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On elliptic modular foliations ☆

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ABSTRACT

In this article we consider the three parameter family of elliptic curves E_t: $y^2 - 4(x-t_1)^3 + t_2(x-t_1) + t_3 = 0, t \in \mathbb{C}^3$, and study the modular holomorphic foliation $\mathcal{F}_\omega$ in $\mathbb{C}^3$ whose leaves are constant locus of the integration of a 1-form ω over topological cycles of E_t. Using the Gauss–Manin connection of the family E_t, we show that $\mathcal{F}_\omega$ is an algebraic foliation. In the case $\omega = \frac{x\,dx}{y}$, we prove that a transcendent leaf of $\mathcal{F}_\omega$ contains at most one point with algebraic coordinates and the leaves of $\mathcal{F}_\omega$ corresponding to the zeros of integrals, never cross such a point. Using the generalized period map associated to the family E_t, we find a uniformization of $\mathcal{F}_\omega$ in T, where $T \subset \mathbb{C}^3$ is the locus of parameters t for which E_t is smooth. We find also a real first integral of $\mathcal{F}_\omega$ restricted to T and show that $\mathcal{F}_\omega$ is given by the Ramanujan relations between the Eisenstein series.

1. INTRODUCTION

A classical way to study an object in algebraic geometry, is to put it inside a family and then try to understand its behavior as a member of the family. In other words, one looks the object inside a certain moduli space. The Abelian integrals which appear in the deformation of holomorphic foliations with a first integral in a complex manifold of dimension two (see [4,8,15,16]), can be studied in this way provided that we consider, apart from the parameter of the first integral, some other parameters. The first natural object to look is the constant locus of integrals. This

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yields to holomorphic foliations in the parameter space, which we call modular foliations. The defining equations of such holomorphic foliations can be calculated using the Gauss–Manin connection and it turns out that they are always defined over $\mathbb{Q}$, i.e. the ingredient of the defining equations are polynomials in the parameters and with coefficients in $\mathbb{Q}$. Modular foliations, apart from topological and dynamical properties, enjoy certain arithmetical properties. They are an important link between the transcendental problems in number theory and their counterparts in holomorphic foliations/differential equations. They are classified as transversely homogeneous foliations (see [6]) and recently some authors have studied examples of such foliations (see [3,12,13,25] and the references there). In this article I want to report on a class of such foliations associated to a three parameter family of elliptic curves. For simplicity, we explain the results of this article for one of such foliations which is important from historical point of view and its transverse group structure is $\mathrm{SL}(2,\mathbb{Z})$.

After calculating the Gauss–Manin connection of the following family of elliptic curves

$$(1) \qquad E_t\colon\ y^2-4(x-t_1)^3+t_2(x-t_1)+t_3=0, \quad t\in\mathbb{C}^3,$$

and considering its relation with the inverse of the period map, we get the following ordinary differential equation:

$$(2) \qquad \mathrm{Ra}\colon \begin{cases} \dot t_1 = t_1^2-\frac{1}{12}t_2,\\ \dot t_2 = 4t_1t_2-6t_3,\\ \dot t_3 = 6t_1t_3-\frac{1}{3}t_2^2, \end{cases}$$

which is called the Ramanujan relations, because he has observed that the Eisenstein series form a solution of (2) (one gets the classical relations by changing the coordinates $(t_1,t_2,t_3)\mapsto(\frac{1}{12}t_1,\frac{1}{12}t_2,\frac{2}{3(12)^2}t_3)$, see [21], p. 4). We denote by $\mathcal{F}(\mathrm{Ra})$ the singular holomorphic foliation induced by (2) in $\mathbb{C}^3$. Its singularities

$$\mathrm{Sing}(\mathrm{Ra}) := \{(t_1, 12t_1^2, 8t_1^3)\mid t_1\in\mathbb{C}\}$$

form a one-dimensional curve in $\mathbb{C}^3$. The discriminant of the family (1) is given by $\Delta=27t_3^2-t_2^3$. For $t\in T:=\mathbb{C}^3\setminus\{\Delta=0\}$, E_t is an smooth elliptic curve and so we can take a basis of the $\mathbb{Z}$-module $H_1(E_t,\mathbb{Z})$, namely $(\delta_1,\delta_2)=(\delta_{1,t},\delta_{2,t})$, such that the intersection matrix in this basis is $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$. Let ω_i, $i=1,2$, be two meromorphic differential 1-forms in $\mathbb{C}^2$ such that the restriction of ω_i to E_t, $t\in T$, is of the second type, i.e it may have poles but no residues around the poles. For instance, take $\omega_1=\frac{dx}{y}$, $\omega_2=\frac{x\,dx}{y}$. Define

$$B_{\omega_i}(t) := \frac{1}{2\pi}\mathrm{Im}\Big(\int_{\delta_1}\omega_i\overline{\int_{\delta_2}\omega_i}\Big), \quad i=1,2,$$

$$B_{\omega_1,\omega_2}(t) := \frac{1}{2\pi}\Big(\int_{\delta_1}\omega_1\overline{\int_{\delta_2}\omega_2}-\int_{\delta_1}\omega_2\overline{\int_{\delta_2}\omega_1}\Big).$$

It is easy to show that the above functions do not depend on the choice of δ_1, δ_2 (see the definition of the period map in Section 2) and hence they define analytic functions on T.

We define

$$K := \left\{ t \in T \;\Big|\; \int_{\delta} \frac{x\,dx}{y} = 0, \text{ for some } 0 \neq \delta \in H_1(E_t, \mathbb{Z}) \right\}$$

and

$$M_r := \{t \in T \mid B_{\frac{x\,dx}{y}}(t) = r\}, \qquad M_{<r} := \bigcup_{s<r} M_s, \quad r \in \mathbb{R}.$$

Using the Legendre relation $\int_{\delta_1} \frac{dx}{y} \int_{\delta_2} \frac{x\,dx}{y} - \int_{\delta_1} \frac{x\,dx}{y} \int_{\delta_2} \frac{dx}{y} = 2\pi i$ one can show that $|B_{\frac{dx}{y}, \frac{x\,dx}{y}}|$ restricted to M_0 is identically 1. We also define

$$N_w := \{t \in M_0 \mid B_{\frac{dx}{y}, \frac{x\,dx}{y}}(t) = w\}, \quad |w| = 1, \ w \in \mathbb{C}.$$

For $t \in \mathbb{C}^3 \setminus \mathrm{Sing}(\mathcal{F}(\mathrm{Ra}))$ we denote by L_t the leaf of $\mathcal{F}(\mathrm{Ra})$ through t. Let $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the Poincaré upper half plane and $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disk.

Theorem 1. *The following is true:*

1. *The leaves of $\mathcal{F}(\mathrm{Ra})$ in a neighborhood of $t \in T$ are given by the level surfaces of $(\int_{\delta_1} \frac{x\,dx}{y}, \int_{\delta_2} \frac{x\,dx}{y}) : (T, t) \to \mathbb{C}^2$. In particular, the function $B_{\frac{x\,dx}{y}}$ is a real first integral of $\mathcal{F}(\mathrm{Ra})$ and for $|w| = 1$, N_w's are $\mathcal{F}(\mathrm{Ra})$-invariant.*
2. *For $t \in X := (M_0 \setminus K) \cup M_{<0}$ the leaf L_t is biholomorphic to $\mathbb{D}$ and for $t \in T \setminus X$ the leaf L_t is biholomorphic to $\mathbb{D} \setminus \{0\}$.*
3. *The set K is $\mathcal{F}(\mathrm{Ra})$-invariant and it is a dense subset of M_0. For all $t \in K$ there is a holomorphic map $\mathbb{D} \to \mathbb{C}^3$, transverse to* Sing(Ra) *at some point p, which is a biholomorphy between $\mathbb{D} \setminus \{0\}$ and L_t.*
4. *For all $t \in T$ the leaf L_t has an accumulation point at T if and only if $t \in M_0$.*
5. *The discriminant variety $\{\Delta = 0\}$ is $\mathcal{F}(\mathrm{Ra})$-invariant and all the leaves in $\{\Delta = 0\}$ are algebraic.*

In Section 6 we have defined an elliptic modular foliation associated to a differential form ω in $\mathbb{C}^2$ such that ω restricted to the fibers of (1) is holomorphic. It is based on the first statement in Theorem 1, Part 1. Such foliations have real first integrals and leave the discriminant variety invariant.

The proof of the above theorem is based on the fact that the foliation $\mathcal{F}(\mathrm{Ra})$ restricted to T is uniformized by the inverse of the period map (see for instance [11] for similar topics). Despite the fact that this theorem does not completely describe the dynamics of $\mathcal{F}(\mathrm{Ra})$, it shows that a modular foliation is not a strange foliation from dynamical/topological point of view. However, such foliations arise some new questions and problems related to holomorphic foliations. For a given algebraic holomorphic foliation $\mathcal{F}$ in $\mathbb{C}^3$ defined over $\bar{\mathbb{Q}}$, the field of algebraic numbers, a transcendent leaf L of $\mathcal{F}$ how frequently crosses points with algebraic coordinates? The set $L \cap \bar{\mathbb{Q}}^3$ can be empty or a one element set. For $\mathcal{F}(\mathrm{Ra})$ these are the only possibilities.

Theorem 2. *The following is true:*

1. *For any point $t \in \mathbb{C}^3 \setminus \{\Delta = 0\}$, the set $\bar{\mathbb{Q}}^3 \cap L_t$ is empty or has only one element. In other words, every transcendent leaf contains at most one point with algebraic coordinates.*
2. *$K \cap \bar{\mathbb{Q}}^3 = \emptyset$, i.e. for all $p \in K$ at least one of the coordinates of p is transcendent number.*

The main idea behind the proof of the above theorem is the first part of Theorem 1 and consequences of the Abelian subvariety theorem on periods of elliptic curves (see [28] and the references there). We will also give an alternative proof for the second part of the above theorem, using a result on transcendence of the values of the Eisenstein series.

I have made a good use of SINGULAR for doing the calculations in this article. The text is written in such a way that the reader can carry out all calculations using any software in commutative algebra. An exception to this is the calculation of the Gauss–Manin connection in Section 2, for which one can use a combination of hand and computer calculations or one must know the general algorithms introduced in [17]. The general definition of a modular foliation can be done using connections on algebraic varieties. The forthcoming text [20] will discuss such foliations, specially those related to the Gauss–Manin connection of fibrations. In the article [18] we have developed the notion of a differential modular form in which we have essentially used the same techniques of this article.

