Hossein Movasati

A Differential Introduction to Modular Forms and Elliptic Curves

February 15, 2024

Publisher

Jalal al-Din Muhammad Balkhi (Rumi), Divane Shams, Rubaiyat, No. 1154. Calligraphy: Tahereh Aladpoosh
I am the grape got crushed under you feet,
I am dragged to any direction you want,
You ask me why I am turning around you,
I am not turning around you but myself.
(Author's translation)

Contents

1 Introduction . . . 1
1.1 Fibonacci sequence . . . 1
1.2 Fermat's last theorem and arithmetic modularity theorem . . . 3
1.3 Beyond elliptic curves . . . 5
1.4 Prerequisites . . . 6
1.5 Organization of the book . . . 7

Part I Elliptic curves and modular forms

2 Modular forms . . . 11
2.1 Introduction . . . 11
2.2 Elliptic functions . . . 11
2.3 The modular group and its action . . . 15
2.4 Weierstrass $\wp$-function . . . 17
2.5 Differential equation of $\wp$ . . . 20
2.6 Eisenstein series . . . 22
2.7 Fourier expansion of Eisenstein series . . . 24
2.8 The Eisenstein series E_2 . . . 26
2.9 The algebra of modular forms . . . 27
2.10 Ramanujan relations between Eisenstein series . . . 29
2.11 The product formula for discriminant . . . 30
2.12 The j function . . . 32
2.13 Growth of coefficients . . . 34
2.14 The numbers e_1, e_2, e_3 . . . 36
2.15 Jacobi's theta functions . . . 36
2.16 Applications of theta series . . . 41
2.17 Poisson summation formula . . . 42
2.18 Some exercises based on Eisenstein's work . . . 43
2.19 Differential equations of theta series . . . 44

3 Elliptic curves and integrals . . . 47
3.1 Introduction . . . 47
3.2 Elliptic integrals . . . 47
3.3 Picard-Lefschetz theory . . . 52
3.4 Weierstrass uniformization theorem I . . . 54
3.5 Period domain and period map . . . 56
3.6 Gauss-Manin connection matrix . . . 60
3.7 Ramanujan and Darboux-Haphen vector field . . . 64
3.8 Picard-Fuchs equation . . . 65
3.9 Hypergeometric functions . . . 66
3.10 Schwarz map . . . 68
3.11 Elliptic integrals and modular forms . . . 70

4 Rudiments of Algebraic Geometry of curves . . . 77
4.1 Introduction . . . 77
4.2 Curves . . . 78
4.3 Schemes . . . 79
4.4 Singularities and smooth curves . . . 81
4.5 Coordinate system on a curve . . . 82
4.6 Discriminant . . . 83
4.7 Curves of genus zero and bigger than one . . . 86
4.8 Elliptic curves in Weierstrass form . . . 87
4.9 Real geometry of elliptic curves . . . 88
4.10 The group law in elliptic curves . . . 89
4.11 Riemann-Roch theorem . . . 91
4.12 Weierstrass form revised . . . 93
4.13 Moduli of elliptic curves . . . 95
4.14 The addition formula . . . 96

5 Mordell-Weil Theorem . . . 99
5.1 Introduction . . . 99
5.2 Descent theorem . . . 100
5.3 The construction of height function . . . 101

6 Torsions and isogeny . . . 107
6.1 Introduction . . . 107
6.2 Torsion points . . . 107
6.3 Isogeny . . . 110
6.4 Nagell-Lutz and Mazur theorems . . . 112

7 Hecke operators . . . 115
7.1 Introduction . . . 115
7.2 Hecke operators . . . 115
7.3 Proof of Theorem 6.1 . . . 122

8 Congruence groups . . . 123
8.1 Introduction . . . 123
8.2 Congruence groups . . . 123
8.3 Weil pairing . . . 124
8.4 Modular forms for congruence groups . . . 126
8.5 Fourier or q-expansion . . . 128
8.6 Transcendence degree of modular forms . . . 130

9 Quasi-modular forms . . . 133
9.1 Introduction . . . 133
9.2 De Rham cohomology of affine varieties . . . 134
9.3 Two incomplete elliptic curves . . . 136
9.4 Algebraic de Rham cohomology of complete curves . . . 139
9.5 Residue calculus, trace map and cup product . . . 139
9.6 Eisenstein modular forms . . . 142
9.7 Ibiporanga: enhanced elliptic curves . . . 143
9.8 Quasi-modular forms . . . 146
9.9 Quasi-modular forms over $\mathbb{C}$ . . . 148
9.10 Generlized period domain and generalized period map . . . 150
9.11 Generalized period map and it inverse . . . 151
9.12 Periods and Ramanujan . . . 154
9.13 Comparision theorem . . . 155
9.14 Partial compactifications . . . 156
9.15 Hecke operators . . . 157

10 Riemann zeta function . . . 159
10.1 Introduction . . . 159
10.2 Riemann zeta function . . . 160
10.3 The big Oh notation . . . 160
10.4 Gamma function . . . 162
10.5 Analytic extension of Riemann's zeta function, I . . . 164
10.6 Second proof for functional equation . . . 166
10.7 Hecke's L-functions . . . 167
10.8 L-function of cusp forms . . . 171
10.9 L-function attached to Gauss hypergeometric equation . . . 174

11 Hasse-Weil zeta function . . . 179
11.1 Introduction . . . 179
11.2 Finite fields . . . 179
11.3 Zeta functions of elliptic curves over finite fields . . . 180
11.4 One dimensional algebraic groups . . . 181
11.5 Reduction of elliptic curves . . . 183
11.6 Zeta functions of elliptic curves over $\mathbb{Q}$ . . . 184
11.7 Hasse-Weil conjecture . . . 185
11.8 Birch Swinnerton-Dyer conjecture . . . 185

11.9 Congruent numbers . . . 186

12 Jacobi forms . . . 191
12.1 Introduction . . . 191
12.2 Jacobi group and Jacobi forms . . . 191
12.3 Relative de Rham cohomology . . . 194
12.4 Meromorphic forms without residues . . . 196
12.5 Mixed Hodge structures . . . 197
12.6 The moduli space T . . . 199
12.7 Self-similar homology . . . 201
12.8 Self-similar cohomology . . . 203
12.9 Gauss-Manin connection of self-similar integrals . . . 205
12.10 Gauss-Manin connection in relative cohomology . . . 208
12.11 Gluing Gauss-Manin connections . . . 209
12.12 Computation of a period matrix . . . 212
12.13 Quasi-Jacobi forms . . . 216

13 Modular curves and differential equations . . . 217
13.1 Introduction . . . 217
13.2 A consequence of Hecke operators . . . 218
13.3 A model of modular curve in dimension four . . . 219
13.4 Moduli space I . . . 221
13.5 Proof of Theorem 13.5 . . . 223
13.6 Dynamics . . . 223
13.7 Picard's curious example . . . 224

Part II Calabi-Yau varieties

14 Calabi-Yau threefolds . . . 231
14.1 Hypersurfaces with a finite group action . . . 231
14.2 Eisenstein series . . . 234

References . . . 237

Index . . . 245

Preface

There are so many books on modular forms and elliptic curves that it might seem useless to add another one. None of these books approach modular forms from the point of view of differential equations and this differentiates the present book from others. Moreover, most of these books prepare the reader for a better understanding of Abelian varieties and Siegel modular forms, whereas in this book we pretend to go in direction of Calabi-Yau varieties, and in particular Calabi-Yau threefolds. This has resulted in a tremendous generalization of modular forms presented in the author's books "Modular and automorphic forms & beyond" published in 2022 and "Gauss-Manin connection in disguise: Calabi-Yau modular forms" published in 2017. Its origin partially comes from many q-expansion computations in theoretical physics and in particular string theory. In one hand we want to collect many classical topics related to elliptic curves, seen as one dimensional compact Calabi-Yau varieties, and (elliptic) modular forms. This includes the arithmetic modularity theorem which relates the L-functions of elliptic curves to those of modular forms. On the other hand we have an eye on the generalization of all these into the framework of arbitrary dimensional Calabi-Yau varieties and the corresponding Calabi-Yau modular forms.

Hossein Movasati
January 2020
Rio de Janeiro, RJ, Brazil

Chapter 1
Introduction

In 2002 I was a post-doc at Max-Planck Institute for Mathematics (MPIM) in Bonn and was working on applications of abelian integrals in holomorphic foliations. Meantime, I was looking for many appearances of periods and multiple integrals in complex and algebraic geometry, and in particular, Hodge theory. At that time I was looking for jobs, and for the first time I saw the word "modular form" in a post-doc announcement at MPIM. I got the post-doc and I never imagined that one day I am going to write my own book and view on this beautiful and elegant theory of mathematics. After twenty years, one of my principal projects has been to put modular forms in a broader context. This has been summarized in the book [Mov22a]. In order to do this, I had to read the founding articles of the classical theory of modular forms which is spread in the last two hundred years. The first part of the present book is mainly the result of such a reading and the second part is dedicated to my own view toward generalizations of modular forms.

Modular forms are holomorphic functions in the upper half plane

$$\mathbb{H} := \{\tau \in \mathbb{C} | \operatorname{Im}(\tau) > 0\}$$

and elliptic curves are Riemann surfaces of genus one (the surface of a donut). With this fast description of our main objects, it is clear that the complex analysis in one variable plays an important role in the present text. Both objects enjoy many arithmetic properties. Elliptic curves can be considered as Diophantine equations and our main interest on modular forms comes from the fact that they are generating functions for many unexpected counting in mathematics. Why generating functions are useful might be explained with the simple example of Fibonacci numbers.

1.1 Fibonacci sequence

The Fibonacci sequence is defined in the following way

$$F_{n+2} = F_{n+1} + F_n, \quad n \geq 0, \quad F_0 = 0,\ F_1 = 1 \tag{1.1}$$

Few elements of this sequence are

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots.$$

Once you have a sequence of natural numbers in mathematics, it is recommended to put it in a generating function:

$$F(q) :\ = q + q^2 + 2q^3 + 3q^4 + \cdots + F_n q^n + \cdots.$$

At the beginning this is just a formal power series, however, soon it will become clear that it is a convergent series, and its radius of convergence carries many information of the sequence F_n itself. For now, let us do the following manipulation:

$$\begin{aligned} F(q) &= \sum_{n=0}^{\infty} F_n q^n = q + \sum_{n=2}^{\infty} (F_{n-1} + F_{n-2}) q^n \\ &= q + q \cdot F(q) + q^2 F(q), \end{aligned}$$

which implies that

$$F(q) = \frac{q}{1 - q - q^2}. \tag{1.2}$$

Therefore, $F(q)$ converges to a rational function. In order to find the radius of convergence of a rational function, we have to find the roots of its denominator:

$$\begin{aligned} F(q) &= \frac{q}{1 - q - q^2} = \frac{q}{(1 - \alpha \cdot q)(1 - \beta \cdot q)} = \frac{(\alpha - \beta)^{-1}}{1 - \alpha \cdot q} - \frac{(\alpha - \beta)^{-1}}{1 - \beta \cdot q} \\ &= \sum_{n=0}^{\infty} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) q^n, \end{aligned}$$

where $\alpha = \frac{1}{2}\left(1 + \sqrt{5}\right)$, $\beta = \frac{1}{2}\left(1 - \sqrt{5}\right)$. We conclude that

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}},$$

which at first glance looks strange because we have found a formula for the integer F_n in terms of square root of 5. Since the radius of convergence of $F(q)$ is $\min\left\{\frac{1}{|\alpha|}, \frac{1}{|\beta|}\right\} = \max\{|\alpha|, |\beta|\} = |\alpha|$, we conclude that

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \lim_{n\to\infty} F_n^{\frac{1}{n}} = \frac{1}{2}\left(1 + \sqrt{5}\right).$$

This number is called the golden ratio or the golden number.

Exercise 1.1 Show that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

1.2 Fermat's last theorem and arithmetic modularity theorem

Modular forms and elliptic curves are firmly rooted in the fertile grounds of number theory. As a proof of the mentioned fact and as an introduction to the present text we mention the following: For $p \geq 2$ prime, the Fermat last theorem ask for a non-trivial integer solution, that is $a, b, c \in \mathbb{Z}$ with $abc \neq 0$, for the Diophantine equation

$$a^p + b^p + c^p = 0.$$

For a hypothetical solution $(A, B, C) = (a^p, b^p, c^p)$ of the Fermat equation with $abc \neq 0$ [Fre86] considered the elliptic curve

$$E_{A,B,C} : y^2 = x(x - A)(x + B).$$

From this, one construct a modular form $f_{A,B,C}$ and a Galois representation with certain properties and then one proves that such objects do not exist. During this passage one encounters the modularity conjecture which claims that every elliptic curve over $\mathbb{Q}$ is modular. Roughly speaking this means that every elliptic curve over $\mathbb{Q}$ appears in the Jacobian of a modular curve of level N. Another formulation of modularity property is by using L-functions which generalizes the famous Riemann's zeta function $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$. Riemann hypothesis claims that all the non-trivial zeros of ζ lies on $\mathrm{Re}(s) = \frac{1}{2}$ and it has strong consequences on the growth of prime numbers. For the L-functions associated to elliptic curves one has the Birch-Swinnerton Dyer conjecture which predicts the rank of an elliptic curve to be the order of vanishing of the corresponding L-function at $s = 1$.

Modular forms as generating functions have many fascinating and mysterious applications. Arithmetic modularity theorem is one of these. In many books and articles we find the expression "Let E be an elliptic curve over $\mathbb{Z}$". This has an intrinsic definition in terms of Grothendieck's theory of schemes, that for now, we don't want to get into its details. We content ourselves with the example

$$E : y^2 + y = x^3 - x^2$$

which the reader might consider it as a Diophantine equation, that is, we are interested to find x and y in the ring of integers, the field of rational numbers, finite fields, etc. Let p be a prime number (don't take the Grothendieck's prime [i]) We count the number of solutions N_p of E modulo the prime p.

[i] A. Grothendieck (1928-2014) is one of the founders of modern Arithmetic Algebraic Geometry. Once he was asked to give an example of a prime number and he answered: 57.

p	Solutions	N_p
2	(0,0),(0,1), (1,0), (1,1)	4
3	(0,0),(0,2), (1,0), (1,2)	4
5	(0,0),(0,4), (1,0), (1,4)	4
7	(0,0),(0,6), (1,0), (1,6),...	9
11	(0,0),(0,10), (1,0), (1,10),...	10

In total we have to substitute p^2 pairs $(x,y)\,,x,y=0,1,\ldots,p-1$ inside E and verify whether modulo prime p, the equality holds or not. The first four solutions in the above table have to do with the fact that over integers E has already four solutions. $(0,0)\,,(0,-1)\,,(1,0)\,,(1,-1)$. A priori, if we have computed $N_2,N_3,N_5,\ldots,N_{11}$, this doesn't give any clue how to find the number N_{13}. We have to check 13^2 cases again. In a modern language, we say that, we are counting the number of $\mathbb{F}_p$-rational points of E and we write

$$N_p :\ = \#E\left(\mathbb{F}_p\right).$$

Here $\mathbb{F}_p := \{0,1,2,\ldots,p-1\}$ is the finite field with p elements. The theory of modular forms, and in particular arithmetic modularity theorem, says that there is a closed formula for the generating function of N_p's. This is as follows. Let

$$\eta\,(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n) \tag{1.3}$$

be the Dedekind eta function.

Exercise 1.2 Show that the radius of convergence of the Dedekind η function is 1.

We consider it as a formal product. Let

$$\begin{aligned}
F\,(q) &= \eta\,(q)^2\ \eta\left(q^{11}\right)^2 \\
&= q^{\frac{2}{24}+\frac{2\cdot 11}{24}} \prod_{n=1}^{\infty} (1-q^n)^2 \prod_{n=1}^{\infty} \left(1-q^{11n}\right)^2 \\
&= q-2q^2-q^3+2q^4+q^5+2q^6-2q^7-2q^9-2q^{10}+q^{11}-2q^{12}+4q^{13}+\cdots \\
&= \sum_{n=1}^{\infty} f_n q^n.
\end{aligned}$$

The arithmetic modularity theorem tells us that

$$N_p = p - f_p \tag{1.4}$$

and f is a modular form.

Exercise 1.3 Find N_p for all $p \leqslant 23$ and verify (1.4). In [MO97] the authors give a list of modular forms f in terms of η, together with the corresponding elliptic curves. This includes our main example in this section. Verify $N_p = p - f_p$ for some of these examples.

More precisely, "f is a weight 2 new form for $\Gamma_0(11)$". One of the aims of the present text is to understand this statement. This phenomena is a part of a general theorem:

Theorem 1.1 (Arithmetic modularity theorem) *([Wil95, BCDT01]) For any elliptic curve E over $\mathbb{Q}$, there is a modular form $f=\sum_{n=1}^{\infty} f_n q^n$ such that* (1.4) *holds for all except a finite number of primes.*

A precise statement, together with other equivalent versions will be presented in this text.

Exercise 1.4 It is a natural question to ask whether f_n for non-prime n has an enumerative meaning or not. For instance, one can define $N_n := E(\mathbb{Z}/n\mathbb{Z})$, that is N_n is the number of solutions of E modulo n. For some small non-prime numbers n show that $N_n = n - f_n$ does not hold. Moreover f_n is multiplicative, that is, $f_n f_m = f_{nm}$ for coprime $n,m \in \mathbb{Z}$ however, $n - N_n$ is not multiplicative. The fact that f_n's are multiplicative follows from the theory of Hecke operators, see Chapter 7. From this theory we can also write formulas for f_{nm} in terms of n and m for arbitrary $n,m \in \mathbb{N}$.

Exercise 1.5 Compute few coefficients of $\Delta = \eta(q)^{24} = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n$ and verify the following equalities for examples of $n,m \in \mathbb{N}$:

$$\begin{aligned} &\tau(n)\tau(m) = \tau(nm) \qquad (n,m)=1, \\ &\tau(p)\tau(p^n) = \tau(p^{n+1}) + p^{11}\tau(p^{n-1}), \quad p \text{ prime.} \end{aligned} \tag{1.5}$$

The proof of these equalities will be done in Chapter 7 using Hecke operators. τ is called the Ramanujan's tau function.

Exercise 1.6 Let $f = \sum_{n=1}^{\infty} f_n q^n$ be a formal power series with $f_n \in \mathbb{Z}$ and $f_1 = 1$. Show that f can be written in the format:

$$f(q) = q\prod_{n=1}^{\infty}(1-q^n)^{g_n}, \;\; g_n \in \mathbb{Z}. \tag{1.6}$$

For many examples of modular forms f which are not necessarily expressed in terms of η see [LMF13]. For some of these compute g_n's and identify those f such that g_n is an increasing sequence of positive integers and with greatest common divisor equal to one. For some examples of such f's see [MN20]. Can you find more?

1.3 Beyond elliptic curves

There is a tremendous amount of effort to generalize the arithmetic modularity theorem beyond elliptic curves, see for instance the expository article [Yui13]. Here we give an example taken from [Sch13, Section 15]. Let us consider the Fermat quartic surface

$$X \subset \mathbb{P}^3 : x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0.$$

We count the number of solutions of this Diophantine equation over the field $\mathbb{F}_p, p \neq 2$, that is,

$$\#X(\mathbb{F}_p) : = \#\left\{[x_0 : x_1 : x_2 : x_3] | x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\right\}.$$

Here, $[x_0 : x_1 : x_2 : x_3]$ is the equivalence class

$$(x_0, x_1, x_2, x_3) \sim (y_0, y_1, y_2, y_3) \Leftrightarrow$$

$$\exists a \in \mathbb{F}_p - \{0\}, \text{ such that } \quad x_i = a y_i, \ \ i = 0, 1, 2, 3, 4.$$

It turns out that for finite fields $\mathbb{F}_p$ with p prime we have

$$\#X(\mathbb{F}_p) = 1 + b_p + h \cdot p + p^2$$

where

$$\eta(4\tau)^6 = q \prod_{n=1}^{\infty} (1 - q^{4n})^6 = \sum_{n=1}^{\infty} b_n q^n$$

$$h = 5 + 3\chi_{-1}(p) + 6 \cdot (\chi_2(p) + \chi_{-2}(p))$$

and $\chi_a(p) : = (\frac{a}{p})$ is the Legendre symbol. Recall that

$$\left(\frac{a}{p}\right) := \begin{cases} 1 & x^2 \equiv_p a \text{ has integer solution and } p \not| a \\ -1 & x^2 \equiv_p a \text{ has no integer solution and } p \not| a \\ 0 & p | a \end{cases}$$

Exercise 1.7 Verify the above affirmation for $p = 3, 5, 7, 9$. What goes wrong for $p = 2$?

There is no arithmetic modularity theorem for a member of the family of Diophantine equations:

$$X : \ x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_0 x_1 x_2 x_3 x_4 = 0, \ \ \psi \in \mathbb{Z}, \ \ \psi \neq 1,$$

that is, if for a fixed $\psi \in \mathbb{Z}, \ \psi \neq 1$, we count the number $\#X(\mathbb{F}_p)$ of solutions of this equation over $\mathbb{F}_p$ then we do not know whether these numbers fit into any formal power series which we understand it well from the complex analysis point of view. The case $\psi = 1$ is different, as in this case X becomes singular, and after resolution of singularities, it is a typical example of a rigid Calabi-Yau threefold. For a list of such Diophantine equations see [Mey05].

1.4 Prerequisites

The most critical prerequisite for following the first part of the present book is complex analysis in one variable. We do not assume that the reader is familiar with algebraic geometry, as we aim to present all the prerequisites of schemes and curves in this book. A basic knowledge of number theory is necessary for a smooth reading of the present text. For this the reader might consult [IR90]. Even though, in the present text we emphasize differential equation aspects of modular forms and elliptic curves, no training in this topic is needed and we cover all the preliminaries. In summary, a mathematics student in the last year of undergraduate must be able to follow the text without major problems.

For the second part of the book we assume that the reader has basic training both in Algebraic Topology and Algebraic Geometry, the first one being more crucial for a smooth reading. However, it is expected that the reader learns the preliminaries alongside the present book.

1.5 Organization of the book

Chapter 2 is dedicated to the classical presentation of modular forms as holomorphic functions in the upper half plane. We mainly follow classical texts in the subject, however, the proof of some of fundamental theorems, such as finite generatedness of modular forms and the functional equation of the Eisenstein series E_2, are either left as exercises to the reader who is supposed to recover them from classical books, or we have postponed them to the next chapters in which we have developed a geometric theory of modular forms based on enhanced elliptic curves.

In Chapter 3 we study elliptic integrals and related objects. This naturally takes us to the theory of elliptic curves over complex numbers and the fact that they are genus one oriented surfaces. After a fast overview of Picard-Lefschetz theory and monodromy groups, which justifies the appearance of $\mathrm{SL}(2,\mathbb{Z})$, we present Weierstrass uniformization theorem, period map, Gauss-Manin connection, Picard-Fuchs equation and Gauss hypergeometric function arising from elliptic integrals. At the end we describe how to construct modular forms with the data of elliptic integrals.

We start Chapter 4 with a basic presentation of algebraic geometry of curves and attempt to convince the reader why using the language of curves as schemes is useful. After a brief study of curves of genus zero, we focus on curves of genus one which together with a marked point are called elliptic curves. There are two fundamental subject in this chapter. First, the group structure of elliptic curves is explained in Section 4.10. Second, in Section 4.11 we explain the fact that any elliptic curve can be written in the Weierstrass format.

Chapter 5 is dedicated to the Mordell-Weil theorem which says that the abelian group of rational points of an elliptic curve is finitely generated. As this is a classical theorem, and we do not have any simplification or a new contribution in understanding it better, we have left many parts of the proof of this theorem as exercises to the reader, who can consult other excellent books on the topic.

Torsion points and isogenies of elliptic curves are explained in Chapter 6. These are the main ingredients for the introduction of modular curves. Two fundamental theorems of arithmetic nature, namely Nagel-Lutz and Mazur theorems, are announced in this chapter.

A fundamental concept responsible for many arithmetic properties of modular forms is the notion of Hecke operators introduced in Chapter 7. The geometric theory of Hecke operators, specially those acting on quasi-modular forms, is not the main focus of classical books in this subject, and this partially justifies our presentation. The first non-trivial application of this theory is the multiplicativity of Ramanujan's τ function.

In Chapter 8 we introduce modular forms for subgroups of $\mathrm{SL}(2,\mathbb{Z})$, and in particular, for congruence groups. Main examples of such modular forms appear in the so-called arithmetic modularity theorem, however, we give many other examples arising from our geometric interpretation of modular forms. In this chapter we also prove that the transcendental degree of the field generated by all these modular forms is two. In this way, the Eisenstein series E_4 and E_6 are the building blocks of the whole theory of modular forms.

In Chapter 9 we start to elaborate the theory of quasi-modular forms in the algebraic geometric framework which requires the concept of algebraic de Rham cohomology and cup products. We briefly describe the incarnation of a quasi-modular form as a holomorphic function on the upper half plane, however this is not the main focus of our attention. The geometric framework has the advantage of a direct generalization to the context of Calabi-Yau varieties. In this chapter we introduce the concept of generalized period map and period domain which is the bridge between holomorphic and algebraic quasi-modular forms.

Part I
Elliptic curves and modular forms

Chapter 2
Modular forms

Am 23. Dezember 1751 wurden Euler die Arbeiten Fagnano's zur Begutachtung vorgelegt und regten ihn zur Entdeckung der Additionstheoreme an, so dass Jacobi den genannten Tag als den Geburtstag der elliptischen Funktionen bezeichnete, ([Fri22, page x]).

2.1 Introduction

In this chapter we present the classical point of view for modular forms, that is, as holomorphic functions in the upper half plane. They appear in a natural way as coefficients in Taylor expansions of elliptic functions. According to [Zag08], the word modular refers to the moduli space of complex curves of genus 1. Historically, elliptic integrals and the lattices obtained by elliptic integrals have been first of interest in mathematics, as they have to do with lengths of many well-known curves such as lemniscate. Therefore, the reader might also start reading this book from Chapter 3. In the prehistory of elliptic functions we also find trigonometric functions for which the reader is invited to read [Wei99, Chapter II].

2.2 Elliptic functions

Definition 2.1 A lattice Λ in $\mathbb{C}$ is a discrete subgroup of $(\mathbb{C},+)$ which generate it as an $\mathbb{R}$-vector space

It follows easily that

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \left\{ n\omega_1 + m\omega_2 \,\middle|\, n,m \in \mathbb{Z} \right\},$$

where $\omega_1, \omega_2 \in \mathbb{C},\ \omega_1, \omega_2 \neq 0,\ \mathrm{Im}(\frac{\omega_2}{\omega_1}) \neq 0$. By changing the order of ω_1, ω_2, if necessary, we can assume that

$$\mathrm{Im}(\tau) > 0, \qquad \tau := \frac{\omega_1}{\omega_2}. \tag{2.1}$$

A lattice in general is equipped with a $\mathbb{Z}$-bilinear map $\Lambda \times \Lambda \to \mathbb{Z}$. In our case it is skew-symmetric, that is, $\langle a, b\rangle = -\langle b, a\rangle \quad \forall a, b \in \Lambda$, and so $\langle \omega_1, \omega_1\rangle = 0, \langle \omega_2, \omega_2\rangle = 0$. Therefore, $\langle \omega_2, \omega_1\rangle := 1$ determines $\langle \cdot, \cdot\rangle$ uniquely. The choice of ω_1, ω_2 with (2.1) and hence with $\langle \omega_2, \omega_1\rangle := 1$ is also called an orientation of Λ. If we choose another basis ω_1', ω_2' with $\langle \omega_2', \omega_1'\rangle := 1$ then

$$\begin{bmatrix} \omega_1' \\ \omega_2' \end{bmatrix} = A \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix},\ \ A \in \mathrm{SL}(2, \mathbb{Z}),$$

where

$$\mathrm{SL}(2,\mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d, \in \mathbb{Z},\ \ ad - bc = 1 \right\}. \tag{2.2}$$

is the modular group. The quotient

$$\mathrm{PSL}(2,\mathbb{Z}) := \mathrm{SL}(2,\mathbb{Z})/\pm I$$

is also called the modular group. Let L be the space of lattices in $\mathbb{C}$. The group

$$\mathbb{C}^* = (\mathbb{C} - \{0\}, \cdot) \tag{2.3}$$

acts on L from the right

$$\begin{gathered} \mathsf{L} \times \mathbb{C}^* \to \mathsf{L} \\ (\Lambda, \lambda) \longmapsto \Lambda \cdot \lambda := \mathbb{Z}\omega_1\lambda + \mathbb{Z}\omega_2\lambda \end{gathered}$$

which is just the rescaling the lattice Λ. For a lattice Λ the associated complex tori is simply $E := \mathbb{C}/\Lambda$. This means that two points $z_1, z_2 \in \mathbb{C}$ are equivalent if $z_1 - z_2 \in \Lambda$. The set $\mathbb{C}/\Lambda$ is an example of a Riemann surface or complex manifold. It is called a real torus of dimension two or a complex torus of dimension one. It has the structure of an abelian group which inherits from $(\mathbb{C}, +)$. Still we do not call it an elliptic curve as this name is reserved for a similar object in algebraic geometry. In mathematics when we have a space, then we start to study the set of its functions. In our case, we are interested on meromorphic functions on $\mathbb{C}/\Lambda$ as we have:

Exercise 2.1 There is no (non-constant) holomorphic function on $E := \mathbb{C}/\Lambda$. Hint: The torus E is compact and any local holomorphic function in an open subset of $\mathbb{C}$ which reaches its maximum is constant (maximum principle for holomorphic functions in one variable).

The pull-back of a meromorphic function by the projection map $\mathbb{C} \to E$ corresponds to a meromorphic function f with

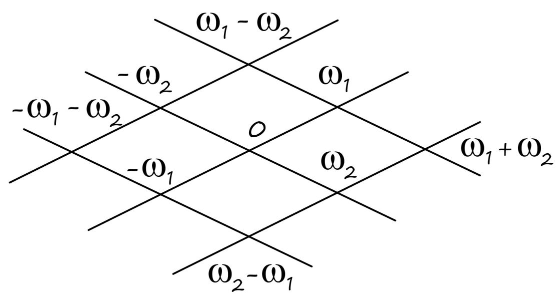


Fig. 2.1 Lattice

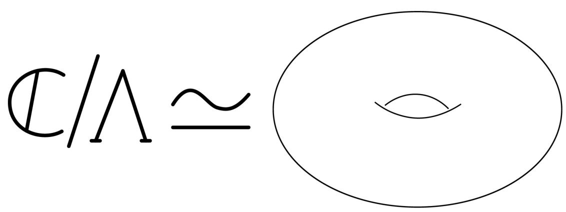


Fig. 2.2 Torus

$$
\begin{aligned}
&f : \mathbb{C} \to \mathbb{C}, \\
&f(z+\omega) = f(z) \quad \forall z \in \mathbb{C},\ \omega \in \Lambda.
\end{aligned}
$$

Since Λ is generated by ω_1, ω_2, the above functional equation of f is equivalent to

$$\begin{aligned} f(z+\omega_1) &= f(z) \\ f(z+\omega_2) &= f(z) \end{aligned} \quad \forall z \in \mathbb{C},$$

that is f is double periodic.We may also view f as a function in both $z \in \mathbb{C}$ and $\Lambda \in \mathsf{L}$. In this case we write $f(z) = f(z, \Lambda)$. Since L is equipped with a $\mathbb{C}^*$-action, it is natural to look for functions f with the functional equation

$$f(\lambda z, \lambda \Lambda) = \lambda^{-a} f(z, \Lambda), \forall \lambda \in \mathbb{C}^*, \tag{2.4}$$

for some fixed $a \in \mathbb{Z}$.

Remark 2.1 In the following, we will use the notion of a meromorphic function on spaces like L which are defined only set theoretically. All these spaces have the structure of an analytic variety and such functions are meromorphic in the classical sense.

Definition 2.2 A meromorphic function f with the property (2.4) is called an elliptic function (of weight a). In other words, an elliptic function f is a meromorphic function in $\mathbb{C}$ such that it is double periodic, that is, there is two $\mathbb{Z}$ linearly independent complex numbers $\omega_1, \omega_2 \in \mathbb{C}$ such that

$$f(z+\omega_1) = f(z), \;\; f(z+\omega_2) = f(z).$$

Let us consider an elliptic function f and write its Laurent series at $z = 0$

$$f(z, \Lambda) = \sum_{n=-\infty}^{+\infty} f_n(\Lambda) z^n$$

The coefficients $f_n(\Lambda)$ are functions of the lattice Λ and it is easy to see that (2.4) is equivalent to the following functional equations for $f_n(\Lambda)$'s

$$f_n(\lambda \Lambda) = \lambda^{-a-n} f_n(\Lambda) \;\; \forall \lambda \in \mathbb{C}^*.$$

This is as follows

$$\begin{aligned} f(\lambda z, \lambda \Lambda) &= \sum_{n=-\infty}^{+\infty} f_n(\lambda \Lambda)(\lambda z)^n \\ &= \lambda^{-a} \left(\sum_{n=-\infty}^{+\infty} f_n(\Lambda) z^n \right). \end{aligned}$$

Definition 2.3 A meromorphic function f on the space L of lattices is called a meromorphic modular form of weight $n \in \mathbb{Z}$ if

$$f(\lambda \Lambda) = \lambda^{-n} f(\Lambda) \quad \forall \lambda \in \mathbb{C}^*, \, \Lambda \in \mathsf{L}.$$

Therefore, from a meromorphic elliptic function of weight a we get meromorphic modular form f_n of weight $n + a$. If we evaluate a meromorphic modular form f of weight n on lattices $\Lambda = \mathbb{Z}\tau + \mathbb{Z}$, let us say $g(\tau) = f(\tau\mathbb{Z} + \mathbb{Z})$, with τ in the upper half plane

$$\mathbb{H} := \{\tau \in \mathbb{C} |\ Im(\tau) > 0\},$$

and regard them as a function in τ, we get a meromorphic function g in $\mathbb{H}$ with the following functional equation:

$$(c\tau + d)^{-n} g\left(\frac{a\tau + b}{c\tau + d}\right) = g(\tau), \quad \forall \tau \in \mathbb{H}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z}). \tag{2.5}$$

2.3 The modular group and its action

The following group acts on $\mathbb{H}$ by Möbius transformation

$$\begin{aligned} &\mathrm{SL}(2,\mathbb{R}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| ad - bc = 1, a, b, c, d, \in \mathbb{R} \right\}, \\ &\mathrm{SL}(2,\mathbb{R}) \times \mathbb{H} \longrightarrow \mathbb{H}, \\ &(A, \tau) \longmapsto A\tau := \tfrac{a\tau + b}{c\tau + d}, \end{aligned}$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{R})$. This follows from

$$\mathrm{Im}(A\tau) = \frac{Im(\tau)\det(A)}{|c\tau + d|^2}. \tag{2.6}$$

An element $A \in \mathrm{SL}(2,\mathbb{R})$ acts as identity on $\mathbb{H}$ if it is $\pm I$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix. Therefore, it is usefull to define

$$\mathrm{PSL}(2,\mathbb{R}) = SL(2,\mathbb{R})/\pm I.$$

The protagonist of the present text is the group $\mathrm{SL}(2,\mathbb{Z})$ defined in (2.2).

Exercise 2.2 Show that the set

$$D := \left\{ \tau \in \mathbb{H} \,\middle|\, -\frac{1}{2} \leq \mathrm{Re}(\tau) \leq \frac{1}{2},\ \ |\tau| \geq 1 \right\}$$

is the closure of a fundamental domain for the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}$, see Figure 2.3. For definitions and details see [Apo90, Section 2.3].

Note that we have to remove some boundary points of D in order to get the classical definition of a fundamental domain. For simplicity we will not do it and call D the classical fundamental domain of the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}$.

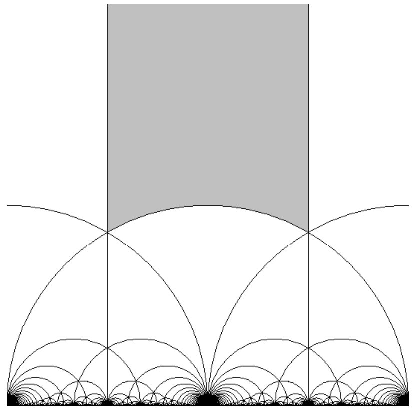


Fig. 2.3 Fundamental domain

Exercise 2.3 Let τ be in the classical fundamental domain and assume that it has non-trivial stablizer under the action of $\mathrm{SL}(2,\mathbb{Z})$, that is, there is $A \in \mathrm{SL}(2,\mathbb{Z})$, $A \neq \pm I$ such that $A\tau = \tau$. Then τ is $\rho_- := \frac{-1+i\sqrt{3}}{2}, i, \rho_+ := \frac{1+i\sqrt{3}}{2}$ and the corresponding A is in the subgroup of $\mathrm{SL}(2,\mathbb{Z})$ given by $\langle S\rangle, \langle R\rangle, \langle S^{-1}RS\rangle$, where

$$S := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, R := \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}. \tag{2.7}$$

Note that the matrix S maps ρ_- to ρ_+ and $S^{-2} = -I, R^{-3} = -I$. Hint: Since $\det A = +1$ and $\mathrm{Im}(\tau) > 0$ we get $|a+d| < 2$ and $\tau = \frac{a-d+\sqrt{(a+d)^2-4}}{2c}$.

Exercise 2.4 Show that the group $\mathrm{SL}(2,\mathbb{Z})$ is generated by the matrices S, R in (2.7). The classical generators of $\mathrm{SL}(2,\mathbb{Z})$ are

$$T := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} =, \quad S := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{2.8}$$

Hint: Note that $T := SR^{-1}$. See [Apo90] Theorem 2.1.

Exercise 2.5 The group $\mathrm{PSL}(2,\mathbb{Z})$ is isomorphic to the free product of the cyclic groups $\langle S\rangle$ and $\langle R\rangle$. Conclude that $\mathrm{SL}(2,\mathbb{Z})$ is a free product with amalgamation

$$\mathrm{SL}(2,\mathbb{Z}) \cong \langle S,R \mid S^4 = R^6 = I, S^2 = R^3 \rangle.$$

Exercise 2.6 Exercise 2.5 implies that the left action of the group generated by S and R on the vector $[1,0]^{\mathrm{tr}}$ consists of all $[c,d]$ with coprime $c,d \in \mathbb{Z}$. Can you describe the left action of the group generated by

$$M_0 := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 5 & 5 & 1 & 0 \\ 0 & -5 & -1 & 1 \end{pmatrix}, \quad M_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.9}$$

on a single vector, let us say $[1,0,0,0]$? In the case of mirror quintic Calabi-Yau threefolds which will be discussed in Part II, instead of $\mathrm{SL}(2,\mathbb{Z})$ we have this group which has actually infinite index in $\mathrm{Sp}(4,\mathbb{Z})$.

2.4 Weierstrass ℘-function

When a group Γ acts on a space M discretely (for instance take the left action) and we want to construct functions on the quotient space

$$\Gamma \backslash M := M/\sim \qquad x \sim y \Leftrightarrow x = Ay \quad \text{for some } A \in \Gamma$$

the first recepie is to start with a function $\tilde{f}$ on M and define the formal sum

$$f(\tau) = \sum_{A\in\Gamma} \tilde{f}(A\tau). \tag{2.10}$$

If we don't care about the convergence of f then we can easily check that it is invariant under the action of Γ: For any $B \in \Gamma$ we have

$$\begin{aligned} f(B\tau) &= \sum_{A\in\Gamma} \tilde{f}(AB\tau) \\ &= \sum_{A\in\Gamma} \tilde{f}(A\tau) = f(\tau). \end{aligned}$$

We have used the fact that the multiplication by B from the right induces a bijection $\Gamma \to \Gamma$. We get the function

$$\check{f} : \Gamma\backslash M \to \mathbb{C},\ \check{f}([\tau]) = f(\tau)$$

which we denote it again by $f = \check{f}$. If Γ is finite then (2.10) is a finite sum and so $\check{f}$ is well-defined, however, in general such a sum might not be convergent. In our case, the lattice Λ as an additive group acts on $\mathbb{C}$

$$\Lambda \times \mathbb{C} \to \mathbb{C}, (\lambda, z) \longmapsto z + \lambda.$$

For our purpose we start with $\tilde{f}(z) = z^{-a},\ a \in \mathbb{Z}$ and define

$$f_a(z) = f_a(z,\Lambda) := \sum_{\omega\in\Lambda} (z+\omega)^{-a} = \sum_{(n,m)\in\mathbb{Z}^2} (z+n\omega_1+m\omega_2)^{-a}.$$

Proposition 2.1 The infinite series $f_a(z)$ converges absolutely for $a \in \mathbb{N}$ with $a \geqslant 3$.

