1 \leqslant i \leqslant m, \tag{12.8}$$

$$f_i,\ i = s+1 \leqslant i \leqslant s+m, \tag{12.9}$$

$$f_i f_j,\ m+1 \leqslant i \leqslant s,\ \ s+m+1 \leqslant j \leqslant 2s, \tag{12.10}$$

$$f_k f_{s+m+j} g_j,\ 1 \leqslant j \leqslant s-m,\ \ \ s+m+1 \leqslant k \leqslant 2s, \tag{12.11}$$

$$f_k f_{m+j} g_j,\ 1 \leqslant j \leqslant s-m,\ \ \ m+1 \leqslant k \leqslant s, \tag{12.12}$$

$$f_i g_j,\ 1 \leqslant j \leqslant s-m,\ \ \ i \in \{s+m+j, m+j\}. \tag{12.13}$$

Proof. The idea is similar to the case of a single complete intersection algebraic cycle treated in Section 12.5. We start with the radical I of the ideal of $Z + \check{Z}$ which contains (12.8) and (12.10). As before, we can replace any polynomial in the ideal of Z and $\check{Z}$ with its companion and add it the ideal to I. Replacing each element in (12.8) with its companion in both Z and $\check{Z}$ we get the elements (12.9). All other companion substitution will give us (12.11) and (12.12). There are elements among these polynomials which are of the form a^2b. Knowing that each time we take radical of the ideal we get elements (12.13). □

Remark 12.6. In Proposition 12.2 we may further claim that $I_{Z+\check{Z}}$ is equal to the ideal generated by (12.8) till (12.13). For this we must prove that the second ideal is radical.

For a complete intersection algebraic cycle as in Corollary 8.1, and for a generic choice of (X, Z), the tangent space of V_Z at 0 is given by:

$$\mathbf{T}_0 V_Z \cong \langle f_1, f_2, \ldots, f_{2s} \rangle_d := \left\{ p_1 f_1 + p_2 f_2 + \cdots + p_{2s} f_{2s} \ \ \middle|\ \ p_i \in \mathsf{k}[x]_{d-a_i} \right\},$$

where $a_i := \deg(f_i)$. We must verify this for our complete intersection algebraic cycles Z and $\check{Z}$ in this section. For a while suppose that this is done. We conclude that both $\mathbf{T}_0(V_Z \cap V_{\check{Z}}) \subset \mathbf{T}_0 V_Z \cap \mathbf{T}_0 V_{\check{Z}}$ are contained in the d-th homogeneous piece of the the ideal $I^*_{Z+\check{Z}}$ defined as bellow. It is the ideal generated by the polynomials (12.8), (12.9), (12.10) with $j - i = s$ and (12.13). These are exactly the companion of single f_i or g_i's in F. We could assume that f_i and g_i's are irreducible and so these are companion of irreducible ingredients of F. We have the inclusions:

$$J^F \subset I^*_{Z+\check{Z}} \subset I_{Z+\check{Z}} \subset I_{[Z]+[\check{Z}]}. \tag{12.14}$$

We reduce the verification of alternative Hodge conjecture in an example to the following purely commutative algebra problem. For simplicity we have only considered the case $m = s$, that is, the cycles Z and $\check{Z}$ do not intersects each other, and it can be formulated easily for arbitrary m.

Problem 12.1. Let $F \in \mathbb{C}[x] = \mathbb{C}[x_0, x_1, \dots, x_{n+1}]$ be a homogeneous polynomial of degree d and of the following format

$$F = f_1 f_{s+1} f_{2s+1} + f_2 f_{s+2} f_{2s+2} + \cdots + f_s f_{2s} f_{3s}, \quad s := \frac{n}{2} + 1$$

where f_i's are homogeneous and we have assumed that n is even. Let I^* be the ideal in $\mathbb{C}[x]$ generated by $f_i f_{s+i}, f_i f_{2s+i}, f_{s+i} f_{2s+i}$, $i = 1, 2, \dots, s$ and J^F be the Jacobian ideal of F, that is, it is generated by $\frac{\partial f}{\partial x_i}$, $i = 0, 1, 2, \dots, n+1$. We have $J^F \subset I^*$ and we assume that $F = 0$ is smooth and hence by Macaulay theorem $\mathbb{C}[x]/J^F$ is Artinian Gorenstein of socle degree $\sigma := (n+2)(d-2)$. Show that if there is a third ideal I such that $J^F \subset I^* \subset I$ and $\mathbb{C}[x]/I^*$ is Artinian Gorenstein of socle degree $\frac{\sigma}{2} = (\frac{n}{2}+1)(d-2)$ then for large d, the degree d pieces of I and I^* are equal.