In the classical theory of elliptic integrals, the parameter t_1 in (1) is equal to zero and one considers the versal deformation of the singularity $y^2 - 4x^3 = 0$. In this article we have generalized the classical Weierstrass theorem and proved that for the inverse of the generalized period map, t_i appears as the Eisenstein series of weight $2i$. The novelty is the appearance of t_1 as the Eisenstein series of weight 2.

The paper is organized as follows: In Section 2 we define the period map, calculate its derivative and the Gauss–Manin connection associated to the family (1). In Section 3 we introduce the action of an algebraic group on $\mathbb{C}^3$ and its relation with the period map. We prove that the period map is a biholomorphism and using its inverse, we obtain the differential equation (2). In Section 4 we describe the uniformization of $\mathcal{F}(\mathrm{Ra})|_T$. In Section 5 we prove Theorem 1. In Section 6 we introduce the general notion of an elliptic modular foliation associated to the family (1). Section 7 is devoted to a theorem on periods of Abelian varieties defined over $\bar{\mathbb{Q}}$ and its corollaries on the periods of elliptic curves. In Section 8 we prove Theorem 2. In Section 9 we study another family of elliptic curves and corresponding modular foliations. Finally in Section 10 we discuss some problems related to limit cycles arising from deformations of the family (1) inside holomorphic foliations.

2. PERIOD MAP AND ITS DERIVATION

For some technical reasons, which will be clear later, it is convenient to introduce a new parameter t_0 and work with the family:

$$(3)\qquad E_t\colon\ y^2-4t_0(x-t_1)^3+t_2(x-t_1)+t_3,\quad t=(t_0,t_1,t_2,t_3)\in\mathbb{C}^4.$$

Its discriminant is $\Delta:=t_0(27t_0t_3^2-t_2^3)$. We will use the notations in the Introduction for this family.

Let

$$\mathcal{P}:=\left\{x=\begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix}\in \mathrm{GL}(2,\mathbb{C})\ \Big|\ \mathrm{Im}(x_1\overline{x_3})>0\right\}.$$

It is well know that the entries of $(\omega_1,\omega_2):=(\frac{dx}{y},\frac{x\,dx}{y})$ restricted to each regular elliptic curve E_t form a basis of $H^1_{\mathrm{dR}}(E_t)$. The associated period map is given by:

$$\mathsf{pm}\colon T\to \mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{P},\ t\mapsto\left[\frac{1}{\sqrt{2\pi i}}\begin{pmatrix}\int_{\delta_1}\omega_1 & \int_{\delta_1}\omega_2\\ \int_{\delta_2}\omega_1 & \int_{\delta_2}\omega_2\end{pmatrix}\right].$$

It is well defined and holomorphic. Here $\sqrt{i}=e^{2\pi i/4}$ and (δ_1,δ_2) is a basis of the $\mathbb{Z}$-module $H_1(E_t,\mathbb{Z})$ such that the intersection matrix in this basis is $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$. Note that $\delta_i=\delta_{i,t},\ i=1,2$, is a continuous family of cycles depending on t. Different choices of δ_1,δ_2 will lead to the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathcal{P}$ from the left. If there is no risk of confusion, we will also use pm for the map from T to $\mathcal{P}$.

Remark 1. A classical way for choosing the cycles δ_1,δ_2 is given by the Picard–Lefschetz theory (see for instance [15] and the references there). For the fixed parameters $t_0\neq 0,\ t_1$ and $t_2\neq 0$, define $f:\mathbb{C}^2\to\mathbb{C}$ as

$$f(x,y)=-y^2+4t_0(x-t_1)^3-t_2(x-t_1).$$

The function f has two critical values given by $\tilde{t}_3,\ \check{t}_3=\pm\sqrt{\frac{t_2^3}{27t_0}}$. In a regular fiber E_t of f one can take two cycles δ_1 and δ_2 such that $\langle\delta_1,\delta_2\rangle=1$ and δ_1 (resp. δ_2) vanishes along a straight line connecting t_3 to $\tilde{t}_3$ (resp. $\check{t}_3$). The corresponding anti-clockwise monodromy around the critical value $\tilde{t}_3$ (resp. $\check{t}_3$) can be computed using the Picard–Lefschetz formula:

$$\delta_1\mapsto\delta_1,\qquad \delta_2\mapsto\delta_2+\delta_1\quad(\text{resp. }\delta_1\mapsto\delta_1-\delta_2,\ \delta_2\mapsto\delta_2).$$

It is not hard to see that the canonical map $\pi_1(\mathbb{C}\setminus\{\tilde{t}_3,\check{t}_3\},t)\to\pi_1(T,t)$ induced by inclusion is an isomorphism of groups and so:

$$\pi_1(T,t) \cong \langle A_1, A_2 \rangle = \mathrm{SL}(2,\mathbb{Z}),$$

where

$$A_1 := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Note that if we define $g_1 := A_2^{-1}A_1^{-1}A_2^{-1} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $g_2 := A_1^{-1}A_2^{-1} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ then we have $\mathrm{SL}(2,\mathbb{Z}) = \langle g_1, g_2 \mid g_1^2 = g_2^3 = -I \rangle$, where I is the identity 2×2 matrix.

Proposition 1. *Consider* pm *as a holomorphic matrix valued function in* T. *We have*

$$d\mathsf{pm}(t) = \mathsf{pm}(t) \cdot A^{\mathrm{tr}}, \quad t \in T, \tag{4}$$

where $A = \frac{1}{\Delta}\sum_{i=1}^{4} A_i dt_i$ *and*

$$\begin{aligned}
A_0 &= \begin{pmatrix} \frac{3}{2}t_0t_1t_2t_3 - 9t_0t_3^2 + \frac{1}{4}t_2^3 & -\frac{3}{2}t_0t_2t_3 \\ \frac{3}{2}t_0t_1^2t_2t_3 + 9t_0t_1t_3^2 - \frac{1}{2}t_1t_2^3 + \frac{1}{8}t_2^2t_3 & -\frac{3}{2}t_0t_1t_2t_3 - 18t_0t_3^2 + \frac{3}{4}t_2^3 \end{pmatrix}, \\
A_1 &= \begin{pmatrix} 0 & 0 \\ 27t_0^2t_3^2 - t_0t_2^3 & 0 \end{pmatrix}, \\
A_2 &= \begin{pmatrix} -\frac{9}{2}t_0^2t_1t_3 + \frac{1}{4}t_0t_2^2 & \frac{9}{2}t_0^2t_3 \\ -\frac{9}{2}t_0^2t_1^2t_3 + \frac{1}{2}t_0t_1t_2^2 - \frac{3}{8}t_0t_2t_3 & \frac{9}{2}t_0^2t_1t_3 - \frac{1}{4}t_0t_2^2 \end{pmatrix}, \\
A_3 &= \begin{pmatrix} 3t_0^2t_1t_2 - \frac{9}{2}t_0^2t_3 & -3t_0^2t_2 \\ 3t_0^2t_1^2t_2 - 9t_0^2t_1t_3 + \frac{1}{4}t_0t_2^2 & -3t_0^2t_1t_2 + \frac{9}{2}t_0^2t_3 \end{pmatrix}.
\end{aligned} \tag{5}$$

Proof. The proof is a mere calculation. The calculation of the derivative of the period map for the differential form $\frac{dx}{y}$ and the case $t_1 = 0$ is classical and can be found in ([24] p. 304, [23]). For the convenience of the reader we explain only the first row of A_3. For $p(x) = 4t_0(x-t_1)^3 - t_2(x-t_1) - t_3$ we have:

$$\Delta = -p' \cdot a_1 + p \cdot a_2,$$

where

$$\begin{aligned}
a_1 = {} & -36t_0^3x^4 + 144t_0^3t_1x^3 + \left(-216t_0^3t_1^2 + 15t_0^2t_2\right)x^2 \\
& + \left(144t_0^3t_1^3 - 30t_0^2t_1t_2\right)x - 36t_0^3t_1^4 + 15t_0^2t_1^2t_2 - t_0t_2^2,
\end{aligned}$$

$$\begin{aligned}
a_2 = {} & \left(-108t_0^3\right)x^3 + \left(324t_0^3t_1\right)x^2 + \left(-324t_0^3t_1^2 + 27t_0^2t_2\right)x \\
& + \left(108t_0^3t_1^3 - 27t_0^2t_1t_2 - 27t_0^2t_3\right).
\end{aligned}$$

Now we consider y as a function in x and make the projection of $H_1(E_t,\mathbb{Z})$ in the x-plane. The derivation with respect to t_3 goes inside of the integral and

$$\frac{\partial}{\partial t_3}\left(\frac{dx}{y}\right) = \frac{1}{2}\frac{dx}{py} = \frac{1}{\Delta}\frac{(-p'a_1 + pa_2)dx}{2py} = \frac{1}{\Delta}\left(\frac{1}{2}a_2 - a_1'\right)\frac{dx}{y}$$
$$= \left(3t_0^2 t_1 t_2 - \frac{9}{2}t_0^2 t_3\right)\frac{dx}{y}$$
$$- 3t_0^2 t_2 \frac{x\,dx}{y} \quad \text{modulo relatively exact 1-forms}$$

(see [22] p. 41 for a description of calculations modulo relatively exact 1-forms). Note that in the third equality above we use $y^2 = p(x)$ and the fact that modulo exact forms we have

$$\frac{p'a_1\,dx}{2py} = \frac{a_1\,dp}{2py} = \frac{a_1\,dy}{p} = -a_1\,d\left(\frac{1}{y}\right) = \frac{a_1'\,dx}{y}. \qquad \square$$

Recall that a meromorphic differential form ω in $\mathbb{C}^2$ is relatively exact for the family (1) if its restriction to each elliptic curve E_t, $\Delta(t) \neq 0$ is an exact form. This is equivalent to say that $\int_\delta \omega = 0$ for all $\delta \in H_1(E_t, \mathbb{Z})$.