Proof. The proof is taken from [Apo90, Lemma 2, page 8]. We can assume that the sum is taken for all $\omega \in \Lambda, |\omega| > R$ and $|z| < R$. There is a constant M depending on R such that

$$\tfrac{1}{|z-\omega|^a} \leqslant \tfrac{M}{|\omega|^a} \qquad \begin{array}{l}\forall \omega \in \Lambda,\ \text{with } |\omega| > R,\\ \forall z \in \mathbb{C},\ \text{with } |z| \leq R.\end{array}$$

In order to see this we observe that $|\frac{z}{\omega} - 1|^a$ as a function in $\{|z| \leq R\} \times \{\omega \in \Lambda |\ |\omega| > R\}$ cannot tend to zero. For any sequence (z_n, ω_n) in its domain of definition, we can replace it with its subsequence such that z_n converges to a point z_0 and ω_n is either constant or it converges to infinity.

It is enough to prove that the sum

$$\sum_{\omega\in\Lambda,\omega\neq 0} \frac{1}{|\omega|^a} \tag{2.11}$$

is convergent. Let r and R be the minimum and maximum distances of 0 from the parallelogram formed by $\pm\omega_1 \pm \omega_2$. Then the parallelogram P_n formed by four vertices $n(\pm\omega_1 \pm \omega_2)$ has the minimum and maximum distances nr and nR, respectively, from 0. Moreover, it has $8n$ points of the lattice. Therefore, the sum S_n in (2.11) corresponding to points of the lattice Λ in all parallelograms $P_1, P_2, \ldots, P_n$ satisfies

$$\frac{8}{R^a}(1+2^{-a+1}+\cdots+n^{-a+1}) \leq S_n \leq \frac{8}{r^a}(1+2^{-a+1}+\cdots+n^{-a+1}).$$

It is no so difficult to show that $\sum_{n=1}^{\infty} n^{-s}$ for $s > 1$ is convergent, see for instance Proposition 10.1.

Exercise 2.7 Show that f_2 does not converge for all $z \in \mathbb{C} - \Lambda$.

Despite the fact that f_2 does not converge, it is possible to correct the term in the infinite sum f_2 and make it convegent.

Proposition 2.2 *The Weierstrass $\wp$ function (read P) is*

$$\wp(z,\Lambda) = \wp(z) := \frac{1}{z^2} + \sum_{\omega\neq 0}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$$

is convergent.

Proof. The proof is taken from [Apo90, Theorem 1.10]. We have

$$\left|\tfrac{1}{(z-\omega)^2}-\tfrac{1}{\omega^2}\right|=\left|\tfrac{z(2\omega-z)}{(z-\omega)^2\omega^2}\right|\leqslant\tfrac{M\cdot R\cdot(2|\omega|+R)}{|\omega|^2|\omega|^2}$$

$$\leqslant\tfrac{M\cdot R\cdot(2+R/|\omega|)}{|\omega|^3}\leqslant\tfrac{3MR}{|\omega|^3},$$

where we used the notation in the proof Proposition (2.2).

If we redifine $f_2 := \wp(z)$ then we have

$$\frac{\partial f_a}{\partial z}=-a\cdot f_{a+1},\;\; a\geq 2.$$

Moreover, it is easy to see that $\wp(-z)=\wp(z)$ that is $\wp$ is an even function.

Exercise 2.8 Show that the number of zeros of a non-constant elliptic function counted in $\mathbb{C}/\Lambda$ is equal to the number of poles, counted with multiplicity, and it is bigger than or equal to 2. Hint: See [Apo90, page 5-6].

Exercise 2.9 Show that there is no function $f(\omega), \omega\in\Lambda$ such that

$$\sum_{\omega\in\Lambda}\frac{1}{z-\omega}+f(\omega)$$

is convergent.

The function $\wp$ has poles at the points of Λ. We write the Laurent expansion of $\wp$ at $z=0$.

Theorem 2.1 *For*

$$0<|z|<r:=\min\{|\omega| \mid \omega\neq 0\}, \tag{2.12}$$

we have

$$\wp(z)=\frac{1}{z^2}+\sum_{n=1}^{\infty}(2n+1)G_{2n+2}\cdot z^{2n},$$

where

$$G_{2n+2}=\sum_{\omega\neq 0}\frac{1}{\omega^{2n+2}}. \tag{2.13}$$

Proof. For z in (2.12) we have $\left|\frac{z}{\omega}\right|<1$ and

$$\frac{1}{(z-\omega)^2}=\frac{1}{\omega^2(1-\frac{z}{\omega})^2}=\frac{1}{\omega^2}\left(1+\sum_{n=1}^{\infty}(n+1)\left(\frac{z}{\omega}\right)^n\right)$$

and so

$$\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}=\sum_{n=1}^{\infty}\frac{n+1}{\omega^{n+2}}\cdot z^n.$$

Summing over $\omega \in \Lambda, \omega \neq 0$,we get the result.

Exercise 2.10 The Weierstrass $\wp$ function can be written in terms of the variables $q := e^{2\pi i\tau}$ and $w := e^{2\pi iz}$:

$$\wp(z,\tau) = (2\pi i)^2 \left(\sum_{m\in\mathbb{Z}} \frac{wq^m}{(1-wq^m)^2} + \frac{1}{12} - \sum_{m\in\mathbb{Z},\, m\neq 0} \frac{q^m}{(1-q^m)^2} \right),$$
$$\wp'(z,\tau) = (2\pi i)^3 \left(\sum_{m\in\mathbb{Z}} \frac{wq^m(1+wq^m)}{(1-wq^m)^3} \right).$$

Hint: See for instance [Hus04, page 192] and [Obe18, Appendix B].

Exercise 2.11 The Weierstrass zeta and sigma functions are

$$\sigma(z,\Lambda) = z \prod_{\omega\in\Lambda^*} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega}+\frac{1}{2}(\frac{z}{\omega})^2} \tag{2.14}$$

$$\zeta(z,\Lambda) = \frac{1}{z} + \sum_{\substack{\omega\in\Lambda \\ \omega\neq 0}} \frac{1}{(z-\omega)} + \frac{1}{\omega} + \frac{z}{\omega^2} \tag{2.15}$$

Show that

$$\frac{d}{dz} \ln \sigma(z) = \zeta(z), \quad \frac{d}{dz}\zeta(z) = -\wp(z) \tag{2.16}$$

$$\zeta(z+\omega) - \zeta(z) = 2\zeta\left(\frac{1}{2}\omega\right), \quad \omega \notin 2\Lambda \tag{2.17}$$

Hint: see [Sil94a, page 40]. Is this sigma function is the same as the function in [Fri16, page 404]?.

Exercise 2.12 Prove the following identity between Weierstrass $\wp$ and ζ:

$$\frac{1}{2}\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} = \zeta(u+v) - \zeta(u) - \zeta(v), \quad u, v \in \mathbb{C}.$$

see [Cha85, page 55].

2.5 Differential equation of $\wp$

In this section we remind the well-known fact that the transcendence degree of elliptic functions is the dimension of the complex torus $\mathbb{C}/\Lambda$ which is one. In particular, we must have a polynomial relation between $\wp$ and any of its derivatives. The first historical example is the following.

Theorem 2.2 The function $\wp$ satisfies the differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 \tag{2.18}$$

where

$$g_2 = 60G_4, \quad g_3 = 140G_6, \tag{2.19}$$

and G_i's are defined in (2.13).

Proof. Let $f(z)$ be the difference of both sides of (2.18). This is clearly an elliptic function with possible poles at $z \in \Lambda$. We show that f is holomorphic at $z = 0$ and so $f = 0$. We have

$$\begin{aligned} \wp'(z) &= \tfrac{-2}{z^3} + 6G_3 \cdot z + 20G_6 \cdot z^3 + \cdots, \\ \wp'(z)^2 &= \tfrac{4}{z^6} - \tfrac{24G_4}{z^2} - 80G_6 + \cdots, \\ 4\wp(z)^3 &= \tfrac{4}{z^6} + \tfrac{36G_4}{z^2} + 60G_6 + \cdots, \end{aligned}$$

and hence

$$\begin{aligned} \wp'(z)^2 - 4\wp(z)^3 &= -\tfrac{60G_4}{z^2} - 140G_6 + \cdots, \\ \wp'(z)^2 - 4\wp(z)^3 + 60G_4\wp(z) &= -140G_6 + \cdots. \end{aligned}$$

Exercise 2.13 If f is a non-constant elliptic function then $\mathrm{Im}(\frac{\omega_1}{\omega_2}) \neq 0$, where ω_1 and ω_2 are periods of f.

Exercise 2.14 Prove that every elliptic function f can be written as

$$R_1[\wp(z)] + \wp'(z)R_2[\wp(z)],$$

where R_1, R_2 are rational functions and $\wp$ has the same set of periods as f. Hint: See [Apo90, page 23, Exercise 5].

Exercise 2.15 Prove that

$$\begin{aligned} \wp(2z) &= \frac{(\wp(z)^2 + \frac{1}{4}g_2)^2 + 2g_3 \cdot \wp(z)}{4\wp^3(z) - g_2\wp(z) - g_3}, \\ &= -2\wp(z) + \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2, \end{aligned}$$

Hint: See [Apo90, page 24, Exercise 9].

Exercise 2.16 Show that

$$\wp''(z) = 6\wp(z)^2 - \frac{1}{2}g_2.$$

The following exercise has been inspired by Picard's curious example in [Mov22b, Section 10], see also Section 13.7.

Exercise 2.17 Show that for all $N \in \mathbb{N}$, $N \geq 2$ there is an elliptic function f such that it has a pole of order N at $[0] \in \mathbb{C}/\Lambda$, a zero of order N at $\frac{1}{N}$ and no other pole or zero. For instance, for $N = 2$ we have $f = \wp(z,\tau) - \wp(\frac{1}{2},\tau)$. Compute f for $N = 3$. Hint: See the hint of Exercise 4.24.

In a more algebraic gemetric framework the above exercise turns out to be Exercise 4.24.

2.6 Eisenstein series

The series

$$G_n := \sum_{\omega\in\Lambda,\ \omega\neq 0} \frac{1}{\omega^n}, \qquad n \quad \text{even}, \quad n \geqslant 4 \tag{2.20}$$

that we have seen Theorem 2.1 are called Eisenstein series. In this theorem we have also proved the convergence of G_n. They satisfy

$$G_n(\lambda\Lambda) = \lambda^{-n} G(\Lambda), \qquad \forall\lambda \in \mathbb{C}^*. \tag{2.21}$$

We usually define

$$G_n(\tau) = G_n(\mathbb{Z}\tau + \mathbb{Z}) = \sum_{(a,b)\in\mathbb{Z}^2,\ (a,b)\neq(0,0)} \frac{1}{(a+b\tau)^n}, \quad \tau \in \mathbb{H},$$

and by abuse of notation use the same letter G_n.

Exercise 2.18 Show also that $G_n \equiv 0$, for n an odd number.

From the functional equation (2.21) we deduce the following: For all $A \in SL(2,\mathbb{Z})$

$$\begin{aligned} G_n(A\tau) &= G_n\left(\frac{a\tau+b}{c\tau+d}\right) \\ &= G_n\left(\frac{a\tau+b}{c\tau+d}\mathbb{Z}+\mathbb{Z}\right) \\ &= (c\tau+d)^{+n} G_n((a\tau+b)\mathbb{Z} + (c\tau+d)\mathbb{Z}) \\ &= (c\tau+d)^n G_n(\tau\mathbb{Z}+\mathbb{Z}) \\ &= (c\tau+d)^n G_n(\tau) \end{aligned}$$

In Section 2.7 we will see that the limit $\lim_{\mathrm{Im}(\tau)\to\infty} G_n(\tau)$ exists. This motivates us to define (holomorphic) modular forms.

Definition 2.4 *Let $k \in \mathbb{Z}$ be an integer and f be a holomorphic function on the upper half plane. Then f is called a modular form for the* $\mathrm{SL}(2,\mathbb{Z})$ *if*

$$(c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right) = f(\tau) \quad \forall \begin{pmatrix} a\ b \\ c\ d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}), \tag{2.22}$$

and f is holomorphic at infinity, that is, $\lim_{\mathrm{Im}(\tau)\to\infty} f(\tau)$ *exists.*

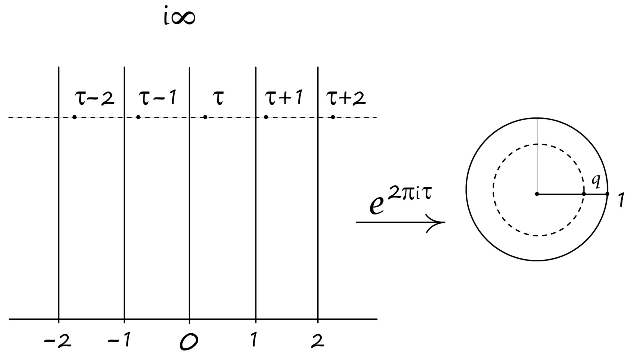


Fig. 2.4 q map

We know that $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, and so, a modular form f satisfies $f(\tau+1) = f(\tau)$. This implies that f defines a meromorphic function $\tilde{f}$ in the punctured disc:

$$D^* = \{z \in \mathbb{C} | \ |z| < 1\} \backslash \{0\}$$

which is defined by

$$f(\tau) = \tilde{f}(q), \text{ where } q := e^{2\pi i \tau}.$$

The map $\mathbb{H} \to \mathbb{D}^*$ is depicted in Figure 2.4. We write the Laurent series of $\tilde{f}$ at $q = 0$.

$$\tilde{f}(q) = \sum_{n=-\infty}^{n=+\infty} f_n \cdot q^n \ , \ f_n \in \mathbb{C} \tag{2.23}$$

This is also called the Fourier expansion of $\tilde{f}$. Since for a holomorhic modular form we have assumed that $\lim_{\mathrm{Im}(\tau)\to\infty} f(\tau)$ exists, we conclude that $f_n = 0$ for all integers $n < 0$.

Definition 2.5 A meromorphic modular form for $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ is a meromorphic function in $\mathbb{H}$ such that apart from the functional equation (2.22) is also meromorphic at $i\infty$, that is, in (2.23) we have some $M \in \mathbb{Z}$ such that $f_n = 0$ for all $n \leqslant M$. It is called weakly holomorphic modular form if it is holomorphic in $\mathbb{H}$ and possibly meromorphic at $i\infty$. It is called (holomorphic) modular form if it is holomorphic in $\mathbb{H} \cup \{i\infty\}$. A meromorphic modular form of weight zero is called a modular function.

We use the letter f for $\tilde{f}$ too, and write the q-expansion of a holomorhic modular form as

$$f = \sum_{n=0}^{\infty} f_n \cdot q^n = \sum_{n=0}^{\infty} f_n \cdot e^{2\pi i \tau}$$

It will be clear from the text whether we consider f as a function of τ or q.

Definition 2.6 We say that a holomorphic modular form f is defined over $\mathbb{Q}$ if the all the coefficients f_n in its Fourier expansion are rational numbers.

2.7 Fourier expansion of Eisenstein series

We know that the Eisenstein series are weakly holomorphic modular forms. In this section we show that they are holomorphic at $i\infty$ and so they are holomorphic modular forms. The computation in this section can be found in [Kob93a, page 110], [Apo90, page 18] and [Ser78, page 91].

Definition 2.7 Bernoulli numbers B_k are defined through the equality

$$\frac{x}{e^x-1}=\sum_{k=0}^{\infty} B_k\cdot\frac{x^k}{k!}.$$

For instance, $B_0=1,\ B_1=\frac{-1}{2},\ B_2=\frac{1}{6},\ B_4=\frac{-1}{30},\ B_6=\frac{1}{42}$. It is easy to see that for any odd $k\geqslant 3$ we have $B_k=0$.

Theorem 2.3 *The Eisenstein series $G_k(\tau),\ k\geq 4$ has the following q-expansion*

$$G_k(\tau)=2\zeta(k)\left(1-\frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\right),$$

where $\zeta(k):=\sum_{n=1}^{\infty}\frac{1}{n^k}$ is the Riemann's zeta function evaluated at k, and

$$\sigma_a(n):=\sum_{d|n}d^a.$$

Let us first state the main ingredient of the proof of Theorem 2.3

Proposition 2.3 *We have*

$$\zeta(k)=-\frac{(2\pi i)^k}{2}\frac{B_k}{k!} \qquad k\geq 2 \text{ and even} \tag{2.24}$$
$$\sum_{n\in\mathbb{Z}}\frac{1}{(a+n)^k}=\frac{(-2\pi i)^k}{(k-1)!}\sum_{n=0}^{\infty}n^{k-1}e^{2\pi ina},\ k\in\mathbb{N},\ k\geq 2\ \ a\in\mathbb{C}-\mathbb{Z}.$$

Proof. We have the following product formula for sine function

$$\sin(\pi a)=\pi a\prod_{n=1}^{\infty}\left(1-\frac{a^2}{n^2}\right),\ \ a\in\mathbb{C}. \tag{2.25}$$

We take the logarithmic derivative of (2.25) and get

$$\pi \cdot cot(\pi a) = \frac{1}{a} + \sum_{n=1}^{\infty} \left(\frac{1}{a+n} + \frac{1}{a-n} \right). \tag{2.26}$$

The left hand side of this equality is

$$\begin{aligned} \pi \cdot cot(\pi a) &= \pi i \frac{e^{\pi i a} + e^{-\pi i a}}{e^{\pi i a} - e^{-\pi i a}} = \pi i + \frac{2\pi i}{e^{2\pi i a} - 1} \\ &= \pi i - 2\pi i \left(\sum_{n=0}^{\infty} e^{2\pi i n a} \right) \end{aligned} \tag{2.27}$$

We get

$$\sum_{n \in \mathbb{Z}} \frac{1}{a+n} = \pi i - 2\pi i \sum_{n=0}^{\infty} e^{2\pi i n a}. \tag{2.28}$$

We differentiate the above equality with respect to a, $k-1$ times, and we get (2.25). In (2.26) multiply both sides with a and set $x := 2\pi i a$.

$$\begin{aligned} \frac{x}{2} + \frac{x}{e^x - 1} &= 1 + \sum_{n=1}^{\infty} \frac{x}{(x + 2\pi i n)} + \frac{x}{(x - 2\pi i n)} \\ &= 1 + \sum_{n=1}^{\infty} \frac{x}{2\pi i n} \left(\sum_{k=0}^{\infty} \left(\frac{-x}{2\pi i n} \right)^k - \left(\frac{x}{2\pi i n} \right)^k \right) \\ &= 1 - 2 \sum_{n=1}^{\infty} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{x^{k+1}}{(2\pi i n)^{k+1}} \\ &= 1 - 2 \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{x^{k+1}}{(2\pi i)^{k+1}} \zeta(k+1). \end{aligned}$$

We get the well-know formula (2.24).

Proof (Proof of Theorem 2.3). Take $a = m\tau, m \in \mathbb{Z}, m \neq 0$, and we have

$$\sum_{n \in \mathbb{Z}} \frac{1}{(m\tau + n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} q^{nm}.$$

The result follows immediately:

$$\begin{aligned} G_k(\tau) &= 2\zeta(k) + 2 \sum_{m=1}^{\infty} \sum_{n=-\infty}^{+\infty} \frac{1}{(m\tau + n)^k} \qquad (2.29) \\ &= 2\zeta(k) \left(1 + \frac{(-2\pi i)^k}{\zeta(k)(k-1)!} \sum_{m,n=1}^{\infty} n^{k-1} q^{nm} \right). \qquad (2.30) \end{aligned}$$

Exercise 2.19 We know that the Eisenstein series $G_k(\tau)$ for $k \geq 1$ odd number is identically zero. However, we can take the equality (2.29) as a new definition of

Eisenstein series of odd weight k. In this case we have still the equality (2.30), and so we know the formula for their Fourier expansions. Note that for k odd, the number $\frac{\pi^k}{\zeta(k)}$ is conjecturally a transcendetal number. Describe the functional equation of $G_k(\tau), k \geq 3$ under the action of $\mathrm{SL}(2,\mathbb{Z})$. For some hint see [Bac12].

Exercise 2.20 Give a direct proof of

$$\lim_{\mathrm{Im}(\tau)\to+\infty} G_k(\tau) = 2\zeta(k), \quad k \geq 4 \text{ even},$$

whitout the computation of the q-expansion of G_k in in this section. This might simplify the proof of Proposition 8.1 without refering to Exercise 2.10.

Exercise 2.21 Prove the product formula for sine in (2.25). Hint: Both sides have the same zero set.

We will use the following new notation

$$E_k = G_k/2\zeta(k) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n, \tag{2.31}$$

$$E_4 = 1 + 240\left(\sum_{n=1}^{\infty} \sigma_3(n)q^n\right), \tag{2.32}$$

$$E_6 = 1 - 504\left(\sum_{n=1}^{\infty} \sigma_5(n)q^n\right), \tag{2.33}$$

$$E_8 = 1 + 480\left(\sum_{n=1}^{\infty} \sigma_7(n)q^n\right). \tag{2.34}$$

It follows that the Eisenstein series E_k are defined over $\mathbb{Q}$.

2.8 The Eisenstein series E_2

For the discussion in this section we follow [Kob93a, page 112]. The proof of the convergence of the Eisenstein series $G_k(\tau)$ in Proposition 2.1 is not valid for $k = 2$ and actually in Exercise 2.7 we have seen that

$$\sum_{(n,m)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(m\tau+n)^2}$$

doesn't converge. Despite this we can define

$$G_2(\tau) := \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}{}' \frac{1}{(m\tau+n)^2},$$

where $'$ means that if $m=0$ then $n\neq 0$. The argument in Section 2.7 shows that the inner sum converge for any m and $\tau\in\mathbb{H}$ and then the other sum converges. Here, the order of summation is important. In a similar way as in Section 2.7 we get

$$G_2(\tau)=2\zeta(2)E_2(\tau),\ \ E_2(\tau)=1-24\ \sum_{n=1}^{\infty}\sigma_1(n)q^n.$$

Theorem 2.4 *We have*

$$(c\tau+d)^{-2}E_2\left(\frac{a\tau+b}{c\tau+d}\right)-E_2(\tau)=\frac{12}{2\pi i}\frac{c}{c\tau+d} \tag{2.35}$$

for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in \mathrm{SL}(2,\mathbb{Z})$.

For the proof of this theorem, first we note that if we define

$$f\,\|_2\, A:=(c\tau+d)^{-2}\,f(A\tau)-c(c\tau+d)^{-1}$$

for a holomorphic function f on $\mathbb{H}$ then

$$(f\,\|_2\,A)\,\|_2\,B=f\,\|_2\,AB.$$

Since $\mathrm{PSL}(2,\mathbb{Z})$ is generated by T and S and (2.35) is trivial for T, it is enough to verify (2.35) for S, that is

$$\tau^{-2}E_2\left(\frac{-1}{\tau}\right)=E_2(\tau)+\frac{12}{2\pi i}\frac{1}{\tau}. \tag{2.36}$$

We will give a more geometric proof of the above equality in Section 9.12.

Exercise 2.22 For an elementary proof of (2.36) using only complex analysis see [Kob93a, page 113]. Reproduce this proof.

2.9 The algebra of modular forms

One of the fundamental theorems in modular forms is the following.

Theorem 2.5 *The Eisenstein series E_4 and E_6 are algebraically independent over $\mathbb{C}$, that is, there is no polynomial $P(X,Y)$ with coefficients in $\mathbb{C}$ such that $P(E_4,E_6)=0$. Moreover any holomorphic modular form f of weight k can be written uniquely as $f=P(E_4,E_6)$, where P is a homogeneous polynomial of degree k in the ring*

$$\mathbb{C}[X,Y],\ \textit{weight}\ (X)=4,\ \textit{weight}\ (Y)=6. \tag{2.37}$$

If f is defined over $\mathbb{Q}$ then P is also defined over $\mathbb{Q}$, that is, $P\in\mathbb{Q}[X,Y]$.

There is a classical proof of Theorem 2.5 which can be found in almost all books on modular forms, see [Kob93a, Proposition 10, page 118], [Apo90, Chapter 6] and [Ser78, Section 3].

Exercise 2.23 Prove Theorem 2.5 using the references above.

In Section 3.5 we will give a new proof which is inspired by the author's study of the generalized period domain in [Mov08]. The proof is based on the study of elliptic integrals and Gauss-Manin connection. Theorem 2.5 for the Eisenstein series $f = E_k,\ k \geq 4$ was known to [Ram16, page 180]. As it was typical to Ramanujan he state this without proof.

Definition 2.8 We denote by $M_k(\mathrm{SL}(2,\mathbb{Z}))$ (resp. $M_k(\mathrm{SL}(2,\mathbb{Z}))_{\mathbb{Q}}$) the space of modular forms of weight k for $\mathrm{SL}(2,\mathbb{Z})$ (resp. further defined over $\mathbb{Q}$). We also denote by $M = M(\mathrm{SL}(2,\mathbb{Z})) := \oplus_{k\in\mathbb{Z}} M_k(\mathrm{SL}(2,\mathbb{Z}))$ and $M(\mathrm{SL}(2,\mathbb{Z}))_{\mathbb{Q}}$ the algebra of modular forms.

By Theorem 2.5, for $k \in \mathbb{Z},\ k \leqslant 2$ or k odd we have $M_k = 0$ and $M_{\mathbb{Q}} = \mathbb{Q}[E_4, E_6]$, $M_k = \mathbb{Q}[E_4, E_6]_k$, where E_4 and E_6 have the weights 4 and 6, respectively.

Exercise 2.24 *Using Theorem 2.5 prove that*

$$E_4^2 = E_8,\ \ E_4 E_6 = E_{10},\ \ E_6 \cdot E_8 = E_{14}$$

and derive the corresponding equalities for $\sigma_k(n)$. For instance

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m).$$

The dimension of the space of modulex forms M_k is listed below:

k	0	2	4	6	8	10	12	14	16	18	k	$k+12$
$\dim(M_k)$	1	0	1	1	1	1	2	1	2	2	d	$d+1$

Note that

$$\dim(M_k) = \sharp\{(x,y) \in \mathbb{N}_0^2 \mid 4x + 6y = k\}$$

and

$$\sum_{k=0}^{\infty} \dim(M_k) q^k = \frac{1}{(1-q^4)(1-q^6)}. \tag{2.38}$$

The last column in the above table contains the following information.

Exercise 2.25 Show that for all $k \in \mathbb{N}$ we have $\dim(M_{k+12}) = \dim(M_k) + 1$ and

$$\dim(M_k) = \begin{cases} 0 & k \text{ is odd} \\ \lfloor \frac{k}{12} \rfloor + 1 & k \text{ even } k \not\equiv_{12} 2 \\ \lfloor \frac{k}{12} \rfloor & k \text{ even } k \equiv_{12} 2 \end{cases}$$

Definition 2.9 A holomorphic modular form f is called a cusp form if in its Fourier expansion $f = \sum_{i=0}^{\infty} f_n q^n$ we have $f_0 = 0$, that is, it has no constant term. We also use the notation $f(i\infty) := f_0$.

2.10 Ramanujan relations between Eisenstein series

The derivation f' of a modular form f of weight k with respect to $\tau \in \mathbb{H}$ is no more a modular form. Its functional equation has three terms

$$(c\tau+d)^{-k-2}f(A\tau) = kc(c\tau+d)^{-1}f(\tau)+f'(\tau).$$

It is possible to correct f with a multiple of E_2 and get a modular form again.

Proposition 2.4 *We have the following map*

$$M_k \to M_{k+2}, \ \ f \mapsto \frac{\partial f}{\partial \tau} - 2\pi i \frac{k}{12} E_2 \cdot f, \tag{2.39}$$

which is called the Serre derivative of f.

Proof. Let g be the Serre derivative of f. We have to show that $g \in M_{k+2}$. Only the functional equation of g is non-trivial:

$$\begin{aligned} g(A\tau) &= f'(A\tau) - k\frac{2\pi i}{12}E_2(A\tau)f(A\tau) \\ &= kc(c\tau+d)^{k+1}f(\tau) + (c\tau+d)^{k+2}\cdot f(\tau) - k((c\tau+d)^2\frac{2\pi i}{12}E_2(\tau) \\ &\quad +c(c\tau+d))(c\tau+d)^k f(\tau) \\ &= (c\tau+d)^{k+2}\cdot g(\tau), \end{aligned}$$

for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$.

We have

$$\frac{\partial}{\partial \tau} := 2\pi i q \frac{\partial}{\partial q}.$$

and sometime it is useful to divide the Serre derivative over $2\pi i$ and redefine it

$$f \mapsto q\frac{\partial f}{\partial q} - \frac{k}{12}E_2 f.$$

Proposition 2.5 *We have the following equalities between the Eisenstein series and their derivatives*

$$\begin{cases} q\frac{\partial E_2}{\partial q} = \frac{1}{12}(E_2^2 - E_4) \\ q\frac{\partial E_4}{\partial q} = \frac{1}{3}(E_2E_4 - E_6) \\ q\frac{\partial E_6}{\partial q} = \frac{1}{2}(E_2E_6 - E_4^2) \end{cases}, \tag{2.40}$$

The differential equation (2.40) is usually called the Ramanujan relations between Eisenstein series, see [Ram16, page 181].

Proof. The proof of the second and third equalities follows from the Serre derivative and $\dim(M_6) = \dim(M_8) = 1$. Further, we must check the equalities for the coefficient of q^0. The proof of the first equality follows in a similar way. We need to prove that $f(\tau) := -\frac{12}{2\pi i}\frac{\partial E_2}{\partial \tau} + E_2^2$ is a modular form of weight 4 and its constant term is 1. For this we use the functional equation of E_2 in Theorem 2.4. Therefore, by Theorem 2.5 it must be E_4.

For a while the reader is invited to forget what he has learned in this section and solve the following problem by elementary methods.

Exercise 2.26 Let $\mathbb{Q}[[q]] := \{a_0 + a_1 q + a_2 q^2 + \cdots + a_n q^n + \cdots | a_i \in \mathbb{Q}\}$ be the ring of formal power series in q and with rational coefficients. Addition and multiplication in $\mathbb{Q}[[q]]$ are defined in a natural way. We have also the derivation:

$$\partial_q : \mathbb{Q}[[q]] \to \mathbb{Q}[[q]],$$

$$\partial_q(a_0 + a_1 q + a_2 q^2 + \cdots + a_n q^n + \cdots) := a_1 + 2a_2 q + \cdots + n a_n q^{n-1} + \cdots.$$

Show that there are unique series $t_1, t_2, t_3 \in \mathbb{Q}[[q]]$ with $t_1 = 1 - 24q + \cdots$ such that

$$\begin{cases} q\partial_q t_1 = \frac{1}{12}(t_1^2 - t_2) \\ q\partial_q t_2 = \frac{1}{3}(t_1 t_2 - t_3) \\ q\partial_q t_3 = \frac{1}{2}(t_1 t_3 - t_2^2) \end{cases}. \tag{2.41}$$

Morover, the coefficients of t_i's are integers. Use the encyclopedia of integer sequences oeis.org and find a closed formula for the coefficients of t_i's.

2.11 The product formula for discriminant

We make a linear combination of weight 12 modular forms E_4^3 and E_6^2 such that the resulting modular form is a cusp form. This can be simplified into the following definition:

$$\Delta := g_2^3 - 27g_3^2 = (2\zeta(4)60E_4)^3 - 27(2\zeta(6)140)^2 E_6^2 = \frac{(2\pi)^{12}}{1728}(E_4^3 - E_6^2).$$

Recall that $\zeta(4) = \frac{\pi^4}{90}$, $\zeta(6) = \frac{\pi^6}{945}$, and hence, we have

$$(2\pi)^{-12}\Delta = \frac{1}{1728}(E_4^3 - E_6^2) = \sum_{n=1}^{\infty} \tau(n) q^n = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + \cdots.$$

The function $\tau(n)$ is called the Ramanujan τ function.

Proposition 2.6 *We have*

$$\frac{1}{1728}(E_4^3 - E_6^2) = q\prod_{n=1}^{\infty}(1 - q^n)^{24}. \tag{2.42}$$

Proof. This follows from Ramanujan relations between Eisenstein series. The logarithmic derivative of both sides in (2.42) is E_2. We also need to check that the coefficient of q^1 in both side of (2.42) is one.

Exercise 2.27 Show that

$$E_{12} - E_6^2 = \frac{(2\pi i)^{-12} \cdot 2^6 \cdot 3^5 \cdot 7^2}{691} \cdot \Delta .$$

From this derive an expression for $\tau(n)$ in terms of σ_{11} and σ_5. Show that

$$\tau(n) \equiv \sigma_{11}(n) (\mod 691)$$

Hint: See [Kob93a, III, Section 2, 4].

For various recursion formulas for τ, see Ramanjan's original article [Ram16, page 195]. The following was conjectured by [Ram16, page 197] and proved by [Del71] as a consequence of his proof for Weil conjectures in [Del73, Del80].

Theorem 2.6 *We have*

$$|\tau(n)| < n^{\frac{11}{12}} \sigma_0(n),$$

where $\sigma_0(n)$ is the number of divisors of n.

See also [Ser69] for an overview of properties of τ. The following conjecture depite being simple is still open.

Conjecture 2.1 (Lehmer's conjecture) *For all $n \in \mathbb{N}$ we have*

$$\tau(n) \neq 0.$$

For information see the Wikipedia webpage on "Ramanujan tau function".

Another interesting function is

$$F(q) := \frac{1728 \cdot q}{E_4^3 - E_6^2} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} = \sum_{n=0}^{\infty} P_n q^n .$$

It can be easily checked that P_n is the unrestricted partition function, that is, P_n is the number of ways a positive integer n can be expressed as a sum of positive integers:

$$n = a_1 + a_2 + \cdots + a_k \ , \ a_k \in \mathbb{N}.$$

There is no restriction on k, order of a_i's, and repetion of a_i's is allowed. For more information see [Apo90, Chapter 5]. According to [Apo90] the Dedekind eta function

$$\eta(\tau) := e^{\frac{2\pi i \tau}{24}} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n \tau}\right)$$

was introduced by Dedekind in 1877. We know that

$$\Delta(\tau) = (2\pi i)^{12} \eta^{24}$$

and the functional equation of Δ with respect to the action of $\mathrm{SL}(2,\mathbb{Z})$. Taking 24-th root of this we get

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \quad \varepsilon(c\tau+d)^{\frac{1}{2}}\eta(\tau)$$

for some ε which is a 24-th root of unity and depends only on A and τ. In fact

Exercise 2.28 (Dedekind functional equation) For all $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z})$ with $c > 0$ we have

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon(A)\Big(-i(c\tau+d)\Big)^{\frac{1}{2}}\eta(\tau)$$

where

$$\varepsilon(A) := \exp\left(\pi i\left(\frac{a+d}{2c} + S(-d,c)\right)\right),$$

$$S(h,k) := \sum_{\gamma=1}^{k-1}\frac{\gamma}{k}\left(\left\{\gamma\cdot\frac{h}{r}\right\} - \frac{1}{2}\right),$$

$$\{a\} := a - [a].$$

Hint: See [Apo90, Theorem 3.4].

For $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ we get

$$\eta\left(\frac{-1}{\tau}\right) = (-i\tau)^{\frac{1}{2}}\eta(\tau).$$

2.12 The j function

The following

$$\begin{aligned} j(\tau) &= 1728\frac{g_2^3}{g_2^3 - 27g_3^2} = \sum_{n=-1}^{\infty} c_n q^n \\ &= 1728\frac{E_4^3}{E_4^3 - E_6^2} = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots \end{aligned}$$

is called the j-function, or Klein's modular function. It is holomorphic in $\mathbb{H}$ and has a pole of order one at $i\infty$. From the functional equation of Eisenstein series it follows that j is invariant under the action of $\mathrm{SL}(2,\mathbb{Z})$:

$$j\left(\frac{a\tau+b}{c\tau+d}\right) = j(\tau), \quad \forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z}). \tag{2.43}$$

Exercise 2.29 Show that the j-function satisfies the differential equation

$$S(j) + Q(j)(j')^2 = 0,$$

where $Q(j) = \frac{36j^2-41j+32}{72(j-1)^2j^2}$ and $S(j)$ is the Schwarzian derivative of j with respect to τ. Hint: From the Ramanujan relations between Eisenstein series we can calculate j, j', j'', j''' as rational functions in E_2, E_4, E_6. Thus, there is a polynomial in four variables which annihilate (j, j', j'', j''').

Theorem 2.7 *The map* $j : \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H} \to \mathbb{C}$ *is one to one and surjective.*

We will give a proof of this theorem in Section 3.5 in which we explicitely construct the inverse of j using elliptic integrals.

Exercise 2.30 Theorem 2.7 can be proved by compactification $\overline{\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}}$ as a Riemann surface, for which we need only to add one more point to $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$, and extending j to a holomorphic map of Riemann surfaces $\overline{\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}} \to \mathbb{P}^1$ which has no crtical points, and hence it is a biholomorphism. Write a proof of this theorem using some ingredients from [Apo90, Section 2.7].

Proposition 2.7 *Any meromorphic modular function f can be written as a rational function in j with coefficients in $\mathbb{C}$. If further f is holomorphic in $\mathbb{H}$ then f can be written as a polynomial in j with coefficients in $\mathbb{C}$.*

Proof. By Theorem 2.7, there is a meromorphic function g in $\mathbb{C}$ with finite number of poles such that $f = g(j)$. Since both j and f are meromorphic at infinity, it follows that g is a rational function in $\mathbb{P}^1$. If f is holomorphic in $\mathbb{H}$ then g has no poles in $\mathbb{C}$ and so it is polynomial.

There is a beautiful history behind the j-function. According to [Apo90, page 22, end of Chapter 1], Berwick in 1916 calculated the first seven coefficients of j, Zuckerman the first 24 in 1939, and Van Wijngaarden the first 100 in 1953, see also [Fri22, page 246] for coefficients of j for q^n, $n \leq 4$. The only reason for computing such numbers, seems to be only the joy of playing with them and their mysteriousness. In [Apo90] we also find some divisibility properties of c_n's due to D.H. Lehmer in 1942 and J. Lehner in 1949. An asymptotic formula due to Petersson in 1932 and Rademacher in 1932 is also reported in this reference. In 1978 MacKay noticed that $196884 = 196883 + 1$ and 196883 is the number of dimensions in which the Monster group can be most simply represented. Based on this observation J.H. Conway and S.P. Norton in 1979 formulated the Monstrous moonshine conjecture which relates all the coefficients in the j-function to the representation dimensions of the monster group. In 1992 R. Borcherds solved this conjecture and got Fields medal, see [Gan06] for more information on this conjecture. The proof does not give any clue why elliptic curves must have something to do with the monster group, and so the mystery involved around it still exists. For instance, in a private conversation J.H. Conway expressed the fact that the proof for him is not satisfactory.

2.13 Growth of coefficients

The growth of coefficients of arithmetic functions, and in particular, the Fourier coefficients of modular forms has been of interest in the early stages of the theory of modular forms. For instance, in [Ram16] we can find many asymptotic behaviour of arithmetic functions. For the content of the present section we basically follow Hecke's original article [Hec37] and [Ser78, Section 4]. For two sequences $f_n, g_n \in \mathbb{C}$ by $f_n = O(g_n)$ we mean that $\frac{f_n}{g_n}$ is bounded. In a similar way, for two complex valued function f and g defined in a neighborhood of $a \in \mathbb{C}$ we write$f = O(g)$ or $f \sim_{x\to a} g$ to say that $\frac{f(x)}{g(x)}$ is bounded near a.