12.7 Sum of two linear cycles

Our main aim in this section is to approach the following conjecture:

Conjecture 12.3. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface containing two linear projective spaces $\mathbb{P}^{\frac{n}{2}}$ and $\check{\mathbb{P}}^{\frac{n}{2}}$ with $\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m$. For*

$$m < \frac{n}{2} - \frac{d}{d-2} \tag{12.15}$$

we have

$$V_{r[\mathbb{P}^{\frac{n}{2}}] + \check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]} = V_{[\mathbb{P}^{\frac{n}{2}}]} \cap V_{[\check{\mathbb{P}}^{\frac{n}{2}}]}.$$

for $r, \check{r} \in \mathbb{Z} - \{0\}$.

Let us consider the space $\check{\mathsf{T}}$ of smooth hypersurface $X \subset \mathbb{P}^{n+1}$ containing two linear projective spaces $\mathbb{P}^{\frac{n}{2}}$ and $\check{\mathbb{P}}^{\frac{n}{2}}$ with $\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m$. An immediate corollary of Conjecture 12.3 is that $\check{\mathsf{T}}$ is a component of the Hodge locus, provided that we have (12.15). Note that by Theorem 12.3 we have $V_{\mathbb{P}^{\frac{n}{2}}} = V_{[\mathbb{P}^{\frac{n}{2}}]}$, and so, $V_{[\mathbb{P}^{\frac{n}{2}}]} \cap V_{[\check{\mathbb{P}}^{\frac{n}{2}}]}$ is a deformation space $V_{r\mathbb{P}^{\frac{n}{2}} + \check{r}\check{\mathbb{P}}^{\frac{n}{2}}}$ of the algebraic cycle $r\mathbb{P}^{\frac{n}{2}} + \check{r}\check{\mathbb{P}}^{\frac{n}{2}}$. In this section we aim to prove the following weaker version of Conjecture 12.3.

Theorem 12.4 (Movasati and Villaflor (2018) and Villaflor (n.d.[a])). *Let $X \subseteq \mathbb{P}^{n+1}$ be the Fermat variety*

$$x_0^d + x_1^d + \cdots + x_{n+1}^d = 0$$

of even dimension n and degree d. Let $\mathbb{P}^{\frac{n}{2}}, \check{\mathbb{P}}^{\frac{n}{2}} \subseteq X$ be the two linear subvarieties such that $\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m$ given by

$$\mathbb{P}^{n-m} := \{x_{n-2m} - \zeta_{2d} x_{n-2m+1} = \cdots = x_n - \zeta_{2d} x_{n+1} = 0\},$$

$$\mathbb{P}^{\frac{n}{2}} := \{x_0 - \zeta_{2d} x_1 = \cdots = x_{n-2m-2} - \zeta_{2d} x_{n-2m-1} = 0\} \cap \mathbb{P}^{n-m},$$

$$\check{\mathbb{P}}^{\frac{n}{2}} := \{x_0 - \zeta_{2d}^{\alpha_0} x_1 = \cdots = x_{n-2m-2} - \zeta_{2d}^{\alpha_{n-2m-2}} x_{n-2m-1} = 0\} \cap \mathbb{P}^{n-m},$$

where $\zeta_{2d} \in \mathbb{C}$ is a primitive $2d$-root of unity, and $\alpha_0, \alpha_2, \ldots, \alpha_{n-2m-2} \in \{3, 5, \ldots, 2d-1\}$. Then Conjecture 12.3 is true in this case.

Note that

$$\mathbb{P}^m := \mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \{x_0 = x_1 = \cdots = x_{n-2m-1} = 0\} \cap \mathbb{P}^{n-m}.$$

Proof. We have the following equality

$$\operatorname{codim} V_{[\mathbb{P}^{\frac{n}{2}}]} \cap V_{[\check{\mathbb{P}}^{\frac{n}{2}}]} = \operatorname{codim} T_0 V_{[\mathbb{P}^{\frac{n}{2}}]} \cap T_0 V_{[\check{\mathbb{P}}^{\frac{n}{2}}]} \tag{12.16}$$

$$= 2\binom{\frac{n}{2}+d}{d} - 2(\frac{n}{2}+1)^2 - \binom{m+d}{d} + (m+1)^2, \tag{12.17}$$

which is proved in Movasati (2021, Proposition 17.8 and Proposition 17.9). Since $T_0 V_{[\mathbb{P}^{\frac{n}{2}}]} \cap T_0 V_{[\check{\mathbb{P}}^{\frac{n}{2}}]} = T_0 (V_{[\mathbb{P}^{\frac{n}{2}}]} \cap V_{[\check{\mathbb{P}}^{\frac{n}{2}}]})$, this implies that $V_{[\mathbb{P}^{\frac{n}{2}}]} \cap V_{[\check{\mathbb{P}}^{\frac{n}{2}}]}$ is smooth. Moreover, $V_{[\mathbb{P}^{\frac{n}{2}}]}$ intersects $V_{[\check{\mathbb{P}}^{\frac{n}{2}}]}$ transversely. Note that by Theorem 12.3 we have $V_{\mathbb{P}^{\frac{n}{2}}} = V_{[\mathbb{P}^{\frac{n}{2}}]}$ which is smooth. Knowing (12.16) and the trivial inclusion $V_{[\mathbb{P}^{\frac{n}{2}}]} \cap V_{[\check{\mathbb{P}}^{\frac{n}{2}}]} \subset V_{r[\mathbb{P}^{\frac{n}{2}}]+\check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]}$, Theorem 12.4 follows from: We have