The matrix A is in fact the Gauss–Manin connection of the family E_t with respect to the basis ω. We consider (3) as an elliptic curve E defined over $\mathbb{Q}(t) = \mathbb{Q}(t_0, t_1, t_2, t_3)$. According to Grothendieck [7], the de Rham cohomology $H^1_{\mathrm{dR}}(E)$ of E is well defined. Any element of $H^1_{\mathrm{dR}}(E)$ can be represented by a meromorphic differential 1-form in $\mathbb{C}^2 = \{(x, y)\}$ whose restriction to a generic elliptic curve E_t is a differential form of the second type i.e. a meromorphic differential form on E_t with no residues around its poles. In the case we are considering, each element in $H^1_{\mathrm{dR}}(E)$ can be represented by a differential form with a unique pole at infinity and $H^1_{\mathrm{dR}}(E)$ is a $\mathbb{Q}(t)$-vector space with the basis $\{[\frac{dx}{y}], [\frac{x\,dx}{y}]\}$. Roughly speaking, the Gauss–Manin connection is a $\mathbb{Q}$-linear operator $\nabla : H_{\mathrm{dR}}(E) \to \Omega^1_T \otimes_{\mathbb{Q}(t)} H_{\mathrm{dR}}(E)$, where Ω^1_T is the set of algebraic differential 1-forms defined over $\mathbb{Q}$ in T. It satisfies the Leibniz rule $\nabla(p\eta) = dp \otimes \eta + p\nabla\eta$, $p \in \mathbb{Q}(t)$, $\eta \in H^1_{\mathrm{dR}}(E)$ and

$$\text{(6)} \qquad d\int_{\delta_t} \eta = \int_{\delta_t} \nabla\eta, \quad \eta \in H^1_{\mathrm{d}R}(E).$$

We write $\nabla(\omega) = B\omega$, $\omega := (\frac{dx}{y}, \frac{x\,dx}{y})^{\mathrm{tr}}$, use (6) and conclude that $B = \frac{1}{\Delta}\sum_{i=0}^{3} A_i$. The Gauss–Manin connection is an integrable connection. For our example, this translates into:

$$dB = B \wedge B \quad \text{equivalently for } B = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix},$$
$$d\omega_{11} = \omega_{12} \wedge \omega_{21}, \; d\omega_{12} = \omega_{12} \wedge \omega_{22} + \omega_{11} \wedge \omega_{12}, \; d\omega_{22} = \omega_{21} \wedge \omega_{12},$$
$$d\omega_{21} = \omega_{21} \wedge \omega_{11} + \omega_{22} \wedge \omega_{21}.$$

For the procedures which calculate the Gauss–Manin connection see [17].

The algebraic group

$$G_0 = \left\{ \begin{pmatrix} k_1 & k_3 \\ 0 & k_2 \end{pmatrix} \;\middle|\; k_3 \in \mathbb{C}, k_1, k_2 \in \mathbb{C}^* \right\} \tag{7}$$

acts on $\mathcal{P}$ from the right by the usual multiplication of matrices. It acts also in $\mathbb{C}^4$ as follows:

$$\begin{aligned} t \bullet g &:= \left(t_0 k_1^{-1} k_2^{-1}, t_1 k_1^{-1} k_2 + k_3 k_1^{-1}, t_2 k_1^{-3} k_2, t_3 k_1^{-4} k_2^2\right) \\ t &= (t_0, t_1, t_2, t_3) \in \mathbb{C}^4, \; g = \begin{pmatrix} k_1 & k_3 \\ 0 & k_2 \end{pmatrix} \in G_0. \end{aligned} \tag{8}$$

The relation between these two actions of G_0 is given by the following proposition:

Proposition 2. *The period* pm *is a biholomorphism and*

$$\mathsf{pm}(t \bullet g) = \mathsf{pm}(t) \cdot g, \quad t \in \mathbb{C}^4, \; g \in G_0. \tag{9}$$

Proof. We first prove (9). Let

$$\alpha : \mathbb{C}^2 \to \mathbb{C}^2, \; (x, y) \mapsto \left(k_2^{-1} k_1 x - k_3 k_2^{-1}, k_2^{-1} k_1^2 y\right).$$

Then

$$\begin{aligned} k_2^2 k_1^{-4} \alpha^{-1}(f) &= y^2 - 4 t_0 k_2^2 k_1^{-4} \left(k_2^{-1} k_1 x - k_3 k_2^{-1} - t_1\right)^3 \\ &\quad + t_2 k_2^2 k_1^{-4} \left(k_2^{-1} k_1 x - k_3 k_2^{-1} - t_1\right) + t_3 k_2^2 k_1^{-4} \\ &= y^2 - 4 t_0 k_1^{-1} k_2^{-1} \left(x - \left(t_1 k_2 k_1^{-1} + k_3 k_1^{-1}\right)\right)^3 \\ &\quad + t_2 k_1^{-3} k_2 \left(x - \left(t_1 k_2 k_1^{-1} + k_3 k_1^{-1}\right)\right) + t_3 k_1^{-4} k_2^2. \end{aligned}$$

This implies that α induces an isomorphism of elliptic curves

$$\alpha : E_{t \bullet g} \to E_t.$$

Now

$$\alpha^{-1} \omega = \begin{pmatrix} k_1^{-1} & 0 \\ -k_3 k_2^{-1} k_1^{-1} & k_2^{-1} \end{pmatrix} \omega = \begin{pmatrix} k_1 & 0 \\ k_3 & k_2 \end{pmatrix}^{-1} \omega,$$

where $\omega = (\frac{dx}{y}, \frac{x\,dx}{y})^{\mathrm{tr}}$, and so

$$\mathsf{pm}(t) = \mathsf{pm}(t \bullet g) \cdot g^{-1}$$

which proves (9).

Let B be a 4×4 matrix and the ith row of B constitutes of the first and second rows of A_i. We use the explicit expressions for A_i's in Proposition 1 and we derive the following equality:

$$\det(B) = \frac{3}{4} t_0 \Delta^3.$$

The matrix B is the derivation of the period map seen as a local function from $\mathbb{C}^4$ to $\mathbb{C}^4$. This shows that pm is regular at each point $t \in T$ and hence it is locally a biholomorphism. The period map pm induces a local biholomorphic map $\bar{\mathsf{pm}}: T/G_0 \to \mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H} \cong \mathbb{C}$. One can compactify $\mathrm{SL}(2,\mathbb{Z}) \setminus \mathbb{H}$ by adding the cusp $\mathrm{SL}(2,\mathbb{Z})/\mathbb{Q} = \{c\}$ (see [9]) and the map $\bar{\mathsf{pm}}$ is continuous at v and sends v to c, where v is the point induced by $t_0 27 t_3^2 - t_2^3 = 0$ in $\mathbb{C}^4/G_0$. Using Picard's Great theorem we conclude that $\bar{\mathsf{pm}}$ is a biholomorphism and so pm is a biholomorphism. □

We denote by

$$F = (F_0, F_1, F_2, F_3) : \mathcal{P} \xrightarrow{\alpha} \mathrm{SL}(2,\mathbb{Z}) \setminus \mathcal{P} \to T$$

the map obtained by the composition of the canonical map α and the inverse of the period map. Taking F of (9) we have

$$\begin{aligned} & F_0(xg) = F_0(x) k_1^{-1} k_2^{-1}, \\ (10) \qquad & F_1(xg) = F_1(x) k_1^{-1} k_2 + k_3 k_1^{-1}, \\ & F_2(xg) = F_2(x) k_1^{-3} k_2, \qquad F_3(xg) = F_3(x) k_1^{-4} k_2^2, \quad \forall x \in \mathcal{L},\ g \in G_0. \end{aligned}$$

By Legendre's theorem $\det(x)$ is equal to one on $\mathsf{pm}(1 \times 0 \times \mathbb{C} \times \mathbb{C})$ and so the same is true for $F_0 \det(x)$. But the last function is invariant under the action of G_0 and so it is the constant function 1. This means that $F_0(x) = \det(x)^{-1}$.

We consider pm as a map sending the vector (t_0, t_1, t_2, t_3) to (x_1, x_2, x_3, x_4). Its derivative at t is a 4×4 matrix whose ith column constitutes of the first and second row of $\frac{1}{\Delta} x A_i^{\mathrm{tr}}$. We use (5) to derive the equality

$$\begin{aligned} (dF)_x &= (d\mathsf{pm})_t^{-1} \\ &= \det(x)^{-1} \\ &\times \begin{pmatrix} -F_0 x_4 & F_0 x_3 \\ \frac{1}{12F_0}(12F_0F_1^2x_3 - 12F_0F_1x_4 - F_2x_3) & -F_1x_3 + x_4 \\ 4F_1F_2x_3 - 3F_2x_4 - 6F_3x_3 & -F_2x_3 \\ \frac{1}{3F_0}(18F_0F_1F_3x_3 - 12F_0F_3x_4 - F_2^2x_3) & -2F_3x_3 \end{pmatrix. \\ &\qquad \begin{matrix} F_0x_2 & -F_0x_1 \\ \frac{1}{12F_0}(-12F_0F_1^2x_1 + 12F_0F_1x_2 + F_2x_1) & F_1x_1 - x_2 \\ -4F_1F_2x_1 + 3F_2x_2 + 6F_3x_1 & F_2x_1 \\ \frac{1}{3F_0}(-18F_0F_1F_3x_1 + 12F_0F_3x_2 + F_2^2x_1) & 2F_3x_1 \end{matrix} \Bigg) . \end{aligned}$$

Define $g_i(z) := F_i\left(\begin{smallmatrix} z & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $z \in \mathbb{H}$. The equalities of the first column of the above matrix imply that $(g_1, g_2, g_3) : \mathbb{H} \to T$ satisfies the ordinary differential equation (2). The equalities (10) imply that g_i's satisfy

$$(11) \qquad (cz+d)^{-2i} g_i(Az) = g_i(z), \quad i = 2, 3,$$

$$(12) \qquad (cz+d)^{-2} g_1(Az) = g_1(z) + c(cz+d)^{-1}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

In fact g_i's, up to some constants, are the Eisenstein series. More precisely, we have the following proposition.