Theorem 2.8 ([Hec37], Satz 5, Satz 6) *If f is a holomorphic cusp form of weight k for the group* $\mathrm{SL}(2,\mathbb{Z})$ *then*

$$f_n = O(n^{\frac{k}{2}}),$$

where $f = \sum_{n=1}^{\infty} f_n q^n$ is the q-expansion of f. Let f be a holomorphic modular form of weight k for $\mathrm{SL}(2,\mathbb{Z})$*. Then*

$$f_n = O(n^{k-1}).$$

Proof. The Cauchy's residue formula implies that

$$f_n = \frac{1}{2\pi i}\int_{\delta} f(q)q^{-n}\,\frac{dq}{q},$$

where δ is a small circle turning around $q = 0 \in \mathbb{C}$ anticlockwise. We write $q = e^{2\pi i\tau}$ and $\tau = x + iy$. The integration in τ is over the path with y constant and x running from 0 to 1. We have

$$f_n = \int f(\tau)e^{-2\pi i n\tau}d\tau = e^{2\pi yn}\int_0^1 f(x+iy)e^{-2\pi inx}dx. \tag{2.44}$$

The function $|f(\tau)\mathrm{Im}(\tau)^{\frac{k}{2}}|$ is invariant under the action of $\mathrm{SL}(2,\mathbb{Z})$, and so, it gives us a function in $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$. Since

$$|f(\tau)| \sim_{q\to 0} q \sim_{y\to+\infty} e^{-2\pi y},$$

this function is bounded when $y \to \infty$, and so, there exists a constant M such that

$$|f(\tau)| \leqslant My^{-\frac{k}{2}} \qquad \forall\tau \in \text{ in a neighborhood of } i\infty \in \mathbb{H}. \tag{2.45}$$

From another side the complement of a neighborhood of $i\infty$ in $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ is a compact subset of $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ and so the equality (2.45) ia true for all $\tau \in \mathbb{H}$ but with possibly larger constant M. We are using the fact that after adding $i\infty$ to $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ it is a compact space. Putting (2.45) in (2.44) we have

$$|f_n| \leqslant e^{2\pi y n} \cdot M \cdot y^{-\frac{k}{2}}$$

Here, y can be any positive number, we put $y = \frac{1}{n}$ and get the desired result.

Let us now prove the second part. Let E_k be the Eisenstein series of weight k, and $\lambda \in \mathbb{C}$, be constant such that $\lambda E_k + f$ is a cusp form. The second part follow from the first part and the asymptotic behaviour of Fourier coefficient $\sigma_{k-1}(n)$ of E_k:

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1} = O(n^{k-1}).$$

Because

$$\begin{aligned} \frac{\sigma_{k-1}(n)}{n^{k-1}} &= \sum_{d|n} \left(\frac{d}{n}\right)^{k-1} \\ &= \sum_{d|n} \left(\frac{1}{d}\right)^{k-1} \leqslant \sum_{d=1}^{\infty} \frac{1}{d^{k-1}} = \zeta(k-1) < \infty. \end{aligned}$$

Since $n^{\frac{k}{2}}$ compared to n^{k-1} is negligible, we get the result.

There is a better result which says that for a cusp form f, we have

$$f_n = O\left(n^{\frac{k}{2}-\frac{1}{2}} \sigma_0(n)\right), \tag{2.46}$$

where $\sigma_0(n)$ is the number of positive divisors of n. This implies that

$$f_n = O\left(n^{\frac{k}{2}-\frac{1}{2}+\varepsilon}\right), \quad \forall \varepsilon > 0$$

This is obtained by P. Deligne as a consequence (see [Del71]) of his proof for Weil conjectures (see [Del73, Del80]). See also [Mil20].

The following simple proposition will be needed in the proof Theorem 2.5.

Proposition 2.8 *A non-zero modular form of weight k for $\mathrm{SL}(2,\mathbb{Z})$ has a zero of order $\leq \frac{k}{12}$ at infinity. The equality happens if and only if $12|k$ and f is a multiple of $\Delta^{\frac{k}{12}}$.*

Proof. Let f be a modular form of weight k with a zero of order $N \in \mathbb{N}$ at infinity. We consider $\frac{f^{12}}{\Delta^k}$ which is a modular function. By Proposition 2.6 it has no poles in $\mathbb{H}$ and by Proposition 2.7 it is a polynomial in j. Its order at infinity is $12N - k$ which is a non-positive integer. If $k = 12N$ it is a polynomial of degree 0 and hence $\frac{f^{12}}{\Delta^k}$ is a constant.

2.14 The numbers e_1, e_2, e_3

Theorem 2.9 in this section is taken from [Apo90, Theorem 1.14] and it would be interesting to trace back the origin of this theorem. Our main reason for presenting this in this section is Exercise 2.38. The equalities (2.63) in this theorem seems to be novel, as I was not able to find them in the literature. For a lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, let

$$e_1 := \wp\left(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2,\ \tfrac{\omega_1}{2}\right),$$

$$e_2 := \wp\left(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2,\ \tfrac{\omega_2}{2}\right),$$

$$e_3 := \wp\left(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2,\ \tfrac{\omega_1+\omega_2}{2}\right).$$

Theorem 2.9 *The numbers e_1, e_2, e_3 are distinct and we have*

$$4\wp^3(z) - g_2 \cdot \wp(z) - g_3 = 4\Big(\wp(z) - e_1\Big)\Big(\wp(z) - e_2\Big)\Big(\wp(z) - e_3\Big) \tag{2.47}$$

Proof. Since $\wp(z)$ is even, $\wp'(z)$ is odd. Therefore,

$$-\wp'\left(\frac{1}{2}\omega\right) = \wp'\left(-\frac{1}{2}\omega\right) = \wp'\left(\omega - \frac{1}{2}\omega\right) = \wp'\left(\frac{1}{2}\omega\right) \qquad \forall \omega \in \Lambda.$$

This implies that $\frac{\omega_1}{2}$, $\frac{\omega_2}{2}$, $\frac{\omega_1+\omega_2}{2}$ are roots of $\wp'(z)$. The function $\wp'(z)$ has a pole of order 3 at $z = 0 \in \mathbb{C}/\Lambda$, and so the mentioned three points, are the only roots of $\wp'(z)$ and they are simple. The differential equation of $\wp(z)$, implies that e_1, e_2, e_3 are roots of the left hand side of $4x^3 - g_2 x - g_3$. We show next that e_1, e_2 and e_3 are distinct, for instance, $e_1 \neq e_2$. The elliptic function $\wp(z) - e_i$ for $i = 1, 2$ has a double root at $\frac{\omega_i}{2}$, because $\wp'\left(\frac{1}{2}\omega_i\right) = 0$. If $e_1 = e_2$ then this function must have pole order $\geqslant 4$ at $z = 0$, which is a contradiction.

2.15 Jacobi's theta functions

Jacobi's theta function (or series) is the following infinite sum

$$\theta(z, \tau) = \sum_{n=-\infty}^{+\infty} e^{2\pi i n z + \pi i n^2 \tau},\ z \in \mathbb{C},\quad \tau \in \mathbb{H}$$

According to [EZ85, page 1] it was introduced in [Jac29, Section 52] and that is the reason why the name Jacobi theta series. In the whole book Jacobi analyses the elliptic integrals $\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$, and it might be worthy to find out the motivation for Jacobi for defining such a series. Our treatment of theta series in this section is taken from [SS03, Chapter 10]. For theta functions attached to lattices see [CS99, Chapter 15], [Ebe94, Chapter 2].

Proposition 2.9 *The Jacobi's theta series is convergent in $\mathbb{C}\times\mathbb{H}$ and satisfies the following functional equations:*

1. $\theta(z+1,\tau)=\theta(z,\tau)$,
2. $\theta(z+\tau,\tau)=\theta(z,\tau)e^{-\pi i\tau}e^{-2\pi iz}$,
3. $\theta(z,\tau)=0$ *for* $z=\frac{1}{2}+\frac{\tau}{2}+n+m\tau,\ n,m\in\mathbb{Z}$.

Note that unlike Weierstrass $\wp$ function, Jacobi's theta function is holomorphic everywhere.

Proof. We first prove the convergence. For a fixed $M,t_0\in\mathbb{R}^+$, and for $|z|<M$ and $\mathrm{Im}(\tau)>t_0$ we have

$$\sum_{n=-\infty}^{\infty}\left|e^{2\pi inz+\pi in^2\tau}\right|\leqslant C\sum_{n=0}^{\infty}e^{2\pi nM}e^{-\pi n^2t_0}$$

for some positive number $C\in\mathbb{R}$. The convergence follows from the fact that for $a,b\in\mathbb{R}$ with $|a|<1$, the series $\sum_{n=0}^{\infty}a^{n^2}b^n$ is always convergent. This shows that θ converges is $\mathbb{C}\times\mathbb{H}$. The proof of item 1 is immediate from the definition of θ. The second item follows from

$$\begin{aligned}\theta(z+\tau,\tau) &= \sum_{n=-\infty}^{+\infty} e^{2\pi inz}\ e^{\pi i(n^2+2n)\tau}\\ &= \sum_{n=-\infty}^{+\infty} e^{2\pi i(n+1)z}\ e^{\pi i(n+1)^2\tau}\ e^{-\pi i\tau}\ e^{-2\pi iz}\\ &= \theta(z,\tau)\cdot e^{-\pi i\tau}\ e^{-2\pi iz}.\end{aligned}$$

The proof of the third item is as follows: Using the first and second items we only need to consider the evaluation at $z=\frac{1}{2}+\frac{\tau}{2}$. We have

$$\theta\left(\frac{1}{2}+\frac{\tau}{2},\tau\right)=\sum_{n=-\infty}^{\infty}(-1)^n\ e^{\pi i(n^2+n)\tau}.$$

For $n\geqslant 0$ the terms corresponding to n and $-n-1$ cancel each other.

Theorem 2.10 *We have*

$$\theta(z,\tau)=\prod_{n=1}^{\infty}(1-q^n)\left(1+q^{n-\frac{1}{2}}\ e^{2\pi iz}\right)\left(1+q^{n-\frac{1}{2}}\ e^{-2\pi iz}\right),\tag{2.48}$$

where $q=e^{2\pi i\tau}$.

Proof. Note that in [SS03] the authors have used $q=e^{\pi i\tau}$. Let $\pi(z,\tau)$ be the right hand side of (2.48). We prove that $\pi(z,\tau)$ is a holomorphic function in $\mathbb{C}\times\mathbb{H}$ and satisfies the same properties as of $\theta(z,\tau)$ in Proposition 2.9. For the convergence we use the criterion for convergence of infinite products. We have

$$(1-q^n)\left(1+q^{n-\frac{1}{2}}\ e^{2\pi i z}\right)\left(1+q^{n-\frac{1}{2}}\ e^{-2\pi i z}\right)=1+q^{n-\frac{1}{2}}\left(e^{2\pi i z}+e^{-2\pi i z}\right)+\cdots$$

and $\sum_{n=1}^{\infty}|q|^n$ converges. The first functional equation for π in Proposition 2.9 is immediate. The second functional equation follows from

$$\begin{aligned}\pi(z+\tau,\tau)&=\prod_{n=1}^{\infty}(1-q^n)\left(1+q^{n+\frac{1}{2}}\ e^{2\pi i z}\right)\left(1+q^{n-\frac{3}{2}}\ e^{-2\pi i z}\right)\\&=\frac{\left(1+q^{-\frac{1}{2}}\ e^{-2\pi i z}\right)}{\left(1+q^{\frac{1}{2}}\ e^{2\pi i z}\right)}\pi(z,\tau)\end{aligned}$$

We have $\frac{1+x}{1+x^{-1}}=x$ for $x\neq -1$ and the second functional equation follows. The product vanishes at a point (z,τ) if $\pm z+\left(n-\frac{1}{2}\right)\tau\in\mathbb{Z}+\frac{1}{2}$ which gives us the third item in Proposition 2.9 for π.

Now, let us prove (2.48). Let $F(z,\tau)=\frac{\theta(z,\tau)}{\pi(z,\tau)}$. This as a function in z is double periodic and has no poles. Therefore, it is constant as a function in z. Therefore $C(\tau)=\theta(z,\tau)/\pi(z,\tau)$. We put $z=\frac{1}{2}$ and $z=\frac{1}{4}$ and respectively get

$$C(\tau)=\frac{\sum_{n=-\infty}^{\infty}(-1)^n\ q^{\frac{1}{2}\ n^2}}{\prod_{n=1}^{\infty}\left(1-q^{\frac{1}{2}n}\right)\left(1-q^{n-\frac{1}{2}}\right)},\tag{2.49}$$

$$C(\tau)=\frac{\sum_{n=-\infty}^{+\infty}(-1)^n\ q^{2n^2}}{\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{4n-2})}.\tag{2.50}$$

The equalities (2.49) and (2.50) imply $C(4\tau)=C(\tau)$ for all $\tau\in\mathbb{H}$. Since $q^{4^k}\to 0$ when $k\to\infty$ we conclude that $C(\tau)=1$.

Theorem 2.11 *For $\tau\in\mathbb{H}$ and $z\in\mathbb{C}$ we have*

$$\theta\left(z,\frac{-1}{\tau}\right)=\sqrt{\frac{\tau}{i}}\ e^{\pi i\tau z^2}\ \theta(z\tau,\tau).$$

Here, we have chosen a branch of $\sqrt{\frac{\tau}{i}}$, $\tau\in\mathbb{H}$ such that for imaginary τ, it is positive.

It is convenient to replace z with $\frac{z}{\tau}$ and rewrite the above formula:

$$\theta\left(\frac{z}{\tau},\frac{-1}{\tau}\right)=\sqrt{\frac{\tau}{i}}\,e^{\pi i\frac{z^2}{\tau}}\;\theta(z,\tau).$$

Proof. It is enough to prove the formula for $z=a\in\mathbb{R}$ and $\tau=it, t\in\mathbb{R}^+$. We have to prove the equality

$$\sum_{n=-\infty}^{\infty}e^{-\frac{\pi n^2}{t}}e^{2\pi ina}=t^{\frac{1}{2}}e^{-\pi ta^2}\sum_{n=-\infty}^{\infty}e^{-\pi n^2t}e^{-2\pi nat}.$$

We write this as

$$\sum_{n=-\infty}^{\infty}e^{-\pi t(n+a)^2}=t^{-\frac{1}{2}}\sum_{n=-\infty}^{\infty}e^{-\frac{\pi n^2}{t}}e^{2\pi ina}.$$

This is exactly the Poisson summation formula that will be proved in Section 2.17. Note that the Fourier transform of $f(x)=e^{-\pi x^2}$ is itself:

$$\int_{-\infty}^{\infty}e^{-\pi x^2}e^{-2\pi ixy}dx=e^{-\pi y^2}.$$

This implies that the Fourier transform of $f(x)=e^{-\pi t(x+a)^2}$ is $g(y)=t^{-\frac{1}{2}}e^{-\frac{\pi y^2}{t}}e^{2\pi iay}$.

We will frequently use the followings:

$$\begin{aligned}
\theta_3(\tau) &:= \theta(0,\tau)=\sum_{n=-\infty}^{+\infty}q^{\frac{1}{2}n^2}=\frac{\eta(\tau)^5}{\eta\left(\frac{1}{2}\tau\right)^2\eta(2\tau)^2},\\
\theta_4(\tau) &:= \theta\left(\frac{1}{2},\tau\right)=\sum_{n=-\infty}^{\infty}(-1)^n\,q^{\frac{1}{2}n^2}=\frac{\eta\left(\frac{1}{2}\tau\right)^2}{\eta(\tau)},\\
\theta_2(\tau) &:= \theta\left(\frac{\tau}{2},\tau\right)=\,q^{-\frac{1}{8}}\sum_{n=-\infty}^{+\infty}q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}=\frac{2\eta(2\tau)^2}{\eta(\tau)}.
\end{aligned}\tag{2.51}$$

Exercise 2.31 Prove the equalities between θ_i and η as above. In particular, prove that $\theta_2\theta_3\theta_4=2\eta(\tau)^3$.

Using Theorem 2.11 we get the following functional equations for $\theta_3,\ \theta_4$

$$\begin{aligned}
\theta_3\left(\frac{-1}{\tau}\right) &= \sqrt{\frac{\tau}{i}}\;\theta_3(\tau)\\
\theta_4\left(\frac{-1}{\tau}\right) &= \sqrt{\frac{\tau}{i}}\;\zeta_8\cdot\theta_2(\tau)
\end{aligned}\tag{2.52}$$

where $\zeta_8=e^{\frac{2\pi i}{8}}$ is the eighth root of unity. Let $f(\tau)=\theta_3(8\tau)^8$. We have $f(\tau+1)=f(\tau)$ and $f\left(\frac{-1}{4\tau}\right)=\left(\frac{\tau}{2}\right)^4f(\tau)$ which says that f is a modular form for the group

$$\left\langle\begin{bmatrix}1&1\\0&1\end{bmatrix},\begin{bmatrix}0&\frac{-1}{2}\\\frac{1}{2}&0\end{bmatrix}\right\rangle\subseteq \mathrm{SL}(2,\mathbb{Q}).$$

Exercise 2.32 For $a,b \in \mathbb{Q}$, we define the following shift of the Jacobi theta functions:

$$\begin{aligned}\theta_{a,b} &: \mathbb{C}\times\mathbb{H}\to\mathbb{C},\\ \theta_{a,b}(z,\tau) &:= \sum_{n\in\mathbb{Z}} e^{2\pi i\left(\frac{1}{2}(n+a)^2\cdot\tau+(n+a)\cdot(z+b)\right)}\\ &= e^{\pi i a^2\tau+2\pi i a(z+b)}\theta(z+a\tau+b,\tau).\end{aligned}$$

In higher dimensions these are called Riemman's theta functions, and they were used by Humbert and Picard to study the double integrals of hyperelliptic surfaces, see [Mov21, Chapter 3]. Such a theta function satisfies the functional equation

$$\theta_{a,b}(z+\tau m+n,\tau) = e^{2\pi i(an-bm-\frac{1}{2}m^2\tau-mz)}\theta_{a,b}(z,\tau). \tag{2.53}$$

Let $\Lambda := \{\tau m+n \mid m,n\in\mathbb{Z}\}$ and $\mathbb{C}/\Lambda$ be the corresponding complex compact torus. In a more geometric language, one says that the exponential factors in (2.53) form a line bundle in $\mathbb{C}/\Lambda$ and $\theta_{a,b}$ is a holomorphic section of this line bundle. Consider the map

$$\mathbb{C}/\Lambda \to \mathbb{P}^{N^2-1},\ z\mapsto [\cdots;\theta_{a,b}(Nz,\tau);\cdots] \tag{2.54}$$

where (a,b) runs over representatives of $\frac{1}{N}\mathbb{Z}/\mathbb{Z}$. Show that for $N>1$ this map is an embedding and for $N=2$, its image is characterized by the intersection of two quartics:

$$A^2x_0^2 = B^2x_1^2+C^2x_2^2,\ \ A^2x_3^2 = C^2x_1^2-B^2x_2^2$$

where

$$x_0=\theta_{0,0}(2z,\tau),\ \ x_1=\theta_{0,\frac{1}{2}}(2z,\tau),\ \ x_2=\theta_{\frac{1}{2},0}(2z,\tau),\ \ x_1=\theta_{\frac{1}{2},\frac{1}{2}}(2z,\tau),$$

and

$$A=\theta_{0,0}(0,\tau)=\theta_3(\tau),\ \ B=\theta_{0,\frac{1}{2}}(0,\tau)=\theta_4(\tau),\ \ B=\theta_{\frac{1}{2},0}(0,\tau)=\theta_2(\tau).$$

We have $A^4=B^4+C^4$ which is called the Jacobi's identity between the theta constants. The main reference for this topic is [Mum91, Chapter 1, Section 5], see also [Hus04, Chapter 10].

Exercise 2.33 Show that the Jacobi's theta function satisfy the heat equation

$$\frac{\partial\theta(z,\tau)}{\partial\tau} = \frac{1}{4\pi i}\frac{\partial^2\theta(z,\tau)}{\partial z^2}$$

This differential equation can be also found in [Fri16, page 414].

Exercise 2.34 Up to some factor, the Jacobi's theta function and Weierstrass σ functions seems to be the same, see for instance [Fri22, page 176] and [DLMF].

Exercise 2.35 One of the ways to justify Jacobi's theta function $\theta_{a,b}$ is through Poincaré series. Recall our notation of a group Γ actting on a complex manifold M. An automorphy factor $j : \Gamma \times M \to \mathbb{C}^*$ is a function which is holomorphic for fixed $A \in \Gamma$ and satisfies the functional equation

$$j(AB,z) = j(A,Bz)j(B,z), \ \ \forall A,B \in \Gamma, \ \ z \in M.$$

A trivial automorphy factor is given by $j(A,z) = \frac{f(Az)}{f(z)}$ for some holomorphic function $f : M \to \mathbb{C}^*$. The set of autmorphy factor is a group and modulo trivial ones it is called the Picard group of $\Gamma\backslash M$. The elements of Picard group are in one to one correspondance with line bundles in $\Gamma\backslash M$. An automorphic form of weight $k \in \mathbb{Z}$ with the automorphy factor j is any meromorphic function f on M such that

$$f(Az) = j(A,z)^k f(z), \ \forall A \in \Gamma, \ z \in M. \tag{2.55}$$

A way to obtain automorphic forms is through Poincaré series. These are convergent series of the form $\sum^*_{A\in\Gamma} j(A,z)^k f(Az)$ for a holomorphic function $f : M \to \mathbb{C}$, where $*$ means that summation is over an equivalence classes in Γ which gives us distinct terms in the series. Show that the theta series $\theta_{a,b}$ can be written as Poincaré series such that (2.53) becomes (2.55). Hint: Consider the lattice action of $\Gamma := \mathbb{Z}\tau + \mathbb{Z}$ on $M := \mathbb{C}$. For more details in general see [Cha14].

2.16 Applications of theta series

We finish this section by a classical application of theta series that can be found in [SS03, Chapter 10, Section 3] and [Zag08, Section 3.1]. For $k \in \mathbb{N}$ and $a = (a_1,\, a_2, \dots a_k) \in \mathbb{Z}^k$ define the number

$$\gamma_{k,a}(n) = \#\{(x_1,\, x_2, \dots, x_k) \in \mathbb{Z}^k \Big| a_1x_1^2 + a_2x_2^2 + \cdots + a_kx_k^2 = n\}.$$

Its generating function can be written in terms of theta series:

$$\sum_{n=0}^{\infty} \gamma_{k,a}(n)q^n = \theta(2a_1\tau)\ \theta(2a_2\tau)\cdots\theta(2a_k\tau), \tag{2.56}$$

where $\theta = \theta_3$. In order to find formulas for $\gamma_{k,a}(n)$ we have to study the analytic function in the right hand side of (2.56). Let $d_1(n)$ denote the number of divisors of n of the form $4k+1$, and $d_3(n)$ the number of divisors of n of the form $4k+3$.

Exercise 2.36 For $n \geqslant 1$ we have

$$\gamma_{2,(1,1)}(n) = 4\bigg(d_1(n) - d_3(n)\bigg). \tag{2.57}$$

Hint: [SS03, page 299].

2.17 Poisson summation formula

Poisson summation formula is the main ingredient of the proof of the functional equation of Jacobi's theta function in Theorem 2.11. In this section we present the first and classical version of this formula for rank one lattices. For the same formula for higher rank unimodular lattices see [CS99, Chapter 15], [Ebe94, Chapter 2]. We mainly follow [Zag08, Appendix A].

Let $\varphi : \mathbb{R} \to \mathbb{C}$ be any continuous function which decreases rapidly, let us say

$$\varphi(x) \sim |x|^{-c}$$

for some $c > 1$ as $x \to \pm\infty$. Then the Fourier transform of φ is

$$\check{\varphi}(y) := \int_{\mathbb{R}} \varphi(x)\, e^{-2\pi ixy}\, dx.$$

Theorem 2.12 *We have*

$$\sum_{n\in\mathbb{Z}} \varphi(n) = \sum_{n\in\mathbb{Z}} \check{\varphi}(n) \tag{2.58}$$

which is called the Poisson summation formula.

Proof. The growth condition on $\varphi(x)$ ensures that $\phi(x) := \sum_{n\in\mathbb{Z}} \varphi(x+n)$ converges to a continuous function ϕ. This function satisfies $\phi(x+1) = \phi(x)$ and so ϕ has Fourier expansion

$$\phi(x) = \sum_{\gamma\in\mathbb{Z}} c_\gamma \cdot e^{2\pi i\gamma x}, \quad \text{where } c_\gamma = \int_0^1 \phi(x)\, e^{-2\pi i\gamma x}\, dx.$$

Substituting $\phi(x)$ in c_γ we get

$$\begin{aligned} c_\gamma &= \int_0^1 \left(\sum_{n=-\infty}^{\infty} \varphi(x+n) \right) e^{-2\pi i\gamma(x+n)}\, dx \\ &= \sum_{n=-\infty}^{\infty} \int_n^{n+1} \varphi(x)\, e^{-2\pi i\gamma x}\, dx \\ &= \int_{-\infty}^{\infty} \varphi(x)\, e^{-2\pi i\gamma x}\, dx = \check{\varphi}(n). \end{aligned}$$

This gives us

$$\sum_{n\in\mathbb{Z}} \varphi(n) = \phi(0) = \sum_{\gamma\in\mathbb{Z}} c_\gamma = \sum_{n\in\mathbb{Z}} \check{\varphi}(n),$$

which is the desired statement.

2.18 Some exercises based on Eisenstein's work

According to [Wei99], [Eis47] fifteen years before Weierstrass did much of the work that is now attributed to Weierstrass. In this section we formulate some exercises in order to estimulate further reading of Weil's book and Eisenstein original article. It has been motivated by the formula in Theorem 2.13, for which the author was able to give a proof using geometric arguments, see Section 9.13. W. Zudilin in a private communication recommended the author to read Weil's book and the present section is the outcome of this reading. We have used the notations of the present book instead of those in [Wei99]. Further, recall that our lattices are oriented, and hence, $\mathrm{Im}(\frac{\omega_1}{\omega_2}) > 0$. Therefore, the nummber δ used in [Wei99] is equal to one. Let us define:

$$\mathfrak{E}_k(z:\omega_1,\omega_2) = \mathfrak{E}_k(z) := \sum_{\omega\in\Lambda} \frac{1}{(z+\omega)^k} = \sum_{\omega\in\Lambda} \frac{1}{(z+n\omega_1+m\omega_2)^k}. \tag{2.59}$$

For $k = 1,2$ we use Eisenstein summation

$$\sum_{\omega\in\Lambda} := \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} := \lim_{N\to\infty}\sum_{n=-N}^{N}\lim_{M\to\infty}\sum_{m=-M}^{M}$$

for which one also uses the notation Σ_e. Note that Eisenstein summation depends on the choice of the basis ω_1,ω_2 for the lattice Λ, and in particular, the order of summation in n and m. By abuse of notations we will also define

$$\mathfrak{E}_k(z,\tau) := \mathfrak{E}_k(z:\tau,1)$$

being clear in the context which is meant.

Exercise 2.37 Prove the following statements:

1. The series $\mathfrak{E}_k,\ k \geq 1$ converges absolutely. For $k \geq 3$ this is already proved in Proposition 2.1.
2. $$\mathfrak{E}_1(z+n\omega_1+m\omega_2) = \mathfrak{E}_1(z) - 2\pi i\frac{n}{\omega_2}. \tag{2.60}$$

   and so
   $$\mathfrak{E}_1(z+1,\tau) = \mathfrak{E}(z,\tau),\ \ \mathfrak{E}_1(z+\tau,\tau) = \mathfrak{E}_1(z,\tau) - 2\pi i.$$
3. Let $\Lambda' = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2' \subset \Lambda$ be a sub lattice and choose a basis $[z_i] \in \Lambda/\Lambda',\ i = 1,2,\ldots,N$. We have

   $$\sum_{i=1}^{N}\mathfrak{E}_1(z+z_i;\omega_1',\omega_2') = \mathfrak{E}_1(z) + \frac{2\pi icz}{\omega_2\omega_2'} - \frac{\pi ie}{\omega_2'},$$

   where $c,e \in \mathbb{Z}$ are defined through the equalities:

$$\begin{bmatrix} \omega_1' \\ \omega_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}, \quad \sum_{i=1}^{N} z_i = e\omega_1 + f\omega_2.$$

In particular, if we put $\Lambda = \Lambda'$ we have

$$\mathfrak{E}_1(z;\omega_1',\omega_2') = \mathfrak{E}_1(z;\omega_1,\omega_2) + \frac{2\pi icz}{\omega_2\omega_2'}.$$

ands so

$$\frac{1}{c\tau+d}\mathfrak{E}_1\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right) = \mathfrak{E}_1(z,\tau) + \frac{2\pi icz}{c\tau+d}.$$

4. We have

$$\mathfrak{E}_1(z) = \frac{1}{z} - \sum_{m=1}^{\infty} G_m z^{m-1},$$

where $G_m = 0$ for m odd and for m even they are classical Eisenstein series (2.20).

5. Recall Weierstrass $\wp$ and ζ functions. We have

$$\wp(z) = \mathfrak{E}_2 - G_2 \tag{2.61}$$

$$\zeta(z) = \mathfrak{E}_1 - G_2 \cdot z, \tag{2.62}$$

Hint: See [Wei99, Chapter III].

Theorem 2.13 *We have the following identity for the Eisenstein series E_2:*

$$\frac{2\pi i}{12}E_2(\tau) = 4 + \sum_{(n,m)\neq(0,0)} \frac{4}{(1-2n\tau-2m)} + \frac{2}{(n\tau+m)} + \frac{1}{(n\tau+m)^2}.$$

Proof. The right hand side of the above equality is

$$2\zeta(\frac{1}{2}) = \zeta(z+1) - \zeta(z) = -G_2 = \frac{2\pi i}{12}E_2.$$

We have used the equalities (2.60) and (2.62).

2.19 Differential equations of theta series

Recall the theta series $\theta_2, \theta_3, \theta_4$ in (2.51).

Exercise 2.38 *We have the following identities between the theta series and the values of the Weierstrass $\wp$ function evaluated at half points*

$$\frac{\frac{\theta_2}{\partial \tau}}{\theta_2} = \frac{12}{2\pi i}\wp\left(\mathbb{Z}\tau+\mathbb{Z}, \frac{1}{2}\right) + E_2,$$
$$\frac{\frac{\theta_3}{\partial \tau}}{\theta_3} = \frac{12}{2\pi i}\wp\left(\mathbb{Z}\tau+\mathbb{Z}, \frac{\tau+1}{2}\right) + E_2, \tag{2.63}$$
$$\frac{\frac{\theta_4}{\partial \tau}}{\theta_4} = \frac{12}{2\pi i}\wp\left(\mathbb{Z}\tau+\mathbb{Z}, \frac{\tau}{2}\right) + E_2.$$

Moreover, these three quantities satisfy the Darboux-Halphen differential equation

$$\mathrm{H}: \begin{cases} \dot{t}_1 = t_1(t_2+t_3) - t_2 t_3, \\ \dot{t}_2 = t_2(t_1+t_3) - t_1 t_3, \\ \dot{t}_3 = t_3(t_1+t_2) - t_1 t_2. \end{cases} \tag{2.64}$$

It is expected to give an elementary proof (using only complex analysis) of Exercise 2.38. The geometric origin of this will be explained in Section 9.13. The theta sereis

$$\theta_{\frac{1}{2},\frac{1}{2}}(z,\tau) := e^{\frac{1}{4}\pi i\tau + \pi i(z+\frac{1}{2})}\theta(z+\frac{1}{2}\tau+\frac{1}{2},\tau). \tag{2.65}$$

is of particular interest. In Chapter 12 we will encounter

$$F(z,\tau) := \frac{\theta_{\frac{1}{2},\frac{1}{2}}}{\eta^3} = i(y^{1/2}-y^{-1/2})\prod_{n=1}^{\infty}\frac{(1-yq^m)(1-y^{-1}q^m)}{(1-q^m)^2}, \tag{2.66}$$

where $y = e^{2\pi i z}$, $q = e^{2\pi i\tau}$ in our geometric setting of enhanced elliptic curves.

Exercise 2.39 Show that F satisfies the functional equation

$$F\left(\frac{z+\lambda\tau+\mu}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (-1)^{\lambda+\mu}(c\tau+d)^{-1}e^{\pi i[\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d} - (\lambda^2\tau+2\lambda z)]}F(\tau,z) \tag{2.67}$$

for $\begin{bmatrix} a & b \\ c & b \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ and $(\lambda,\mu)\in\mathbb{Z}^2$. We will see that F^2 is a Jacobi form of weight -2 and index 1.

Exercise 2.40 Show that

$$\frac{\partial}{\partial z}\ln(F(z,\tau)) = \zeta(z,\tau) + \frac{(2\pi i)^2}{12}E_2(\tau)\cdot z.$$

We will give a geometric framework for the following:

Exercise 2.41 The quantities

$$a := (-2\pi i)^{-1}\wp(z_0)\ ,\ b := (-2\pi i)^{-\frac{3}{2}}\wp'(z_0)\ ,\ c := -(-2\pi i)^{-\frac{1}{2}}\frac{\partial\ln(F)}{\partial z},\ d = -2\ln(F),$$

$$t_1 := \frac{2\pi i}{12}E_2(\tau)\ ,\ t_2 := 12\left(\frac{2\pi i}{12}\right)^2 E_4(\tau)\ ,\ t_3 := 8\left(\frac{2\pi i}{12}\right)^3 E_6(\tau),$$

satisfies the equality $t_3 = 4a^3 - t_2 a - b^2$ and the ordinary differential equations

$$\begin{cases} \dot{a} &= -2a^2 + 2at_1 + bc + \frac{t_2}{3} \\ \dot{b} &= 6a^2c - \frac{ct_2}{2} - 3ab + 3bt_1 \\ \dot{c} &= ac + ct_1 - \frac{b}{2} \\ \dot{d} &= c^2 - a + 2t_1 \\ \dot{t}_1 &= t_1^2 - \frac{t_2}{12} \\ \dot{t}_2 &= 4t_1t_2 - 6t_3 \end{cases} \quad \text{where} \quad \dot{x} := \frac{\partial x}{\partial \tau} \tag{2.68}$$

$$\begin{cases} \dot{a} &= b \\ \dot{b} &= 6a^2 - \frac{t_2}{2} \\ \dot{c} &= a + t_1 \\ \dot{d} &= 2c \\ \dot{t}_1 &= 0 \\ \dot{t}_2 &= 0 \end{cases} \quad \text{where} \quad \dot{x} := (-2\pi i)^{-\frac{1}{2}} \frac{\partial x}{\partial z} \tag{2.69}$$

Hint: In (2.69) all the equalities are trivial except for the second and third one which follow from Exercise 2.16 and Exercise 2.40, respectively. The main trick for proving (2.68) is as follows. We explain it for the first equality. We use (2.39) to have the functional equation of c and observe that $\frac{\partial a}{\partial \tau} - bc$ is an elliptic function of weight 4. We compute its Laurent expansion at $z = 0$ and observe that the coefficients of $\frac{1}{z^i}$, $i = 0, 1, 2, 3, 4$ coincide with those in $-2a^2 + 2at_1 + \frac{t_2}{3}$. Therefore, the difference of two quantities is a holomorphic elliptic function vanishing at $z = 0$, and hence it must be zero.

Chapter 3
Elliptic curves and integrals

Although most of the seminars I couldn't understand, after 10 times I started to get something and that something could be very useful for my development in mathematics or even to physics eventually, (S.-T. Yau in Kavli IPMU News No. 33 March 2016).

3.1 Introduction

In this chapter we study elliptic curves over complex numbers and the corresponding elliptic integrals. Our discussion in this chapter closely follows [Mov21, Chapter 3] and [Mov12] which aims to find the origin of Hodge theory in the study of elliptic integrals. Our approach to elliptic integrals starts with Weierstrass familly of elliptic curves, however, historically elliptic integrals of the Jacobi family $y^2 = (1-x^2)(1-k^2x^2)$ (the name borrowed from [Hus04, Chapter 4, Section 3]) have appeared in the literature first. For instance, these integrals have been the main object of study in the treatise [Jac29]. In order to have a precise historical account on elliptic integrals, the reader might have a look at Fricke's three volumes books [Fri16, Fri22, Fri11]. Note that Fricke died in 1930 and the manuscript of the third volume was only published in 2011. We assume the reader is familiar with the projective space $\mathbb{P}^2$ and curves inside it, otherwise, the reader might read the first few sections of Chapter 4.

3.2 Elliptic integrals

We start with an elliptic integral of the form

$$\int_a^b \frac{dx}{\sqrt{p(x)}}, \tag{3.1}$$

where $p(x)$ is a polynomial of degree 3 and with three distinct real roots, and a,b are two consecutive elements among the roots of p and $\pm\infty$. For instance, the polynomial $p(x) := 4x^3 - t_2x - t_3, \quad t_2,t_3 \in \mathbb{C}$ has three distinct roots if and only if $\Delta := 27t_2^3 - t_3^2 \neq 0$. If $p(x)$ has repeated roots one can compute it easily.

Exercise 3.1 Compute the indefinite integral

$$\int \frac{dx}{\sqrt{p(x)}}, \tag{3.2}$$

where p is a polynomial of degree 1 and 2. Compute it also when p is of degree 3 but it has double roots. These integrals are computable because $y^2 = P(x)$ is a rational curve! Let p be a polynomial of degree 3 and with three real roots $t_1 < t_2 < t_3$. Show that two of the four integrals

$$\int_{-\infty}^{t_1} \frac{dx}{\sqrt{p(x)}},\ \int_{t_1}^{t_2} \frac{dx}{\sqrt{p(x)}},\ \int_{t_2}^{t_3} \frac{dx}{\sqrt{p(x)}},\ \int_{t_3}^{+\infty} \frac{dx}{\sqrt{p(x)}},$$

can be computed in terms of the other two.

In many calculus books we find tables of integrals and there we never find a formula for elliptic integrals. Already in the 19th century, it was known that if we choose p randomly (in other words for generic p) such integrals cannot be calculated in terms of until then well-known functions. For particular examples of p we have some formulas calculating elliptic integrals in terms of the values of the Gamma function on rational numbers.

Exercise 3.2 For particular examples of polynomials p of degree 3, there are some formulas for elliptic integrals (3.2) in terms of the values of the Gamma function on rational numbers. For instance, verify the equality

$$\int_7^{+\infty} \frac{dx}{\sqrt{x^3-35x-98}} = \frac{\Gamma(\frac{1}{7})\Gamma(\frac{2}{7})\Gamma(\frac{4}{7})}{2\pi i\sqrt{-7}}. \tag{3.3}$$

In [Wal06, page 439] we find also the formulas

$$\begin{aligned} \int_0^1 \frac{dx}{\sqrt{1-x^3}} &= \frac{\Gamma(\frac{1}{3})^2}{2^{\frac{4}{3}}3^{\frac{1}{2}}\pi}, \\ \int_0^1 \frac{dx}{\sqrt{x-x^3}} &= \frac{\Gamma(\frac{1}{4})^2}{2^{\frac{3}{2}}\pi^{\frac{1}{2}}}. \end{aligned}$$

These formulas can be also derived using the software Mathematica. The Chowla-Selberg theorem, see for instance Gross's articles [Gro78, Gro79], describes this phenomenon in a complete way. The right hand side of (3.3) can be written in terms of the Beta function which is more natural when one deals with the periods of algebraic differential forms.

The fact that we only need two of the integrals in (3.1) in order to calculate the others, can be easily seen by considering the integration in the complex domain $x \in \mathbb{C}$, in which we may discard the assumption that p has only real roots. The integration is done over a path γ in the $x \in \mathbb{C}$ domain which connects two points in the set of roots of p and ∞, and avoids other roots except at its start and end points. An amazing fact that we learn in a complex analysis course is that if the path γ moves smoothly, without violating the properties as before, then the value of the integral does not change. This is certainly the origin of homotopy theory, or at least one of them. The next step in the study of elliptic integrals is the invention of the y variable which is basically the square root of $p(x)$:

$$E := \left\{(x,y) \in \mathbb{C}^2 \;\middle|\; y^2 = p(x) \right\}. \tag{3.4}$$

This is called an elliptic curve in Weierstrass form.

Exercise 3.3 Up to multiplication by a constant which can be computed explicitly, the integral (3.1) can be written as

$$\int_\delta \frac{dx}{y},$$

where δ is a closed path in E.