$$T_0 V_{[\mathbb{P}^{\frac{n}{2}}]} \cap T_0 V_{[\check{\mathbb{P}}^{\frac{n}{2}}]} = T_0 V_{r[\mathbb{P}^{\frac{n}{2}}]+\check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]}$$

if and only if $m < \frac{n}{2} - \frac{d}{d-2}$. By Theorem 12.2, Proposition 11.2 and Corollary 8.3, this is equivalent to the following algebraic equality

$$(J^F : P_1)_d \cap (J^F : P_2)_d = (J^F : P_1 + P_2)_d, \tag{12.18}$$

where $P_1 = R_1 Q$, $P_2 = R_2 Q$ with

$$Q := \prod_{k \geqslant n-2m \text{ even}} \frac{(x_k^{d-1} - (\zeta_{2d} x_{k+1})^{d-1})}{(x_k - \zeta_{2d} x_{k+1})},$$

$$R_1 := c_1 \cdot \prod_{k<n-2m \text{ even}} \frac{(x_k^{d-1} - (\zeta_{2d} x_{k+1})^{d-1})}{(x_k - \zeta_{2d} x_{k+1})},$$

$$R_2 := c_2 \cdot \prod_{k<n-2m \text{ even}} \frac{(x_k^{d-1} - (\zeta_{2d}^{\alpha_k} x_{k+1})^{d-1})}{(x_k - \zeta_{2d}^{\alpha_k} x_{k+1})},$$

for some $c_1, c_2 \in \mathbb{C}^\times$. We claim that

$$(J^F : P_1)_e \cap (J^F : P_2)_e = (J^F : P_1 + P_2)_e, \tag{12.19}$$

if and only if $e < (d-2)(\frac{n}{2} - m)$ or $e > (d-2)(\frac{n}{2}+1)$. In fact, for $e > (d-2)(\frac{n}{2}+1)$ the claim follows from the fact that $(d-2)(\frac{n}{2}+1)$ is the socle of the three ideals appearing in (12.19). For $e < (d-2)(\frac{n}{2} - m)$, consider any $q \in (J^F : P_1 + P_2)_e$. Write

$$q = r + s,$$

where $r \in \mathbb{C}[x_0, \dots, x_{n-2m-1}]_e$ and $s \in \langle x_{n-2m}, \dots, x_{n+1}\rangle_e \subseteq \mathbb{C}[x_0, \dots, x_{n+1}]_e$. Noting that

$$(J^F : Q) = \langle x_0^{d-1}, \dots, x_{n-2m-1}^{d-1}, x_{n-2m}, \dots, x_{n+1}\rangle,$$

it is clear that $s \in (J^F : P_i) = ((J^F : Q) : R_i)$ for every $i = 1, 2$. In consequence $r \in ((J^F : Q) : R_1 + R_2)$. Since $r \cdot (R_1 + R_2)$ does not depend on $x_{n-2m}, \dots, x_{n+1}$ we conclude that

$$r \in (I : R_1 + R_2)_e \subseteq \mathbb{C}[x_0, \dots, x_{n-2m-1}]_e$$

for $I = \langle x_0^{d-1}, \dots, x_{n-2m-1}^{d-1}\rangle$. Using Proposition 11.6 for $r = \frac{n}{2} - m$, we conclude that $r \in (I : R_i)_e$ for $i = 1, 2$, and so $q \in (J^F : P_i)_e$ for $i = 1, 2$ as claimed.

Finally, if $(d-2)(\frac{n}{2}-m) \leqslant e \leqslant (d-2)(\frac{n}{2}+1)$, we know from Proposition 11.6 for $r = \frac{n}{2} - m$, that there exist some $p \in \mathbb{C}[x_0, \dots, x_{n-2m}]$ such that

$$p \in (J^F : R_1 + R_2)_{(d-2)(\frac{n}{2}-m)} \setminus (J^F : R_1)_{(d-2)(\frac{n}{2}-m)},$$

and so

$$p \in (J^F : P_1 + P_2)_{(d-2)(\frac{n}{2}-m)} \setminus (J^F : P_1)_{(d-2)(\frac{n}{2}-m)}.$$

Since $(J^F : P_1)$ is Artinian Gorenstein with socle $(d-2)(\frac{n}{2}+1)$, we conclude that there exist some polynomial $q \in \mathbb{C}[x_0, \dots, x_{n+1}]_{e-(d-2)(\frac{n}{2}-m)}$ such that

$$pq \in (J^F : P_1 + P_2)_e \setminus (J^F : P_1)_e,$$

as desired. $\square$

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