Proposition 3. *We have*

$$(13) \qquad g_k(z) = a_k \left(1 + (-1)^k \frac{4k}{B_k} \sum_{n \geqslant 1} \sigma_{2k-1}(n) e^{2\pi i z n}\right), \quad k = 1, 2, 3, \ z \in \mathbb{H},$$

where B_k is the kth Bernoulli number ($B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, ...), $\sigma_i(n) := \sum_{d|n} d^i$,

$$(14) \qquad p_\infty := (a_1, a_2, a_3) = \left(\frac{2\pi i}{12}, 12\left(\frac{2\pi i}{12}\right)^2, 8\left(\frac{2\pi i}{12}\right)^3\right).$$

Proof. The statement for g_2 and g_3 follows from the Weierstrass Uniformization theorem (see for instance [23]). Note that in our definition of the period map the factor $\frac{1}{\sqrt{2\pi i}}$ appears. The functions g_k, $k = 1, 2, 3$, have finite growth at infinity, i.e. $\lim_{\mathrm{Im}(z) \to +\infty} g_k(z) = a_k < \infty$. For g_1 this follows from the Ramanujan relations (2) and the equality $\frac{d}{dz} = 2\pi i q \frac{d}{dq}$, where $q = e^{2\pi i z}$. The set M of holomorphic functions on $\mathbb{H}$ which have finite growth at infinity and satisfy (12) contains only one element. The reason is as follows: The difference of any two elements of M has finite growth at infinity and satisfy (11) with $i = 1$. Such a holomorphic function is a modular form of weight 2 which does not exist (see [9]). Now the function g_1 and its corresponding series in (13) have finite growth at infinity and satisfy (12) (see [1], p. 69). Therefore, they must be equal. □

4. UNIFORMIZATION OF $\mathcal{F}$(RA)

From this section on, we set $t_0 = 1$ and work again with the family (1). We use the same notations for pm, $\mathcal{P}$, G_0, T, Δ and so on. For instance, redefine

$$\mathcal{P} := \left\{ x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in GL(2, \mathbb{C}) \;\middle|\; \mathrm{Im}(x_1 \overline{x_3}) > 0, \ \det(x) = 1 \right\}$$

and

$$G_0 = \left\{ \begin{pmatrix} k & k' \\ 0 & k^{-1} \end{pmatrix} \;\middle|\; k' \in \mathbb{C}, k \in \mathbb{C}^* \right\}.$$

The action of G_0 on $\mathbb{C}^3$ is given by

$$t \bullet g := \left(t_1 k^{-2} + k' k^{-1}, t_2 k^{-4}, t_3 k^{-6}\right),$$
$$t = (t_1, t_2, t_3) \in \mathbb{C}^3, g = \begin{pmatrix} k & k' \\ 0 & k^{-1} \end{pmatrix} \in G_0.$$

We also define

$$g = (g_1, g_2, g_3) : \mathbb{H} \to T \subset \mathbb{C}^3,$$
$$\mathrm{Ra} := \left(t_1^2 - \frac{1}{12} t_2\right) \frac{\partial}{\partial t_1} + (4t_1 t_2 - 6t_3) \frac{\partial}{\partial t_2} + \left(6t_1 t_3 - \frac{1}{3} t_2^2\right) \frac{\partial}{\partial t_3},$$
$$\eta_1 := \left(t_1^2 - \frac{1}{12} t_2\right) dt_2 - (4t_1 t_2 - 6t_3)\, dt_1,$$
$$\eta_2 := (4t_1 t_2 - 6t_3)\, dt_3 - \left(6t_1 t_3 - \frac{1}{3} t_2^2\right) dt_2,$$
$$\eta_3 := \left(t_1^2 - \frac{1}{12} t_2\right) dt_3 - \left(6t_1 t_3 - \frac{1}{3} t_2^2\right) dt_1,$$
$$\eta_4 = 3t_3\, dt_2 - 2t_2\, dt_3.$$

The foliation $\mathcal{F}(\mathrm{Ra})$ is induced by $\eta_i, i = 1, 2, 3$. We have

$$\begin{aligned} d\Delta(\mathrm{Ra}) &= \left(2.27 t_3 dt_3 - 3t_2^2 dt_2\right)(\mathrm{Ra}) \\ &= 2.27 t_3 \left(6t_1 t_3 - \frac{1}{3} t_2^2\right) - 3t_2^2 (4t_1 t_2 - 6t_3) \\ &= 12 t_1 \Delta. \end{aligned}$$

This implies that the variety $\Delta_0 := \{\Delta = 0\}$ is invariant by the foliation $\mathcal{F}(\mathrm{Ra})$. Inside Δ_0 we have the algebraic leaf $\{(t_1, 0, 0) \in \mathbb{C}^3\}$ of $\mathcal{F}(\mathrm{Ra})$. We parameterize Δ_0 by $(3t^2, t^3)$, $t \in \mathbb{C}$ and conclude that (2) restricted to Δ_0 is given by

$$\mathcal{F}(\mathrm{Ra})|_{\Delta_0} : \begin{cases} \dot{t} = 2t_1 t - t^2 \\ \dot{t}_1 = t_1^2 - \frac{1}{4} t^2. \end{cases} \tag{15}$$

It has the first integral $\frac{t_1^2}{t} - t_1 + \frac{1}{4} t$. This implies that the leaves of $\mathcal{F}(\mathrm{Ra})$ inside Δ_0 are given by:

$$t_3^{1/3} - 2\left((t_1 + c)^2 - t_1^2\right)^{1/2} = 2(t_1 + c), \quad c \in \mathbb{C}.$$

Proposition 4. *The following is a uniformization of the foliation $\mathcal{F}(\mathrm{Ra})$ restricted to T:*

$$u : \mathbb{H} \times \left(\mathbb{C}^2 \setminus \{(0,0)\}\right) \to T,$$
$$(z, c_2, c_4) \to g(z) \bullet \begin{pmatrix} (c_4 z - c_2)^{-1} & c_4 \\ 0 & c_4 z - c_2 \end{pmatrix}$$

(16)

$$= \big(g_1(z)(c_4 z - c_2)^2 + (c_4 z - c_2), g_2(z)(c_4 z - c_2)^4,$$
$$g_3(z)(c_4 z - c_2)^6\big).$$

Proof. One may check directly that for fixed c_2, c_4 the map induced by u is tangent to (2) which implies the proposition. We give another proof which uses the period map: From (5) we have

$$d(\mathsf{pm})(t) = \frac{1}{\Delta}\mathsf{pm}(t)\begin{pmatrix} \frac{3}{4}\eta_2 & \frac{3}{2}\eta_4 \\ \frac{9}{2}t_3\eta_1 - 3t_2\eta_3 + \frac{3}{2}t_1\eta_2 & -\frac{3}{4}\eta_2 \end{pmatrix}^{\mathrm{tr}}.$$

Therefore,

$$d(\mathsf{pm}(t))(\mathrm{Ra}(t)) = \mathsf{pm}(t)\begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} = \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}.$$

This implies that the x_2 and x_4 coordinates of the pull forward of the vector field Ra by pm are zero. Therefore, the leaves of $\mathcal{F}(\mathrm{Ra})$ in the period domain are of the form

$$\begin{pmatrix} z(c_4 z - c_2)^{-1} & c_2 \\ (c_4 z - c_2)^{-1} & c_4 \end{pmatrix} = \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} (c_4 z - c_2)^{-1} & c_4 \\ 0 & c_4 z - c_2 \end{pmatrix}. \qquad \square$$

5. PROOF OF THEOREM 1

We follow the notations introduced in Section 4. In particular we work with the family (3) with $t_0 = 1$.

Proof of 1. The first part follows from Proposition 4. The leaves of the pull-forward of the foliation $\mathcal{F}(\mathrm{Ra})$ by the period map pm have constant x_2 and x_4 coordinates. By definition of $B_{\frac{x\,dx}{y}} := \mathrm{Im}(x_2\overline{x_4})$ in the period domain, we conclude that M_r's are $\mathcal{F}(\mathrm{Ra})$-invariant. On M_0 an $x \in \mathcal{P}$ can be written in the form $\left(\begin{smallmatrix} x_1 & x_4 r \\ x_3 & x_4 \end{smallmatrix}\right)$, $r \in \mathbb{R}$, $x_4(x_1 - rx_3) = 1$. Then

$$B_{\frac{dx}{y},\frac{x\,dx}{y}}(x) = \overline{x_4}(x_1 - rx_3) = \frac{\overline{x_4}}{x_4}, \tag{17}$$

which implies that N_w's are $\mathcal{F}(\mathrm{Ra})$-invariants. $\square$

Proof of 2. Let us define

$$L_{c_2,c_4} := \left\{\begin{pmatrix} z(zc_4 - c_2)^{-1} & c_2 \\ (zc_4 - c_2)^{-1} & c_4 \end{pmatrix} \,\middle|\, z \in \mathbb{H} \setminus \left\{\frac{c_2}{c_4}\right\}\right\}.$$

We look at a leaf L_{c_2,c_4} of $\mathcal{F}(\mathrm{Ra})$ at the period domain $\mathcal{P}$. The leaf $[L_{c_2,c_4}] \subset \mathrm{SL}(2,\mathbb{Z}) \setminus \mathcal{P}$ may not be biholomorphic to $\mathbb{H} \setminus \{\frac{c_2}{c_4}\}$ if there exists $A \in \mathrm{SL}(2,\mathbb{Z})$ which maps a point of L_{c_2,c_4} to another point in L_{c_2,c_4}. This implies that $A[c_2, c_4]^{\mathrm{tr}} = [c_2, c_4]^{\mathrm{tr}}$ and hence $\frac{c_4}{c_2} \in \mathbb{Q}$. After taking another representative for the leaf $[L_{c_2,c_4}]$, we can assume that $c_4 = 0$. Now, the only elements of $\mathrm{SL}(2,\mathbb{Z})$ which maps $[c_2, 0]$ to itself are of the form $\left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right)$, $b \in \mathbb{Z}$. This implies that the corresponding leaf in T is biholomorphic to $\mathbb{D} \setminus \{0\}$. If $B_{\frac{x\,dx}{y}}(t) \leqslant 0$ and $t \notin K$, then $\frac{c_2}{c_4} \notin \mathbb{H}$ and L_t is biholomorphic to $\mathbb{H}$. If $B_{\frac{x\,dx}{y}}(t) > 0$ then $\frac{c_2}{c_4} \in \mathbb{H}$ and L_t is biholomorphic to $\mathbb{H} \setminus \{\frac{c_2}{c_4}\}$. □

Proof of 3. Take $t \in K$ and a cycle $\delta \in H_1(E_t, \mathbb{Z})$ such that $\int_\delta \frac{x\,dx}{y} = 0$ and δ is not of the form $n\delta'$ for some $2 \leqslant n \in \mathbb{N}$ and $\delta' \in H_1(E_t, \mathbb{Z})$. We choose another $\delta' \in H_1(E_t, \mathbb{Z})$ such that (δ', δ) is a basis of $H_1(E_t, \mathbb{Z})$ and the intersection matrix in this basis is $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$. Now $\mathsf{pm}(t)$ has zero x_4-coordinate and so its $B_{\frac{x\,dx}{y}}$ is zero. This implies that $K \subset M_0$. It is dense because an element $\left(\begin{smallmatrix}x_1 & x_4 r\\ x_3 & x_4\end{smallmatrix}\right) \in M_0 \subset \mathcal{L}$ can be approximated by the elements in M_0 with $r \in \mathbb{Q}$.