We add another point O to E and will call it the point at infinity. We write $\bar{E} = E \cup \{O\}$ and sometimes by abuse of notation use the same letter E for $\bar{E}$. If we write the equation of E in homogeneous coordinates $[x : y : z] \in \mathbb{P}^2$ then

$$O = [0 : 1 : 0].$$

see for instance Chapter 4 for definition of the projective space $\mathbb{P}^2$. We define $H_1(E,\mathbb{Z})$ as the abelization of the fundamental group of E, that is, the quotient of the fundamental group of E by its subgroup generated by commutators:

$$H_1(E,\mathbb{Z}) := \pi_1(E,b)/[\pi_1(E,b),\pi_1(E,b)], \tag{3.5}$$

where for a group G, $[G,G]$ is the subgroup of G generated by the commutators $aba^{-1}b^{-1}$, $a,b \in G$. It turns out that the integrals

$$\int_\delta \frac{dx}{y},\ \delta \in H_1(E,\mathbb{Z}).$$

are well-defined.

Proposition 3.1 The abelian group $H_1(E,\mathbb{Z})$ is free of rank 2, and hence, it is isomorphic to $(\mathbb{Z}^2,+)$.

Proof. We prove that the non-abelian group $\pi_1(E,b)$ is free and it is generated by two elements. Let $\pi : E \to \mathbb{C}$ be the projection into x-coordinate and a be the x-coordinate of b. Let also $p_i = (t_i,0) \in E$. Our claim follows from the following purely topological statement.

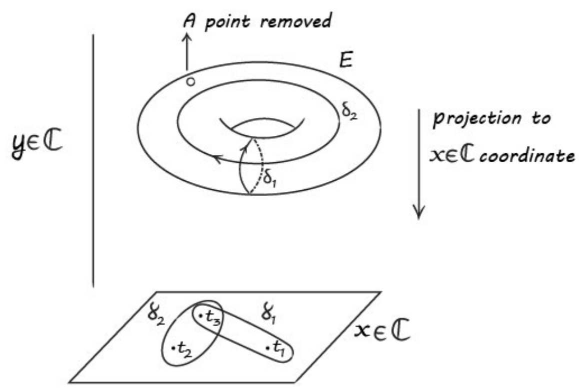


Fig. 3.1 The elliptic curve $y^2 = p(x)$ in the four dimensional space $\mathbb{C}^2$.

Exercise 3.4 Let E be a connected real surface and $\pi : E \to \mathbb{C}$ be a continuous map which is a 2 to 1 covering outside three points $p_i = \pi^{-1}(t_i),\ i = 1,2,3$. Moreover, assume that near these points π is topologically equivalent to $(\mathbb{C},0) \to (\mathbb{C},0),\ z \to z^2$. Then $\pi_1(E)$ is freely generated by two elements.

Proof (Continuation of the proof of Proposition 3.1). An element δ of the homotopy group $\pi_1(E,b)$ can be identified with $\gamma := \pi(\delta) \in \pi_1(\mathbb{C}\backslash\{t_1,t_2,t_3\},a)$ which has this property that the multivalued function $y := \sqrt{p(x)}$ along γ is one valued. The closed paths γ_1 and γ_2 in Figure 3.1 have this property, and hence are in the image of π, let us say $\pi(\delta_i) = \gamma_i,\ \ i = 1,2$. We claim that δ_1, δ_2 generate $\pi_1(E,b)$ freely. In order to see this, consider a system of 3 paths $\lambda_i,\ i = 1,2,3$ starting from a and ending at a point near t_i, turning around t_i anti-clockwise, and returning to a in the same way, and such that: 1. each path λ_i has no self intersection points, except at a which is the starting and end point 2. two distinct paths λ_i and λ_j meet only at a. This system of paths is also called a distinguished set of paths. The homotopoy group $\pi_1(\mathbb{C}\backslash\{t_1,t_2,t_3\},a)$ is freely generated by $\lambda_1,\lambda_2,\lambda_3$. The subgroup of $\pi_1(\mathbb{C}\backslash\{t_1,t_2,t_3\},a)$ consisting of elements γ such that $y|_\gamma$ is one valued, consists of elements of the form $\lambda_1^{n_1}\lambda_2^{n_2}\lambda_3^{n_3}\cdots$, where $n_1+n_2+n_3+\cdots$ is an even number. This is generated by $\gamma_1 := \lambda_1\lambda_3^{-1}, \gamma_2 := \lambda_2\lambda_3^{-1}$ and $\lambda_i^2,\ i = 1,2,3$. The last three elements gives us loops around p_i's and so they are homotopically zero in E. The proof of the fact that between δ_1 and δ_2 in $\pi_1(E,b)$ there are no relations is left to the reader in Exercise 3.4.

Theorem 3.1 *The set E as a toplogical space is a compact torus minus one point, see Figure 3.1.*

Proof. We need to show that $\bar{E} = E \cup \{O\}$ is a torus. This is an oriented compact surface. Its orientation comes from the canonical orientation of $\mathbb{C}$. It is known that the genus g of an oriented compact surfaces S (the number of holes) classifies them. Moreover, for a point $p \in S$, $H_1(S - \{p\}, \mathbb{Z}) \cong \mathbb{Z}^{2g}$. We use Proposition 3.1 and conclude that in our case $g = 1$, and hence, $\bar{E}$ is a torus.

Exercise 3.5 *Prove Theorem 3.1 in the framework of Exercise 3.4.*

Another important ingredient of $H_1(E,\mathbb{Z})$ is the skew symmetric intersection form or bilinear map

$$H_1(E,\mathbb{Z}) \times H_1(E,\mathbb{Z}) \to \mathbb{Z}. \tag{3.6}$$

For two paths $\delta_1, \delta_2 \in H_1(E,\mathbb{Z})$ we can assume that they intersect each other transversely. At each intersection point p we can attach a number $\varepsilon(p)$ which is $+1$ or -1, depending on whether near such a point δ_1 together with δ_2 give us the canonical orientation of E or not. Then we define

$$\langle \delta_1, \delta_2 \rangle := \sum_{p \in \text{ intersection of } \delta_1 \text{ and } \delta_2} \varepsilon(p).$$

Exercise 3.6 Show that the intersection pairing (3.6) is well-defined.

The generators δ_1 and δ_2 of $H_1(E,\mathbb{Z})$ which are explicitly constructed in the proof of Proposition 3.1, can be choosen in such a way that $\langle \delta_1, \delta_2 \rangle = -1$. The following proposition might have been the historical reason behind the interest on the upper half plane $\mathbb{H}$.

Proposition 3.2 *For any $\delta \in H_1(E,\mathbb{Z})$, $\delta \neq 0$ we have $\int_\delta \frac{dx}{y} \neq 0$. Moreover, let δ_1 and δ_2 be generators of $H_1(E,\mathbb{Z})$ with $\langle \delta_1, \delta_2 \rangle = -1$. Then the quotient*

$$\tau := \frac{\int_{\delta_1} \frac{dx}{y}}{\int_{\delta_2} \frac{dx}{y}}$$

has positive imaginary part.

Proof. It might be interesting to give an elementary proof of this proposition in terms of the mathematics of 19th century, and trace back its first appearance in the literature. One way to do this is to compute elliptic integrals in terms of Gauss hypergeometric function, see Section 3.9. We give the following not so elementary proof. The first statement follows from the second statement. First, we note that $\omega := \frac{dx}{y}$ restricted to E is holomorphic even at $y = 0$ and the infinity point O. Next we observe that

$$\sqrt{-1}\left(\int_{\delta_2} \omega \overline{\int_{\delta_1} \omega} - \int_{\delta_1} \omega \overline{\int_{\delta_2} \omega}\right) = \sqrt{-1} \int_{\overline{E}} \omega \wedge \bar{\omega} > 0,$$

where $\omega = \frac{dx}{y}$. The equality follows from Stokes theorem for the complement of δ_1 and δ_2 in E. In order to see this we observe that in local holomorphic coordinate system $z = x_1 + \sqrt{-1}x_2$ in E we have $\omega = dz$. Therefore,

$$\begin{aligned}\int_{\overline{E}} dz\wedge d\bar{z} &= \int_{\overline{E}} d(zd\bar{z})\\ &= \int_{\delta_2} zd\bar{z} - \int_{\delta_2}(z+\int_{\delta_1}\omega)d\bar{z} + \int_{\delta_1} zd\bar{z} - \int_{\delta_1}(z-\int_{\delta_2}\omega)d\bar{z}\\ &= \int_{\delta_2}\omega\overline{\int_{\delta_1}\omega} - \int_{\delta_1}\omega\overline{\int_{\delta_2}\omega}.\end{aligned}$$

Moreover,

$$\sqrt{-1}\omega\wedge\bar{\omega} = \sqrt{-1}dz\wedge d\bar{z} = dx_1\wedge dx_2,$$

whose integration over a domain is always positive.

Definition 3.1 The lattice of elliptic integrals is

$$\int_{H_1(E,\mathbb{Z})}\frac{dx}{y} = \mathbb{Z}\int_{\delta_1}\frac{dx}{y} + \mathbb{Z}\int_{\delta_2}\frac{dx}{y},$$

where δ_1,δ_2 is a basis of $H_1(E,\mathbb{Z})$ with $\langle\delta_1,\delta_2\rangle = -1$.

In the forthcoming sections we will mainly consider

$$E_t = E_{t_2,t_3} = \left\{(x,y)\in\mathbb{C}^2 \middle| y^2 = 4x^3 - t_2x - t_3\right\},\quad t\in \mathsf{S} \tag{3.7}$$

$$\mathsf{S} := \mathbb{C}^2 - \{(t_2,t_3)\in\mathbb{C}^2 \mid \Delta = 0\},\quad \Delta := t_2^3 - 27t_3^2. \tag{3.8}$$

The curve E_t is called an elliptic curve in the Weierstrass format.

3.3 Picard-Lefschetz theory

The appearnace of the group $\mathrm{SL}(2,\mathbb{Z})$ in Section 2.2 does not reveal the nature of this group in algebraic geometry. In this section we prove that this is actually the monodromy group of the family of elliptic curves in Weierstrass format and in order to explain this we sketch the Picard-Lefschetz theory and formula. The content of the present section might not be as elementary as the rest of the text, however, we hope that at least the reader get a taste of this beautiful topological theory.

Let us consider the family of elliptic curves in the Weierstrass format in (3.7). By Ehresmann's theorem the fibration $E_t, t\in\mathsf{S}$ is a C^∞ bundle over S, that is, it is locally trivial. This is the basic stone for the Picard-Lefschetz theory (see for instance [Mov21, Chapter 6] and the references therein). It gives us the following linear action:

$$\pi_1(\mathsf{S},b)\times H_1(E_b,\mathbb{Z})\to H_1(E_b,\mathbb{Z}),\ \ (\gamma,\delta)\mapsto h_\gamma(\delta),$$

where $b\in\mathsf{S}$ is a fixed point. The action of $\pi_1(\mathsf{S},b)$ on $H_1(E_b,\mathbb{Z})$ is called the monodromy action. We have in a natural way a morphism of groups

$$\pi_1(\mathsf{S},b) \to \mathrm{Aut}(H_1(E_b,\mathbb{Z})), \gamma \mapsto h_\gamma$$

and its image is called the monodromy group. The intuition behind $h_\gamma(\delta)$ is the following. Let us consider δ a closed path in E_b. As $b=\gamma(0)$ moves on the path γ to $\gamma(t)$, δ can be also lifted to δ_t in $E_{\gamma(t)}$. This lifiting is up to homotopy unique. For instance, if δ is an oval in the real elliptic curve E_b as in Figure 3.2 and γ is part of the real axis then this lifting can be seen in $\mathbb{R}^2$. As t varies from 0 to 1 we get a new closed path in $E_b := E_{\gamma(1)}$ which we call it $h_\gamma(\delta)$.

In order to calculate the monodromy group we proceed as follows: First we choose two cycles $\delta_1,\delta_2 \in H_1(E_b,\mathbb{Z})$. For instance, we can take as in the proof of Proposition 3.1, see also Figure 3.1. Picard-Lefschetz theory gives another recipe in order to choose such cycles. This is as follows. For the fixed parameter $t_2 \neq 0$, define the function f in the following way:

$$f:\mathbb{C}^2 \to \mathbb{C},\ (x,y) \mapsto -y^2+4x^3-t_2x.$$

The function f has two critical values given by $\tilde{t}_3,\ \check{t}_3 = \pm\sqrt{\frac{t_2^3}{27}}$. Let $b=(b_2,b_3)$. In a regular fiber $E_t = f^{-1}(b_3)$ of f one can take two cycles δ_1 and δ_2 such that $\langle \delta_2,\delta_1\rangle = 1$ and δ_1 (resp. δ_2) vanishes along a straight line connecting b_3 to $\tilde{t}_3$ (resp. $\check{t}_3$). These are called vanishing cycles. For the proof of the fact that δ_1,δ_2 form a basis of $H_1(E_b,\mathbb{Z})$ see [Mov21, Theorem 6.4].

The corresponding clockwise monodromy around the critical value $\tilde{t}_3$ (resp $\check{t}_3$) can be computed using the Picard-Lefschetz formula:

$$\delta_1 \mapsto \delta_1,\ \delta_2 \mapsto \delta_2+\delta_1\ (\text{ resp. } \delta_1 \mapsto \delta_1-\delta_2,\ \delta_2 \mapsto \delta_2).$$

It is not hard to see that the canonical map $\pi_1(\mathbb{C}\backslash\{\tilde{t}_3,\check{t}_3\},b) \to \pi_1(\mathsf{S},t)$, for $t_2 \neq 0$, induced by inclusion is an isomorphism of groups and so the image of the monodromy group written in the basis δ_1 and δ_2 is:

$$\langle A_1,A_2\rangle = \mathrm{SL}(2,\mathbb{Z}),\ \ \text{where } A_1 := \begin{bmatrix}1&0\\1&1\end{bmatrix},\ A_2 := \begin{bmatrix}1&-1\\0&1\end{bmatrix}.$$

Note that $g_1 := A_2^{-1}A_1^{-1}A_2^{-1} = \begin{bmatrix}0&1\\-1&0\end{bmatrix}$, $g_2 := A_1^{-1}A_2^{-1} = \begin{bmatrix}1&1\\-1&0\end{bmatrix}$ and $\mathrm{SL}(2,\mathbb{Z}) = \langle g_1,g_2 \mid g_1^2 = g_2^3 = -I\rangle$, where I is the identity 2×2 matrix. We conclude that

Theorem 3.2 *The monodromy morphism of groups $h:\pi_1(\mathsf{S},b) \to \mathrm{SL}(2,\mathbb{Z})$ is surjective.*

Proof. This follows from Exercise 3.4 and the computation of the monodromy as above.

Let us explain the above topological picture by the following one parameter family of elliptic curves:

$$E_\psi : y^2-4x^3+12x-4\psi = 0. \tag{3.9}$$

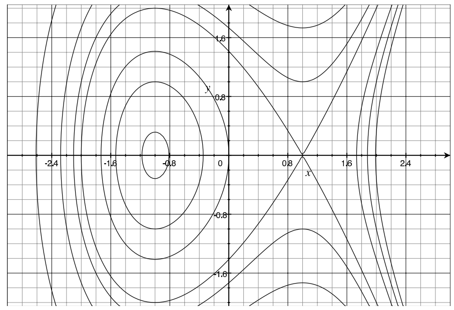


Fig. 3.2 Elliptic curves: $y^2-x^3+12x-4\psi,\ \psi=-1.9,-1,0,2,3,5,10$

For b a real number between 2 and -2 the elliptic curve E_b intersects the real plane $\mathbb{R}^2$ in two connected pieces which one of them is an oval and we can take it as δ_2 with the anti-clockwise orientation. In this example as ψ moves from -2 to 2, δ_2 is born from the point $(-1,0)$ and ends up in the α-shaped piece which is the intersection of E_2 with $\mathbb{R}^2$. The cycle δ_1 lies in the complex domain and it vanishes on the critical point $(1,0)$ as ψ moves to 2. It intersects each connected component of $E_b\cap\mathbb{R}^2$ once and it is oriented in such away that $\langle\delta_1,\delta_2\rangle=-1$.

Exercise 3.7 If we fix t_1 and t_3 and let t_2 vary then we get three critical curves. Describe the intersection number between the corresponding vanishing cycles $\delta_i,\ i=1,2,3$, linear relations between δ_i's and the monodromy around each critical fiber.

Exercise 3.8 Discuss the Picard-Lefschetz theory as above for the Legendre family of elliptic curves:

$$y^2=x(x-1)(x-\lambda)$$

More precisely, compute the monodromy group of this family and its index in $\mathrm{SL}(2,\mathbb{Z})$.

Exercise 3.9 Let X be a simply connected manifold and G be a group acting on X discretely and without non-identity stabilizers. Show that X/G is also a manifold and for any point $b\in X/G$ we have a canonical isomorphism $\pi_1(X/G,b)\cong G$.

3.4 Weierstrass uniformization theorem I

Let

$$t:=(g_2,g_3)=(60G_4(\Lambda),140G_6(\Lambda)),$$

where G_4 and G_6 are complex numbers defined in (2.13). From Theorem 2.9 it follows that $g_2^3-27g_3^2$ is never zero in $\mathbb{H}$, and so, t gives us a point in S, where S defined in (3.8). Note that we are not allowed to use the product formula for the discriminant in (2.42), as the proof of this goes through Section 3.5 which we have not yet been proved. Let E_t be the corresponding elliptic curve in (3.7).

Theorem 3.3 (Weierstrass uniformization theorem) *We have a well-defined map*

$$\begin{aligned} f &: \mathbb{C}/\Lambda \longrightarrow \overline{E}_t, \\ f(z) &:= [\wp(\Lambda, z) : \wp'(\Lambda, z) : 1], \\ f(0) &:= O = [0 : 1 : 0], \end{aligned} \tag{3.10}$$

which is an isomorphism of sets. Its inverse is given by:

$$\begin{aligned} f^{-1} &: \overline{E}_t \to \mathbb{C}/\Lambda, \\ f^{-1}(P) &:= \int_O^P \frac{dx}{y}. \end{aligned} \tag{3.11}$$

Actually, f is an isomorphism of Riemann surfaces. Moreover, we will see that $\overline{E}_t$ has a structure of a group and it is also a morphism of groups.

Proof. The fact that f is well-defined follows from the differential equation of the Weierstrass $\wp$ function, see Theorem 2.2. The heart of the proof is Exercise 2.8. Let $(x, y) \in E_t$. The elliptic function $\wp(z) - x$ has a pole of order two at $z = 0$. Therefore, it must have two zeros z_1, z_2 in $\mathbb{C}/\Lambda$. By the differential equation of $\wp$, we know that $\{\wp'(z_1), \wp'(z_2)\} = \{y, -y\}$. If $y \neq 0$ then there is exactly one of z_i's, let us say z_1, such that $\wp'(z_1) = y$. If $y = 0$ then $\wp$ has a zero of multiplicity 2 at z_1 and hence $z_1 = z_2$ in $\mathbb{C}/\Lambda$. This argument proves that f is one to one and surjective.

Let δ_1 and δ_2 be closed paths in $\mathbb{C}/\Lambda$ which are the images of the vectors $\omega_1, \omega_2 \in \mathbb{C}$ under the canonical map $\mathbb{C} \to \mathbb{C}/\Lambda$. We also use the same notation for their images in E_t under the map (3.10). We have

$$\int_{\delta_i} \frac{dx}{y} = \omega_i, \quad i = 1, 2. \tag{3.12}$$

In particular, the lattice of elliptic integrals for E_t as above is Λ. The integration in (3.11) can be interpreted in the following way. We take a path in the x-plane which connects the infinity to the x-coordinate of P. We also choose a branch of $\frac{dx}{y} = \frac{dx}{\sqrt{P(x)}}$. In geometric terms, that is to say, we connect P to the point at infinity O of E_t and we integrate $\frac{dx}{y}$ over this path. The map (3.11) is well-defined. By the first part f is surjective and it is enough to prove the equality $f^{-1} \circ f = \mathrm{Id}$ (if $f^{-1} \circ f = \mathrm{Id}$ and f is surjective then $f \circ f^{-1} = \mathrm{Id}$). This follows from

$$\begin{aligned} f^{-1} \circ f(\tilde{z}) &= \int_O^{f(\tilde{z})} \frac{dx}{y} \\ &= \int_0^{\tilde{z}} \frac{d\wp(z)}{\wp'(z)} = \int_0^{\tilde{z}} dz = \tilde{z}. \end{aligned}$$

Proposition 3.3 *Let E be an elliptic curve in the Weierstrass format and $P \in E$ be a point. We have*

$$x(P) = \left(\widehat{\int_O^P}\frac{dx}{y}\right)^{-2} + \sum\left(\left(\int_O^P \frac{dx}{y}\right)^{-2} - \left(\int_O^P \frac{dx}{y} - \widehat{\int_O^P}\frac{dx}{y}\right)^{-2}\right)$$

$$y(P) = (-2)\sum\left(\int_O^P \frac{dx}{y}\right)^{-3}$$

where $P = (x(P), y(P))$ and the sum is taken over all, except one, non-homotopic paths in E which connect O to P and $\int_O^P$ means integration over this path. The integration over the exceptional path is denoted by $\widehat{\int_O^P}$.

Proof. The proof follows from the equality $f \circ f^{-1} = \mathrm{Id}$. It is easy to see that the formula for $x(P)$ and $y(P)$ as above, doesn't depend on the choice of the exceptional path.

Remark 3.1 The Eisenstein series can be written in the following way. Let $E = E_{t_2,t_3}$ be an elliptic curve in the Weierstrass format. Let also $\delta_0 \in H_1(E,\mathbb{Z})$ be a primitive element, that is, it is not divisable by an integer. We have

$$b_k \cdot \zeta(2k) \cdot \sum_{\delta \text{ a monodromy of } \delta_0} \left(\int_\delta \omega\right)^{-2k} = t_k,\ \ k = 2,3, \tag{3.13}$$

where $b_2 = 60,\ b_3 = 140$ and the sum runs in all monodromies $\delta \in H_1(E,\omega)$ of δ_0 in the Weierstrass family. Since the monodromy group of the Weierstrass family is $\mathrm{SL}(2,\mathbb{Z})$, we can also take the sum over all primitive elements of $H_1(E,\mathbb{Z})$. The sum

$$\sum_{\delta \text{ a monodromy of } \delta_0,\ \ \langle\delta,\delta_0\rangle > 0} \left(\int_\delta \omega\right)^{-k},\ k \geq 3 \tag{3.14}$$

is related to the discussion in Exercise 2.19. These functions seem to give an embedding of the universal cover of $\mathbb{C}^2 \backslash \{27t_2^3 - t_3^2 = 0\}$ inside some affine space. The following sum might be also interesting for one parameter families of elliptic curves:

$$\sum_{\delta \text{ a clockwise monodromy of } \delta_0} \left(\int_\delta \omega\right)^{-k}. \tag{3.15}$$

3.5 Period domain and period map

Let L be the space of oriented lattices $\Lambda \subset \mathbb{C}$. After choosing a basis ω_1, ω_2 for Λ with $\langle \omega_1, \omega_2 \rangle = -1$, we have

$$\mathsf{L} \cong \mathrm{SL}(2,\mathbb{Z}) \Bigg\backslash \left\{ \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \Bigg|, \omega_1, \omega_2 \in \mathbb{C}\ \mathrm{Im}\left(\frac{\omega_1}{\omega_2}\right) > 0 \right\}, \tag{3.16}$$

where the action is given by multiplication of matrices. In this way, L is the first notion of a period domain that appears in the present text. In the following we will use the coordinate system (t_2,t_3) for $\mathbb{C}^2$ and $\mathsf{S} := \mathbb{C}^2 \backslash \{t_2^3 - 27t_3^2 = 0\}$.

Theorem 3.4 *The map given by*

$$\mathsf{p} : \mathsf{S} \to \mathsf{L}$$

$$(t_2,t_3) \mapsto \int_{H_1(E_t,\mathbb{Z})} \frac{dx}{y}$$

is well-defined and it is a biholomorphism which satisfies

$$\mathsf{p}(t_2k^{-4}, t_3k^{-6}) = k\mathsf{p}(t_2,t_3), \ \ k \in \mathbb{C}^*. \tag{3.17}$$

Its inverse p^{-1} *is given by the Eisenstein series:*

$$\Lambda \to (g_2(\Lambda), g_3(\Lambda)) = (60G_4(\Lambda), 140G_6(\Lambda)).$$

The map p is also called the period map. We will see other versions of the period map in Section 3.10 and Chapter 9.

Proof. The fact that p is well-defined follows from Proposition 3.2. The proof of the equality (3.17) is as follows. Let

$$\alpha : \mathbb{C}^2 \to \mathbb{C}^2,\ (x,y) \mapsto (k^2x, k^3y)$$

and $f = y^2 - 4x^3 + t_2x + t_3$. We have

$$k^{-6}\alpha^*(f) = y^2 - 4x^3 + t_2k^{-4}x + t_3k^{-6}.$$

This implies that α induces an isomorphism of elliptic curves

$$(E_{t_2k^{-4}, t_3k^{-6}}, k^{-1}\frac{dx}{y}) \to (E_{(t_2,t_3)}, \frac{dx}{y})$$

and the result follows. The equality (3.12) and the construction of δ_i's before this equality imply that $\mathsf{p} \circ \mathsf{p}^{-1} = \mathrm{Id}$, and so, p^{-1} is injective. We now prove that p^{-1} is surjective, or equivalently p is injective. We have the commutative diagram

$$\begin{tikzcd} \pi_1(\mathsf{S},b) \arrow[r, "\mathsf{p}_*"] \arrow[dr, "h"'] & \pi_1(\mathsf{L},\tilde{b}) \arrow[d, "f"] \\ & \mathrm{SL}(2,\mathbb{Z}) \end{tikzcd},$$

where $\tilde{b} := \mathsf{p}(b)$, p_* is the induced map in homotopy groups and h is the monodromy map discussed in Section 3.3. We write L as the quotient (3.16) and the action of $\mathrm{SL}(2,\mathbb{Z})$ in this quotient has no non-identity stabilizer. If it has, let us say $\omega = [\omega_1, \omega_2]^{\mathrm{tr}}$ with $A \in \mathrm{SL}(2,\mathbb{Z}), A \neq I$ and $A\omega = \omega$, then $A\bar{\omega} = \bar{\omega}$ too, and so, the 2×2 matrix formed by ω and $\bar{\omega}$ has zero determinant. This implies that $\frac{\omega_2}{\omega_1}$ is a real number which is a contradiction. Therefore, by Exercise 3.9 f is an isomorphism: $\pi_1(\mathsf{L},\tilde{b}) \cong \mathrm{SL}(2,\mathbb{Z})$. Later, in Proposition 3.6 we will compute the derivative of the period map p, and in particular, we will prove that p is a local biholomorphism. This implies that p is a covering and if it is not injective then $\pi_1(\mathsf{S},b) \overset{\mathsf{p}_*}{\to} \pi_1(\mathsf{L},\tilde{b})$ is injective but not surjective. Since h is surjective (Theorem 3.2), p_* is also surjective, therefore, p must be injective.

Proof (Proof of Theorem 2.7). We give two proofs. The first uses the fact that the inverse of j is a period map. The second proof will be give in Section 3.10. Theorem 3.4 implies that we have a bijection $\mathsf{S}/\mathbb{C}^* \to \mathsf{L}/\mathbb{C}^*$. Under the identifications

$$\mathsf{S}/\mathbb{C}^* \cong \mathbb{C}, \ \ (t_2,t_3) \mapsto \frac{1728t_2^3}{t_2^3 - 27t_3^2},$$
$$\mathsf{L}/\mathbb{C}^* \cong \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}, \ \ \begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix} \mapsto \frac{\omega_1}{\omega_2},$$

the inverse of this map is the map $j : \mathbb{H}/\mathrm{SL}(2,\mathbb{Z}) \to \mathbb{C}$ in Theorem 2.7.

Proof (Proof of Theorem 2.5). If there is a polynomial $P \in \mathbb{C}[X,Y]$ such that $P(E_4,E_6) = 0$ then the image of the inverse p^{-1} of the period map lies in the curve $P(X,Y) = 0$ (up to multiplication of X and Y with constants). This contradicts the fact that p is a biholomorphism. We regard modular forms as functions on the space of lattices L. Under the bijection p any modular form of weight k becomes a holomorphic function f in S with the property

$$f(t_2\lambda^4, t_3\lambda^6) = \lambda^k f(t_2,t_3), \ \ \lambda \in \mathbb{C}^*. \tag{3.18}$$

We use the growth condition of f and prove that f is a polynomial of degree k in the weighted ring $\mathbb{C}[t_2,t_3]$, weight $(t_2) = 4$, weight $(t_3) = 6$. Since the lattices associated to τ and $\tau + 1$ are the same, we get a map

$$\mathbb{D} - \{0\} \to \mathsf{L}, \ q = e^{2\pi i\tau} \mapsto \mathbb{Z}\tau + \mathbb{Z},$$

where $\mathbb{D}$ is the disc of radius 1 and center 0 in $\mathbb{C}$. We compose it with p^{-1} and get the map

$$i : \mathbb{D} \to \mathbb{C}^2, \ \ q \mapsto (g_2(q), g_3(q)).$$

Note that this map is even defined at $0 \in \mathbb{D}$ and the image of 0 is in the discriminant locus $\{\Delta = 0\}$. The growth condition of f implies that $f|_{\mathrm{Im}(i)}$ extends as a holomorphic function at $i(0)$. Now, under the $\mathbb{C}^*$-action in $\mathbb{C}^2$ any point near to $i(0)$ is equivelent to a point in $\mathrm{Im}(i)$. This implies that f is bounded near $i(0)$. By Riemann's extension theorem, see Exercise 3.10, f extends to a holomorphic function

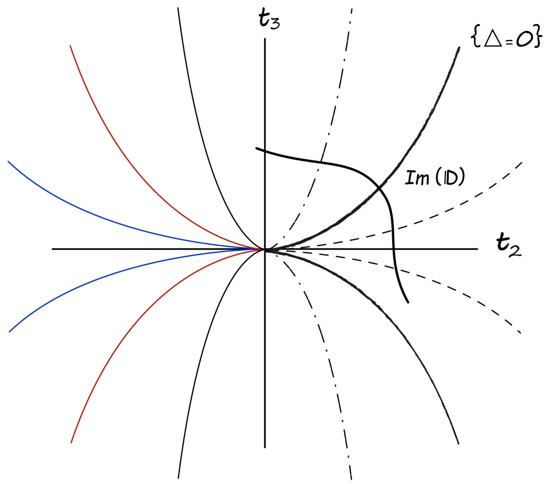


Fig. 3.3 Discriminant

in $\mathbb{C}^2\backslash\{(0,0)\}$ and by Hartogs extension theorem, see Exercise 3.11, we conclude that f is holomorphic in $\mathbb{C}^2$. We write the Taylor series of f at $(0,0)$ and (3.18) implies the desired polynomiality of f, see Figure 3.3.

Let us now prove the second part which claims that if f is defined over $\mathbb{Q}$ then P has rational coefficients. Let $a=\dim_{\mathbb{C}}[E_4,E_6]_k$. We choose a basis of monomials $m_i=E_4^{\alpha_i}E_6^{\beta_i}$, $i=1,2,\ldots,a$ for $\mathbb{C}[E_4,E_6]_k$ and define A to be the $a\times n$ matrix such that A_{ij} is the j-th Fourier coefficient of m_i. By Proposition 2.8, for $n>\frac{k}{12}$ the rows of A are linearly independent. Therefore, for $n>a$ (which is automatic by Exercise 2.25) we can choose an $a\times a$ minor B of A with non-zero determinant. We can derive from the equality $f=P(E_4,E_6)$ another equality $C_f=C_PB$, where the entries of the $1\times a$ matrix C_f are collected from the Fourier coefficients of f and the entries of the $1\times a$ matrix C_P is formed by the coefficients of P. We have $C_P=C_fB^{-1}$ and the proof is finished.

Exercise 3.10 Let us take a connected neighborhood U of $0\in\mathbb{C}^2$, remove the x-axis $\{y=0\}$ from it and call it $U-\{y=0\}$. Show that any bounded holomorphic function in $U-\{y=0\}$ extends to a holomorphic function in U. The same affirmation is valid if instead of boundedness of f we assume that f extends to a neighborhood of a point in $\{y=0\}$. Hint: This follows from Riemann's extension theorem, see [Gun90, Vol. I, Chapter D].

Exercise 3.11 Let us take a connected neighborhood U of $0\in\mathbb{C}^2$, remove 0 from it and call it $U-\{0\}$. Show that any holomorphic function in $U-\{0\}$ extends to a holomorphic function in U. Hint: This is Hartogs extension theorem, see [Gun90, Vol. I, Chapter D].

3.6 Gauss-Manin connection matrix

Elliptic integrals which depend on a parameter satisfy linear differential equations which are called Picard-Fuchs equations. In many examples elliptic integrals depend on more than one parameter and the description of the differential equations of such integrals is done under the name Gauss-Manin connection. In this section we explain these concepts for the Weierstrass family of elliptic curves. For more examples and the algorithms which compute such differential equations see [Mov21, Chapter 12]. For our main statement in this section we need the follwoing:

Exercise 3.12 For an arbitrary $\delta \in H_1(E_{t_2,t_3}, \mathbb{Z})$, verify the following equalities

$$\begin{aligned}
\int_\delta \frac{x^2dx}{y} &= \frac{1}{12}t_2 \int_\delta \frac{dx}{y}, \\
\int_\delta \frac{x^3dx}{y} &= \frac{3}{20}t_2 \int_\delta \frac{xdx}{y} + \frac{1}{10}t_3 \int_\delta \frac{dx}{y}, \\
\int_\delta \frac{x^4dx}{y} &= \frac{1}{7}t_3 \int_\delta \frac{xdx}{y} + \frac{5}{336}t_2^2 \int_\delta \frac{dx}{y}, \\
\int_\delta \frac{x^5dx}{y} &= \frac{7}{240}t_2^2 \int_\delta \frac{xdx}{y} + \frac{1}{30}t_2t_3 \int_\delta \frac{dx}{y}.
\end{aligned}$$

Hint: Restricted to the elliptic curve E_{t_2,t_3} we have the equality

$$d(x^a y) = \left((4a+6)x^{a+2} - (a+\frac{1}{2})t_2x^a - at_3x^{a-1} \right) \frac{dx}{y}.$$

Proposition 3.4 *Let us consider the Weierstrass family of elliptic curves $E_t : y^2 = p(x),\ \ p := 4x^3 - t_2x - t_3$. We have*

$$\begin{pmatrix} d\left(\int \frac{dx}{\sqrt{p(x)}}\right) \\ d\left(\int \frac{xdx}{\sqrt{p(x)}}\right) \end{pmatrix} = \begin{pmatrix} -\frac{1}{12}\frac{d\Delta}{\Delta}, & \frac{3}{2}\frac{\alpha}{\Delta} \\ -\frac{1}{8}t_2\frac{\alpha}{\Delta}, & \frac{1}{12}\frac{d\Delta}{\Delta} \end{pmatrix} \begin{pmatrix} \int \frac{dx}{\sqrt{p(x)}} \\ \int \frac{xdx}{\sqrt{p(x)}} \end{pmatrix}, \tag{3.19}$$

where

$$\Delta := 27t_3^2 - t_2^3,\ \alpha := 3t_3dt_2 - 2t_2dt_3.$$

The above data is the Gauss-Manin connection of the family of elliptic curves $y^2 = p(x)$ before the invention of cohomology theories. The manipulations needed in its proof are widely present in the works of many mathematicians of 19th century.

Definition 3.2 The two by two matrix A in (3.19) is called the Gauss-Manin connection matrix of the family of elliptic curves $y^2 = 4x^3 - t_2x - t_3$ and written in $\frac{dx}{y}, \frac{xdx}{y}$.

The period matrix is

$$\mathsf{P} := \begin{bmatrix} \int_{\delta_1} \frac{dx}{y} & \int_{\delta_1} \frac{xdx}{y} \\ \int_{\delta_2} \frac{dx}{y} & \int_{\delta_2} \frac{xdx}{y} \end{bmatrix} \tag{3.20}$$

and it follows from Proposition 3.4 that it satisfies the following differential equation:

$$d\mathsf{P}^{\mathrm{tr}} = A\mathsf{P}^{\mathrm{tr}}. \tag{3.21}$$

The first application of Proposition 3.4 is the following:

Proposition 3.5 *Let E be an elliptic curve in the Weierstrass format $y^2 = 4x^3 - t_2x - t_3,\ t_2, t_3 \in \mathbb{C},\ 27t_3^2 - t_2^3 \neq 0$ and let δ_1, δ_2 be a basis of $H_1(E, \mathbb{Z})$ with $\langle \delta_1,\ \delta_2 \rangle = -1$. We have*

$$\int_{\delta_2} \frac{dx}{y} \int_{\delta_1} \frac{xdx}{y} - \int_{\delta_1} \frac{dx}{y} \int_{\delta_2} \frac{xdx}{y} = 2\pi\sqrt{-1}. \tag{3.22}$$

Equation (3.22) is called the Legendre relations between elliptic integrals.

Proof. We have to show that $\det(\mathsf{P}) = -2\pi i$. From (3.21) it follows that P satisfies

$$d\left(\det(\mathsf{P})\right) = \mathrm{Trace}(A)\det(\mathsf{P}),$$

and from the explicit expression of A in (3.19) we know that $\mathrm{Trace}(A) = 0$. Therefore, $\det(\mathsf{P})$ is a constant independent of t_2, t_3. It remains to compute it for a value of t_2, t_3. We will do this for $t_2 = 12, t_3 = 8$ in Section 3.9.

Proposition 3.6 *The period map $\mathsf{p} : \mathsf{S} \to \mathsf{L}$ defined in Theorem 3.4 is a local biholomorphism.*

Proof. We need to prove that the derivative of p at any point is an isomorphism. For a basis $\delta_1, \delta_2 \in H_1(E_{t_2,t_3}, \mathbb{Z})$, it is enough to verify this for the multivalued function

$$\tilde{\mathsf{p}} : \mathsf{S} \to \mathbb{C}^2,\ \ (t_2, t_3) \mapsto \left(\int_{\delta_1} \frac{dx}{y}, \int_{\delta_2} \frac{dx}{y} \right).$$

Let $\mathsf{P} := \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ be the period matrix and A be the Gauss-Manin connection matrix in (3.19). Let us write it as $A = A_2 dt_2 + A_3 dt_3$. By Proposition 3.4 the derivative of $\tilde{\mathsf{p}}$ at a point is

$$\begin{bmatrix} \frac{\partial x_1}{\partial t_2} & \frac{\partial x_3}{\partial t_2} \\ \frac{\partial x_1}{\partial t_3} & \frac{\partial x_3}{\partial t_3} \end{bmatrix} = B \begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix}, \quad B = \begin{bmatrix} -\frac{1}{12}\frac{-3t_2^2}{\Delta} & \frac{3}{2}\frac{3t_3}{\Delta} \\ -\frac{1}{12}\frac{2\cdot 27t_3}{\Delta} & \frac{3}{2}\frac{-2t_2}{\Delta} \end{bmatrix},$$

where the first and second row of B are the first row of A_2 and A_3, respectively. Using Proposition 3.5 we conclude that the determinant of this matrix is not zero.

For the proof of Proposition 3.4 we need the following. We first write down the equation which says that Δ is the resultant of p and p' (the derivation of p with respect to x):

$$\Delta = -p' \cdot a_1 + p \cdot a_2,$$

where

$$a_1 = -36x^4 + 15t_2x^2 - t_2^2,\ \ a_2 = -108x^3 + 27t_2x - 27t_3$$

see Section 4.6 for an algorithm of this computation.