The image of the map g is the locus of the points t in T such that $\mathsf{pm}(t)$ is of the form $\left(\begin{smallmatrix}z & -1\\ 1 & 0\end{smallmatrix}\right)$ in a basis of $H_1(E_t, \mathbb{Z})$. We look g as a function of $q = e^{2\pi i z}$ and we have

$$g(0) = p_\infty, \quad \frac{\partial g}{\partial q}(0) = (-24a_1, 240a_2, -504a_3),$$

where a_i's are defined in (14). This implies that the image of g intersects Sing(Ra) transversely. For $t \in K$ the x_4-coordinate of pm is zero and the leaf through t, namely L_t, has constant x_2-coordinate, namely c_2. By (16) L_t is uniformized by

$$u(z) = \big(c_2^2 g_1(z), c_2^4 g_2(z), c_2^6 g_3(z)\big), \quad z \in \mathbb{H}.$$

This implies that L_t intersects Sing(Ra) transversely at $(c_2^2 a_1, c_2^4 a_2, c_2^6 a_3)$.

Note that the leaf space $\mathrm{SL}(2, \mathbb{Z}) \setminus (\mathbb{C}^2 \setminus \{\mathrm{Im}(c_2\bar{c}_4) > 0\})$ of the foliation $\mathcal{F}(\mathrm{Ra})$ in $M_{r>0}$ is biholomorphic to the quasi affine set $\mathbb{C}^2 \setminus \{27t_3^2 - t_2^3 = 0\}$ using the Eisenstein series. The same is true for $M_{r<0}$. The leaf space in M_0 is isomorphic to $\mathbb{C}^* \times \mathrm{SL}(2, \mathbb{Z}) \setminus \mathbb{R}$ as a set and so has no reasonable structure. □

Proof of 4. Let $t \in T$ and the leaf L_t through t have an accumulation point at $t_0 \in T$. We use the period map pm and look $\mathcal{F}(\mathrm{Ra})$ in the period domain. For $(c_2, c_4) \in \mathbb{C}^2 \setminus \{0\}$ the set $S = \{A(c_2, c_4)^{\mathrm{tr}} \mid A \in \mathrm{SL}(2, \mathbb{Z})\}$ has an accumulation point in $\mathbb{C}^2$ if and only if $\frac{c_2}{c_4} \in \mathbb{R} \cup \infty$ or equivalently $B_{\frac{x\,dx}{y}}(t) = 0$. □

Proof of 5. It is already proved in Section 4. □

6. ELLIPTIC MODULAR FOLIATIONS

Let η be any meromorphic differential 1-form in $\mathbb{C}^2$ whose restriction to a smooth elliptic curve E_t gives us a differential form of the second type. For instance, one can take $\eta = \frac{p(x,y)\,dx}{y}$ or $p(x, y)(3x\,dy - 2y\,dx)$, where p is a polynomial in x, y. Such a 1-form can be written in the form

$$\eta = p_1(t)\frac{dx}{y} + p_2(t)\frac{x\,dx}{y} \quad \text{modulo relatively exact 1-forms,} \tag{18}$$

where p_1 and p_2 are two meromorphic functions in t with poles along $\Delta = 0$ (a meromorphic one form η in $\mathbb{C}^2$ is called relatively exact if its restriction to each smooth elliptic curve E_t is an exact form).

An elliptic modular foliation $\mathcal{F}_\eta$ associated to η is a foliation in $\mathbb{C}^3 = \{(t_1, t_2, t_3)\}$ given locally by the constant locus of the integrals $\int_{\delta_t} \eta$, $\delta_t \in H_1(E_t, \mathbb{Z})$, i.e. along the leaves of $\mathcal{F}_\eta$ the integral $\int_{\delta_t} \eta$ as a function in t is constant. The algebraic description of $\mathcal{F}_\eta$ is as follows: We write $\eta = p\omega$, where $\omega = (\frac{dx}{y}, \frac{x\,dx}{y})^{\mathrm{tr}}$ and $p = (p_1, p_2)$. If $\nabla\omega = B\omega$ is the Gauss–Manin connection of the family (1) with respect to the basis ω (see Section 2) then

$$\nabla(\eta) = \nabla(p\omega) = (dp + pB)\omega$$

and it is easy to see that

$$\text{(19)}\qquad \mathcal{F}_\eta\colon dp_1 + p_1\omega_{11} + p_2\omega_{21} = 0, \qquad dp_2 + p_1\omega_{12} + p_2\omega_{22} = 0,$$

where $B = \left(\begin{smallmatrix}\omega_{11} & \omega_{12}\\ \omega_{21} & \omega_{22}\end{smallmatrix}\right)$. By the first part of Theorem 1 we know that $\mathcal{F}_{\frac{x\,dx}{y}} = \mathcal{F}(\mathrm{Ra})$. Using the above expression for $\mathcal{F}_\eta$ one can show that $\{\Delta = 0\}$ is $\mathcal{F}_\eta$-invariant and every leaf of $\mathcal{F}_\eta$ inside $\{\Delta = 0\}$ is algebraic.

Example 1. For s a fixed complex number, the foliation $\mathcal{F}_{\frac{(s+x)dx}{y}}$ is given by the vector field:

$$\left(t_1^2 + 2t_1 s - \frac{1}{12}t_2 + s^2\right)\frac{\partial}{\partial t_1} + (4t_1t_2 + 4t_2 s - 6t_3)\frac{\partial}{\partial t_2}$$
$$+ \left(6t_1t_3 - \frac{1}{3}t_2^2 + 6t_3 s\right)\frac{\partial}{\partial t_3}.$$

For $s = 0$ this is the foliation $\mathcal{F}(\mathrm{Ra})$ discussed in the previous sections and for $s = \infty$ this is the trivial foliation $\mathcal{F}_{\frac{dx}{y}}$: $dt_2 = 0$, $dt_3 = 0$.

Example 2. We have $\frac{x^2 dx}{y} = (-t_1^2 + \frac{1}{12}t_2)\frac{dx}{y} + 2t_1\frac{x\,dx}{y}$ modulo relatively exact forms and so $\mathcal{F}_{\frac{x^2 dx}{y}}$ is given by:

$$\left(-48t_1^4 + 24t_1^2 t_2 - 48t_1t_3 + t_2^2\right)\frac{\partial}{\partial t_1}$$
$$+ \left(-384t_1^3 t_2 + 1728t_1^2 t_3 - 96t_1t_2^2 + 48t_2t_3\right)\frac{\partial}{\partial t_2}$$
$$+ \left(-576t_1^3 t_3 + 96t_1^2 t_2^2 - 144t_1t_2t_3 - 8t_2^3 + 288t_3^2\right)\frac{\partial}{\partial t_3}.$$

7. ABELIAN SUBVARIETY THEOREM

In this section we are going to state a consequence of the Abelian subvariety theorem on periods of an Abelian variety defined over $\bar{\mathbb{Q}}$. For the convenience of the reader, we recall some basic facts about Abelian varieties. For further information the reader is referred to [10] for the analytic theory and [14] for the arithmetic theory of Abelian varieties.

An Abelian variety A viewed as a complex manifold is biholomorphic to $\mathbb{C}^g/\Lambda$, where Λ is a lattice of rank $2g$ in $\mathbb{C}^g$. In addition we have an embedding of A in some projective space which makes sense to say that A is defined over $\bar{\mathbb{Q}}$. From now on, we work only with the category of Abelian varieties defined over $\bar{\mathbb{Q}}$. According to Grothendieck [7] the de Rham cohomology $H^1_{\mathrm{dR}}(A)$ can be constructed in the context of algebraic geometry and it is a $\bar{\mathbb{Q}}$-vector space of dimension $2\dim(A)$. Every $[\omega] \in H^1_{\mathrm{dR}}(A)$ is represented by a differential form ω of the first or second type defined over $\bar{\mathbb{Q}}$. A differential 1-form ω on A is called to be of the first type if it is holomorphic on A and it is called to be of the second type if it is meromorphic with poles but no residues around the poles. Let A_1, A_2 be two Abelian varieties of the same dimension defined over $\bar{\mathbb{Q}}$. An isogeny between A_1 and A_2 is a surjective morphism $f: A_1 \to A_2$ of algebraic varieties defined over $\bar{\mathbb{Q}}$ with $f(0_{A_1}) = 0_{A_2}$. It is well known that every isogeny is a group homomorphism and there is another isogeny $g: A_2 \to A_1$ such that $g \circ f = n_{A_1}$ for some $n \in \mathbb{N}$, where n_{A_1} is the multiplication by n map in A_1. The isogeny f induces an isomorphism $f_*: H_1(A_1, \mathbb{Q}) \to H_1(A_2, \mathbb{Q})$ (resp. $f^*: H^1_{\mathrm{dR}}(A_2) \to H^1_{\mathrm{dR}}(A_1)$) of $\mathbb{Q}$-vector spaces (resp. $\bar{\mathbb{Q}}$-vector spaces). For $A = A_1 = A_2$ simple, it turns out that $\mathrm{End}_0(A) = \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a division algebra, i.e. it is a ring, possibly non-commutative, in which every non-zero element has an inverse. An Abelian variety is called simple if it does not contain a non trivial Abelian subvariety. Every Abelian variety is isogenous to the direct product $A_1^{k_1} \times A_2^{k_2} \times \cdots \times A_n^{k_n}$ of simple, pairwise non-isogenous Abelian varieties A_i, all defined over $\bar{\mathbb{Q}}$ and this decomposition is unique up to isogeny and permutation of the components. For an Abelian variety A defined over $\bar{\mathbb{Q}}$ the period set

$$P(A) := \left\{ \int_{\delta} \omega \;\middle|\; \delta \in H_1(A, \bar{\mathbb{Q}}),\ [\omega] \in H^1_{\mathrm{dR}}(A) \right\}$$

is a $\bar{\mathbb{Q}}$-vector space of dimension at most $(2\dim A)^2$. We are going to state the precise description of $\dim_{\bar{\mathbb{Q}}} P(A)$.

Let A be a simple Abelian variety. The division algebra $k := \mathrm{End}_0(A)$ acts both on $H_1(A, \mathbb{Q})$ and $H^1_{\mathrm{dR}}(A)$ and we have

$$\int_{a\cdot\delta} \omega = \int_{\delta} a \cdot \omega, \quad a \in k,\ [\omega] \in H^1_{\mathrm{dR}}(A).$$

This means that the periods of $a \cdot \delta$ reduces to the periods of δ. Let $H_1(A, \mathbb{Q}) = \bigoplus_{j=1}^{s} k\dot{\delta}_j$ be the decomposition of $H_1(A, \mathbb{Q})$ under the action of k. Each $k \cdot \delta_j$ is a $\mathbb{Q}$-vector space of dimension $\dim_{\mathbb{Q}} k$ and so $s = \frac{\dim_{\mathbb{Q}} H_1(A,\mathbb{Q})}{\dim_{\mathbb{Q}} k} = \frac{2\dim(A)}{\dim_{\mathbb{Q}}(\mathrm{End}_0(A_i))}$. Considering $r = 2\dim A$ differential forms $\omega_1, \omega_2, \ldots, \omega_r$ which form a basis of $H^1_{\mathrm{dR}}(A)$, we obtain $s_A := \frac{4\dim(A)^2}{\dim_{\mathbb{Q}}(\mathrm{End}_0(A_i))}$ periods $\int_{\delta_j} \omega_i$, $i = 1, 2, \ldots, r$, $j =$

$1, 2, \ldots, s$, which span the $\bar{\mathbb{Q}}$-vector space $P(A)$ and may be $\bar{\mathbb{Q}}$-independent. If A is isogenous to the direct product $A_1^{k_1} \times A_2^{k_2} \times \cdots \times A_n^{k_n}$ of simple, pairwise non-isogenous Abelian varieties A_i, all defined over $\bar{\mathbb{Q}}$, then we obtain $\sum_{i=1}^{n} s_{A_i}$ periods which span the $\bar{\mathbb{Q}}$-vector space $P(A)$. In fact, they form a basis and there is no more relation between the periods of A.

Theorem 3. *Let A be an Abelian variety defined over $\bar{\mathbb{Q}}$ and isogenous to the direct product $A_1^{k_1} \times A_2^{k_2} \times \cdots \times A_n^{k_n}$ of simple, pairwise non-isogenous Abelian varieties A_i, all defined over $\bar{\mathbb{Q}}$. Then the $\bar{\mathbb{Q}}$-vector space V_A generated by $1, 2\pi i$ together with all periods $\int_\delta \omega$, $\delta \in H_1(A, \mathbb{Q})$, $[\omega] \in H^1_{dR}(A)$, has dimension*

$$\dim_{\bar{\mathbb{Q}}}(V_A) = 2 + 4 \sum_{i=1}^{n} \frac{\dim(A_i)^2}{\dim_{\mathbb{Q}}(\mathrm{End}_0(A_i))}.$$

Note that the above theorem says a little bit more: The collection of s_A periods which we described before are $\bar{\mathbb{Q}}$-linear independent among themselves and even with the numbers $1, \pi$. The above theorem is a consequence of Wüstholz analytic subgroup theorem (see for instance [27], Lemma 1). It is stated and proved in Theorem 6.1 of [26] (Appendix). Similar theorems are stated and used by many authors (see [28], Satz 1, Satz 2; [27], Proposition 2; [19], Corollary 1). In this text we need the following corollaries of the above theorem.

Corollary 1. *Let A_1 and A_2 be two Abelian varieties over $\bar{\mathbb{Q}}$ with a common non-zero period, i.e. there exist $[\omega_i] \in H^1_{\mathrm{dR}}(A_i)$, $\delta_i \in H_1(A_i, \mathbb{Q})$, $i = 1, 2$, such that $\int_{\delta_1} \omega_1 = \int_{\delta_2} \omega_2 \neq 0$. Then there is sub Abelian varieties B_1 of A_1 and B_2 of A_2 with B_1 isogenous to B_2. In particular, if A_1 and A_2 are simple then A_1 is isogenous to A_2. In this case, we have an isogeny $a: A_1 \to A_2$ such that $a^*[\omega_2] = n[\omega_1]$ and $a_*\delta_1 = n\delta_2$ for some $n \in \mathbb{N}$, where $a^*: H^1_{\mathrm{dR}}(A_2) \to H^1_{\mathrm{dR}}(A_1)$ and $a_*: H_1(A_1, \mathbb{Q}) \to H_1(A_2, \mathbb{Q})$ are the induced maps in the first cohomology, respectively homology.*

Note that all the Abelian varieties and isogenies in the above corollary are defined over $\bar{\mathbb{Q}}$.

Proof of Corollary 1. If there is no common factor in the decomposition of A_1 and A_2 into simple Abelian varieties then applying Theorem 3 to A_1 and A_2 and $A_1 \times A_2$ we conclude that $\dim_{\bar{\mathbb{Q}}} P(A_1 \times A_2) = \dim_{\bar{\mathbb{Q}}} P(A_1) + \dim_{\bar{\mathbb{Q}}} P(A_2)$. This implies that $P(A_1) \cap P(A_2) = \{0\}$ which contradicts the hypothesis.

Now, let us prove the second part. Choose an isogeny $b: A_1 \to A_2$ and let $\tilde{\delta}_2 = b_*^{-1}\delta_2$ and $\tilde{\omega}_2 = b^*\omega_2$. Since $\int_{\tilde{\delta}_2} \tilde{\omega}_2 = \int_{\delta_1} \omega_1 \neq 0$, there must be $c \in \mathrm{End}_0(A_1)$ with $c \cdot \delta_1 = \tilde{\delta}_2$, otherwise by our hypothesis and Theorem 3 applied for A_1, we will get less dimension for $P(A_1)$. We choose $n \in \mathbb{N}$ such that $d := n \cdot c \in \mathrm{End}(A_1)$ and so we have $d_*\delta_1 = n\tilde{\delta}_2$. By our hypothesis we have

$$\int_{\delta_1} d^*\tilde{\omega}_2 = \int_{d_*\delta_1} \tilde{\omega}_2 = n \int_{\delta_1} \omega_1$$

and so by Theorem 3 we must have $d^*\tilde{\omega}_2 = n\omega_1$ (for this one can also use [28], Satz 2). Now, $e = b \circ d : A_1 \to A_2$ has the properties: $e_*\delta_1 = n\delta_2$, $e^*[\omega_2] = n[\omega_1]$. $\square$

I do not know whether Corollary 1 is true for $n = 1$ or not. To obtain $n = 1$ we have to make more hypothesis.

Corollary 2. *Let A_i, $i = 1, 2$, be two simple Abelian varieties defined over $\bar{\mathbb{Q}}$ and $0 \neq [\omega_i] \in H^1_{\mathrm{dR}}(A_i)$ such that the $\mathbb{Z}$-modules $\{\int_\delta \omega_i \mid \delta \in H_1(A_i, \mathbb{Z})\}$ coincide. Then there is an isomorphism $a : A_1 \to A_2$ such that $a^*[\omega_2] = [\omega_1]$.*

Proof. We fix $\delta_i \in H_1(A_i, \mathbb{Z})$, $i = 1, 2$, such that $\int_{\delta_1} \omega_1 = \int_{\delta_2} \omega_2 \neq 0$, apply Corollary 1 and obtain an isogeny $a : A_1 \to A_2$ with $a^*[\omega_2] = n[\omega_1]$ and $a_*\delta_1 = n\delta_2$ for some $n \in \mathbb{N}$. We claim that $a_* H_1(A_1, \mathbb{Z}) = n H_1(A_2, \mathbb{Z})$. For an arbitrary $\delta \in H_1(A, \mathbb{Z})$ we have

$$\int_{a_*\delta} \omega_2 = \int_\delta n\omega_1 = n \int_{\delta'} \omega_2 \quad \text{for some } \delta' \in H_1(A_2, \mathbb{Z}).$$

Therefore, we have $\int_{a_*\delta - n\delta'} \omega_2 = 0$. Since A_2 is simple, by Theorem 3 we have $a_*\delta = n\delta'$ and so $a_* H_1(A_1, \mathbb{Z}) \subset n H_1(A_2, \mathbb{Z})$. In the same way we prove that $n H_1(A_2, \mathbb{Z}) \subset a_* H_1(A_1, \mathbb{Z})$.

Let $A_{1,n} := \{x \in A_1 \mid nx = 0\}$ be the n-torsion points of A_1. There is an isomorphism $b : A_1 \to A_2$ such that $b \circ n_{A_1} = a$. To construct b we proceed as follows: For a moment assume that $a^{-1}(0_{A_2}) = A_{1,n}$. The quotient $B := A_1/A_{1,n}$ is a well-defined Abelian group defined over $\bar{\mathbb{Q}}$ and the isogenies a and n_{A_1} induce isomorphisms $\tilde{a} : B \to A_2$ and $\tilde{n} : B \to A_1$ of Abelian varieties. The isomorphism $b := \tilde{a} \circ \tilde{n}^{-1}$ satisfies $b \circ n_{A_1} = a$. In fact it is the one which we want: we have $b_*\delta_1 = \frac{1}{n} b_*(n\delta_1) = \frac{1}{n} a_*\delta_1 = \delta_2$ and $b^*\omega_2 = \frac{1}{n} n b^*\omega_2 = \frac{1}{n} a^*\omega_2 = \omega_1$.

Let us prove $a^{-1}(0_{A_2}) = A_{1,n}$. It is enough to prove this equality in the analytic context. We identify $t_{A_1} \cong t_{A_2} \cong \mathbb{C}^g$, where the first isomorphism is given by the derivative of a at 0_{A_1}, $H_1(A_i, \mathbb{Z}) \cong \Lambda_i \subset \mathbb{C}^g$ and obtain a $\mathbb{Z}$-linear map $\mathsf{a} : \Lambda_1 \to \Lambda_2$ which induces a $\mathbb{C}$-linear isomorphism $\mathbb{C}^g \to \mathbb{C}^g$ (we identify A_i with $\mathbb{C}^g/\Lambda_i$, $i = 1, 2$, and a with a). We have $\mathsf{a}(\Lambda_1) = n\Lambda_2$ and $A_{1,n} = \frac{\Lambda_1}{n}/\Lambda_1$. Therefore $\mathsf{a} A_{1,n} = 0 \mod \Lambda_2$. If $\mathsf{a}(x) = 0 \mod \Lambda_2$ then $\mathsf{a}(nx) = n\delta = \mathsf{a}(\delta')$ for some $\delta \in \Lambda_2$, $\delta' \in \Lambda_1$. Since a is injective we have $nx = \delta'$ and so $x \in A_{1,n}$. $\square$