Proposition 3.7 *For a polynomial $A \in \mathbb{C}[x]$ we have the following equality restricted to the elliptic curve E_{t_2,t_3}:*

$$\frac{Adx}{y^3} = \frac{1}{\Delta}\left(a_2A - 2\frac{\partial}{\partial x}(Aa_1)\right)\frac{dx}{y} + \frac{1}{\Delta}d\left(\frac{2Aa_1}{y}\right).$$

Proof.

$$\begin{aligned}\frac{Adx}{y^3} &= \frac{1}{\Delta}\frac{A(-p'a_1+pa_2)dx}{py}\\ &= \frac{1}{\Delta}\left(a_2A\frac{dx}{y} - \frac{Aa_1}{y}\frac{dp}{p}\right)\\ &= \frac{1}{\Delta}(a_2A\frac{dx}{y} + 2Aa_1d(\frac{1}{y}))\end{aligned}$$

Proof (Proof of Proposition 3.4). The proof is a mere calculation which is classical and can be found in ([Sas74] p. 304, [Sai01]). In the following we write $y = \sqrt{P(x)}$ and eliminate the integral sign from our computations. P. Deligne in a personal communication (January 31, 2016) writes "When reading old literature, I find it useful to mentally replace "integral" by "differential form". It is often what they are really concerned with, even if they had not the language to say so. I doubt that using the "integral" terminology helps".

We explain only the calculation of $\frac{\partial}{\partial t_3}(\frac{dx}{y})$. We have

$$\begin{aligned}\frac{\partial}{\partial t_3}(\frac{dx}{y}) &= \frac{1}{2}\frac{dx}{py}\\ &= \frac{1}{\Delta}\frac{(-p'a_1+pa_2)dx}{2py} = \frac{1}{\Delta}(\frac{1}{2}a_2 - a_1')\frac{dx}{y}\\ &= \frac{1}{\Delta}(90x^3 - \frac{33}{2}t_2x - \frac{27}{2}t_3)\frac{dx}{y}\\ &= \frac{1}{\Delta}\left(-\frac{9}{2}t_3\frac{dx}{y} - 3t_2\frac{xdx}{y}\right).\end{aligned}$$

The equalities are written modulo exact forms, whose integration over closed paths are zero. Note that in the third equality above we use $y^2 = p(x)$ and Proposition 3.7.

Exercise 3.13 In a similar way as in the proof of Proposition 3.4 calculate $\frac{\partial}{\partial t_i}(\frac{x^jdx}{y}),\ j = 1,2,\ \ i = 2,3.$

We will write (3.19) in the following format

$$\begin{pmatrix} \nabla(\frac{dx}{y}) \\ \\ \nabla(\frac{xdx}{y}) \end{pmatrix} = \begin{pmatrix} -\frac{1}{12}\frac{d\Delta}{\Delta}, & \frac{3}{2}\frac{\alpha}{\Delta} \\ \\ -\frac{1}{8}t_2\frac{\alpha}{\Delta}, & \frac{1}{12}\frac{d\Delta}{\Delta} \end{pmatrix} \begin{pmatrix} \frac{dx}{y} \\ \\ \frac{xdx}{y} \end{pmatrix} \tag{3.23}$$

even though we have not defined what is this Gauss-Manin connection ∇. The reader might simply put integral sign behind differential forms and replace ∇ with the differential d in parameters, in order to go back to the hisorical version of Gauss-Manin connection.

It is observed in [CMY21] that the inverse of the Gauss-Manin connection is simpler than the Gauss-Manin connection itself. This is also the case in our main example above

$$A = \begin{bmatrix} -4t_2 & -\frac{72t_3}{t_2} \\ 6t_3 & 4t_2 \end{bmatrix}^{-1} dt_2 + \begin{bmatrix} -6t_3 & -4t_2 \\ \frac{t_2^2}{3} & 6t_3 \end{bmatrix}^{-1} dt_3.$$

Despite the fact the proof of Proposition 3.4 is a tedious elementary calculus manipulation that can be performed by hand, one might seek for general algorithms and their implementations which does this job for us. This is done in [Mov21, Chapter 12].

Exercise 3.14 Show that the Gauss-Manin connection matrix of the family of elliptic curves $y^2 - 4(x-t_1)(x-t_2)(x-t_3) = 0$ and written in $\frac{dx}{y}, \frac{xdx}{y}$ is

$$\frac{dt_1}{2(t_1-t_2)(t_1-t_3)} \begin{bmatrix} -t_1 & 1 \\ t_2t_3 - t_1(t_2+t_3) & t_1 \end{bmatrix} + \tag{3.24}$$

$$\frac{dt_2}{2(t_2-t_1)(t_2-t_3)} \begin{bmatrix} -t_2 & 1 \\ t_1t_3 - t_2(t_1+t_3) & t_2 \end{bmatrix} + \frac{dt_3}{2(t_3-t_1)(t_3-t_2)} \begin{bmatrix} -t_3 & 1 \\ t_1t_2 - t_3(t_1+t_2) & t_3 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} -t_1 & 1 \\ -(t_1t_2+t_1t_3-t_2t_3) & t_1 \end{bmatrix}^{-1} dt_1 + \frac{1}{2} \begin{bmatrix} -t_2 & 1 \\ -(t_1t_2-t_1t_3+t_2t_3) & t_2 \end{bmatrix}^{-1} dt_2$$

$$+\frac{1}{2} \begin{bmatrix} -t_3 & 1 \\ (t_1t_2-t_1t_3-t_2t_3) & t_3 \end{bmatrix}^{-1} dt_3.$$

Exercise 3.15 Show that the Gauss-Manin connection matrix of the family of elliptic curves $y^2 - (x-t_1)(x-t_2)(x-t_3)(x-t_4) = 0$ and written in $\frac{dx}{y}, \frac{xdx}{y}, \frac{x^2dx}{y}$ is

$$\begin{bmatrix} -2t_1 & 2 & 0 \\ 0 & -2t_1 & 2 \\ (t_1t_2t_3+t_1t_2t_4+t_1t_3t_4-t_2t_3t_4) & -(2t_1t_2+2t_1t_3+2t_1t_4) & (t_1+t_2+t_3+t_4) \end{bmatrix}^{-1} dt_1 + \cdots$$

where $\cdots$ mean that we change the role of t_i with t_1 and write the Gauss-Manin connection matrix of dt_i using the symmetry between t_i's.

Exercise 3.16 The j-invariant of the the family of elliptic curves

$$E:\, y^2+xy-x^3+\frac{36}{j-1728}x+\frac{1}{j-1728}=0, \quad j\neq 0,1728,$$

is the parameter j. Note that this family misses the elliptic curve with $j=1728$. The Gauss-Manin connection matrix of E in the basis $[\frac{dx}{2y+x},\ \frac{dx}{2y+x}]^{\mathrm{tr}}$ is

$$\frac{1}{j(j-1728)}\begin{pmatrix} -432 & -60 \\ -(j-1728) & 432 \end{pmatrix}.$$

Other families with the same property are $y^2=x^3+x^2-\frac{1}{j}$ and $y^2+xy=x^3-\frac{1}{j}$ (see [Hus04, Proposition 5.3, page 76]).

3.7 Ramanujan and Darboux-Haphen vector field

Our main observation in this section is the following:

Proposition 3.8 *In the parameter space of the family of elliptic curves $y^2=4(x-t_1)^3-t_2(x-t_1)-t_3$ there is a unique vector field R, such that*

$$\nabla_{\mathsf{R}}(\frac{dx}{y})=-\frac{xdx}{y},\ \nabla_{\mathsf{R}}(\frac{xdx}{y})=0. \tag{3.25}$$

The vector field R is given by

$$\mathsf{R}=(t_1^2-\frac{1}{12}t_2)\frac{\partial}{\partial t_1}+(4t_1t_2-6t_3)\frac{\partial}{\partial t_2}+(6t_1t_3-\frac{1}{3}t_2^2)\frac{\partial}{\partial t_3}. \tag{3.26}$$

Proof. This follows from the computation of Gauss-Main connection in Proposition 3.4 and explicit calculations.

The vector field R is called the Ramanujan vector field. Let us consider the family of elliptic curves considered in Exercise 3.15. . A vector field with the properties (3.25) is given by

$$\mathrm{H}=(t_1(t_2+t_3)-t_2t_3)\frac{\partial}{\partial t_1}+(t_2(t_1+t_3)-t_1t_3)\frac{\partial}{\partial t_2}+(t_3(t_1+t_2)-t_1t_2)\frac{\partial}{\partial t_3}.$$

This is called the Darboux-Halphen vector field.

Exercise 3.17 Perform the calculations leading to a proof of Proposition 3.8. Perform also similar calculations leading to the Darboux-Halphen vector field.

3.8 Picard-Fuchs equation

Before the invention of Gauss-Manin connection, the term Picard-Fuchs equation was mainly used to denote the differential equation of elliptic and abelian integrals. In this section we explain this with explicit families of elliptic curves.

Let us consider the following one parameter family of elliptic curves $E_\psi : y^2 - 4x^3 + 12x - 4\psi = 0$ discussed in Section 3.3. We also consider a basis δ_1, δ_2 of $H_1(E_\psi, \mathbb{Z})$, It follows from Proposition 3.4 that the matrix

$$Y = \begin{bmatrix} \int_{\delta_1} \frac{dx}{y} & \int_{\delta_2} \frac{dx}{y} \\ \int_{\delta_1} \frac{xdx}{y} & \int_{\delta_2} \frac{xdx}{y} \end{bmatrix} \tag{3.27}$$

forms a fundamental system of the linear differential equation:

$$Y' = \frac{1}{\psi^2 - 4} \begin{bmatrix} \frac{-1}{6}\psi & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{6}\psi \end{bmatrix} Y, \tag{3.28}$$

that is, any solution of (3.28) is a linear combination of the columns of Y with $\mathbb{C}$ coefficients. This example shows a little bit the historical aspects of the Gauss-Manin connection. From (3.28) it follows that the elliptic integral $I := \int_\delta \frac{dx}{y}$ (resp. $I := \int_\delta \frac{xdx}{y}$) for all $\delta \in H_1(E_\psi, \mathbb{Z})$ satisfies the differential equation

$$\frac{5}{36} I + 2\psi I' + (\psi^2 - 4) I'' = 0 \qquad (\text{ resp. } \frac{-7}{36} I + 2\psi I' + (\psi^2 - 4) I'' = 0) \tag{3.29}$$

where $' = \frac{\partial}{\partial \psi}$. These are called Picard-Fuchs equations. We give more (historical) examples of Picard-Fuchs equations.

Exercise 3.18 Prove that for the Legendre, resp. Weierstrass, family of elliptic curves $E_t : y^2 - x(x-1)(x-t)$, resp. $E_t : y^2 - x^3 + 3tx - 2t$, the periods $I(t) := \int_{\delta_t} \frac{dx}{y},\ \delta_t \in H_1(E_t, \mathbb{Z})$ satisfy the Picard-Fuchs equation $L(I) = 0$, where

$$L := 1 + (8t - 4)\frac{\partial}{\partial t} + 4t(t-1)\frac{\partial^2}{\partial^2 t}, \tag{3.30}$$

resp.

$$L := (27t + 4) + 144t(2t - 1)\frac{\partial}{\partial t} + 144t^2(t-1)\frac{\partial^2}{\partial^2 t}, \tag{3.31}$$

see [KZ01] for some discussion on these Picard-Fuchs equations.

Exercise 3.19 For the family of elliptic curves $y^2 = (1 - x^2)(1 - k^2 x^2)$ (Jacobi family) we must choose the differential forms $\frac{dx}{y}, \frac{xdx}{y}$ and $\frac{x^2 dx}{y}$ in order to compute the Gauss-Manin connection matrix. This is

$$\begin{bmatrix} \frac{k}{k^2-1} & 0 & \frac{k}{k^2-1} \\ 0 & -\frac{1}{k} & 0 \\ -\frac{1}{k^3-k} & 0 & -\frac{k^2-2}{k^3-k} \end{bmatrix}.$$

The Picard-Fuchs equation of these three differential forms are respectively:

$$\begin{aligned} &k+(3k^2-1)\partial+(k^3-k)\partial^2 \\ &1+k\partial \\ &3k+(5k^2-3)\partial+(k^3-k)\partial^2 \end{aligned}$$

where $\partial=\frac{\partial}{\partial k}$. In particular, $\int_\delta \frac{xdx}{y}=\frac{c_\delta}{k}$, where c_δ is a constant which only depends on δ. In [Hus04, page 92] we also find the family $y^2=(1-\sigma^2x^2)(1-\frac{x^2}{\sigma^2})$ which is called the Jacobi family.

3.9 Hypergeometric functions

In this section we compute explicitly elliptic integrals. In most of our discussion we have taken the domain of integration any $\delta\in H_1(E,\mathbb{Z})$ without specifying it. For the computation of elliptic integrals we need to fix such cycles. Let us consider the family of elliptic curves (3.9) and $\delta_1,\delta_2\in H_1(E_\psi,\mathbb{Z})$ described in Section 3.3: for ψ a real number between -2 and 2, δ_2 is the closed curve inside $E_\psi\cap\mathbb{R}^2$ which encircles $(-1,0)$ and $\delta_1\in H_1(E_\psi,\mathbb{Z})$ vanishes on the nodal point $(1,0)$. Whenever we need to emphasize that $\delta_i, i=1,2$ depends on ψ we write $\delta_i=\delta_{i,\psi}$. Recall that the cycles $\delta_i,\ i=1,2$ form a basis of $H_1(E_\psi,\mathbb{Z})$ and $[\int_{\delta_i}\frac{dx}{y},\int_{\delta_i}\frac{xdx}{y}]^{\mathrm{tr}},\ \ i=1,2$ are solutions of (3.28). We make a linear transformation

$$z=\frac{\psi+2}{4}$$

which sends the singularities $\psi=-2,2$ of (3.28) to $z=0,1$. We write (3.28) in the variable z. The integrals $\int_{\delta_2}\frac{dx}{y}$ and $\int_{\delta_2}\frac{xdx}{y}$ are holomorphic around $z=0$. We write $X:=[\int_{\delta_2}\frac{dx}{y},\int_{\delta_2}\frac{xdx}{y}]^{\mathrm{tr}}$ as a formal power series in z $X=\sum_{i=0}^{\infty}Y_iz^i$, substitute it in (3.28) and obtain a recursive formula for Y_i's. The constant term turns out to be of the form $Y_0=[a_0,-a_0]^{\mathrm{tr}}$, where a_0 is the value of $\int_{\delta_2}\frac{dx}{y}$ at $\psi=-2$. This must be calculated separately. The intersection of the elliptic curve $E_\psi,-2<\psi<2$ with the real plane $\mathbb{R}^2$ has two connected component, one of them is δ_2 and the other $\tilde\delta_2$ is a closed path in E_ψ which crosses the point at infinity $[0;1;0]$. It turns out that if we give the clockwise orientation to $\tilde\delta_2$ then it is homotopic to δ_2 in E_ψ and

$$a_0=\int_{\tilde\delta_2}\frac{dx}{y}\bigg|_{\psi=-2}=2\int_2^\infty\frac{dx}{2(x+1)\sqrt{x-2}}=\frac{2\mathrm{tang}^{-1}(\frac{\sqrt{x-2}}{\sqrt{3}})}{\sqrt{3}}\bigg|_2^\infty=\frac{\pi}{\sqrt{3}}.$$

Note that for ψ a real number near -2, by Stokes formula we have $\int_{\delta_2}\frac{dx}{y}=\int_{\Delta_2}\frac{dx\wedge dy}{y^2}>0$, where Δ_2 is the region in $\mathbb{R}^2$ bounded by δ_2, and so we already knew that $a_0\geq 0$. This explain the fact that why δ_2 is homotopic to clockwise oriented $\tilde{\delta}_2$. The result of all these calculations is:

$$\int_{\delta_2}\frac{dx}{y}=\frac{\pi}{\sqrt{3}}F(\frac{1}{6},\frac{5}{6},1|\frac{\psi+2}{4}), \tag{3.32}$$

$$\int_{\delta_2}\frac{xdx}{y}=-\frac{\pi}{\sqrt{3}}F(\frac{-1}{6},\frac{7}{6},1|\frac{\psi+2}{4}),$$

where

$$F(a,b,c|z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n,\ c\notin\{0,-1,-2,-3,\dots\}, \tag{3.33}$$

is the Gauss hypergeometric function and $(a)_n:=a(a+1)(a+2)\cdots(a+n-1)$.

Let us now calculate the integrals $\int_{\delta_1}\frac{x^i dx}{y},\ i=0,1$. We have the isomorphism $E_{-\psi}\to E_{\psi},\ (x,y)\mapsto(-x,iy)$ which sends the cycle $\delta_{2,-\psi}$ to $\delta_{1,\psi}$ and $\delta_{1,-\psi}$ to $-\delta_{2,\psi}$. This gives us the equalities:

$$\int_{\delta_{1,\psi}}\frac{x^j dx}{y}=(-1)^j i\int_{\delta_{2,-\psi}}\frac{x^j dx}{y}$$

Finally, we have calculated all the entries of the Fundamental system Y in (3.27):

$$Y=\begin{bmatrix}\frac{\pi i}{\sqrt{3}}F(\frac{1}{6},\frac{5}{6},1|\frac{-\psi+2}{4}) & \frac{\pi}{\sqrt{3}}F(\frac{1}{6},\frac{5}{6},1|\frac{\psi+2}{4})\\ \frac{\pi i}{\sqrt{3}}F(\frac{-1}{6},\frac{7}{6},1|\frac{-\psi+2}{4}) & -\frac{\pi}{\sqrt{3}}F(\frac{-1}{6},\frac{7}{6},1|\frac{\psi+2}{4})\end{bmatrix}$$

The monodromy around $z=0$ leaves δ_2 invariant and takes δ_1 to $\delta_1+\delta_2$. From this it follows that for a fixed complex number a:

$$\int_{\delta_1}\frac{dx}{y}=\frac{\ln(az)}{2\pi i}(\int_{\delta_2}\frac{dx}{y})+\frac{1}{2i\sqrt{3}}f(z)=\frac{1}{2i\sqrt{3}}(F(\frac{1}{6},\frac{5}{6},1|z)\ln(az)+f(z)), \tag{3.34}$$

where f is a one valued function in a neighborhood of $z=0$. From Exercise 3.20, Item 4 it follows that f is holomorphic at $z=0$. We choose a in such a way that the value of f at $z=0$ is 0. This is equivalent to the following formula for a:

$$a=\exp(2\pi i(\lim_{z\to 0}\int_{\delta_1}\frac{dx}{y}-\frac{\ln z}{2\pi i}\int_{\delta_2}\frac{dx}{y})).$$

According to Exercise 3.20, Item 4 we have

$$a=\frac{1}{432}.$$

We write $f=\sum_{i=1}^{\infty} f_n z^n$ and substitute (3.34) in the Picard-Fuchs equation (3.32) and we obtain the following recursion for f_n's:

$$f_{n+1}=\frac{(n-\frac{1}{6})(n-\frac{5}{6})}{(n+1)^2}f_n+\frac{(\frac{1}{6})_n(\frac{5}{6})_n}{(n!)^2}\frac{2n+1}{(n+1)^2}-\frac{2}{n+1}\frac{(\frac{1}{6})_{n+1}(\frac{5}{6})_{n+1}}{((n+1)!)^2},\ f_0=0.$$

We will need the value $f_1=\frac{13}{18}$.

Exercise 3.20 1. Deduce (3.29) from (3.28).
2. The integrals $\int_{\delta_2}\frac{dx}{y}$ and $\int_{\delta_2}\frac{xdx}{y}$ are holomorphic at $z=0$.
3. Do the details of the calculations which lead to the equalities (3.32).
4. Prove

$$\lim_{z\to 0}\ \frac{\pi i}{\sqrt{3}}F(\frac{1}{6},\frac{5}{6},1|1-z)-\frac{\pi}{\sqrt{3}}F(\frac{1}{6},\frac{5}{6},1|z)\frac{\ln(z)}{2\pi i}=\frac{\ln(432)}{2\pi i}.$$

Remark 3.2 Historically, the following identities are proved first:

$$\int_{-\infty}^{0}\frac{dx}{\sqrt{x(x-1)(x-\lambda)}}=F(\frac{1}{2},\frac{1}{2},1|1-\lambda),$$

$$\int_{1}^{\infty}\frac{dx}{\sqrt{x(x-1)(x-\lambda)}}=F(\frac{1}{2},\frac{1}{2},1|\lambda),$$

see [Hus04, Theorem 6.1, page 184].

3.10 Schwarz map

In Theorem 3.4 we can quotient both domain and image of the period map p by the $\mathbb{C}^*$ action and obtain

$$\mathbb{C}\to \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H},\ j\mapsto \left[\frac{\int_{\delta_1}\frac{dx}{y}}{\int_{\delta_2}\frac{dx}{y}}\right] \tag{3.35}$$

where $\mathsf{S}/\mathbb{C}^*\cong\mathbb{C}$, $\quad (t_2,t_3)\mapsto 1728\frac{t_2^3}{t_2^3-27t_3^2}$. This is the inverse of the j-function discussed in Section 2.12. We do not have a family of elliptic curves over $j\in\mathbb{C}$. In other words, the universal family of elliptic curves does not exist. Instead we have the family in Exercise 3.16 which misses the elliptic curve with $j=1728$. We will therefore consider the family of elliptic curves (3.9) which at least contains all elliptic curves, but repeated, becasue

$$j(E_\psi)=\frac{432}{z(1-z)}$$

Recall the computation of elliptic integrals in this case in terms of Gauss hypergeometric function. The multivalued function

$$\mathsf{p}:\mathbb{C}\to\mathbb{H},\ z\mapsto\frac{\int_{\delta_1}\frac{dx}{y}}{\int_{\delta_2}\frac{dx}{y}}=i\frac{F(\frac{1}{6},\frac{5}{6},1|1-z)}{F(\frac{1}{6},\frac{5}{6},1|z)}$$

after composing with the projection is $\mathbb{H}\to\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ is just the map (3.35) and one might also call it a period map. However, historically this p is called the Schwarz map. In the context of mirror symmetry it is also called the mirror map. We summarize its global behavior in the following proposition:

Proposition 3.9 *Let*

$$U:=\{z\in\mathbb{C}\mid \mathrm{Re}(z)<\frac{1}{2}\}\backslash\{z\in\mathbb{R}\mid z\le 0\}.$$

and consider the branch of the Schwarz map in U which has pure imaginary values in $0<z<\frac{1}{2}$. Its image is the interior of the classical fundamental domain of the action of $\mathrm{SL}(2,\mathbb{Z})$ in $\mathbb{H}$ depicted in Figure 2.3. Its analytic continuation result in the triangulation of $\mathbb{H}$ as in Figure 2.3.

Basic ingredients of the proof are the global injectivity of the period map (3.35) proved in Theorem 3.4 and the following exercise:

Exercise 3.21 *Let p be the branch of the Schwarz map described in Proposition 3.9. Prove the following:*

1.
$$\lim_{z\in\mathbb{R},\ z\to 0^+}\mathsf{p}(z)=+\infty.$$

2.
$$|\mathsf{p}(\frac{1}{2}+ix)|=1,\ x\in\mathbb{R}.$$

3.
$$\lim_{x\in\mathbb{R},\ x\to\pm\infty}\mathsf{p}(\frac{1}{2}+ix)=\pm\frac{1}{2}+\frac{\sqrt{3}}{2}.$$

4. The analytic continuation of p from the upper half (resp. lower half) of $\mathbb{C}$ to $\mathbb{R}^-$ has the constant real part $\frac{1}{2}$ (resp. $-\frac{1}{2}$).

The hypergeometric functions F_1 and F_2 appearing in the denominator and nominator of the mirror map are a basis of solutions for the Picard-Fuchs equation

$$y''+\frac{2z-1}{z^2-z}y'+\frac{5}{36}y=0. \tag{3.36}$$

Therefore, the Wronskian $W=F_1'F_2-F_1F_2'$ satisfies the differential equation $\frac{W'}{W}=-\frac{2z-1}{z^2-1}$ and so

$$F_1'F_2-F_1F_2'=c\frac{1}{z(1-z)}, \tag{3.37}$$

where c is a constant. It can be computed for instance by asymptotic behaviour of F_1 and F_2 at $z=0$ or $z=1$, or the evaluation at $z=1/2$. In particular,

$$d\tau = c\frac{1}{z(1-z)F_2^2}dz.$$

Remark 3.3 In Hodge theory the global (resp. local) injectivity of the period map is known as global (resp. local) Torelli problem. It might be useful to give a proof of the injectivity of the period map p in Theorem 3.4 without using the explicit construction of its inverse by Eisenstein series. This is as follows. After taking quotient by $\mathbb{C}^*$ action, we need to prove that the map $\mathsf{p}: \mathbb{C} \to \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ given by (3.35) is injective. First, we observe that the quotient $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ has a canonical structure of a Riemann surface such that the map p is a local biholomorphism. Let U be a subset of $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ containing all τ with $\mathrm{Im}(\tau)>1$. The map

$$U \to D(0,e^{-2\pi}),\ \tau \mapsto q = e^{2\pi i\tau},$$

where $D(0,r)$ is a disk in $\mathbb{C}$ with center 0 and radius r, is a coordinate system around each point of U. Using this map $\bar{S} := \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H} \cup \{\infty\}$ becomes a compact Riemann surface, where the value of the above coordinate at ∞ is $q=0$. From another side $\mathsf{S}/\mathbb{C}^* \cong \mathbb{C}$ admits also the canonical compactification $\overline{\mathsf{S}/\mathbb{C}^*}$ which is obtained by adding the single point $a := \{\Delta = 0\}/\mathbb{C}^*$. A coordinate system around a for $\overline{\mathsf{S}/\mathbb{C}^*}$ is given by $(\mathbb{C},0) \to \overline{\mathsf{S}/\mathbb{C}^*},\ z \mapsto (12,-4(4z-2))$. Note that we do not choose the natural coordinate $j^{-1} = \frac{1}{432}z(1-z)$. For this recall the one parameter family of elliptic curves in Section 3.9. The map p written in these coordinates is:

$$z \mapsto q = e^{2\pi i\frac{\int_{\delta_1}\frac{dx}{y}}{\int_{\delta_2}\frac{dx}{y}}} = \frac{1}{432}ze^{\frac{f(z)}{F(\frac{1}{6},\frac{5}{6},1|z)}} = \frac{1}{432}ze^{\frac{\frac{13}{18}z+\cdots}{1+\frac{5}{36}z+\cdots}} \tag{3.38}$$

This is an invertible map at $z=0$. All these imply that p extends to a local biholomorphism $\overline{\mathsf{S}/\mathbb{C}^*} \to \bar{S}$ without critical points. Since both the image and domain of this map are compact Riemann surfaces of genus zero, we conclude that p is a global biholomorphism.

3.11 Elliptic integrals and modular forms

In this section we explain a way to get modular forms working with elliptic integrals. This method is old and goes back to Jacobi, Legendre, Klein, Fricke and Ramanujan among many others, however, it seems that it is neglected in the modern treatment of modular forms, as rarely a classical book on modular forms covers this topic. This tendency has persisted until physicists, and in particular string theoretist, see for instance [CdlOGP91], produced q-expansions which encode Gromov-Witten invariants, and for this they used periods of Calabi-Yau varieties. Calabi-Yau one folds are elliptic curves, and such q-expansions are actually quasi-modular forms.

This method is a systematic way to see the historical equalities:

$$\sqrt[4]{E_4(\tau)} = F(\frac{1}{12}, \frac{5}{12}, 1; \frac{1728}{j(\tau)}), \tag{3.39}$$

due to [KF17a, KF17b], and

$$\theta_3(\tau)^2 = F(\frac{1}{2}, \frac{1}{2}, 1; t(\tau)), \quad t(\tau) := 16\frac{\eta(\tau/2)^8\eta(2\tau)^{16}}{\eta(\tau)^{24}}, \tag{3.40}$$

due to [Ram00, page 23-39], see also [Coo09]. It seems to the author that this was known to Jacobi in the following format:

$$K(k) = \frac{\pi}{2}\theta_3^2\left(i\frac{K(\sqrt{1-k^2})}{K(k)}\right), \quad K(k) := \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \tag{3.41}$$

noticing that $K(k) := \frac{\pi}{2}F(\frac{1}{2}, \frac{1}{2}, 1; k^2)$. These are called inversion formulas. In the followng we explain how to get these and similar formulas starting from a family of elliptic curves. However, one can also modify the presentation here starting from the underlying Picard-Fuchs equations, and in general, Fuchsian linear differential equations. For some examples see [DGMS13, Sti88]. For a brief history of inversion formulas and more examples see [Coo09, Section 1.2].

Let $E_z, z \in C$ be a family of elliptic curve over a curve C of an arbitrary genus. For simplicity, we can take the base $C = \mathbb{P}^1$. Let $A \subset C$ be the set of critical values, that is, those t with E_t singular, and fix one element $0 \in A$ and a point b near to 0. Our main example of this situation is the family (3.9) with $\psi = 4z - 2$.

$$E_\psi : y^2 - 4x^3 + 12x - 4(4z-2) = 0. \tag{3.42}$$

We assume that the anti-clockwise monodromy $H_1(E_z, \mathbb{Z}) \to H_1(E_z, \mathbb{Z})$ for z near b and in a basis $\delta_1, \delta_2 \in H_1(E_z, \mathbb{Z}), \langle\delta_1, \delta_2\rangle = -1$ is given by

$$\begin{bmatrix}\delta_1\\ \delta_2\end{bmatrix} \to \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}\delta_1\\ \delta_2\end{bmatrix}.$$

In Section 3.3 we have seen that this condition is valid for (3.42). We fix this basis, and hence, we can identify the monodromy group $\Gamma := \mathrm{Im}(\pi_1(C\backslash A, b) \to \mathrm{Aut}(H_1(E_b, \mathbb{Z})))$ with a subgroup of $\mathrm{SL}(2,\mathbb{Z})$. Moreover, for a holomorphic differential form ω (for our example $\omega = \frac{dx}{y}$) we have

$$\begin{aligned}
\omega_{2,z} &:= \int_{\delta_2}\omega = \text{ holomorphic in } (C,b) \text{ and } \omega_{2,z}(0) \neq 0,\\
\omega_{1,z} &:= \int_{\delta_1}\omega = \frac{\ln(az)\omega_{2,z} + \widetilde{\omega_{1,z}}}{2\pi i},\\
&\quad \widetilde{\omega_{1,z}} \text{ is holomorphic in } (C,0) \text{ and } \widetilde{\omega_{1,z}}(0) = 0,
\end{aligned}$$

where z is a holomorphic coordinate system around $0 \in C$. Here, the complex number a is taken in such a way that the condition $\widetilde{\omega_{1,z}}(0) = 0$ holds (for our example $a = \frac{1}{432}$). These conditions for (3.42) are verfied in Section 3.9. The Schwarz map is defined in the following way

$$\tau = \frac{\omega_{1,z}}{\omega_{2,z}} = \frac{1}{2\pi i}\left(\ln(az) + \frac{\tilde{\omega}_{1,z}}{\omega_{2,z}}\right)$$
$$q = e^{2\pi i\tau} = aze^{\frac{\tilde{\omega}_{1,z}}{\omega_{2,z}}}.$$

It follows that $q(0) = 0$ and it is regular at this point. Therefore, we can use it as a coordinate system in $(C,0)$ and holomorphic functions f in $(C,0)$ can be written as functions in q. In other words, we use the inverse $z \mapsto q(z)$ function and consider the composition $f(z(q))$. For simplicity, we will denote this and $f(z(e^{2\pi i\tau}))$ by $f(q)$ and $f(\tau)$ respectively. Now, consider a function f which has analytic continuation in $C\backslash A$ along any path, and possibly with ramification points in A, then $f(\tau)$ has a chance to be a holomorphic (one valued) function in the upper half plane. Let γ be a set of paths in C which connectes critical values of C to each other and $C\backslash\gamma$ is simply connected. In our example, we take γ to be the real line $\mathbb{P}^1(\mathbb{R})$ minus the interbal $(0,1)$. For z near to 0 and in the set $C\backslash\gamma$, we take a branch of $\ln(az)$ with $0 < \mathrm{Im}(\ln(az)) < 2\pi$. Then τ is near $i\infty$ and it is in the band

$$\{\tau \in \mathbb{C} \big| 0 < \mathrm{Re}(\tau) < 1, \quad \mathrm{Im}(\tau) > r\}$$

form some $r \in \mathbb{R}^+$. Therefore, the image D of the restriction of the Schwarz map to $C\backslash\gamma$ is a polygon-type shape in $\mathbb{H}$ with one vertice at $i\infty$. The analytic continuation of τ will result on a triangulation of $\mathbb{H}$ with polygons. The domain D is the fundamental domain of the action of the monodromy group Γ on $\mathbb{H}$.

Theorem 3.5 *For a rational function g on C and $k \in \mathbb{Z}$, the function*

$$f = \left(\int_{\delta_2} \omega\right)^k \cdot g$$

regarded as a function in τ is a meromorphic modular form of weight k for the monodromy group Γ.

Proof. We only need to verify the functional equation. Let $\omega_i = \omega_{i,z},\ i = 1,2$. After a monodromy $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have the transformations $\omega_1 \mapsto a\omega_1 + b\omega_2$ and $\omega_2 \mapsto c\omega_1 + d\omega_2$. Therefore, τ transforms to $A(\tau) := \frac{a\tau+b}{c\tau+d}$ and f transforms to

$$(c\omega_1 + d\omega_2)^k \cdot g = (c\tau + d)^k \omega_2^k \cdot g.$$

Regarding $\omega_2^k \cdot g$ as a function in $A(\tau)$ we get the result.

It might be interesting to formulate a finer version of Theorem 3.5 in which we specify rational functions g such that f becomes a holomorphic modular form. We will do this only in our main example with some flavour of quasi modular forms. In this way, we might also prove that any modular form for the monodromy group Γ is obtained as in Theorem 3.5 and then characterize the algebra of modular forms for Γ in terms of elliptic integrals (for an example see [Sti88, Theorem 5]).

Theorem 3.6 *We have*

$$F(-\frac{1}{6},\frac{7}{6},1\mid z)F(\frac{1}{6},\frac{5}{6},1\mid z)=E_2(i\frac{F(\frac{1}{6},\frac{5}{6},1|1-z)}{F(\frac{1}{6},\frac{5}{6},1|z)}), \tag{3.43}$$

$$F(\frac{1}{6},\frac{5}{6},1\mid z)^4=E_4(i\frac{F(\frac{1}{6},\frac{5}{6},1|1-z)}{F(\frac{1}{6},\frac{5}{6},1|z)}),$$

$$(1-2z)F(\frac{1}{6},\frac{5}{6},1\mid z)^6=E_6(i\frac{F(\frac{1}{6},\frac{5}{6},1|1-z)}{F(\frac{1}{6},\frac{5}{6},1|z)}).$$

Theorem 3.6 is proved in [Mov12, Section 8.7] as a byproduct of the geometric interpretation for quasi-modular forms. During the preparation of the present text, we realized that similar formulas for E_4 and E_6 were known in [Sti88, Theorem 3, Theorem 4]:

$$E_4(\tau)=F(\frac{1}{12},\frac{5}{12},1\mid\frac{1728}{j(\tau)})^4$$
$$E_6(\tau)=(1-\frac{1728}{j(\tau)})^{\frac{1}{2}}F(\frac{1}{12},\frac{5}{12},1\mid\frac{1728}{j(\tau)})^6$$

In [Fri22, Section 10, page 336] many computations are carried in this direction, however, direct relations with Eisenstein series is missing. In [Fri16, page 311] we can find a formula which is basically equivalent to our formula for E_2.

Proof. We can give three proofs. The first one which is in [Mov12] and will be explained in Chapter 9 uses the notion of generalized period domain and map. This is the way the author got these identities. For the second proof we take the left hand side (3.43) and write their q-expansions. We only need to find the three constant terms and one coefficient of q^1. We then check that these constants coincide with those in E_2,E_4,E_6. We next verify that the left hand side of (3.43) as function in τ satisfy the Ramanujan differential equation. For the third proof we must verify that the left hand side of (3.43) as function in τ are holomorphic even at $i\infty$. For E_4,E_6 we then get the identity using Theorem 3.6 and the fact the space of modular forms of weight 4 and 6 is one dimensional (Theorem 2.5). For E_2, a similar as in Theorem 3.6 we verify that the left hand side of (3.43) satisfy the same functional equation as E_2. The difference of this function with E_2 is a modular form of weight 2, and hence, by Theorem 2.5 it is zero.

Exercise 3.22 Write down the details and computations for the the last two proofs of Theorem (3.6).

Since the j-invariant of the family elliptic curves E_z is $j(E_z)=\frac{432}{z(1-z)}$ we can rewrite (3.43) in the followig historical format

$$F(-\frac{1}{6},\frac{7}{6},1\mid z)F(\frac{1}{6},\frac{5}{6},1\mid z)=E_2(\tau), \tag{3.44}$$

$$F(\frac{1}{6},\frac{5}{6},1\mid z)^4=E_4(\tau),$$

$$(1-2z)F(\frac{1}{6},\frac{5}{6},1\mid z)^6=E_6(\tau).$$

where $z=\sqrt{\frac{1}{4}-\frac{432}{j}}+\frac{1}{2}$.

Exercise 3.23 Prove the Fricke-Klein (3.39), Ramanujan (3.40) and Jacobi (3.41) inversion formulas.

The n-th Fourier coefficient of the modular form f in Theorem 3.5 can be computed in the following way: Since $2\pi i\frac{dq}{q}=\frac{\partial}{\partial\tau}$, we have

$$\begin{aligned}
f_n &= \frac{1}{2\pi i}\int f(q)q^{-n}\frac{dq}{q}\\
&= \int_\gamma f\left(\frac{\omega_{1,z}}{\omega_{2,z}}\right)e^{-2\pi i n\frac{\omega_{1,z}}{\omega_{2,z}}}\,d\left(\frac{\omega_{1,z}}{\omega_{2,z}}\right)\\
&= \int_\gamma g(z)\omega_{2,z}^k(az)^{-n}e^{-n\frac{\tilde{\omega}_{1,z}}{\omega_{2,z}}}\frac{\tilde{g}(z)}{\omega_{2,z}^2}\frac{dz}{z}\\
&= \int_\gamma g(z)\tilde{g}(z)\omega_{2,z}^{k-2}\sum_{m=0}^{\infty}\frac{(az)^{-n}}{m!}\left(-n\frac{\tilde{\omega}_{1,z}}{\omega_{2,z}}\right)^m\frac{dz}{z}\\
&= \sum_{m=0}^{\infty}\frac{a^{-n}(-n)^m}{m!}\text{The coefficient of } z^n \text{ in } g_m(z),
\end{aligned}$$

where $\tilde{g}=\theta\omega_{1,z}\omega_{2,z}-\omega_{1,z}\tilde{\omega}_{2,z}$ turns out to be a rational function in z and

$$g_m(z):=g(z)\tilde{g}(z)\omega_{2,z}^{k-2}\left(\frac{\tilde{\omega}_{1,z}}{\omega_{2,z}}\right)^m.$$

Exercise 3.24 S. Ramanujan was aware of the relation of Eisenstein series and elliptic integrals. In [Ram16, pages 180, 187] in the footnote he writes the following identities without proof:

$$\begin{aligned}
P &= \frac{12\eta\omega}{\pi^2}=(\frac{2K}{\pi})^2(\frac{3E}{K}+k^2-2),\\
Q &= \frac{12g_2\omega^4}{\pi^4}=(\frac{2K}{\pi})^4(1-k^2+k^4),\\
R &= \frac{216g_3\omega^6}{\pi^6}=(\frac{2K}{\pi})^6(1+k^2)(1-2k^2)(1-\frac{1}{2}k^2).
\end{aligned}$$

Write down the missing definitions in these equalities.

Remark 3.4 It is of historical interest to trace back the origin of inversion formulas presented in this section. The first appearance of the q variable as the exponential of the quotient of two elliptic integrals seems to go back to [Jac29]. From Section 35, page 84 on, he starts doing many computations and writting many elliptic integral expressions in terms of q, see [Jac29, Section 36, page 89,90] . It is not clear for the author his motivation. Legendre's comments on Jacobi's work also indicate this. "...it is regrettable that the author fulfills the aim which he has imposed to himself by a sort of divination, without sharing with us the secret whose conception has progressively led him to the form for $1-y$ which is required in order to satisfy the conditions of the problem", see [Cog14, page 530]. Jacobi's theta function, see Section 47 and 51 of Jacobi's book, is derived by similar inversion formulas.