8. PROOF OF THEOREM 2

For a modular foliation $\mathcal{F}_\eta$ we define:

$$K_\eta = \left\{ t \in T \;\middle|\; \int_\delta \eta = 0 \text{ for some } \delta \in H_1(E, \mathbb{Z}) \right\},$$

$$P_\eta : \mathbb{C}^3 \to \mathbb{C}^3, \quad P_\eta(t) := t \bullet \begin{pmatrix} p_2^{-1} & p_1 \\ 0 & p_2 \end{pmatrix} = \left(t_1 p_2^2 + p_1 p_2, t_2 p_2^4, t_3 p_2^6\right),$$

and

$$\tilde{\Delta} = \det(D_t P_\eta),$$

where p_i, $i = 1, 2$, are given by (18). Using the commutative diagram

$$\begin{array}{ccc} T \setminus \{p_2 = 0\} & \overset{\mathsf{pm}}{\to} & \mathrm{SL}(2,\mathbb{Z}) \setminus \mathcal{P} \\ P_\eta \downarrow & & \downarrow \tilde{P}_\eta \\ T & \overset{\mathsf{pm}}{\to} & \mathrm{SL}(2,\mathbb{Z}) \setminus \mathcal{P} \end{array},$$

where $\tilde{P}_\eta$ is the map given by the action of $\left(\begin{smallmatrix} p_2^{-1} & p_1 \\ 0 & p_2 \end{smallmatrix}\right)$ from left on $\mathrm{SL}(2,\mathbb{Z}) \setminus \mathcal{P}$, one can show that P_η maps every leaf of $\mathcal{F}_\eta$ to a leaf of $\mathcal{F}(\mathrm{Ra})$ (possibly a point) and so

$$D_t P_\eta\big(X(t)\big) = \check{\Delta} \cdot \mathrm{Ra}\big(P(t)\big) \quad \text{for some } \check{\Delta} \in \mathbb{C}[t],$$

where $X = \sum_{i=1}^{3} X_i \frac{\partial}{\partial t_i}$ is a polynomial vector field tangent to $\mathcal{F}_\eta$ and X_i's have no common factors.

Theorem 4. *Let $\mathcal{F}_\eta$ be an elliptic modular foliation associated to the family* (1) *and η, where η is defined over $\bar{\mathbb{Q}}$. The following is true:*

1. *For any point $a \in \bar{\mathbb{Q}}^3 \cap (T \setminus \{p_1 = p_2 = 0\})$ we have:*

$$\bar{\mathbb{Q}}^3 \cap L_a \subset P_\eta^{-1} P_\eta(a).$$

In particular, for $a \in T \setminus (\{\tilde{\Delta} = 0\} \cup \{p_2 = 0\})$ the intersection $\bar{\mathbb{Q}}^3 \cap L_a$ is finite.

2. *$K_\eta \cap \bar{\mathbb{Q}}^3$ is the $\bar{\mathbb{Q}}$-rational points of the algebraic set*

$$\big\{t \in T \mid 0 = [\eta|_{E_t}] \in H^1_{\mathrm{dR}}(E_t)\big\}. \tag{20}$$

Proof. Since η is define over $\bar{\mathbb{Q}}$, we have $p_1, p_2 \in \bar{\mathbb{Q}}(t_1, t_2, t_3)$. If a leaf L of $\mathcal{F}_\omega$ contains two distinct $a_i \in T$, $i = 1, 2$ points with algebraic coordinates then by the definition of a modular foliation, the period $\mathbb{Z}$-modules $\{\int_\delta \eta \mid \delta \in H_1(E_{a_i}, \mathbb{Z})\}$, $i = 1, 2$, coincide. We apply Corollary 2 and conclude that there is an isomorphism $b : E_{a_1} \to E_{a_2}$ with $b^*[\omega] = [\omega]$. Let $b^*[p_2(a_2)^{-1} \frac{dx}{y}] = k \frac{dx}{y}$, $k \in \mathbb{C}$. We have

$$\mathsf{pm}(a_1) \begin{pmatrix} k & p_1(a_1) \\ 0 & p_2(a_1) \end{pmatrix} = \mathsf{pm}(a_2) \begin{pmatrix} p_2(a_2)^{-1} & p_1(a_2) \\ 0 & p_2(a_2) \end{pmatrix}.$$

Taking determinant of the above equality we get $k = p_2(a_1)^{-1}$ and using (9), we conclude that $P_\eta(a_1) = P_\eta(a_2)$.

If for some $0 \neq \delta \in H_1(E_t, \mathbb{Z})$ we have $\int_\delta \eta = 0$ then using Theorem 3 we conclude that $0 = [\eta|_{E_t}] \in H^1_{\mathrm{dR}}(E_t)$. Note that the set (20) is equal to $\{p_1(t) = p_2(t) = 0\}$. □

Theorem 2 follows from Theorem 4 for $\eta = \frac{x\,dx}{y}$. The map $P_{\frac{x\,dx}{y}}$ is identity and the set (20) is empty.

For the second part of Theorem 2 we give another proof. Recall the notations in Section 4. Suppose that there is a parameter $t \in T \cap \bar{\mathbb{Q}}^3$ such that $\int_\delta \frac{x\,dx}{y} = 0$, for some $\delta \in H_1(E_t, \mathbb{Z})$. We can assume that δ is not a multiple of another cycle in $H_1(E_t, \mathbb{Z})$. The corresponding period matrix of t in a basis (δ', δ) of $H_1(E_t, \mathbb{Z})$ has zero x_4-coordinate and so the numbers

$$t_i = F_i\begin{pmatrix} x_1 & x_2 \\ x_3 & 0 \end{pmatrix} x_3^{-2i} g_i\left(\frac{x_1}{x_3}\right), \quad i = 2, 3,$$

$$t_1 = F_1\begin{pmatrix} x_1 & x_2 \\ x_3 & 0 \end{pmatrix} = x_3^{-2} g_1\left(\frac{x_1}{x_3}\right)$$

are in $\bar{\mathbb{Q}}$. This implies that for $z = \frac{x_1}{x_3} \in \mathbb{H}$ we have

$$\frac{g_3}{g_1^3}(z), \frac{g_2}{g_1^2}(z), \frac{g_3^2}{g_2^3}(z) \in \bar{\mathbb{Q}}.$$

This is in contradiction with the following theorems.

Theorem (Nesterenko 1996, [21]). *For any $z \in \mathbb{H}$, the set*

$$e^{2\pi i z}, \frac{g_1(z)}{a_1}, \frac{g_2(z)}{a_2}, \frac{g_3(z)}{a_3}$$

contains at least three algebraically independent numbers over $\mathbb{Q}$.

9. THE FAMILY $Y^2 - 4T_0(X - T_1)(X - T_2)(X - T_3)$

In this section we consider the family

$$(21) \qquad E_t\colon\ y^2 - 4t_0(x - t_1)(x - t_2)(x - t_3), \quad t \in \mathbb{C}^4,$$

with the discriminant $\Delta = \frac{-16}{27}(t_0(t_1 - t_2)(t_2 - t_3)(t_3 - t_1))^2$. First, let us identify the monodromy group associated to this family. Fix a smooth elliptic curve E_t. In $H_1(E_t, \mathbb{Z})$ we distinguish three cycles as follows: In the x-plane, we join t_{i-1} to t_{i+1}, $i = 1, 2, 3$, $t_4 = t_1$, $t_{-1} = t_3$ by a straight line $\tilde{\delta}_i$ and above it in E_t, we consider the closed cycle $\delta_i = \delta_{i,1} - \delta_{i,2}$ which is a double covering of $\tilde{\delta}_i$, where by definition $\delta_4 = \delta_1$. We assume that the triangle formed by $\tilde{\delta}_1, \tilde{\delta}_2$ and $\tilde{\delta}_3$ in the x-plane is oriented anti-clockwise and so we have:

$$(22) \qquad \langle \delta_i, \delta_{i+1} \rangle = 1, \quad i = 1, 2, 3.$$

Since $H_1(E_t, \mathbb{Z})$ is of rank 2, we have $n_1\delta_1 + n_2\delta_2 + n_3\delta_3 = 0$ in $H_1(E_t, \mathbb{Z})$ for some $n_1, n_2, n_3 \in \mathbb{Z}$ which are not simultaneously zero. The equalities (22) imply that $n_1 = n_2 = n_3$ and so we have $\delta_1 + \delta_2 + \delta_3 = 0$. A topological way to see this is to assume that the oriented triangles $\delta_{1,j} + \delta_{2,j} + \delta_{3,j}$, $j = 1, 2$, are homotop to zero in E_t (for a better intuition take the paths $\tilde{\delta}_i$ such that the triangle formed by them has almost zero area).

We choose $\delta = (\delta_1, \delta_2)$ as a basis of $H_1(E_t, \mathbb{Z})$. Let us now calculate the monodromy group in the basis δ. Since for fixed t_1, t_2, t_3 the elliptic curves E_t with t_0 varying are biholomorphic to each other, the monodromy around $t_0 = 0$ is trivial. For calculating other monodromies we assume that $t_0 = 1$. It is not difficult to see that the monodromy around the hyperplane $t_{i-1} = t_{i+1}$ is given by

$$\delta_i \mapsto \delta_i, \qquad \delta_{i-1} \mapsto \delta_{i-1} - 2\delta_i, \qquad \delta_{i+1} \mapsto \delta_{i+1} + 2\delta_i.$$

We conclude that the monodromy group Γ in the basis $(\delta_1, \delta_2)^{\mathrm{tr}}$ is generated by:

$$A_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix},$$

where A_i is the monodromy around the hyperplane $t_{i-1} = t_{i+1}$. This is the congruence group $\Gamma(2) = \{A \in \mathrm{SL}(2, \mathbb{Z}) \mid A \equiv_2 I\}$ which is isomorphic to the permutation group in three elements. Now, we consider the period map $\mathsf{pm}: T \to \Gamma \setminus \mathcal{P}$, where $T := \mathbb{C}^4 \setminus \{\Delta = 0\}$. The calculation of the Gauss–Manin connection of the family (21) in the basis $\omega = (\frac{dx}{y}, \frac{x\,dx}{y})^{\mathrm{tr}}$ and hence the derivative of pm can be done using the map which sends the family (21) to (3). We have

$$\begin{aligned} B = {} & \frac{dt_1}{2(t_1 - t_2)(t_1 - t_3)} \begin{pmatrix} -t_1 & 1 \\ t_2 t_3 - t_1(t_2 + t_3) & t_1 \end{pmatrix} \\ & + \frac{dt_2}{2(t_2 - t_1)(t_2 - t_3)} \begin{pmatrix} -t_2 & 1 \\ t_1 t_3 - t_2(t_1 + t_3) & t_2 \end{pmatrix} \\ & + \frac{dt_3}{2(t_3 - t_1)(t_3 - t_2)} \begin{pmatrix} -t_3 & 1 \\ t_1 t_2 - t_3(t_1 + t_2) & t_3 \end{pmatrix}, \end{aligned}$$

where $\nabla\omega = B\omega$. As before we can prove that the period map is a global biholomorphism. We look at its inverse $F = (F_0, F_1, F_2, F_3)$ which satisfies:

$$(DF)_x = \frac{1}{\det(x)} \left(\begin{matrix} -F_0 x_4 & F_0 x_3 \\ F_1F_2x_3 + F_1F_3x_3 - F_1x_4 - F_2F_3x_3 & -F_1x_3 + x_4 \\ F_1F_2x_3 - F_1F_3x_3 + F_2F_3x_3 - F_2x_4 & -F_2x_3 + x_4 \\ -F_1F_2x_3 + F_1F_3x_3 + F_2F_3x_3 - F_3x_4 & -F_3x_3 + x_4 \end{matrix} \right. \qquad (23)$$

$$\left. \begin{matrix} F_0 x_2 & -F_0 x_1 \\ -F_1F_2x_1 - F_1F_3x_1 + F_1x_2 + F_2F_3x_1 & F_1x_1 - x_2 \\ -F_1F_2x_1 + F_1F_3x_1 - F_2F_3x_1 + F_2x_2 & F_2x_1 - x_2 \\ F_1F_2x_1 - F_1F_3x_1 - F_2F_3x_1 + F_3x_2 & F_3x_1 - x_2 \end{matrix} \right).$$

It is easy to see that $F_0(x) = \det(x)^{-1}$. In a similar way as in Section 3 we define the action of G_0 on $\mathbb{C}^4$ by

$$t \bullet g := \left(t_0 k_1^{-1} k_2^{-1}, t_1 k_1^{-1} k_2 + k_3 k_1^{-1}, t_2 k_1^{-1} k_2 + k_3 k_1^{-1}, t_3 k_1^{-1} k_2 + k_3 k_1^{-1}\right), \qquad (24)$$
$$t \in \mathbb{C}^4, g = \begin{pmatrix} k_1 & k_3 \\ 0 & k_2 \end{pmatrix} \in G_0,$$

and it turns out that $\mathsf{pm}(t \bullet g) = \mathsf{pm}(t) \cdot g,\ t \in \mathbb{C}^4,\ g \in G_0$. Taking F of this equality we conclude that $F_i,\ i = 1, 2, 3$, satisfies

$$F_i(xg) = F_i(x) k_1^{-1} k_2 + k_3 k_1^{-1}, \quad i = 1, 2, 3. \tag{25}$$

We define $\theta_i,\ i = 1, 2, 3$, to be the restriction of F_i to $x = \left(\begin{smallmatrix} z & -1 \\ 1 & 0 \end{smallmatrix}\right),\ z \in \mathbb{H}$ and consider it as a function in z. Now, the equalities (25) imply that

$$(cz+d)^{-2} \theta_i(Az) = \theta_i(z) + c(cz+d)^{-1}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,\ i = 1, 2, 3. \tag{26}$$

The first column of (23) implies that $(\theta_1, \theta_2, \theta_3) : \mathbb{H} \to \mathbb{C}^3$ satisfies the differential equation:

$$\begin{cases} \dot{t}_1 = t_1(t_2 + t_3) - t_2 t_3, \\ \dot{t}_2 = t_2(t_1 + t_3) - t_1 t_3, \\ \dot{t}_3 = t_3(t_2 + t_3) - t_1 t_2. \end{cases} \tag{27}$$

The foliation induced by the above equations in $\mathbb{C}^3$ has the axis t_1, t_2 and t_3 as a singular set. It leaves the hyperplanes $t_i = t_j$ invariant and is integrable there. For instance, a first integral in $t_1 = t_2$ is given by $\frac{t_1 - t_2}{t_2^2}$. Considering the map from the family (21) to (3), we conclude that:

$$\begin{aligned} g_1 &= \frac{1}{3}(\theta_1 + \theta_2 + \theta_3), \\ g_2 &= 4 \sum_{1 \leqslant i < j \leqslant 3} (g_1 - \theta_i)(g_1 - \theta_j), \\ g_3 &= 4(g_1 - \theta_1)(g_1 - \theta_2)(g_1 - \theta_3). \end{aligned}$$

We can write the Taylor series of θ_i's in $q = e^{2\pi i z}$. I do not know statements similar to Proposition 3 for θ_i's.

10. Another basis

Let us consider the family (3). Sometime it is useful to use the differential forms

$$\eta_1 := \frac{-2}{5}(2x\,dy - 3y\,dx) \quad \text{and} \quad \eta_2 := \frac{-2}{7}x(2x\,dy - 3y\,dx). \tag{28}$$

They are related to ω_1, ω_2 by:

$$\frac{d\eta_1}{df} = \frac{dx}{y}, \qquad \frac{d\eta_2}{df} = \frac{x\,dx}{y}, \tag{29}$$

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} t_1 t_2 - \frac{6}{5} t_3 & -\frac{4}{5} t_2 \\ \frac{1}{105 t_0}(84 t_0 t_1^2 t_2 - 36 t_0 t_1 t_3 - 5 t_2^2) & -\frac{4}{5} t_1 t_2 - \frac{6}{7} t_3 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}. \tag{30}$$

Note that the above matrix has determinant $\frac{4}{105 t_0}\Delta$ and so $\eta_i,\ i = 1, 2$, restricted to a smooth elliptic curve E_t form a basis of $H^1_{\mathrm{dR}}(E_t)$. The calculation of the Gauss–Manin connection with respect to the basis $\eta = (\eta_1, \eta_2)^{\mathrm{tr}}$ leads to:

$$
A_0 = \begin{pmatrix} \frac{21}{2}t_0t_1t_2t_3 - 9t_0t_3^2 + \frac{3}{4}t_2^3 & -\frac{21}{2}t_0t_2t_3 \\ \frac{21}{2}t_0t_1^2t_2t_3 + 9t_0t_1t_3^2 - \frac{1}{2}t_1t_2^3 - \frac{5}{8}t_2^2t_3 & -\frac{21}{2}t_0t_1t_2t_3 - 18t_0t_3^2 + \frac{5}{4}t_2^3 \end{pmatrix},
$$
$$
A_1 = \begin{pmatrix} 0 & 0 \\ 27t_0^2t_3^2 - t_0t_2^3 & 0 \end{pmatrix},
$$
$$
A_2 = \begin{pmatrix} -\frac{63}{2}t_0^2t_1t_3 - \frac{5}{4}t_0t_2^2 & \frac{63}{2}t_0^2t_3 \\ -\frac{63}{2}t_0^2t_1^2t_3 + \frac{1}{2}t_0t_1t_2^2 + \frac{15}{8}t_0t_2t_3 & \frac{63}{2}t_0^2t_1t_3 - \frac{7}{4}t_0t_2^2 \end{pmatrix},
$$
$$
A_3 = \begin{pmatrix} 21t_0^2t_1t_2 + \frac{45}{2}t_0^2t_3 & -21t_0^2t_2 \\ 21t_0^2t_1^2t_2 - 9t_0^2t_1t_3 - \frac{5}{4}t_0t_2^2 & -21t_0^2t_1t_2 + \frac{63}{2}t_0^2t_3 \end{pmatrix},
$$

where $\nabla\eta = (\frac{1}{\Delta}\sum_{i=1}^4 A_i dt_i)\eta$. We have

$$
\mathcal{F}_{\eta_1}\colon \frac{\partial}{\partial t_1},
$$
$$
\mathcal{F}_{\eta_2}\colon \left(-60t_1^2 + 5t_2\right)\frac{\partial}{\partial t_1} + (48t_1t_2 - 72t_3)\frac{\partial}{\partial t_2} + (72t_1t_3 - 4t_2^2)\frac{\partial}{\partial t_3}.
$$

Remark 2. Both differential forms ω_1 and η_1 are invariant under the morphism $(x, y) \mapsto (x + s, y)$ and this is the reason why $\mathcal{F}_{\frac{dx}{y}} = \mathcal{F}_{\eta_1}$ is given by $dt_2 = 0$, $dt_3 = 0$. This and the first row of the equality (30) implies that for constant t_2, t_3 the integral $\int_\delta \frac{x\,dx}{y}$ is a degree one polynomial in t_1 and hence $\nabla^2_{\frac{\partial}{\partial t_1}} \frac{x\,dx}{y} = 0$. This equality can be also checked directly from the Gauss–Manin connection (5).

Remark 3. Consider the weighted ring $\mathbb{R}[x, y]$, $\deg(x) = 2$, $\deg(y) = 3$. One can extend the definition of the degree to the differential 1-forms ω in $\mathbb{R}^2$ by setting $\deg(dx) = 2, \deg(dy) = 3$. Any real holomorphic foliation $\mathcal{F}(\omega)$ in $\mathbb{R}^2$ with $\deg(\omega) = 6$ has no limit cycles. In fact, we can write $\omega = df - a\eta_1,\ a \in \mathbb{R}$ (up to multiplication by a constant and a linear change of coordinates), where f is the polynomial in (1) with $t \in \mathbb{R}^4$, and if $\mathcal{F}(\omega)$ has a limit cycle δ then $0 = \int_\delta df = a\int_\delta \eta_1 = (-2a)\int_\delta dx \wedge dy$, which is a contradiction. Considering $\mathcal{F}(\omega)$, $\deg(\omega) = 7$, we can write $\omega = df - a\eta_1 - b\eta_2,\ a, b \in \mathbb{R}$, and such a foliation can have limit cycles because the integral $\int_{\delta_s} \eta_2$ may have zeros, where δ_s is a continuous family of real vanishing cycles parameterized by the image s of f. To count the zeros of $\int_{\delta_s} \eta_2$ we may do as follows: We choose another cycle $\tilde{\delta}_s$ such that δ_s and $\tilde{\delta}_s$ form a basis of $H_1(\{f = s\}, \mathbb{Z})$ with $\langle\delta_s, \tilde{\delta}_s\rangle = 1$. The real valued function $B_2(s) = \mathrm{Im}(\int_{\delta_s} \eta_2 \overline{\int_{\tilde{\delta}_s} \eta_2})$ is analytic in $\mathbb{C} \setminus \{c_1, c_2\}$, where c_1 and c_2 are critical values of f. It is continuous and zero in c_1, c_2 . The intersection of the real curve $B_2 = 0$ with the real line $\mathbb{R}$ is a bound for the number of zeros of $\int_{\delta_s} \eta_2$.

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