Chapter 4
Rudiments of Algebraic Geometry of curves

If we agree with him [Hilbert] that problems are the lifeblood of mathematics, then certainly we may say that algebraic geometry and number theory always have had more open problems than solved ones, and that each progress towards their solution has always brought with it a host of new and exciting methods, (J. Dieudonné in [Die72]).

4.1 Introduction

Throughout the present text we work with a field k of arbitrary characteristic and not necessarily algebraically closed. By $\bar{\mathsf{k}}$ we mean the algebraic closure of k. The main examples that we have in mind are

$$\mathsf{k} = \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F}_p := \frac{\mathbb{Z}}{p\mathbb{Z}},$$

and a number field. A number field k is a field that contains $\mathbb{Q}$ and has finite dimension, when considered as a vector space over $\mathbb{Q}$. We also consider function fields $\tilde{\mathsf{k}}(t) = \tilde{\mathsf{k}}(t_1, t_2, \ldots, t_s)$ over a field $\tilde{\mathsf{k}}$ (in the list above). It is the field of rational functions $\frac{a(t_1,t_2,\cdots,t_s)}{b(t_1,t_2,\ldots,t_s)}$, where a and b are polynomials in indeterminates $t_1, t_2, \ldots, t_s$ and with coefficients in $\tilde{\mathsf{k}}$. It is recommended to the reader to read O. Zariski and A. Weil article's [Zar52, Wei52] in the international congress of mathematics 1950, where they describe their own view of how general one must take the base field in algebraic geometry. We do not assume that the reader is familiar with algebraic geometry. Exceptions are Section 4.11 and Section 4.12 in which we use Riemann-Roch theorem in order to prove that any abstract elliptic curve can be given in the Weierstrass form. The reader might skip theses sections.

4.2 Curves

Let k be a field and $\mathsf{k}[x_1,x_2,\dots,x_n]$ be the space of polynomials in n variables and with coefficients in k. The n dimensional affine space over k is by definition

$$\mathbb{A}^n(\mathsf{k}) = \mathsf{k}\times\mathsf{k}\times\cdots\times\mathsf{k},\ n \text{ times}$$

and the projective n dimensional space is

$$\mathbb{P}^n(\mathsf{k}) := \mathbb{A}^{n+1}(\mathsf{k}) - \{(0,0,\cdots,0)\}/\sim$$

$$a\sim b \text{ if and only if } \exists\lambda\in\mathsf{k}, a=\lambda b.$$

We will consider the following inclusion

$$\mathbb{A}^n(\mathsf{k})\to\mathbb{P}^n(\mathsf{k}),\ (x_1,x_2,\cdots,x_n)\mapsto[x_1;x_2;\cdots;x_n;1]$$

and call $\mathbb{P}^n(\mathsf{k})$ the compactification of $\mathbb{A}^n(\mathsf{k})$. The projective space at infinity is defined to be

$$\mathbb{P}^{n-1}_\infty(\mathsf{k}) = \mathbb{P}^n(\mathsf{k})-\mathbb{A}^n(\mathsf{k}) = \{[x_1;x_2;\cdots;x_n;x_{n+1}] \mid x_{n+1}=0\}.$$

For simplicity, in the case $n=1,2$ and 3 we use x, (x,y) and (x,y,z) instead of $x_1,x_2,\dots$. The notation $\mathbb{P}^n$ and $\mathbb{A}^n$ are reserved for the same concepts as schemes, see Section 4.3. Any polynomial $f\in\mathsf{k}[x,y]$ defines an affine curve

$$C(\mathsf{k}) := \{(x,y)\in\mathsf{k}^2 \mid f(x,y)=0\}.$$

One of the most famous curves is given by the polynomial $f=x^n+y^n-1$ which we call it Fermat curve. The set $C(\mathsf{k})$ may be empty, for instance take $\mathsf{k}=\mathbb{Q},\ f=x^2+y^2+1$. This means that the identification of a curve with its points in some field is not a good treatment of curves. One of the starting points of the theory of schemes is this simple observation. We will handle this issue in Section 4.3.

For $f\in\mathsf{k}[x,y]$ we define the homogenization of f

$$F(x,y,z) = z^d f(\frac{x}{z},\frac{y}{z}),\ d:=\deg(f).$$

The polynomial F defines a projective plane curve in $\mathbb{P}^2(\mathsf{k})$:

$$\bar{C}(\mathsf{k}) := \{[x;y;z]\in\mathbb{P}^2(\mathsf{k}) \mid F(x,y,z)=0\}.$$

Note that

$$\forall c\in\mathsf{k}, (x,y,z)\in\mathsf{k}^3,\ F(cx,cy,cz) = c^dF(x,y,z).$$

One has the injection

$$C(\mathsf{k})\to\bar{C}(\mathsf{k}),\ (x,y)\mapsto[x;y;1]$$

and for this reason one sometimes says that $\bar{C}(\mathsf{k})$ is the compactification of $C(\mathsf{k})$. Let g be the last homogeneous piece of the polynomial f. By definition it is a homogeneous polynomial of degree d. The points in

$$\bar{C}(\mathsf{k})-C(\mathsf{k})=\{[x;y]\in\mathbb{P}^1_\infty \mid g(x,y)=0\}$$

are called the points at infinity of $C(\mathsf{k})$. The set of points at infinity of the Fermat variety over $\mathsf{k}=\mathbb{Q}$ is empty if n is even and it is $\{[1;-1;0]\}$ if n is odd. Over $\bar{\mathbb{Q}}$ it has n elements $[\zeta;1;0]$, $\zeta^n=-1$.

From now on we use the notation $C(\mathsf{k})$ to denote the curve $\bar{C}(\mathsf{k})$ in the previous section. We simply say that the curve $C(\mathsf{k})$ in an affine chart is given by $f(x,y)=0$. The projective space $\mathbb{P}^2(\mathsf{k})$ is covered by three canonical charts:

$$\alpha_i:\mathbb{A}^2(\mathsf{k})\hookrightarrow\mathbb{P}^2(\mathsf{k})$$

$$\alpha_1(x,y)=[x;y;1],\ \alpha_2(x,z)=[x;1;z],\ \alpha_3(y,z)=[1;y;z].$$

and the curve in each chart is respectively given by

$$f_1(x,y):=F(x,y,1)=0,\ \ f_2:=F(x,1,z)=0,\quad \text{and } f_3:=F(1,y,z)=0.$$

We are also going to use the notion of an arbitrary curve over k from algebraic geometry of schemes. Roughly speaking, a curve C over k means C over $\bar{\mathsf{k}}$ and the ingredient polynomials of C are defined over k. The reader who is not familiar with those general objects may follow the text for affine and projective curves as above.

4.3 Schemes

We defined $\mathbb{P}^2(\mathsf{k})$ and $C(\mathsf{k})$ without defining $\mathbb{P}^2$ and C. In this section we fill this gap and we explain the rough idea behind the definition of the schemes $\mathbb{P}^2$ and C. By the affine scheme $\mathbb{A}^2$ we simply think of the ring $\mathsf{k}[x,y]$. Open subsets of $\mathbb{A}^2$ are given by the localization of $\mathsf{k}[x,y]$. We will need two open subsets of $\mathbb{A}^2$ given respectively by

$$\mathsf{k}[x,y,\frac{1}{y}] \text{ and } \mathsf{k}[x,y,\frac{1}{x}].$$

By the projective scheme $\mathbb{P}^2$ we mean three copies of $\mathbb{A}^2$, namely

$$\mathsf{k}[x,y],\ \mathsf{k}[x,z],\ \mathsf{k}[y,z]$$

together with the isomorphism of affine subsets:

$$\mathsf{k}[x,y,\frac{1}{y}]\cong\mathsf{k}[x,z,\frac{1}{z}],\ x\mapsto\frac{x}{z},\ y\mapsto\frac{1}{z}, \tag{4.1}$$

$$\mathsf{k}[x,y,\frac{1}{x}] \cong \mathsf{k}[y,z,\frac{1}{z}],\ x \mapsto \frac{1}{z},\ y \mapsto \frac{y}{z},$$
$$\mathsf{k}[x,z,\frac{1}{x}] \cong \mathsf{k}[y,z,\frac{1}{y}],\ x \mapsto \frac{1}{y},\ z \mapsto \frac{z}{y}.$$

The best way to see these isomorphisms is, for instance: we look at an element of $\mathsf{k}[x,y]$ as a function on the first chart $\mathbb{A}^2(\mathsf{k})$ and for (a,b) in this chart we use the identities

$$[a;b;1] = [\frac{a}{b};1;\frac{1}{b}] = [1;\frac{b}{a};\frac{1}{a}].$$

We sometimes write $\mathbb{P}^2_{\mathsf{k}}$ or $\mathbb{P}^2/\mathsf{k}$ in order to emphasize that it is defined over k. We think of the the scheme C or C/k in the same way as $\mathbb{P}^2$, but replacing $\mathsf{k}[x,y]$ with $\mathsf{k}[x,y]/\langle f_1\rangle$ and so on. We read C/k as "C is defined over k". Here $\langle f_1\rangle$ is the ideal in $\mathsf{k}[x,y]$ generated by a single element f_1. We can also think of C in the same way as $\mathbb{P}^2$ but with the following additional relations between variables:

$$f_1(x,y) = 0 \text{ in } \mathsf{k}[x,y],$$

$$f_2(x,z) = 0 \text{ in } \mathsf{k}[x,z],$$

and

$$f_3(y,z) = 0 \text{ in } \mathsf{k}[y,z].$$

Remark 4.1 The above discussion does not use the fact that k is a field. In fact, we can use an arbitrary ring R instead of k. In this way, we say that we have a scheme C over the ring R.

Definition 4.1 The function field of the projective space $\mathbb{P}^2$ is defined to be

$$\mathsf{k}(\mathbb{P}^2) := \mathsf{k}(x,y) \cong \mathsf{k}(x,z) \cong \mathsf{k}(y,z),$$

where the isomorphisms are given by (4.1). The field of rational function on the curve C is the field of fractions of the ring $\mathsf{k}[x,y]/\langle f_1\rangle$. Using the isomorphism (4.1), this definition does not depend on the chart with (x,y) coordinates. We can also think of $\mathsf{k}(C)$ as $\mathsf{k}(x,y)$ but with the relation $f_1(x,y) = 0$ between the variables x,y. Any $f \in \mathsf{k}(C)$ induces a map $C(\mathsf{k}) \to \mathsf{k}$ that we denote it by the same letter f.

An algebraic curve C over $\bar{\mathsf{k}}$ can be identified with its $\bar{\mathsf{k}}$-rational points and it might be helpful to have the following in mind.

Exercise 4.1 Let $f,g \in \mathsf{k}[x,y]$ (resp. $F,G \in \mathsf{k}[x,y,z]$ homogeneous) and assume that f (resp. F) is irreducible over $\bar{\mathsf{k}}$. Let C be the curve in $\mathbb{A}^2_{\mathsf{k}}$ (resp. $\mathbb{P}^2_{\mathsf{k}}$) given by $f = 0$ (resp. $F = 0$). If g (resp. G) evaluaed at the points of $C(\bar{\mathsf{k}})$ is zero then f (resp. F) divides g (resp. G) in the ring $\mathsf{k}[x,y]$ (resp. $\mathsf{k}[x,y,z]$).

A rational function f in C can be identified with the restriction of $\frac{P(x,y,z)}{Q(x,y,z)}$ to $C(\bar{\mathsf{k}}) \to \bar{\mathsf{k}}$, where P and Q are two homogeneous polynomial of the same degree. If two such quotients $\frac{P_1}{Q_1}$ and $\frac{P_2}{Q_2}$ give the same function $C(\bar{\mathsf{k}})$ then by Exercise 4.1, $P_1Q_2 - P_2Q$ is divisable by the equation of C.

Exercise 4.2 A full definition of of a scheme can be found in [Har77, Chapter 2]. For a scheme X over $\mathrm{Sepc}(R)$, where R is a ring, the set of R-points of X is denoted by $X(R)$ and it consists of all scheme morphisms $\mathrm{Sepc}(R) \to X$. Show that

$$\mathbb{P}^n_{\mathbb{Z}}(\mathbb{Z}) \cong \{(m_0, m_1, \cdots, m_n) \in \mathbb{Z}^{n+1} \mid \gcd(m_0, m_1, \cdots, m_n) = 1\}/\{\pm 1\}.$$

One of the fundamental observation in Grothendieck's revolution of Algebraic Geometry, replacing varieties with schemes, is that

$$\mathrm{Sepc}(\mathbb{Z}) := \{ \text{ prime ideals of } \mathbb{Z}\} \simeq \{2, 3, 5, \ldots, p, \ldots\}$$

is like a parameter, for instance, the parameter λ in the Legendre family of elliptic curves $E_\lambda : y^2 = 4x(x-1)(x-\lambda),\ \lambda \in \mathbb{C}$. We consider E_λ as a curve over the ring $\mathbb{C}[\lambda]$. The prime ideals of $\mathbb{C}[\lambda]$ are $\mathrm{Sepc}(\mathbb{C}[\lambda]) := \left\{(\lambda - \lambda_0)\mathbb{C}[\lambda] \,\middle|\, \lambda_0 \in \mathbb{C}\right\} \simeq \mathbb{C}$. For a prime ideal $P \subseteq \mathbb{C}[\lambda]$, the residue field is naturally isomorphic to $\mathbb{C}$

$$\mathbb{C}[\lambda]/(\lambda - \lambda_0)\mathbb{C}[\lambda] \cong \mathbb{C},\ \ p(\lambda) \mapsto P(\lambda_0).$$

The process of substituting λ with λ_0, can be interpreted as considering E_λ over the residue field. In a similar way for an elliptic curve over $\mathbb{Z}$, for instance $E : y^2 = x^3 + 1$, the process of working modulo a prime number p is the same as considering E over the residue field $\mathbb{F}_p$.

Another reason for using schemes is that defining ideal of affine varieties have more data than the underlying variety. For instance, the underlying variety of the ideal $I = \langle x, xy\rangle \subset \mathsf{k}[x, y]$ is just $\{x = 0\}$, however, we have $I = \langle x\rangle \cap \langle x, y\rangle$ which means that we must look at the underlying variety as a union of $\{x = y = 0\}$ and $\{x = 0\}$.

4.4 Singularities and smooth curves

Definition 4.2 We say that an affine curve given by $f(x, y) = 0$ is singular if there is a point $(a, b) \in \bar{\mathsf{k}}^2$ such that

$$f(a, b) = f_x(a, b) = f_y(a, b) = 0,$$

where f_x is the derivation of f with respect to x and so on. The point (a, b) is called a singularity of the affine curve. A projective curve is singular in one of its affine charts it has a singularity. For a curve C, affine or projective, we denote by $\mathrm{Sing}(C) \subset C(\bar{\mathsf{k}})$ the set of singular points of C.

Exercise 4.3 Show that the singularities of a projective curve C given by the homogeneous polynomial $F(x,y,z)$ are given by

$$\mathrm{Sing}(C) := \left\{ [x;y;z] \in \mathbb{P}^2_{\mathsf{k}} \;\middle|\; \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0 \right\}.$$

.

Exercise 4.4 Let C be an affine curve in $\mathbb{A}^2_{\mathsf{k}}$ given by the polynomial $f \in \mathsf{k}[x,y]$ and let $f = f_d + f_{d-1} + \cdots$ be its decomposition into homogeneous pieces, that is, $f_i \in \mathsf{k}[x,y]$ is homogebeous of degree i. For k an algebraically closed field we can write $f_d = \prod_{i=1}^{d}(a_i x - b_i y),\ \ a_i, b_i \in \mathsf{k}$. Show that the points at infinity of C are given by $[b_i;a_i;0] \in \mathbb{P}^2, i = 1,2,\ldots,d\}$. The point $[b_i;a_i;0]$ is singular if and only if for some j different from i we have $[b_i;a_i;0] = [b_j;a_j;0]$ and $f_{d-1}(b_i,a_j) = 0$.

4.5 Coordinate system on a curve

In Chapter 2 we have seen that an elliptic function g can be considered as a meromorphic function on the torus $\mathbb{C}\backslash\Lambda$ and at each point p of this torus we have a coordinate system given by the coordinate system z of $\mathbb{C}$, and we can write the Taylor or Laurant series of g at p. In this section we aim to reproduce all these replacing the torus with an algebraic curve defined over k, point p with a smooth k-rational point p of C and g with a rational function on C. We would like to highlight that k can be a field of arbitrary characteristic and not necessarily algebraically closed.

Let $C \subset \mathbb{P}^2$ be a curve defined over a field k and $p \in C(\mathsf{k})$ be a smooth k-rational point. For simplicity, we take an affine chart (x,y) containing the point p and assume that C is given by $f(x,y) = 0$ for $f \in \mathsf{k}[x,y]$.

Definition 4.3 For a rational function g in C, we say that g is regular at p of if there is $P,Q \in \mathsf{k}[x,y]$ with $g = \frac{P(x,y)}{Q(x,y)}, Q(p) \neq 0$. We say that g has a pole at p if g is not regular at p.

Note that if $f(x,y)$ is the equation of C and we have $g = \frac{P(x,y)}{Q(x,y)}$ with $P(p) = 0$ and $Q(p) = 0$ then there might be a different P and Q such that $Q(p) \neq 0$. For instance, for the curve $y^2 = 4x^3 - t_2^2 x, t_2 \in \mathsf{k},\ t_2 \neq 0$ we have the point $p = (\frac{t_2}{2}, 0) \in C(\mathsf{k})$ and

$$g = \frac{4x^2 - t_2}{y} = \frac{y}{x}$$

The first expression of g cannot be used for evaluation at p, but the second expresion implies that $g(p) = 0$. For $p = (0,0)$, the first expression implies that p is a pole of g.

Definition 4.4 The germ of regular functions in a neighborhood of p is defined

$$\mathscr{O}_{C,p} := \{g \in \mathsf{k}(C) \mid g \text{ is regular at } p\}.$$

This is a ring and it has the maximal ideal

$$\mathfrak{m}_{C,p} := \left\{ g \in \mathscr{O}_{C,p} \;\mid\; g(p) = 0 \right\}.$$

The quotient $\mathfrak{m}_{C,p}/\mathfrak{m}_{C,p}^2$ is called the cotangent space of C at p.

Exercise 4.5 Show that p is a smooth point of C if and only if the cotangent space of C at p is a one dimensional k-vector space.

Definition 4.5 A coordinate system t in a neighborhood of a smooth point p in C is any generator of the one dimensional cotangent vector space of C at p.

Exercise 4.6 If p is smooth point of C show that for all $n \in \mathbb{N}$, $\mathfrak{m}_{C,p}^n/\mathfrak{m}_{C,p}^{n+1}$ is generated by t^n, where t is a coordinate system arround p. For a rational function g on C with a pole at p, we have $a \in \mathbb{N}$ such that $t^a g$ is regular at p, that is, we can write $g = \frac{\tilde{g}}{t^a}$ for some $\tilde{g} \in \mathscr{O}_{C,p}$. The smallest a with this property is called the pole order of g at p.

Exercise 4.7 [Algebraic Taylor series] Let g be a rational function in a curve C defined over k, $p \in C(\mathsf{k})$ be a smooth point of C with g regular at p, and t be a coordinate system around p. We have

$$g = \sum_{i=0}^{+\infty} g_i t^i, \; f_i \in \mathsf{k}. \tag{4.2}$$

This means that there is a formal power series as in the right hand side of (4.2) such that for all $n \in \mathbb{N}_0$ we have

$$g - \sum_{i=0}^{n} g_i t^i \in \mathfrak{m}_{C,p}^{n+1}.$$

If $g_0 = g_1 = \cdots = g_{a-1} = 0$ and $g_a \neq 0$ then we say that g has a zero of order a at p. If g has a pole at p then in a similar way we can write the Laurant series of g at p:

$$g = \sum_{i=-a}^{+\infty} g_i t^i, \; g_i \in \mathsf{k}, \; g_a \neq 0, \tag{4.3}$$

where $a \in \mathbb{N}$ is the pole order of g at p.

Exercise 4.8 Let g be a rational function on a curve C defined over an algebraically closed field $\bar{\mathsf{k}}$. The number of zeros and poles of g, counted with multiplicity, are equal.

4.6 Discriminant

In this section we define the discriminant of a polynomials $f \in \mathsf{R}[x]$ following [Mov21, Section 10.9]. Let R be a ring and k be the field of fractions of R.

Definition 4.6 Let us be given a polynomial $f \in \mathsf{R}[x, y, \cdots]$, where $(x, y, \cdots)$ is a multi variable. The discriminant ideal of f contains all element $\Delta \in \mathsf{R}$ such that

$$\Delta = f a_1 + f_x a_2 + f_y a_3 + \cdots, \quad \text{for some } a_1, a_2, \ldots \in \mathsf{R}[x, y, \ldots].$$

When the discriminant ideal is principal we denote by Δ its generator and we call it the discriminant of the polynomial f. It is defined up to units of the ring R, and hence in the case $\mathsf{R} = \mathbb{Z}$ it is defined up to sign. In this case we fix this ambiguity by assuming that Δ is positive.

Our main example is the following:

Proposition 4.1 *Let $f = y^2 - 4x^3 + t_2 x + t_3$ defined over $\mathsf{R} := \mathbb{Z}[\frac{1}{6}, t_2, t_3]$ and $V := \mathsf{R}[x, y]/\langle f_x, f_y \rangle$. The discriminant Δ is the determinant of the multiplication by f in V.*

Proof. Using the explicit form of f, we can easily verify that the R-module V is freely generated by $1, x$ (here we use the fact that 2 and 3 are invertible in R). Let $M : V \to V$, $M(\omega) = f\omega$. We write M in the basis $1, x$:

$$M = \begin{pmatrix} t_3 & \frac{1}{18} t_2^2 \\ \frac{2}{3} t_2 & t_3 \end{pmatrix}.$$

Let $p(z) := z^2 - \mathrm{tr}(M) z + \det(M) = \det(M - z I_{2\times 2})$ be the characteristic polynomial of M. We have $P(f)V = 0$ and this implies that $\det(M) = \frac{-1}{27}\Delta$ is in the discriminant ideal. The rest of the proof is in Exercise 4.9.

Exercise 4.9 In Proposition 4.1, Δ generates the discriminant ideal is left to the reader in

Exercise 4.10 For

$$f = y^2 - x^3 - t_4 x - t_6, \; t_4, t_6 \in \mathsf{R}; \tag{4.4}$$

show that the discriminant ideal is generated by

$$\Delta = 2(4t_4^3 + 27t_6^2),$$

The corresponding a_1, a_2 and a_3 in this case are given by

$$a_1 = 2(27x^3 - 27y^2 + (27t_4)x + (-27t_6)),$$

$$a_2 = 2(-9x^4 + (-15t_4)x^2 + (-4t_4^2)), \; a_3 = -54x^3 y + 27y^3 + (-54t_4)xy.$$

In this cases, Δ is the resultant of the polynomials P and $\frac{\partial P}{\partial x}$, where $P = x^3 + t_4 x + t_6$.

In all the case below, the discriminant ideal is principal and we have calculated its generator.

$$f = y^2 - x^3 - t_4 x - t_6 - t_2 x^2 + t_1 xy + t_3 y. \tag{4.5}$$

$$\Delta = (t_1^6 t_6 - t_1^5 t_3 t_4 + t_1^4 t_2 t_3^2 + 12 t_1^4 t_2 t_6 - t_1^4 t_4^2 - 8 t_1^3 t_2 t_3 t_4 - t_1^3 t_3^3 - 36 t_1^3 t_3 t_6 + 8 t_1^2 t_2^2 t_3^2 + 48 t_1^2 t_2^2 t_6 - 8 t_1^2 t_2 t_4^2$$

$$+30 t_1^2 t_3^2 t_4 - 72 t_1^2 t_4 t_6 - 16 t_1 t_2^2 t_3 t_4 - 36 t_1 t_2 t_3^3 - 144 t_1 t_2 t_3 t_6 + 96 t_1 t_3 t_4^2 + 16 t_2^3 t_3^2 + 64 t_2^3 t_6 - 16 t_2^2 t_4^2$$

$$
\begin{gathered}
-72t_2t_3^2t_4-288t_2t_4t_6+27t_3^4+216t_3^2t_6+64t_4^3+432t_6^2),\\
a_1=432x^3-432y^2+(-432t_1)xy+(432t_2)x^2+(-432t_3)y+(432t_4)x+(-t_1^6-12t_1^4t_2+36t_1^3t_3-48t_1^2t_2^2\\
+72t_1^2t_4+144t_1t_2t_3-64t_2^3+288t_2t_4-216t_3^2-432t_6),\\
a_2=-144x^4+(-48t_1^2-192t_2)x^3+(-t_1^4-8t_1^2t_2-120t_1t_3-16t_2^2-240t_4)x^2+(-t_1^5)y+(6t_1^4t_2-20t_1^3t_3+\\
24t_1^2t_2^2-40t_1^2t_4-80t_1t_2t_3+32t_2^3-160t_2t_4)x+(t_1^4t_4+4t_1^3t_2t_3+8t_1^2t_2t_4-16t_1^2t_3^2\\
+8t_1t_2^2t_3-64t_1t_3t_4+16t_2^2t_4-64t_4^2),\\
a_3=-432x^3y+216y^3+(-144t_1)x^4+(324t_1)xy^2+(54t_1^2-432t_2)x^2y+(-3t_1^3-120t_1t_2-216t_3)x^3+\\
(324t_3)y^2+(108t_1t_3-432t_4)xy+(-t_1^5+4t_1^3t_2-21t_1^2t_3+8t_1t_2^2-96t_1t_4-216t_2t_3)x^2+(t_1^6+6t_1^4t_2-18t_1^3t_3+24t_1^2t_2^2\\
-36t_1^2t_4-72t_1t_2t_3+32t_2^3-144t_2t_4+162t_3^2)y+(-t_1^5t_2+t_1^4t_3+2t_1^3t_4+4t_1^2t_2t_3+8t_1t_2t_4-27t_1t_3^2-216t_3t_4)x\\
+(-t_1^5t_4+t_1^4t_2t_3-4t_1^3t_2t_4-t_1^3t_3^2+8t_1^2t_2^2t_3+14t_1^2t_3t_4-8t_1t_2^2t_4-36t_1t_2t_3^2+32t_1t_4^2+16t_2^3t_3-72t_2t_3t_4+27t_3^3).
\end{gathered}
$$

This modulo 2 is:

$$
\begin{gathered}
\Delta=t_1^4t_2t_3^2+t_1^5t_3t_4+t_1^6t_6+t_1^3t_3^3+t_1^4t_4^2+t_3^4,\\
a_1=t_1^6,\ a_2=t_1^4x^2+t_1^5y+t_1^4t_4,\\
a_3=t_1^5x^2+t_1^3x^3+t_1^6y+t_1^5t_2x+t_1^4t_3x+t_1^2t_3x^2+t_1^4t_2t_3+t_1^5t_4+t_1^3t_3^2+t_1t_3^2x+t_3^3.
\end{gathered}
$$

For the case

$$
f=y^2-x^3-t_4x-t_6-t_2x^2, \tag{4.6}
$$

we have

$$
\begin{gathered}
\Delta=2(4t_2^3t_6-t_2^2t_4^2-18t_2t_4t_6+4t_4^3+27t_6^2),\\
a_1=2(27x^3-27y^2+(27t_2)x^2+(27t_4)x+(-4t_2^3+18t_2t_4-27t_6)),\\
a_2=2(-9x^4+(-12t_2)x^3+(-t_2^2-15t_4)x^2+(2t_2^3-10t_2t_4)x+(t_2^2t_4-4t_4^2)),\\
a_3=-54x^3y+27y^3+(-54t_2)x^2y+(-54t_4)xy+(4t_2^3-18t_2t_4)y.
\end{gathered}
$$

Modulo 3 this is:

$$
\begin{gathered}
\Delta=t_2^2t_4^2-t_2^3t_6-t_4^3,\\
a_1=t_2^3,\ a_2=t_2^2x^2+t_2^3x+t_2t_4x-t_2^2t_4+t_4^2,\ a_3=t_2^3y.
\end{gathered}
$$

The main property of the discriminant ideal is:

Exercise 4.11 Let I be any maximal ideal of R and so R/I is a field. The affine variety $f=0$ is singular over the field R/I if and only if the discriminant ideal is a subset of I. Hint: the proof is a slight modification of [Mov21, Proposition 10.8].

Let us describe our main examples for Exercise 4.11.

1. $\mathsf{R}=\mathsf{k}$ and so $I=\{0\}$.
2. $\mathsf{R}=\mathbb{Z}$. This is a principal ideal domain and so the discriminat ideal is generated by some $\Delta\in\mathbb{N}$ and I is generated by some prime $p\in\mathbb{N}$. In this case, $f=0$ is singular over $\mathbb{F}_p$ if and only if $p\mid\Delta$.
3. $\mathsf{R}=\mathsf{k}[t]$ and for $a\in\mathsf{k}^s$, I is the ideal of R generate by $t_i-a_i,\ i=1,2,\dots,s$. Let also assume that the discriminant ideal is generated by Δ. In this case, the

curve $f=0$ with the evaluation of the parameters $t=a$ is singular if and only if $\Delta(a)=0$.

Definition 4.7 Let $C\subset\mathbb{P}^2$ be a curve over the ring R. We define its discriminant ideal to be the ideal generated by all discriminant ideals of C in affine charts.

Exercise 4.12 Let us take the curve $y^2z-x^3-17z^3=0$ over $\mathbb{Z}$. Its discriminant in the affine coordinates $z=1$, respectively $y=1$, is $2\cdot 3\cdot 17^2$, respectively 2^4. Its discriminant is $2^4\cdot 3\cdot 7^2$.

4.7 Curves of genus zero and bigger than one

Let $f\in\mathsf{k}[x,y]$ and let C be the curve induced by $f=0$ in $\mathbb{P}^2_{\mathsf{k}}$. Let us assume that f is of degree 2 and it is irreducible over $\bar{\mathsf{k}}$, that is, it is not the product of two linear polynomials in $\bar{\mathsf{k}}[x,y]$. Further, assume that $C(\mathsf{k})$ has at least one point P. Note that if we look C in an affine chart then this point can be a point at infinity. The following procedure finds all the points of $C(\mathsf{k})$. We fix a line L in $\mathbb{P}^2_{\mathsf{k}}$ defined over k, for instance take $y=0$. For any point $X\in L(\mathsf{k})$, we connect X to P by a line L' and find the second intersection $g(X)$ of L' with C. Since $P\in C(\mathsf{k})$, we have $g(X)\in C(\mathsf{k})$ and we get a bijection

$$g: L(\mathsf{k})\to C(\mathsf{k}).$$

Exercise 4.13 Use the above geometric argument and find all k-rational points of the Diophantine equation

$$tx^2+sy^2=t+s.$$

for some $s,t\in\mathsf{k}$. Take for instance $t=s=1$.

The argument discussed in the previous paragraph works in the following case:

Exercise 4.14 Let $f\in\mathsf{k}[x,y]$ be a degree 3, irreducible polynomial over $\bar{\mathsf{k}}$ and the induced curve C in $\mathbb{P}^2_{\mathsf{k}}$ is singular. Show that $\mathrm{Sing}(C)$ consists of only one point which is a k-rational point of C.

Therefore, if C is singular we have an automatically a unique singular point $P\in C(\mathsf{k})$. This point serves us as the point with the same name in the previous paragraph.

Exercise 4.15 Find all the k-rational points of the Diophantine equation

$$y^2-x^3-t_4x-t_6=0,$$

for some $t_4,t_6\in\mathsf{k}$ with $4t_4^3+27t_6^2=0$.

Let $C/\mathbb{Q}$ be a smooth projective curve of degree d in $\mathbb{P}^2$, that is, its defining polynomial is of degree d. Its genus is given by $g(C):=\frac{(d-1)(d-2)}{2}$. The main objective of the Diophantine theory is to describe the set $C(\mathbb{Q})$ for the curves defined over $\mathbb{Q}$. The most famous example is the Fermat curve given by the polynomial $f = x^d + y^d - 1$. The machinery of algebraic geometry is very useful to distinguish between various types of Diophantine equations. For instance, one can describe the rational points of genus zero curves, that is, $d = 1, 2$. We have already discussed this at the beginning of this section. The genus one curves ($d = 3$) are called elliptic curves and the study of their rational points is the objective of the present text. For higher genus we have a conjecture of Mordell around 1922 which is proved by Faltings in 1982: A non-singular projective curve of genus > 1 and defined over $\mathbb{Q}$ has only finitely many $\mathbb{Q}$-rational points. In fact, the above theorem is true even for number fields. For instance, the above theorem says that the Fermat curve has a finite number of $\mathbb{Q}$-rational points. However, it does not say something about the nature of its rational points. Mordell's conjecture for function fields was proved by in [Man63, Man64], and interestingly enough, this is the origin of the name Gauss-Manin connection discussed in the earlier chapter. It was invented by A. Grothendieck after reading Manin's paper, see [Pha79].

4.8 Elliptic curves in Weierstrass form

Let E be a complete smooth curve of genus one over the field k. If the reader is not familiar with the notion of a curve over a field, he can use the curves in $\mathbb{P}^2$ which we worked out in the previous section, and hence, E is given by a degree 3 homogeneous polynomial $f(x, y, z)$.

Definition 4.8 An elliptic curve over k is a pair (E, O), where E is a genus one complete smooth curve and O is a k-rational point of E, that is, $O \in E(\mathsf{k})$.

Therefore, by definition an elliptic curve over k has at least one k-rational point. A smooth projective curve of degree 3 is therefore an elliptic curve if it has a k-rational point. For instance, the Fermat curve

$$F_3 : x^3 + y^3 = z^3$$

is an elliptic curve over $\mathbb{Q}$ in many different ways, depending on the choice of the $\mathbb{Q}$-rational point, such as $[0, 1, 1]$ or $[1, 0, 1]$. However

Exercise 4.16 The curve

$$E : 3x^3 + 4y^3 + 5z^3 = 0$$

has not $\mathbb{Q}$-rational points and so it is not an elliptic curve defined over $\mathbb{Q}$. It is an interesting fact to mention that $E(\mathbb{Q}_p)$ for all prime p and $E(\mathbb{R})$ are not empty. This example is due to Selmer, see [Cas66, Sel51].

Definition 4.9 An elliptic curve in the Weierstrass form E is the affine curve given by the polynomial

$$E_{t_2,t_3} : y^2 - x^3 - t_2x - t_3,\ t_2,t_3 \in \mathsf{k}, \quad \Delta := 2(4t_2^3 + 27t_3^2) \neq 0. \tag{4.7}$$

Note that we have assumed that k is not of characteristic 2 and so E_{t_2,t_3} is smooth. In homogeneous coordinates it is written in the form

$$\bar{E}_{t_2,t_2} : \ zy^2 - x^3 - t_2xz^2 - t_3z^3 = 0.$$

It has only one point at infinity, namely $[0;1;0]$, which is considered as the marked point in the definition of an elliptic curve.

The point $[0:1:0]$ is in fact a smooth point of $\bar{E}_{t_2,t_3}$ which is tangent to the projective line at infinity of order 3 and it is the only intersection point of the line at infinity with $\bar{E}_{t_2,t_2}$. If $\mathrm{char}(\mathsf{k}) = 2$ then the curve E_{t_2,t_3} is always singular. We have already seen in Exercise 4.11 that $\Delta = 0$ if and only if the corresponding curve is singular.

4.9 Real geometry of elliptic curves

For a projective smooth curve C defined over $\mathbb{R}$ the set $C(\mathbb{R}) \subset \mathbb{P}^2(\mathbb{R})$ has many connected components, all of them topologically isomorphic to a circle. We call each of them an oval. For an elliptic curve E defined over $\mathbb{R}$ we want to analyze the topology of $E(\mathbb{R})$. For simplicity (in fact because of Proposition 4.3 which will be presented later) we assume that $E = E_{t_2,t_3}$ is in the Weierstrass form. For $(t_2,t_3) \in \mathbb{R}^2$ let $\Delta = 2(4t_2^3 + 27t_3^2)$ be the discriminant of the elliptic curve E. We have:

1. If $\Delta < 0$ then $E(\mathbb{R})$ has two connected components, one is a closed path in $\mathbb{R}^2$, which we call it an affine oval, and the other a closed path in $\mathbb{P}^2(\mathbb{R})$. We call it a projective oval.
2. If $\Delta > 0$ then $E(\mathbb{R})$ has only one component which is a projective oval.
3. If $\Delta = 0$ and $t_3 < 0$ then $E(\mathbb{R})$ is an α-shaped path in $\mathbb{R}^2$ (∞-shaped path in $\mathbb{P}^2(\mathbb{R})$). In this case, we say that E has a real nodal singularity.
4. If $\Delta = 0$ and $t_3 > 0$ then $E(\mathbb{R})$ is a union of a point and a projective oval. In this case, we say that E has a complex nodal singularity.
5. If $t_2 = t_3 = 0$ then $E(\mathbb{R})$ look likes a broken line in $\mathbb{R}^2$. In this case, we say that E has a cuspidal singularity.

Note that $E(\mathbb{R})$ intersects the line at infinity only at $[0;1;0]$. To see/prove all the topological statements above, it is enough to take an example in each class and draw the corresponding $E(\mathbb{R})$. Note that in the (t_2,t_3)-space each set defined by the above items is connected and the topology of $E(\mathbb{R})$ does not change in each item (see Figure 4.1 and Equation (4.5), the correspondence between the values of t_2,t_3 and $E_{t_2,t_3}(\mathbb{R})$ are done by colours).

Exercise 4.17 For a smooth elliptic curve E over $\mathbb{R}$ and in the Weierstrass form describe the real curves $E(\mathbb{R})$ inside the torus $E(\mathbb{C})$. Hint: Use the Riemann-Hurwitz formula.

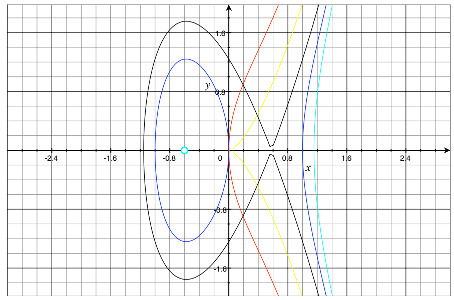


Fig. 4.1 Elliptic curves: $y^2 - x^3 - t_2 x - t_3 = 0$.

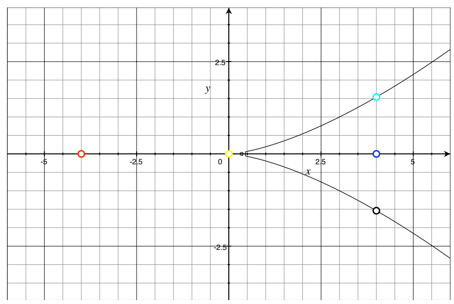


Fig. 4.2 The discriminant curve $4t_2^3 + t_3^2 = 0$.

4.10 The group law in elliptic curves

In this section we define the group structure of an elliptic curve. According to [Hus04, Section 5, page 13] "It was Jacobi [1835] in Du usu Theoriae Integralium Ellipticorum et Integralium Abelianorum in Analysi Diophantea who first suggested the use of a group law on a projective cubic curve". Let E be a smooth cubic curve in $\mathbb{P}^2$. Let also $P, Q \in E(\mathsf{k})$ and L be the line in $\mathbb{P}^2$ connecting two points P and Q. If $P = Q$ then L is the tangent line to E at P. The line L is defined over k and it is easy to verify that the third intersection $R := PQ$ of $E(\bar{\mathsf{k}})$ with $L(\bar{\mathsf{k}})$ is also in $E(\mathsf{k})$. Fix a point $O \in E(\mathsf{k})$ and call it the zero element of $E(\mathsf{k})$. Define

$$P + Q = O(PQ).$$

that is, in order to find $P+Q$ we connect P to Q by a line and find its third interstion point PQ with E. Then we connect PQ to O by a line and find its intersection point with E and call it $P+Q$.

Remark 4.2 For an elliptic curve in the weierstrass form $y^2 = 4x^3 - t_2 x - t_3$, $27t_3^2 - t_2^3 \neq 0$, one usually take $O = [0;1;0]$, that is the point at infinity. In this way th line connecting a point P to O is the perpendicular line to the x axis in the (x, y) chart.

The point O is a smooth point of E and the line at infinity intersects E only at O, therefore, it has a tangency of order 3 with E. This is also called an inflection point of E. Note that we do not need this property of O. For instance, by our definition we can verify $O+O=O$ easily: we draw the tangent line to E at O and find its third intersction point OO with E. The line connecting OO to O (the same line as before) intersects E in the third point O as this line is tangent to E at O. If O is not an inflection point of E then the line L tangent to E at O intersects E at the third point OO. In this case, by the definition of addition we know that $P,-P$ and OO lies in the same line (and not $P,-P,O$). Therefore, in this case for two points $P,Q\in E(\mathsf{k})$ we have $-PQ\ \not P+Q$. In general we have the following statement: O is an inflection point of E if and only if for any line L intersecting E at three points P,Q,R we have $P+Q+R=O$.

Theorem 4.1 *The above construction turns $E(\mathsf{k})$ into a commutative group.*

Proof. The only non-trivial piece of the proof is the associativity property of $+$:

$$(P+Q)+R = P+(Q+R).$$

The proof constitutes of three pieces:

1. Let $P_i=[x_i;y_i;z_i]$ be 8 points in $\mathbb{P}^2(\bar{\mathsf{k}})$ such that the vectors $(x_i^3,\cdots,z_i^3)\in\bar{\mathsf{k}}^{10}$ of monomials of degree 3 in x_i,y_i,z_i are linearly independent. A cubic polynomial F passing through all P_i's corresponds to a vector $a\in\bar{\mathsf{k}}^{10}$ such that $P_i\cdot a=0$ and so the space of such cubic polynomials is two dimensional. This means that there is two cubic polynomial F and G such that any other cubic polynomial passing through P_i's is of the form $\lambda F+\mu G$ and so it crosses a ninth point too.

2. We apply the first part to the eight points

$$O,P,Q,R,PQ,QR,P+Q,Q+R, \tag{4.8}$$

and conclude that $(P+Q)R=P(Q+R)$. Here, we take three generic points P,Q,R. We have to show that these 8 points satisfy the hypothesis of the first item, see Exercise 4.18. Note that from these 8 points it crosses three cubic polynomials: E, the product of lines through $(0,PQ,P+Q)$, (R,Q,QR), $(P(Q+R),P,Q+R)$ and the product of the lines $(0,QR,Q+R)$, (PQ,Q,P), $(P+Q,R,(P+Q)R)$:

$$\begin{bmatrix} P+Q & PQ & O \\ R & Q & QR \\ (P+Q)R, P(Q+R) & P & Q+R \end{bmatrix}$$

Each column or row corresponds to a line.

3. The morphisms $E\times E\times E\to E,\ (P,Q,R)\mapsto(P+Q)+R, P+(Q+R)$ coincides in a Zariski open subset and so they are equal.

Exercise 4.18 Show that for all triples $P,Q,R\in E(\bar{\mathsf{k}})$, except for a finite number, the vectors in $\bar{\mathsf{k}}^{10}$ attached to eight points (4.8) are linearly independent.

Exercise 4.19 On the elliptic curve

$$E : y^2 = x^3 + 17$$

over $\mathbb{Q}$, we have the points

$$P_1 = (-2,3),\ P_2 = (-1,4),\ P_3 = (2,5),\ P_4 = (4,9),\ P_5 = (8,23),$$

$$P_6 = (43,282),\ P_7 = (52,375),\ P_8 = (5234,378661).$$

Verify the following identities:

$$-P_5 = 2P_1,\ P_4 = P_1 - P_3,\ 3P_1 - P_3 = P_7.$$

Prove that $E(\mathbb{Q})$ is freely generated by P_1 and P_3 and there are only 16 integral points $\pm P_i,\ i = 1,2,\ldots,8$. Hint: See [Nag35] and [Sil92a, page 60].

Exercise 4.20 For the elliptic curve $E_n : y^2 = x^3 - n^2 x$ find an explicit formula for the x coordinates of inflection points, see [Kob93b, page 35, problem 4b].

Exercise 4.21 How many elements of $E_n(\mathbb{R})$ are of order $2,3$ and 4? Describe geometrically where these points are located, see [Kob93b, page 36, problem 7].

Exercise 4.22 For an elliptic curve over $\mathbb{R}$ prove that $E(\mathbb{R})$ (as a group) is isomorphic to $\mathbb{R}/\mathbb{Z}$ or $\mathbb{R}/\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, see [Kob93b, page 36, problem 9].

4.11 Riemann-Roch theorem

Let C be a smooth curve over a field k. If the reader is not familiar with the definition of an abstract curve, then he can take C a smooth curve of degree d in $\mathbb{P}^2$. Recall the pole and zero order of a rational function on C presnted in Section 4.5.

Definition 4.10 A divisor in C and defined over k is a formal finite sum $D := \sum_i n_i p_i,\ n_i \in \mathbb{Z},\ p_i \in C(\bar{\mathsf{k}})$ such that p_i's are poles (resp. zeros) of a rational function f (defined over k) on C and n_i is the pole order (resp. zero order) at p.

Note that in the above definition it does not make any difference if we take zeros or poles of f (instead of f we use $\frac{1}{f}$).

Exercise 4.23 A formal finite sum $D := \sum_i n_i p_i,\quad n_i \in \mathbb{Z},\ p_i \in C(\bar{\mathsf{k}})$ is a divisor defined over k if and only if it is invariant under the Galois group $\mathrm{Gal}(\bar{\mathsf{k}}/\mathsf{k})$, that is,

$$\sigma(D) = D,\ \forall \sigma \in \mathrm{Gal}(\bar{\mathsf{k}}/\mathsf{k}),$$

where

$$\sigma(\sum_i n_i p_i) := \sum_i n_i \sigma(p_i).$$

The set of divisors over k, let us denote it by $\mathrm{Div}(C/\mathsf{k})$ form an abelian group in a natural way. For any rational function $f \in \mathsf{k}(C)$ we define

$$\mathrm{div}(f) := \sum_i n_i p_i,$$

where f is of order n_i at p_i. It is a divisor defined over k. The set of such divisors form an abelian group which we denote it by $\mathrm{Div}(\mathsf{k}(C))$. The Picard group of C is defined to be

$$\mathrm{Pic}(C) := \mathrm{Div}(C)/\mathrm{Div}(\mathsf{k}(C)).$$

The Chern class map is defined in the following way

$$c : \mathrm{Pic}(C) \to \mathbb{Z}, \quad c(\sum_i n_i P_i) := \sum_i n_i.$$

Note that by Exercise 4.8, c evaluated on $\mathrm{Div}(\mathsf{k}(C))$ is zero, and so our definition is well-defined. We define

$$\mathrm{Pic}_0(C) := \ker(\mathrm{Pic}(C) \to \mathbb{Z}).$$

Now, assume that C is an elliptic curve E. Recall that by Definition 4.8 one has to take a k-rational point $O \in E(\mathsf{k})$. Take, for instance a degree 3 smooth curve in $\mathbb{P}^2$ with a k-rational point O. We have a canonical map

$$E(\mathsf{k}) \to \mathrm{Pic}_0(E), \quad P \mapsto P - O \tag{4.9}$$

Proposition 4.2 *The map* (4.9) *is an isomorphism of groups.*

Proof. First of all we notice that it is a group morphism. Just for this proof we denote by $\oplus$ the addition structure in $E(\mathsf{k})$. Let L_1, respectively L_2, be the equation of the line in $\mathbb{P}^2$ passing through P, Q, PQ, respectively $O, PQ, P \oplus Q$. We have $\frac{L_1}{L_2} \in \mathsf{k}(E)$ with the divisor

$$P + Q + PQ - O - PQ - P \oplus Q$$

and so in $\mathrm{Pic}_0(E)$ we have $P - O + Q - O = P \oplus Q - O$. Now we prove that our map is surjective. For $\sum_i n_i P_i$ with $\sum_i n_i = 0$ we have $\sum_i n_i P_i = \sum_i n_i (P_i - O)$ which is the image of $\oplus_i n_i P_i$. For the injectivity we note that for $P \in E(\mathsf{k})$ different from O, there is no rational function f on E with $\mathrm{div}(f) = P - O$. This can be easily checked for curves

The following is the algebraic counterpart of Exercise 2.17.

Exercise 4.24 Let (E, O) be an elliptic curve over a field k of characteristic zero. For a torsion point of order N the line bundle associated to the divisor $N[P] - N[O]$ is trivial, that is, there is a rational function f on E defined over k such that $\mathrm{div}(f) = N[P] - N[O]$. For instance, if E is written in the Weierstrass format $y^2 = p(x)$ then for $N = 2$ we have $f = x - a$, where a is a root of p. Compute f for $N = 3$. Hint: The rational function f is a product of $\frac{L_1}{L_2}$ attached to the the equality $P + aP = (a+1)P$ in the proof of Proposition 4.2.

Originally, I formulated Exercise 4.24 after reformulating Picard's differential equation in [Mov22b], see also Chapter 13. Later I realized that [Mil04], [Sil92a, Theorem XI.8.1] give algorithms to compute f.

Definition 4.11 We say that a divisor $D=\sum_i n_i p_i$ is positive and write $D \geq 0$ if all coefficients n_i are non negative integers. In a similar way we define $D \leq 0$.

For a divisor D on a curve C/k define the linear system

$$\mathscr{L}(D)=\{f \in \mathsf{k}(C),\ f \neq 0 \mid \operatorname{div}(f)+D \geq 0\} \cup \{0\}$$

and

$$l(D)=\dim_{\mathsf{k}}(\mathscr{L}(D)).$$

Theorem 4.2 *(Riemann-Roch theorem) Let C be a smooth curve over k.*

$$l(D)-l(K-D)=\deg(D)-g+1,$$

where K is the canonical divisor and g is the genus of C.

We only need to know that the canonical divisor satisfies:

$$\deg(K)=2g-2$$

and so for $\deg(D)>2g-2$, equivalently $\deg(K-D)<0$, we have

$$l(D)=\deg(D)-g+1. \tag{4.10}$$

4.12 Weierstrass form revised

In this section we prove that any elliptic curve can be realized as a certain curve in $\mathbb{P}^2$ which is a generalization of Weierstrass format. The following proposition is proved in [Sil92a, III, Proposition 3.1].

Proposition 4.3 *Let E be an elliptic curve over a field k. There exist functions $x, y \in \mathsf{k}(E)$ such that the map*

$$E \rightarrow \mathbb{P}^2,\ a \mapsto [x(a); y(a); 1],$$

give an isomorphism of E/k onto a curve given by

$$y^2+a_1 xy+a_3 y=x^3+a_2 x^2+a_4 x+a_6,\ a_1, \cdots, \ldots, a_6 \in \mathsf{k},$$

sending O to $[0;1;0]$. If further $\operatorname{char}(\mathsf{k}) \neq 2,3$ we can assume that the image curve is given by

$$y^2=4x^3-t_2 x-t_3,\ t_2, t_3 \in \mathsf{k}, t_2^3-27t_3^2 \neq 0.$$

We call x and y the Weierstrass coordinates of of E.

Proof. Using Riemann-Roch theorem and in particular (4.10) with $g=1$ and $D=nO$ we get $l(D)=n$. For $n=2$ we can choose $x,y\in \mathsf{k}(E)$ such that $1,x$ form a basis of $\mathscr{L}(2O)$ and $1,x,y$ form a basis of $\mathscr{L}(3O)$. The function x (resp. y) has a pole of order 2 (resp. 3) at O. Now $\mathscr{L}(6O)$ has dimension 6 and $1,x,y,x^2,xy,y^2,x^3\in\mathscr{L}(6O)$. It follows that there is a relation

$$ay^2+a_1xy+a_3y=bx^3+a_2x^2+a_4x+a_6,\ a_1,\cdots,\ldots,a_6,a,b\in\mathsf{k}.$$

Note that $ab\neq 0$, otherwise every term would have a different pole order at O and so all the coefficients would vanish. Multiplying x,y with some constants and dividing the whole equation with another constant, we get the desired equation. The map induced by x and y is the desired map (check the details).

If $\mathrm{char}(\mathsf{k})\neq 2,3$ we make the change of variables $x'=x, y'=y-\frac{a_1x}{2}$ and we eliminate xy term. A change of variables $x'=x-\frac{a_2}{3},\ y'=y-\frac{a_3}{2}$ will eliminate x^2 and y terms.

Exercise 4.25 Write the following elliptic curves in the Weierstrass form:

$$y^2=x^4-1,\ O=[0;1;0]$$

$$x^3+y^3=1,\ O=[0;1;1]$$

Exercise 4.26 The intersection of two quadrics in $\mathbb{P}^3$ turns out to be a genus one curve, and hence, if we pick a point O in it, it is an elliptic curve. Discuss the Weierstrass form of these elliptic curves. Hint: Have a look at [Hus04, Section 8, page 21].

We will need the following refinement of Proposition 4.3.

Proposition 4.4 *Let E be an elliptic curve over a field k of characteristic $\neq 2,3$ and let ω be a regular differential form on E. There exist unique functions $x,y\in\mathsf{k}(E)$ such that the map*

$$E\to\mathbb{P}^2,\ a\mapsto[x(a);y(a);1]$$

gives an isomorphism between the curve E and the curve in $\mathbb{P}^2$ given by

$$y^2=4x^3-t_2x-t_3,\ t_2,t_3\in\mathsf{k}$$

Under this isomorphism O is identified with $[0;1;0]$ and $\omega=\frac{dx}{y}$.

We call x and y the Weierstrass coordinates of E. Since $x,y\in\mathsf{k}(E)$ the above isomorphism is defined over k. Note that $\frac{x}{y}$ has a zero of order one at O and hence the map $E\to\mathbb{P}^2$ is well-defined at O and it takes the value $[\frac{x}{y}(O);1;\frac{1}{y}(O)]=[0;1;0]$.

Proof. This is a consequence of Proposition 4.3.

4.13 Moduli of elliptic curves

Now, we can state what is the moduli of elliptic curves. Recall an elliptic curve E_{t_2,t_3} in Weierstrass form (4.7).

Proposition 4.5 *Assume that* $\mathrm{char}(\mathsf{k}) \neq 2,3$. *Two elliptic curves* E_{t_2,t_3} *and* $E_{t'_2,t'_3}$ *are isomorphic if and only if there exists* $\lambda \in \mathsf{k},\ \lambda \neq 0$ *such that*

$$t'_2 = \lambda^4 t_2,\ t'_3 = \lambda^6 t_3.$$

The isomorphism is given by

$$(x,y) \mapsto (\lambda^2 x, \lambda^3 y).$$

Proof. Let (x,y) and (x',y') be two sets of Weierstrass coordinate functions on an elliptic curve E_{t_2,t_3}. It follows that $\{1,x\}$ and $\{1,x'\}$ are both bases of $\mathscr{L}(2O)$, and similarly $\{1,x,y\}$ and $\{1,x',y'\}$ are both bases for $\mathscr{L}(3O)$. Writing x',y' in terms of x,y and substituting in the equation of $E_{t'_2,t'_3}$ we get the first affirmation of the proposition. The second affirmation is easy to check.

Combining Proposition 4.3 and Proposition 4.5 we conclude that the moduli space of elliptic curves over a field of characteristic $\neq 2,3$ is

$$\mathscr{M}_1(\mathsf{k}) := \left(\mathbb{A}^2(\mathsf{k}) - \{(t_2,t_3) \mid 4t_2^3 + 27t_3^2 = 0\}\right) / \sim,$$

where

$$(t_2,t_3) \sim (t'_2,t'_3) \text{ if and only if } \exists \lambda \in \mathsf{k},\ \lambda \neq 0,\ (t'_2,t'_3) = (\lambda^4 t_2, \lambda^6 t_3).$$

If k is algebraically closed then this is the set of k-rational points of the weighted projective space $\mathbb{P}^{2,3}(\mathsf{k})$ minus a point induced in $\mathbb{P}^{2,3}$ by $\Delta = 0$. In this case the j-invariant of elliptic curves

$$j : \mathscr{M}_1(\mathsf{k}) \to \mathbb{A}(\mathsf{k}),\ j[t_2;t_3] = \frac{1728 \cdot 4t_2^3}{4t_2^3 + 27t_3^2}$$

is an isomorphism and so the moduli of elliptic curves over k is $\mathbb{A}^1(\mathsf{k})$. However, note that if k is not algebraically closed then j has non-trivial fibers. For instance, all the elliptic curves

$$y^2 = x^3 - t_3,\ t_3 \in \mathbb{Q}$$

are isomorphic over $\bar{\mathbb{Q}}$ but not over $\mathbb{Q}$.

Exercise 4.27 For $j_0 \neq 0, 1728$, the elliptic curve:

$$E_{j_0} : y^2 + xy = x^3 - \frac{36}{j_0 - 1728} x - \frac{1}{j_0 - 1728}.$$

has the j-invariant $j(E) = j_0$.

4.14 The addition formula

In this section we give formulas for the addition in an elliptic curve in the Weierstrass format and defined over a field. We also derive similar formulas for the Weierstrass $\wp$ function. The comparison between algebraic and transcendental methods in order to describe algebraic structures has been the core tool in the developement of elliptic curves from the old time, see for instance [Fri22, Chapter 1, Section 1, page 156].

Recall the Weierstrass uniformization theorem Theorem 3.3 which says that

$$f : \mathbb{C}/\Lambda \to E_t(\mathbb{C}), \;\; f(z) := [\wp(\Lambda, z) : \wp'(\Lambda, z) : 1]$$

is a bijection and described its inverse. The torus $\mathbb{C}/\Lambda$ is equipped with an abelian group structure induced from $(\mathbb{C}, +)$ and in this chapter we have seen than $E_t(\mathbb{C})$ also enjoys a an abelian group structure. The main goal of the present section is to prove that

Theorem 4.3 *The map f in Weierstrass uniformization theorem is a morphism of abelian groups.*

Proof. Since the inverse of z (resp. (x,y)) in $\mathbb{C}/\Lambda$ (resp. E_t) is $-z$ (resp. $(x,-y)$), and we have $\wp(-z) = \wp(z)$ and $\wp'(-z) = -\wp'(z)$, we know that f respects the inverse element. Therefore, it is enough to prove that

$$f(-z_1 - z_2) + f(z_1) + f(z_2) = O, \; \forall z_1, z_2 \in \mathbb{C}/\Lambda.$$

By definition of the group structure of E_t this is equivalent to say that $P := f(z_1),\; Q := f(z_2)$ and $R = f(-z_1 - z_2)$ are colinear. In other words, if we intersect a line $ax + by + c = 0,\; b \neq 0$ in $\mathbb{C}^2$ with E_t and get three points P, Q, R then by definition of group structure of E_t we know that $P + Q + R = O$ (the case $b = 0$ reduces to the discussion of f and inverse elements at the begining of the proof). We need to prove that if z_1, z_2, z_3 are three roots of $a\wp(z) + b\wp'(z) + c = 0$ then we must have $z_1 + z_2 + z_3 = 0$. The equation of line in $\mathbb{C}^2$ with (x,y) coordinate system and passing through the points $(\wp(z_1), \wp'(z_1))$ and $(\wp(z_2), \wp'(z_2))$ is given by

$$\begin{vmatrix} \wp(z_1) & \wp'(z_1) & 1 \\ \wp(z_2) & \wp'(z_2) & 1 \\ x & y & 1 \end{vmatrix} = 0.$$

We need to prove that $(\wp(z_3), \wp'(z_3)) = (\wp(z_1 + z_2), -\wp'(z_1 + z_2))$ lies in this line. The theorem follows from Proposition 4.6.

Proposition 4.6 *The Weierstrass $\wp$-function satisfies*

$$\begin{vmatrix} \wp(z), & \wp'(z) & 1 \\ \wp(y), & \wp'(y) & 1 \\ \wp(z+y), & -\wp'(z+y), & 1 \end{vmatrix} = 0, \;\; z, y \in \mathbb{C}. \tag{4.11}$$

Proof. For a fixed $y \in \mathbb{C},\ y \neq 0$, let us consider $\wp(z+y)$ and $\wp'(z+y)$ as functions in z. Let f be the left hand side of 4.11. It is an elliptic function with possible poles at $z = 0, -y$. It is enough to prove that f is holomorphic at these points and it vanishes at $z = 0$. This follows from the Laurant series of $\wp$ at $z = 0$ and its derivation, see Theorem 2.1. We only need to know

$$\wp(z) = \frac{1}{z^2} + 3g_2 z^2 + O(z^4),\ \ \wp'(z) = \frac{-2}{z^3} + 6g_2 z + O(z^3).$$

Proposition 4.7 *We have*

$$\wp(z+y) = \frac{1}{4}\left(\frac{\wp'(z) - \wp'(y)}{\wp(z) - \wp(y)}\right)^2 - \wp(z) - \wp(y). \tag{4.12}$$

Proof. For a fixed $y \in \mathbb{C},\ y \neq 0$, let us consider $\wp(z+y)$ which is double periodic in z, and hence, it is a rational function in $\wp$ and $\wp'$. This means that we (4.12) is expected. For fixed y, let $f(z)$ be the difference between the left and the right hand sides of (4.12). Its only possible poles are in

$$z = 0,\ \pm y.$$

We examine the Laurent expansion of $f(z)$ at the point $z = 0$ and see that it is holomorphic at $z = 0$ and there it vanishes. In a similar way, it has no poles at $z = y$ and so, at worst it has a simple pole at $z = -y$. Since f is double periodic we get the result.

Exercise 4.28 Write down the details of the proof of Proposition 4.7.

In (4.12) we let y go to z and we get

$$\wp(2z) = \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 - 2\wp(z). \tag{4.13}$$

We can state the same formulas as in (4.11),(4.12) and (4.13) in a purely algebraic context. Consider an elliptic curve E_t over the field k of characteristic $\neq 2$ with the Weierstrass coordinates x and y and the equation $y^2 = 4x^3 - t_2 x - t_3$. The field can be of an arbitrary characteristic $\neq 2$ and it is not necessarily algebraically closed. For a point $P \in E_t(\mathsf{k})$ we write $P = (x(P), y(Q))$, as x, y ara rational functions in E with poles at O, and hence, they can be evaluated at P.

Proposition 4.8 *We have*

$$\begin{vmatrix} x(P), & y(P) & 1 \\ x(Q), & y(Q) & 1 \\ x(P+Q) & -y(P+Q), & 1 \end{vmatrix} = 0, \tag{4.14}$$

$$x(P+Q) = \frac{1}{4}\left(\frac{y(P)-y(Q)}{x(P)-x(Q)}\right)^2 - x(P) - x(Q), \tag{4.15}$$

$$x(2P) = \frac{1}{4}\left(\frac{6x(P)^2 - \frac{1}{2}t_2}{y(P)}\right)^2 - 2x(P). \tag{4.16}$$

Proof. The proof of (4.14) is trivial as the equality says that the three points P, Q, PQ are colinear. Note that $x(PQ) = x(P+Q)$ and $y(PQ) = -y(P+Q)$. The proof of the others is left to the reader.

Exercise 4.29 Prove the equalities (4.15) and (4.16) for an elliptic curve E over a field of arbitrary characteristic and using the definition of the group structure of E in Section 4.10.

Chapter 5
Mordell-Weil Theorem

Mathematicians have been familiar with very few questions for so long a period with so little accomplished in the way of general results, as that of finding the rational solntions, or say for shortness, the solutions of indeterminate equations of genus unity of the forms ... (L. J. Mordell in [Mor22, page 179]).

5.1 Introduction

In this chapter we prove the Mordell-Weil theorem. Some parts of the proof is left to the reader so that he reproduce them using classical references such as [Sil92b].

Theorem 5.1 (Mordell-Weil theorem) *For an elliptic curve E over a number field $\mathbb{Q}$ the group $E(\mathbb{Q})$ of $\mathbb{Q}$-rational points is finitely generated abelian group.*

The above theorem is proved in [Mor22]. Its generalization for an arbitrary number field is proved in [Wei29] and it is known as the Mordell-Weil theorem. For the proof we follow [Lan78a, Hus04]. The book [Ser97] is also recommended for those who wants to understand this theorem for abelian varieties. Theorem 5.1 implies that the torsion subgroup of $E(\mathbb{Q})$

$$E(\mathbb{Q})_{\text{tors}} := \{P \in E(\mathbb{Q}) \mid nP = 0, \text{ for some } n \in \mathbb{N}\},$$

is finite and $E(\mathbb{Q})_{\text{free}} := E(\mathbb{Q})/E(\mathbb{Q})_{\text{tors}}$ is a freely generated $\mathbb{Z}$-module of finite rank, let us say $r \in \mathbb{N}$.

Definition 5.1 The non-negative integer r is called the rank of $E(\mathbb{Q})$.

The free part of $E(\mathbb{Q})$ is mysterious. We do not know whether there exists an elliptic curve of arbitrary rank or not. The proof of Mordell's theorem consist of two steps. 1: $E(\mathbb{Q})/2E(\mathbb{Q})$ is finite group. This is sometimes called the weak Mordell theorem. 2. The existence of a height function on $E(\mathbb{Q})$.

Exercise 5.1 Show that for an elliptic curve defined over $\mathbb{C}$ (resp. $\bar{\mathbb{Q}}$) $E(\mathbb{C})$ (resp. $E(\bar{\mathbb{Q}})$) is not finitely generated

5.2 Descent theorem

Method of infinite descent, which is also called Fermat's method of descent, has been extensively used in elementray number theory in order to prove statements like:

Exercise 5.2 The square root of two is irrational and $x^4 + y^4 = w^2$ has no non-zero solutions in pairwise coprime integers x, y, z.

Exercise 5.3 Show that the only solutions of $2y^2 = x^4 + 1$ over rational numbers are $(\pm 1, \pm 1)$. This example is due to Euler, see [SS96, page 61].

This method seems to have originated the height notion.

Theorem 5.2 *(Descent theorem). Let Γ be a commutative group. Suppose that there is a function $h : \Gamma \to [0, \infty)$ such that*

1. *For any real number M, the set $\{P \in \Gamma \mid h(P) \leqslant M\}$ is finite.*
2. *For every $P_0 \in \Gamma$, there is a constant k_0 such that*

$$h(P + P_0) \leq 2h(P) + k_0 \quad \forall P \in \Gamma.$$

3. *There is a constant k such that*

$$h(2P) \geq 4h(P) - k \quad \forall P \in \Gamma.$$

4. $|\Gamma/2\Gamma| < \infty$.

Then Γ is finitely generated.

Definition 5.2 The function h in Theorem 5.2 with the properties $1, 2, 3$ and 4 is called the height funtion.

Proof. Let $\Gamma/2\Gamma = Q_1,\ Q_2, \ldots,\ Q_n$. For every $P \in \Gamma$ there exists Q_{i_1}, depending on P such that $P - Q_{i_1} = 2P_1 \in 2\Gamma$. We repeat this for P_2 and get:

$$\begin{cases} P - Q_{i_1} = 2P_1, \\ P_1 - Q_{i_2} = 2P_2, \\ \qquad \vdots \\ P_{m-1} - Q_{i_m} = 2P_m. \end{cases}$$

This implies that

$$P = Q_{i_1} + 2Q_{i_2} + 2^2 Q_{i_3} + \cdots + 2^{m-1} Q_{i_m} + 2^m P_m.$$

Now we use the height function . We have

$$h(P-Q_i) \leqslant 2h(P)+k' \qquad \forall P \in \Gamma \qquad i=1,\dots,n$$

where k' is the maximum of k_i's attached to each Q_i in property 2 of h. We use property 3 of h and we have

$$\begin{aligned} 4h(P_j) &\leqslant h(2P_j)+k = h\left(P_{j-1}-Q_{i_j}\right)+k \\ &\leqslant 2h(P_{j-1})+k'+k. \end{aligned}$$

We conclude that

$$\begin{aligned} h(P_j) &\leqslant \frac{1}{2}\,h\left(P_{j-1}\right)+\frac{k+k'}{4} \\ &= \frac{3}{4}\,h(P_{j-1}) - \frac{1}{4}\bigg(h(P_{j-1})-(k+k')\bigg). \end{aligned}$$

Therefore, if $h(P_{j-1}) \geqslant k+k'$ then

$$h(P_j) \leqslant \frac{3}{4}\,h(P_{j-1}).$$

We do this process until for some $m \quad h(P_m) \leqslant k+k'$ and so by property 1 of h the set of such P_m is finite. We conclude that Γ is generated by

$$Q_1,\, Q_2,\dots,\, Q_n,\ \{P \in \Gamma \,\Big|\, h(P) \leqslant k+k'\}.$$

For more on descent procedure see [Sil92b, page 199 chapter VIII].

5.3 The construction of height function

For a rational number $x := \frac{m}{n}$ with $(m,n)=1$ let us define $H(x)$ to be the maximum of the norms of its nominator and denominator:

$$H(x) = \max\{|m|,\, |n|\}.$$

Let E be an elliptic curve over $\mathbb{Q}$ in the Weierstrass format $y^2 = x^3 - t_2x - t_3,\ t_2,t_3 \in \mathbb{Q}$. By a linear change of the form $(x,y) \mapsto (a^2x, a^3y),\ \ a \in \mathbb{Q}$, we can assume that $t_2,t_3 \in \mathbb{Z}$. We construct a height function for $E(\mathbb{Q})$. For $P=(x,\,y) \in E(\mathbb{Q})$ we define

$$\begin{aligned} &H(P) := H(x) = \max\{|m|,\, |n|\}, \\ &h(P) := \log H(P). \end{aligned}$$

Let

$$P = (x,y) = \left(\frac{m}{e^2},\ \frac{n}{e^3}\right) \in E(\mathbb{Q}), \quad m,n,e \in \mathbb{Z}. \tag{5.1}$$

and so $n^2 = m^3 + t_2 e^4 m + t_3 e^6$. From this equality, we can assume that that $(m,e^2) = 1$. If there is a prime p such that $p|m$ and $p|e$, this equality implies that $p^2|m, p^3|n$ and we can replace m,n,e with $\frac{m}{p}$, $\frac{n}{p}$ and $\frac{e}{p}$. We have

$$\begin{aligned} |n|^2 &\leqslant |m|^3 + |t_2|\, e^4 |m| + |t_3|\, e^6 \\ &\leqslant k^2\, H(P)^3 \end{aligned}$$

and so $|n| \leqslant kH(P)^{\frac{3}{2}}$, where k is a constant number which only depends on E. Let $P_0 = (x_0, y_0) \in E(\mathbb{Q})$ be another point. We would like to estimate $h(P+P_0)$ from above in terms of P_0 and the data of E. We first use the addition formula, see Proposition 4.7:

$$\begin{aligned} x(P+P_0) &= \left(\frac{y-y_0}{x-x_0}\right)^2 - x - x_0 \\ &= \frac{(y-y_0)^2 - (x-x_0)^2(x+x_0)}{(x-x_0)^2} \\ &= \frac{Ay + Bx^2 + Cx + D}{Ex^2 + Fx + G}. \end{aligned}$$

Here, we have used $y^2 = x^3 - t_2 x - t_3$ and $A,\ B, \ldots,\ G$ are constants depending only on E and P_0. We write P in the format (5.1) and we have

$$H(P+P_0) \leqslant \max\left\{ \left|Ane + Bm^2 + Cme^2 + De^4\right|,\ \left|Em^2 + Fme^2 + Ge^4\right| \right\}.$$

But, we know that

$$|e| \leqslant H(P)^{\frac{1}{2}},\ |n| \leqslant k \cdot H(P)^{\frac{3}{2}},\ |m| \leqslant H(P),$$

which implies that

$$H(P+P_0) \leqslant K_0 H(P)^2,$$

where K_0 is a constant term depends only on E and P_0. Taking the logarithm of this, we have the property 2 of h. Let us prove the property 3 of h. We use again the addition formula and let x goes to x_0. We get

$$x(2P) = \frac{f'(x)^2}{4f(x)} - 2x,$$

where $f(x) := x^3 - t_2 x - t_3$ and $y^2 = f(x)$ is the elliptic curve E. We assume that $2P \neq 0$

$$x(2P) = \frac{P(x)}{Q(x)},\ \deg P(x) = 4,\ \deg Q(x) = 3,$$

where the coefficients of P, Q only depends on the elliptic curve. Let $\Delta := 4(27t_3^2 - 4t_2^3)$ be the discriminant of E. This is the resultant of $P(x)$ and $Q(x)$. This means that

$$f_1(x)P(x) + f_2(x)Q(x) = \Delta,$$

$$f_1,\ f_2 \in \mathbb{Z}[x],\ \deg\ f_1,\ \deg\ f_2 \leqslant 3.$$

We need also

$$g_1(x)P(x) + g_2(x)Q(x) = \Delta \cdot x^7,$$

$$g_1,\ g_2 \in \mathbb{Z}[x],\ \deg\ g_1,\ \deg\ g_2 \leqslant 3.$$

This can be considered as the resultant of $x^4 P\left(\frac{1}{x}\right),\ x^4 Q\left(\frac{1}{x}\right)$, see Exercise 4.10. From now one we replace a polynomial $p(x)$ of degree d and in one variable x, with its homogenization $P(x,y) := y^d p(\frac{x}{y})$ and use the capital letter P. With this notation, let $x = \frac{a}{b}$, $(a,b) = 1$ and

$$x(2P) = \frac{F(a,b)}{G(a,b)} \quad , \quad \delta := \gcd\Big(F(a,b),\ G(a,b)\Big).$$

Therefore,

$$\begin{aligned} F_1(a,b)\, F(a,b) + F_2(a,b)\, G(a,b) &= 4\Delta b^7, \\ G_1(a,b)\, F(a,b) + G_2(a,b)\, G(a,b) &= 4\Delta a^7. \end{aligned} \tag{5.2}$$

This gives $\delta \big| 4\Delta$ and $\big|\delta\big| \leqslant \big|4\Delta\big|$. Therefore,

$$H(2P) \geqslant \max\ \{\ \big|F(a,b)\big|,\ \big|G(a,b)\big|\ \}\ /\ \big|4\Delta\big|.$$

On the other hands

$$\begin{aligned} \big|4\Delta\ b^7\big| &\leqslant 2\max\{\ \big|F_1\big|,\ \big|F_2\big|\ \}\max\ \{\ \big|F\big|,\ \big|G\big|\}, \\ \big|4\Delta\ a^7\big| &\leqslant 2\max\{\ \big|G_1\big|,\ \big|G_2\big|\ \}\max\ \{\ \big|f\big|,\ \big|G\big|\}, \end{aligned} \tag{5.3}$$

where $F_1,\ F_2 \dots$ are evaluated at (a,b). Now $F_1,\ F_2,\ G_1,\ G_2$ are polynomials of degree $\leqslant 3$ and so

$$\max\ \{\ \big|F_1\big|,\ \big|F_2\big|,\ \big|G_1\big|,\ \big|G_2\big|\ \} \leqslant C\max\ \{\ \big|a\big|^3,\ \big|b\big|^3\ \}, \tag{5.4}$$

where C is a constant which only depends on E. Combining (5.3) and (5.4) we get

$$\big|4\Delta\big|\ \max\ \{\ \big|a^7\big|,\ \big|b^7\big|\ \} \leqslant 2C \cdot \max\{\ \big|a^3\big|,\ \big|b^3\big|\ \}\ \max\ \{F(a,b),\ G(a,b)\}$$

and so

$$H(2P) = \tfrac{\max\{\left|F(a,b)\right|,\left|G(a,b)\right|\}}{\delta} \geqslant \tfrac{\max\{\left|F(a,b)\right|,\left|G(a,b)\right|\}}{4\Delta}$$

$$\geqslant (2C)^{-1}\max\{\big|a\big|,\big|b\big|\} = (2C)^{-1}H(P).$$

Exercise 5.4 Write a summary of a different proof of Mordell-Weil theorem in the literature.

Exercise 5.5 *Let E be an elliptic curve over $\mathbb{Q}$. For all $P_1, P_2 \in E(\mathbb{Q})$ we have*

$$h(P_1+P_2)+h(P_1-P_2) \leqslant 2h(P_1)+2h(P_2)+k, \tag{5.5}$$

where k only depends on E. Hint: see [Sil92b, page 216].

Proposition 5.1 . *Let E be an elliptic curve over $\mathbb{Q}$ and $P \in E(\mathbb{Q})$. The limit*

$$\hat{h}(P) := \frac{1}{2} \lim_{N\to\infty} 4^{-N} h(2^N P)$$

exists. The function $\hat{h}$ is called the canonical or Neron-Tate height of E.

Proof. The proof is taken from from [Sil92b, Proposition 9.1, page 228]. We show that the sequence in the limit is a Cauchy sequence. In (5.5) we put $P = P_1 = P_2$

$$\left| h(2P) - 4h(P) \right| \leqslant k,$$

where k is a constant depending only on k. For $N \geqslant M \geqslant 0$ integers, we have

$$\begin{aligned}
\left| 4^{-N} h(2^N P) - 4^{-M} h(2^M P) \right| &= \left| \sum_{n=M}^{N-1} 4^{-n-1}\, h(2^{n+1}P) - 4^{-n}\, h(2^n P) \right| \\
\leqslant \sum_{n=M}^{N-1} 4^{-n-1} \left| h(2^{n+1}P) - 4h(2^n P) \right| &\leqslant \sum_{n=M}^{N-1} 4^{-n-1}\, k \\
&\leqslant \tfrac{k}{4^{M+1}}.
\end{aligned}$$

Exercise 5.6 The canonical height $\hat{h}$ satisfies

1. For all $P, Q \in E(\mathbb{Q})$
$$\hat{h}(P+Q) + \hat{h}(P-Q) = 2\hat{h}(P) + 2\hat{h}(Q).$$
2. For all $P \in E(\mathbb{Q})$ and $m \in \mathbb{Z}$
$$\hat{h}(mP) = m^2 \hat{h}(P).$$
3. $\hat{h}$ is a quadratic form in $E(\mathbb{Q})$, that is, $\hat{h}$ is even and the pairing
$$\begin{aligned} &\langle \cdot\,,\, \cdot \rangle : E(\mathbb{Q}) \times E(\mathbb{Q}) \to \mathbb{R}, \\ &\langle P, Q \rangle = \hat{h}(P,Q) - \hat{h}(P) - \hat{h}(Q). \end{aligned}$$
is bilinear.

4. For $P \in E(\mathbb{Q})$ we have $\hat{h}(P) \geqslant 0$ and $\hat{h}(P) = 0$ if and only if P is a torsion point.
5. $2\hat{h} - h$ is bounded on $E(\mathbb{Q})$.

Hint: See [Sil92b, Theorem 9.3 page 229].

Definition 5.3 Let $P_1, P_2, \ldots P_\gamma$ be a basis of the free part of $E(\mathbb{Q})$ The regulator of $E/\mathbb{Q}$ is defined

$$R_{E/\mathbb{Q}} := \det \left| \langle P_i, P_j \rangle \right|.$$

Exercise 5.7 (Weak Mordell-Weil theorem) Let E be an elliptic curve over a number field k. Then $E(\mathsf{k})/mE(\mathsf{k})$ is finite for all $m \in \mathbb{N}$. Hint: For a proof see [Hus04, Chapter 6 Section 3]. The proof uses an explicit 2-isogeny, see [Hus04][Chapter 4, Section 5].

Elliptic curves over rational numbers and of high rank are rare, and searching for them is of major interest in the theory of elliptic curves. For an overview of the results and also the last rank record see [Elk07].

Exercise 5.8 For a table of rank of elliptic curves of the format $y^2 = x^3 + ax,\ y^2 = x^3 + a,\ \ a \in \mathbb{Z}$ see [Hus04, Section 3, page 37]. Check at least one entry of this table.

Remark 5.1 The rank of elliptic curves over function fields, such as $\mathbb{C}(t)$, has many links to algebraic geometry of elliptically fibered surfaces and their Hodge theory. In [Shi86, Shi92] it is proved that the elliptic curve $E_d : y^2 + x^3 + z^d - 1,\ \ d \geq 2$ over $\mathbb{C}(z)$ has rank ≤ 68. The equality happens if and only if d is divisible by 360. Using a computer implementation of Hodge cycles, one can show, for instance

$$\mathrm{rank}(E_{90}) = 36,\ \mathrm{rank}(E_{120}) = 56,\ \mathrm{rank}(E_{180}) = 60,\ \ \mathrm{rank}(E_{360}) = 68,$$

see [Mov21, Section 15.13]. There is an analogy between a Hodge cycle and a rational point of an elliptic curve, for instance, the self-intersection of a Hodge cycle is parallel to the canonical height of a rational point. For the geometric context we have the notion of dimension of the tangent space of a Hodge locus. The author does not know any parallel of this in the arithmetic side.

Chapter 6
Torsions and isogeny

...number theory swarms with bugs, waiting to bite the tempted flower-lovers who, once bitten, are inspired to excesses of effort! (B. Mazur in [Maz91]).

6.1 Introduction

Torsion points from the transcendental point of view are related to the computability of elliptic integrals. We have formulated this in Exercise 6.3. For instance, it says that for any integer $m \in \mathbb{N},\ m \geq 2$ and $t \in \mathbb{R},\ t > 1$, there is a point $b \in \mathbb{R},\ b > t$ such that we have an equality of the form

$$m \int_{b}^{+\infty} \frac{dx}{\sqrt{x(x-1)(x-t)}} = \int_{0}^{1} \frac{dx}{\sqrt{x(x-1)(x-t)}}. \tag{6.1}$$

Both integrations are taking place in the real line. For instance, for $m = 2$ we can take $b = t$. In general b will be a x coordinate of a m-torsion point of the elliptic curve $y^2 = x(x-1)(x-t)$. It is a matter of studying history of mathematics to find out this aspect of torsion points in the huge amount of literature produced in the 19th century on elliptic integrals. On the other hand the arithmetic study of torsion points seems to be initiated by Beppo Levi, who produced a series of paper from 1906 to 1911. This has been glorified with a complete classification of the group of torsion points of elliptic curves over rationals in [Maz77, Maz78]. For a historical account on this see [SS96].

6.2 Torsion points

Using the period map in Theorem 3.4 we have seen a correspondence between the space of lattices $\Lambda \subset \mathbb{C}$ and the space of pairs (E, ω), where E is an elliptic curve over $\mathbb{C}$ and ω is a regular differential 1-form. Since we have not defined these objects intrinsically, we may take E in the Weierstrass format $y^2 = 4x^3 - t_2x - t_3$, and $\omega = \frac{dx}{y}$. Note that in one side of this correspondance, we can talk about pairs (E, ω) defined over an arbitrary field k.

Definition 6.1 A pair (E, ω) is called an enhanced elliptic curve. There will be other enhancements of elliptic curves, and in order to reduce confusion, we say that E is enhanced with a regular differential 1-form.

Definition 6.2 Let E be an elliptic curve over a field k. The set of m-torsions of E is defined as

$$E[m] := \{P \in E(\mathsf{k}) \mid mP = O\}.$$

This is a subgroup of $E(\mathsf{k})$.

Proposition 6.1 *Let E be an elliptic curve over a field k of characteristic zero. We have an embedding of groups*

$$E[m] \hookrightarrow \frac{\mathbb{Z}}{m\mathbb{Z}} \times \frac{\mathbb{Z}}{m\mathbb{Z}}$$

and for k an algebraically closed field, this is an isomorphism.

Proof. For k an arbitrary field, since we have the embedding $E(\mathsf{k}) \hookrightarrow E(\bar{\mathsf{k}})$ which is a morphism of groups, it is enough to prove only the second part of the proposition. We first prove it for $\mathsf{k} = \mathbb{C}$. By Proposition 4.3, we can write E in the Weierstrass format. By Weierstrass uniformization theorem Theorem 3.3 and the fact that it is a morphism of groups, see Theorem 4.3, we can assume that E is the complex torus $\mathbb{C}/\Lambda$. In this case

$$E[m] \simeq \frac{1}{m}\Lambda/\Lambda \simeq \frac{\mathbb{Z}}{m\mathbb{Z}} \times \frac{\mathbb{Z}}{m\mathbb{Z}}.$$

Now, assume that k is an arbitrary algebraically closed field of characteristic zero. The argument is a typical example of the so called Lefschetz principle. Since E uses a finite number of elements of k, we replace k with its subfield generated over $\mathbb{Q}$ by the coefficients of E, and we can assume that there is an embedding of fields $\sigma : \mathsf{k} \hookrightarrow \mathbb{C}$, see Exercise 6.1. For instance, if E is in the Weierstrass format $y^2 = 4x^3 - t_2x - t_3,\ t_2, t_3 \in \mathsf{k}$, we replace k with $\overline{\mathbb{Q}(t_2, t_3)} \subset \mathsf{k}$. Let E_σ be the elliptic curve over $\mathbb{C}$ obtained from E and regarding its coefficients as complex numbers. We have an embedding of groups

$$E[m] \hookrightarrow E_\sigma[m] \simeq \frac{\mathbb{Z}}{m\mathbb{Z}} \times \frac{\mathbb{Z}}{m\mathbb{Z}}.$$

Now, let us take E over a characteristic zero and algebraicaly closed field k. We have a subfield $\check{\mathsf{k}} \subset \mathsf{k}$ such that E is defined over $\check{\mathsf{k}}$ and we call it $\check{E}$. We have $(\mathbb{Z}/m\mathbb{Z})^2 \cong \check{E}[m] \hookrightarrow E[m]$ and we have to show that $E[m]$ has no more points that $\check{E}[m]$. Let us consider the polynomial equations of $mP = O$ together with equations of the curve E. In the case of E in the Weierstrass format this is in total 3 polynomial equations in x, y. This system of polynomial equations has m^2 distinct solution over $\check{\mathsf{k}}$, and it is defined over $\check{\mathsf{k}}$. Since both $\check{\mathsf{k}}$ is algebraically closed, there will be no more points if we enlarge $\check{\mathsf{k}}$.

Remark 6.1 Let E an elliptic curve over k. For $n,m \in \mathbb{N}$, both $E[n]$ and $E[m]$ are subsets of $E[nm]$ and so $E(\mathsf{k})_{\mathrm{tors}}$ is a direct limit of $E[n]$'s according to these inclusions. If if $\mathsf{k} = \mathbb{C}$ then by Weierstrass uniformization E is biholomorphic to a torus $\mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, and so, $E(\mathbb{C})_{\mathrm{tors}} = \frac{\mathbb{Q}\omega_1+\mathbb{Q}\omega_2}{\mathbb{Z}\omega_1+\mathbb{Z}\omega_2}$. For n and m as above we have also the map $E[nm] \to E[n]$, $P \mapsto mP$ and so we can take inverse limit of $E[n]$'s and define a new object. For instance, for $p \in \mathbb{N}$ we have

$$\cdots \to E[p^{n+1}] \to E[p^n] \to \cdots \to E[p].$$

and the inverse limit of this is called the Tate module T_pE.

Exercise 6.1 Show that any field k of characteritic zero contains $\mathbb{Q}$ and if it is finitely generated over $\mathbb{Q}$, that is $\mathsf{k} = \mathbb{Q}(t_1,t_2,\ldots,t_s)$ for some $t_i \in \mathsf{k}$, then we have always an embedding $\mathsf{k} \hookrightarrow \mathbb{C}$ of fields.

Exercise 6.2 For the elliptic curve $E : y^2 = (x-t_1)(x-t_2)(x-t_3)$, where t_i's are three distinct elements in k, show that

$$E[2] = \{(t_1,0),(t_2,0),(t_3,0),O\}.$$

For families of elliptic curves with a 3-torsion point see [Hus04, Chapter 4, Section 2].

Exercise 6.3 Let E be an elliptic curve over $\mathbb{Q}$ and ω be a regular differential 1-form on E. For a torsion point of order m of E and γ a path from O to P in $E(\mathbb{C})$ show that

$$m\int_O^P \omega = \int_\delta \omega,$$

for some $\delta \in H_1(E,\mathbb{Z})$ which depends on γ. If this language looks fancy (you do not understand it) then try to prove the equality (6.3) which is a mere calculus.

Exercise 6.4 Show that for the elliptic curve

$$y^2(x-z) - ax^2y + (a+b)xyz - bxz^2 = 0, \quad a \in \mathbb{Q}, \quad b = -(a-1)(a-2), \tag{6.2}$$

with the neutral element $O := [1:1:1]$, $P = [1:0:0]$ is a torsion point of order 8. Conversely, any elliptic curve with rational torsion point of order 8 can be written in the above format. This family of elliptic curves is due to B. Levi, see [SS96]. For which values of a, the above curve is singular?

Exercise 6.5 Show that there is no elliptic curve over $\mathbb{Q}$ with a rational torsion point of order 16. This result is due to B. Levi, see [SS96, page 61].

6.3 Isogeny

Let $\check{\Lambda} \subseteq \Lambda \subseteq \mathbb{C}$ be two lattices and let $N := \#\Lambda/\check{\Lambda}$. Let $E = \mathbb{C}/\Lambda$ and $\check{E} = \mathbb{C}/\check{\Lambda}$ be the corresponding tori. The identity map $\mathbb{C} \to \mathbb{C}$ induces a map of tori

$$f : \check{E} \to E \tag{6.3}$$

This is actually a holomorphic map between two Riemann surfaces. It is also a morphism of groups. We have also $f^*\omega = \check{\omega}$, where ω and $\check{\omega}$ are the differential form dz induced in E and $\check{E}$, respectively, where $z \in \mathbb{C}$ is the coordinate function. Furthermore, $f^{-1}([z]) = [z] + \frac{\Lambda}{\check{\Lambda}}$ which means that f is a N to 1 map with no ramification points.

Definition 6.3 The map f as in (6.3) is called an isogeny of degree N. We have $N\Lambda \subseteq \check{\Lambda} \subseteq \Lambda$ and this gives us the maps

$$\begin{array}{rcl} E & \xrightarrow{g} \quad \check{E} & \xrightarrow{f} E \\ [z] & \longmapsto [Nz] & \\ & [z] & \longrightarrow [z] \end{array}$$

The map g is called the dual isogeny of f. Note that both $f \circ g,\ \ g \circ f$ are multiplication by N map that we denote it by $[N]$. Note also that in the level of differential forms $f^*\omega = \check{\omega}$ and $g^*\check{\omega} = N\omega$.

Exercise 6.6 Let G be a finite abelian group generated by at most two elements. There are unique $d_1,\ d_2 \in \mathbb{N}$ such that

$$G \simeq \frac{\mathbb{Z}}{d_1\mathbb{Z}} \oplus \frac{\mathbb{Z}}{d_1 d_2\mathbb{Z}}.$$

Conclude that any isogeny $E_1 \to E_2$ can be uniquely written as

$$E_1 \xrightarrow{[d_1]} E_1 \xrightarrow{\beta} E_2,$$

where $[d_1]$ is a multiplication by d_1 and kernel of β is cyclic of order d_2.

Remark 6.2 Let $a \in \mathbb{C}^*$ amd $\Lambda \subseteq \mathbb{C}$ be a lattice. Let also $\check{E} := \mathbb{C}/a\Lambda,\ \ E := \mathbb{C}/\Lambda$. We have the map

$$f_a : E \to \check{E}, \quad z \mapsto [az],$$

which is a bijection and its inverse is given by $f_{a^{-1}}$. Moreover, $f_a^*\omega = a\check{\omega}$. Under the isomorphism between lattices and enhanced elliptic curves, the lattices $a\Lambda$ and Λ corresponds to $(E, a\omega), (E, \omega)$, respectively.

Exercise 6.7 Show that the number of sublattices $\check{\Lambda} \subseteq \Lambda$ of a fixed lattice Λ with $\#\Lambda/\check{\Lambda} = n$ is $\sigma(n) = \sum_{d|n} d$. Hint: Take a basis ω_1, ω_2 of Λ and show that

$$\begin{array}{ll} \mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{Mat}_n(2,\mathbb{Z}) \cong & \left\{\check{\Lambda} \subseteq \Lambda,\ \#\Lambda/\check{\Lambda} = n\right\} \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} & \mapsto \check{\Lambda} \text{ generated by } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \end{array}$$

The quotient in the left hand side has representatives

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \quad 0 \leqslant b \leqslant d-1,\ a\cdot d = n.$$

Next, we would like to describe isogeny of elliptic curves over a field k. Like in the 19th century such that Algebraic Geometry was only a chapter in the theory of transcendental functions, we use Weierstrass $\wp$ function and uniformization theorem, see Theorem 3.3, in order to write the tori E and $\check{E}$ in the Weierstrass format, and then explain the algebraic expression of an isogeny. Let us assume that E is defined over $\bar{\mathbb{Q}}$, that is, the corresponding t_2, t_3 in its Weierstrass format are algebraic numbers. Let also $G := \Lambda/\check{\Lambda}$. We know that G is a subgroup of $E[N]$ and by Proposition 6.1 we have $E[N] \subset E(\overline{\mathbb{Q}})$. We may think of $\check{E}$ as the quotient E/G. The pull-back of $\wp(z,\Lambda)$ and $\wp'(z,\Lambda)$ by the isogeny map (6.3) are elliptic functions with respect to the lattice $\check{\Lambda}$. Therefore, by Exercise 2.14 they can be written as rational functions in $\wp(z,\check{\Lambda})$ and $\wp'(z,\check{\Lambda})$. More precisely, we have

$$\begin{aligned} &\wp(z,\Lambda) = P(\wp(z,\check{\Lambda})), \\ &\wp'(z,\Lambda) = \wp'(z,\check{\Lambda})\cdot Q(\wp(z,\check{\Lambda})), \end{aligned}$$

where $P(x)$ and $Q(x)$ are rational functions in x and with coefficients in $\mathbb{C}$. This follows from the fact that $\wp(z,\Lambda)$ and $\wp'(z,\Lambda)$ are respectively even and odd elliptic functions.

Theorem 6.1 *If E is defined over $\overline{\mathbb{Q}}$ then the isogeny*

$$\check{E} \to E \quad (x,y) \to \Big(P(x), yQ(x)\Big)$$

is also defined over $\overline{\mathbb{Q}}$, that is, $\check{E}$ and P,Q are defined over $\overline{\mathbb{Q}}$.

This will be proved in Section 7.3 in which we introduce Hecke operator. The reader can consult more Algebro-Geometric oriented books in elliptic curves, such as [Sil94b, Sil92a], for a purely algebraic notion of an isogeny.

Exercise 6.8 Show that the Dwork family $x^3+y^3 = 1-3\alpha xy$ is 3 isogeneous to the Hessian family $y^2+3\alpha xy+y = x^3$. Hint: See [Hus04, page 91]. Despite the name "Dwork family", this family was in use much before Dwork, see for instance [Fri11, page 93].

Exercise 6.9 According to A. Weil in [Wei52], L. Kronecker studied the family of elliptic curves $E_\rho : y^2 = 1-\rho x^2+x^4$. He shows that for a ρ' algebraic over $\mathbb{Q}(\rho)$, we have an isogeny $E_\rho \to E_{\rho'}$ of the form $(x,y) \mapsto \left(\frac{x^n F(\frac{1}{x})}{F(x)}, \frac{G(x)}{F(x)^2}\right)$, where F,G are polynomials with coefficients in $\mathbb{Q}(\sigma,\sigma')$ and σ,σ' are roots of $1-\rho x^2+x^4$ and $1-\rho' x^2+x^4$, respectively. Discuss this isogeny in more details.

We might define an isogeny in a more general context of complex analysis. We avoid the general definition of complex manifolds and holomorphic maps between them and present this in the following way.

Definition 6.4 We say that a map $f: E \to \check{E}$ between two complex tori is holomorphic if there are holomorphic maps $f_i: U_i \to V_i$, where U_i, V_i are open subsets of $\mathbb{C}$ for $i = 1,2,\ldots,k$, such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & \check{E} \\ \uparrow & & \uparrow \\ U_i & \xrightarrow{f_i} & V_i. \end{array}$$

commutes and the image of U_i's under the canonical projection $\mathbb{C} \to E$ form a covering of E. We say that it is an isogeny if it is not a constant map and it is a homomorphism of groups. The endomorphism group of a torus E is the set of all holomorphic maps $f: E \to E$ which are also homomorphism of groups.

Exercise 6.10 Show that any isogeny $f: E \to \check{E}$ is of the form in Remark 6.2, and hence, if we change the coordinate function from z to az, it is of the form in Definition 6.3. Show that for any endomorphism $f: E \to E$ there is a complex number a such that $a\Lambda \subset \Lambda$ and f is induced by $\mathbb{C} \to \mathbb{C}$, $z \mapsto az$.

It turns out that we have a natural isomorphism

$$\mathrm{End}(E) \cong \{\alpha \in \mathbb{C} \mid \alpha\Lambda \subset \Lambda\},$$

where $\mathrm{End}(E)$ is the group of endomorphisms of E.

Definition 6.5 We have $\mathbb{Z} \subset \mathrm{End}(E)$ and we say that E is CM (short for complex multiplication) if the inclusion is strict.

Later, we will encounter two spcial CM elliptic curves as follows:

Exercise 6.11 Classify all elliptic curves E with $\alpha \in \mathrm{End}(E)$ which is not multiplication by ± 1 and is an isomorphism. More precisely, show that we have only two such elliptic curve

$$E = E_{\langle \tau, 1\rangle}, \quad \tau = i, \quad \frac{-1+i\sqrt{3}}{2}.$$

6.4 Nagell-Lutz and Mazur theorems

Two important arithmetic results on torsion points of elliptic curves over $\mathbb{Q}$ are Nagell-Lutz theorem and Mazur theorem. In this section we state both theorems and leave the proofs to the readers as there are many excellent textbooks on the topic.

Nagell-Lutz theorem was independently proved by Trygve Nagell in 1935 and by Elisabeth Lutz in 1937, see [Cas90, Lut37]. It gives a finite set of possibilities for a torsion point of an elliptic curve defined over $\mathbb{Z}$, and so, it is a useful tool to compute torsion points.

Theorem 6.2 (Nagell-Lutz Theorem) *Let E be an elliptic curve with the Weierstrass equation:*

$$y^2 = x^3 + t_2 x + t_3, \quad t_2, t_3 \in \mathbb{Z}, \quad \Delta := 4t_2^3 + 27t_3^2 \neq 0.$$

Then for all non-zero torsion points $P = (a,b) \in E(\mathbb{Q})$ we have:

1. *The coordinates of P are in $\mathbb{Z}$, that is, $a, b \in \mathbb{Z}$.*
2. *If P is of order greater than 2, then b^2 divides Δ.*
3. *If P is of order 2 then $b = 0$ and $a^3 + t_2 a + t_3 = 0$.*

A proof of this theorem can be found in [Sil92a, page 221] or [ST92, page 56].

Exercise 6.12 Compute the torsion subgroup of the elliptic curve.

$$y^2 = x^3 + 2, \ \cdots$$

see [Mil20, Exercise 8.11] for a list of other elliptic curves.

Next, we state Mazur theorem. ´

Theorem 6.3 ([Maz77, Maz78]) *Let E be an elliptic curve over $\mathbb{Q}$. Then the torsion subgroup $E(\mathbb{Q})_{\mathrm{tors}}$ is one of the following fifteen groups:*

$$\mathbb{Z}/N\mathbb{Z}, \ 1 \leq N \leq 10, \ \ \textit{or } N = 12$$

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2N\mathbb{Z}, \ 1 \leq N \leq 4.$$

In particular,

$$\#(E(\mathbb{Q})_{\mathrm{tors}}) \leq 16. \tag{6.4}$$

The early history of Mazur theorem goes back to Beppo Levi, who produced a series of paper from 1906 to 1911 and for instance, he proved Exercise 6.5. He also formulated the boundedness of $\#(E(\mathbb{Q})_{\mathrm{tors}})$ which is known as torsion conjecture or uniform boundedness conjecture. It was also reformulated in [Nag52] and in [Ogg71]. The torsion conjecture for abelian varieties is still open. It is natural to conjecture that if E is an elliptic curve over a number field k then the order of the torsion subgroup of $E(\mathsf{k})$ is bounded by a constant which depends only on the degree of k over $\mathbb{Q}$. It is proved in [Kam92] for all quadratic fields and in [Mer96] for all number fields. For the proof of these statements one needs the notion of modular curve $X_0(N)$ and modular forms which will be introduced in the forthcoming chapters.

Exercise 6.13 Let E be the elliptic curve defined by the equation $y^2 = x^3 + ax$, where a is a fourth-power free integer. Then

$$E(\mathbb{Q})_{\text{tors}} \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & -a \text{ is square} \\ \mathbb{Z}/4\mathbb{Z}, & a = 4 \\ \mathbb{Z}/2\mathbb{Z}, & \text{otherwise.} \end{cases}$$

Hint: See [Hus04, Section 3, page 34].

Exercise 6.14 Let E be the elliptic curve defined by the equation $y^2 = x^3 + a$, where a is a sixth-power free integer. Then

$$E(\mathbb{Q})_{\text{tors}} \cong \begin{cases} \mathbb{Z}/6\mathbb{Z}, & a = 1 \\ \mathbb{Z}/4\mathbb{Z}, & \text{if } a \text{ is a square different from 1, or } a = -432 \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } a \text{ is a cube different from 1} \\ 0, & \text{otherwise.} \end{cases}$$

Hint: See [Hus04, Section 3, page 37].

Exercise 6.15 The following exericise is the first theorem announced in the treatise [Jac29, Section 15, page 28]. Let us consider the Jacobi's family of elliptic curves $E_{u^2} : y^2 = (1-x^2)(1-u^2x^2)$. Show that for (u,v) in the curve

$$u^6 - v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4) = 0$$

we have the following isogeny

$$f : E_{u^8} \to E_{v^8}, \quad (x,y) \mapsto (X,Y)$$

$$X := \frac{v(v-u^5)x + u^3(u^2+v^2)(v-u^5)x^3 + u^{10}(1-uv^3)x^5}{v^2(1-uv^3) + uv^2(u^2+v^2)(v-u^5)x^2 + u^6v^3(v-u^5)x^4},$$
$$Y := y\frac{v(1-uv^3)}{v-u^5}\frac{\partial X}{\partial x},$$

such that $f^*\frac{dx}{y} = \frac{v-u^5}{v(1-uv^3)}\frac{dx}{y}$. In Jacobi's book this is written as:

$$\frac{v(1-uv^3)dX}{\sqrt{(1-X^2)(1-v^8X^2)}} = \frac{(v-u^5)dx}{\sqrt{(1-x^2)(1-u^8x^2)}}$$

Find out Jacobi's motivation which resulted in this computation, see also [Cog14]

Chapter 7
Hecke operators

The final scene shows the 83-year-old Alexander von Humboldt following Eisenstein's coffin at the cemetery at Blucherplatz. He had obtained money from the King for Eisenstein to go to Sicily for a cure, but it was already too late. The plague of the nineteenth century had taken yet another distinguished victim, (N. Schappacher in [BKK$^+$98a, page 60]).

7.1 Introduction

In this chapter we introduce one of the fundamental features of modular forms which is responsible for many arithmetic properties. This is namely the Hecke operators acting on the space of modular forms. There are many text books covering this topic perfectly, see for instance [Apo90, Chapter 6], [Zag08] and [Ser78]. We will adopt a more geometric approach suitable for the same topic in the context of algebraic geometry of elliptic curves, see Section 9.15. The first application of Hecke operators is the following.

Theorem 7.1 *Let τ be the Ramanujan's τ function in Section 2.11. The numbers $\tau(n)$ are multiplicative, that is, for all $n,m \in \mathbb{N}$ with $(n,m) = 1$ we have*

$$\tau(n \cdot m) = \tau(n) \cdot \tau(m).$$

7.2 Hecke operators

For $A \in \mathrm{GL}(2,\mathbb{R})$ and a modular form f of weight k we define the slash operator

$$f|_k A := (\det A)^{k-1}(c\tau + d)^{-k} f(A\tau), \quad A = \begin{bmatrix} * & * \\ c & d \end{bmatrix}.$$

In some books, the power $k-1$ of $\det A$ is different. For instance, in Chapter 7 the slash operator is different. For $A \in \mathrm{SL}(2,\mathbb{R})$ this will not make any difference.

Proposition 7.1 *If $B \in \mathrm{GL}(2,\mathbb{R})$and f is a meromorphic (resp. holomorphic or weakly holomorphic) modular form of weight k for some group $\Gamma \in \mathrm{GL}(2,\mathbb{R})$ then $f|_k B$ is a meromorphic (resp. holomorphic or weakly holomorphic) modular form of the same weight for the group $B^{-1}\Gamma B$.*

Proof. This follows from

$$(f|_k B)|_k B^{-1}AB = (f|_k A)|_k B = f|_k B$$

For a fixed integer k, there is a one to one correspondance between the following functions:

1. Holomorphic functions $f : \mathbb{H} \to \mathbb{C}$ such that

$$(c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right) = f(\tau), \quad \forall \tau \in \mathbb{H}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z}). \tag{7.1}$$

2. Holomorphic functions in the space L of lattices such that $f(\lambda\Lambda) = \lambda^{-k} f(\Lambda)$.
3. Holomorphic functions in the space of enhanced elliptic curves (E,ω) such that $f(E,\lambda\omega) = \lambda^{-k} f(E,\omega)$. We say that f is a homogeneous function.

Inserting the growth condition in each item above, we get three different encarnation of modular forms. For the first item we know already what growth condition is, for others we do not know how to define it without translating the modular form to the one in the first item. We will discuss this for the third item for quasi modular forms in Section 9.8.

Let $\mathrm{M}_k = \mathrm{M}_k(\mathrm{SL}(2,\mathbb{Z}))$ be the vector space of modular forms of weight k. The Hecke operator

$$T_n : \mathrm{M}_k \to \mathrm{M}_k$$

is a linear map and it is given by one of the following equivalent definitions:

1. For $f : \mathbb{H} \to \mathbb{C}$ a modular form of weight k we have

$$T_n(f) = \sum_{[A] \in \mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{Mat}_n(2,\mathbb{Z})} f|_k A = \sum_{i=1}^{s} f|_k A_i,$$

   where $\{[A_1],[A_2],\cdots [A_s]\} = \mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{Mat}_n(2,\mathbb{Z})$.
2. For $f : \mathsf{L} \to \mathbb{C}$ a modular form of weight k we have

$$T_n(f)(\Lambda) = n^{k-1} \sum_{\Lambda' \subset \Lambda,\ \#\Lambda/\Lambda' = n} f(\Lambda'),$$

   where Λ' runs through all sublattices $\Lambda' \subset \Lambda$ of index n.
3. For f a homogeneous function on the sapce of enhanced elliptic curves (E,ω), we have

$$T_n f(E,\omega) = n^{k-1} \sum f(\check{E},\check{\omega}),$$

where $\check{E},\check{\omega}$ runs though all isogenies $\alpha : \check{E} \to E$ such that $\alpha^*(\omega) = \check{\omega}$ and $\deg(\alpha) = n$. In this case f can be considered as a homogeneous polynomial of degree k in $\mathbb{C}[t_2,t_3]$, $\deg(t_2) = 4$, $\deg(t_3) = 6$ and we can rewrite

$$T_n(f)(t_2,t_3) = n^{k-1} \sum_{t'} f(t'),$$

where $t' = (t_2', t_3')$ runs through all parameters for which there is an isogeny $\alpha : E_{t'} \to E_t$ such that $\alpha^*(\frac{dx}{y}) = \frac{dx}{y}$ and $\deg(\alpha) = n$.

Exercise 7.1 Prove the equivalence of the above three definitions.

Exercise 7.2 Prove that each equivalence class in $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{Mat}_n(2,\mathbb{Z})$ is represented exactly by one of the matrices

$$\begin{bmatrix} d & b \\ 0 & \frac{n}{d} \end{bmatrix},\ d|n,\ 0 \le b < \frac{n}{d}.$$

Using the Exercise 7.2 we know that the action of the Hecke operator T_n on a modular form of weight k is given by

$$T_n(f)(\tau) = n^{k-1} \sum_{a\cdot d = n,\ 0 \le b \le d-1} d^{-k} f\left(\frac{a\tau + b}{d}\right). \tag{7.2}$$

Theorem 7.2 *For two natural numbers n and m and Hecke operators $T_n, T_m \in \mathrm{M}_k \to \mathrm{M}_k$ we have*

$$T_n \circ T_m = \sum_{d|(n,m)} d^{k-1} T_{\frac{nm}{d^2}}. \tag{7.3}$$

In particular, for n and m coprime we have

$$T_n \circ T_m = T_{nm}$$

and for p a prime number

$$T_p \circ T_{p^e} = T_{p^{e+1}} + p^{k-1} T_{p^{k-1}}. \tag{7.4}$$

Proof. For a proof see [Sil94b, page 62]. We give two essentially same proofs, one for modular forms as holomorphic functions on $\mathbb{H}$ and the other for modular forms as functions on pairs (E,ω). Let $K_n := \mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{Mat}_n(2,\mathbb{Z})$. Let us take representatives $A_i,\ i = 1,2,\cdots,\sigma(n)$, $B_i,\ i = 1,2,\cdots,\sigma(m)$ for K_n and K_m, respectively. For a fixed $d|(n,m)$ we also take representatives $C_{d,r},\ r = 1,2,\cdots,\sigma(\frac{nm}{d^2})$ for $K_{\frac{nm}{d^2}}$. It is not hard to show that for any $C_{d,r}$ there is exactly d pairs (A_i, B_j) such that

$$A_i B_j = \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix} C_{d,r}.$$

see the second part of Exercise 7.3. Now we have

$$\begin{aligned} T_n \circ T_m f(\tau) &= \sum_{i=1}^{\sigma(n)} \sum_{j=1}^{\sigma(m)} f|_k A_i B_j \\ &= \sum_{d|(n,m)} \sum_{r=1}^{\sigma(\frac{nm}{d^2})} d \cdot f|_k \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix} C_{d,r} \\ &= \sum_{d|(n,m)} \sum_{r=1}^{\sigma(\frac{nm}{d^2})} d^{k-1} \cdot f|_k C_{d,r}. \end{aligned}$$

Now we reproduce the proof for modular forms as functions on the sapce of (E,ω). Let $n, m \in \mathbb{N}$ and $d|(n,m)$ be fixed. We prove that for pairs of isogenies.

$$E_1 \xrightarrow{\alpha} E_2 \xrightarrow{\beta} E, \quad \deg \alpha = n, \quad \deg \beta = m$$

there is a unique isogeny $E_1 \xrightarrow{\gamma} E$, $\deg \gamma = \frac{nm}{d^2}$ such that

$$\gamma \circ [d] \; = \; \beta \circ \alpha$$

and for γ fixed we have d pairs of such isogenies α, β. Here $[d] : E_1 \to E_1$ is the multiplication by d map. This decomposition is inspired by the identity

$$\sigma(n) \cdot \sigma(m) = \sum_{d|(n,m)} d \cdot \sigma\left(\frac{nm}{d^2}\right).$$

If this is the case, then

$$\begin{aligned} T_n \circ T_m f(E,\omega) &= (nm)^{k-1} \sum_{E_1 \xrightarrow{\alpha} E_2 \xrightarrow{\beta} E} f\Big(E_1, (\beta \circ \alpha)^* \omega\Big) \\ &= \sum_{d|(n,m)} (nm)^{k-1}\, d \sum_{E_1 \xrightarrow{\gamma} E} f\Big(E_1, (\gamma \circ [d])^* \omega\Big) \\ &= \sum_{d|(n,m)} d^{k-1}\, T_{\frac{nm}{d^2}}. \end{aligned}$$

We have used $[d]^*\omega = d\omega$ and $f(E, d*) = d^{-k} f(E,*)$. In order to prove the affirmation on isogenies we have to prove the corresponding affirmation on lattices. Let $\Lambda \subseteq \mathbb{C}$ be a fixed lattices and

$$\Lambda_1 \subseteq \Lambda_2 \subseteq \Lambda, \quad \#\frac{\Lambda_2}{\Lambda_1} = n, \; \#\frac{\Lambda}{\Lambda_2} = m.$$

For $(n,m)=1$, Λ_2 is uniquely characterized by $\Lambda_2=\{x\in\Lambda \mid nx\in\Lambda_1\}$ and the affirmation is trivial. For a subgroup $G_d:=\mathbb{Z}/d\mathbb{Z}\times\mathbb{Z}/d\mathbb{Z}\subseteq\Lambda/\Lambda_1$ we define

$$\Lambda_3= \text{ pull-back of } G_d \text{ by } \quad \Lambda\to\Lambda/\Lambda_1.$$

We have $d\Lambda_3\subseteq\Lambda_1\subseteq\Lambda_3$ and the index in both inclusions $d\Lambda_3\subseteq\Lambda_3$ and $\Lambda_1\subseteq\Lambda_3$ is d^2. Therefore, $d\Lambda_3=\Lambda_1$, and hence, $E_1\cong\mathbb{C}/\Lambda_3$. The proof follows from Exercise 7.3.

Exercise 7.3 Let G be a finite abelian group generated by at most two elements, $\#G=nm$ and $d:=(n,m)$. Show that the number of subgroups $\check{G}\subset G$ with $\#\check{G}=n$ is d times the number of subsgroups of G isomorphic to $\mathbb{Z}/d\mathbb{Z}\times\mathbb{Z}/d\mathbb{Z}$. This problem is equivalent to the following. Let $K_n:=\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{Mat}_n(2,\mathbb{Z})$. Let us take representatives $A_i,\ i=1,2,\cdots,\sigma(n)$, $B_i,\ i=1,2,\cdots,\sigma(m)$ for K_n and K_m, respectively. For a fixed $d|(n,m)$ we also take representatives $C_{d,r},\ r=1,2,\cdots,\sigma(\frac{nm}{d^2})$ for $K_{\frac{nm}{d^2}}$. For any $C_{d,r}$ there is exactly d pairs (A_i,B_j) such that

$$A_iB_j=\begin{bmatrix} d & 0\\ 0 & d\end{bmatrix}C_{d,r}.$$

The formula (7.3) is summarized in the following formal equality:

$$\sum_{n=1}^{\infty}T_nn^{-s}=\prod_p(1-T_pp^{-s}+p^{k-1-2s})^{-1}.$$

Proposition 7.2 *Let f be a modular form with the Fourier expansion:*

$$f(\tau)=\sum_{n=0}^{\infty}a_nq^n,\ q=e^{2i\pi\tau}.$$

For $m\in\mathbb{N}$, we have $T_mf(\tau)=\sum_{n=0}^{\infty}b_nq^n$, where

$$b_n=\sum_{d|\gcd(m,n)}d^{k-1}a_{mn/d^2}.$$

In particular, if f is defined over $\mathbb{Q}$ then T_nf is also defined over $\mathbb{Q}$.

Proof. This follows from Equation (7.2):

$$\begin{aligned}
T_m(f)(q) &= m^{k-1} \sum_{a\cdot d=m,\ \ 0\le b\le d-1} d^{-k} \sum_{n=0}^{\infty} a_n q^{\frac{an}{d}} e^{\frac{2\pi i b n}{d}} \\
&= m^{k-1} \sum_{n=0}^{\infty} \sum_{a\cdot d=m} d^{-k} a_n q^{\frac{an}{d}} \left(\sum_{b=0}^{d-1} e^{\frac{2\pi i b n}{d}} \right) \\
&= m^{k-1} \sum_{n=0}^{\infty} \sum_{d|(n,m)} d^{-k+1} a_n q^{\frac{mn}{d^2}} \\
&= m^{k-1} \sum_{n=0}^{\infty} \sum_{d|(n,m)} (\frac{m}{d})^{-k+1} a_{\frac{nm}{d^2}} q^n .
\end{aligned}$$

In the last line we have made the change of variables $\tilde{n} := \frac{nm}{d^2},\ \ \tilde{d} = \frac{m}{d}$, and then we have removed the tilde.

Exercise 7.4 Show that the Eisenstein series E_k is an eigenform for all Hecke operators T_n with eigenvalue $\sigma_{k-1}(n)$:

$$T_n E_k = \sigma_{k-1}(n) E_k.$$

Hint: This follows directly from the definition of E_k and T_n, see [Sil94b, 1.25, page 92].

Proposition 7.3 *A modular form of weight k with $f_0 = 1$ is an eigenform for all Hecke operators if and only if f is the Eiseinstein series E_k.*

Proof. Let f be an eigenform for all T_n's. We have $T_n f = \lambda_n \cdot f$ and so

$$\lambda_n f_m = (T_n f)_m = \sum_{d|(n,m)} d^{k-1} f_{\frac{nm}{d^2}} \tag{7.5}$$

We put $m = 0$ and get $\lambda_n f_0 = \sigma_{k-1}(n) f_0$, and $\lambda_n = \sigma_{k-1}(n)$. If we put $m = 1$ we get also $\lambda_n f_1 = f_n$.

Theorem 7.3 *Let f be a cusp form of weight k and suppose that f is an eigen form for all Hecke operators and $f_1 = 1$. Then*

$$T_n f = f_n \cdot f$$

Proof. We have $T_n f = \lambda_n \cdot f$ and so in (7.5) we put $m = 1$ and get $\lambda_n = f_n$.

Definition 7.1 *A normalized eigenform is a modular form f with*

1. *f is a cusp form with $f_1 = 1$.*
2. *For all $n \in \mathbb{N}$ we have $T_n f = f_n \cdot f$.*

Theorem 7.4 *Let $f(\tau) = \sum_{n=1}^{\infty} f_n q^n$ be a normalized eigenfunction of weight k. Then*

$$f_n f_m = \sum_{d|(n,m)} d^{k-1} f_{\frac{nm}{d^2}} .$$

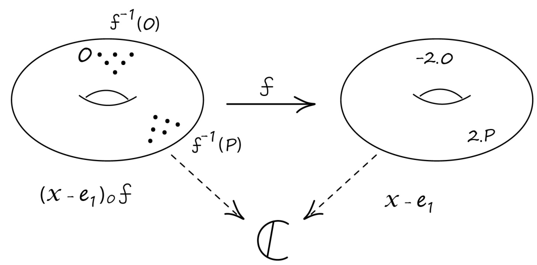


Fig. 7.1 Isogeny

In particular,

$$\begin{aligned} &f_{nm} = f_n f_m \qquad (n,m)=1, \\ &f_{p^n}\cdot f_p = f_{p^{n+1}} + p^{k-1} f_{p^{n-1}} \qquad p \ \ prime. \end{aligned}$$

Proof. We just apply Equation (7.3) to f.

Now we are ready to prove Theorem 7.1.

Theorem 7.5 Let

$$\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24} \; = \; \sum_{n=1} \tau(n)q^n.$$

We have

$$\begin{aligned} \tau(n)\tau(m) &= \tau(nm) \qquad (n,m)=1, \\ \tau(p)\tau(p^n) &= \tau(p^{n+1}) + p^{11}\tau(p^{n-1}), \quad p \text{ prime.} \end{aligned} \tag{7.6}$$

Proof. This follows from the fact that $S_{12}\left((\mathrm{SL}(2,\mathbb{Z})\right)$ is generated by Δ. The identities were conjectured by Ramanujan and proved by Mordell.

Exercise 7.5 Find a basis of $S_n\left(\mathrm{SL}(2,\mathbb{Z})\right)$, $n=14$, 16, 18 which are normalized eigenforms.

7.3 Proof of Theorem 6.1

First, we prove that $\check{E}$ is defined over $\overline{\mathbb{Q}}$. Let $f \in M_k$ be a modular form defined over $\mathbb{Q}$. We consider f as a function on the sapce of lattices and define

$$P_f(x) = \prod_{\check{\Lambda} \subset \Lambda, \quad \#\Lambda/\check{\Lambda}=n} \left(x - f(\check{\Lambda})\right).$$

Let us write $P(x) = \prod_{i=1}^{N}(x - x_i)$, where $N := \sigma(n)$. The coefficients of $P(x)$ are symmetric polynomials with $\mathbb{Q}$ coefficients in the quantities

$$T_n f^m = \sum_{i=1}^{N} x_i^m, \quad m = 1, 2, \ldots, N.$$

By Proposition 7.2 we know that Hecke operators send a modular form defined over $\mathbb{Q}$ and weight k to a modular form defined over $\mathbb{Q}$ and weight k, and hence,

$$P_f(x) = x^N + \sum_{i=1}^{N} f_i x^{N-i}, \;\; f_i \in M_{ik}(\mathrm{SL}(2,\mathbb{Z}))_{\mathbb{Q}}.$$

By Theorem 2.5, f_i can be written as a polynomial of degree ik in $\mathbb{Q}[E_4, E_6]$ with $\deg(E_4) = 4$, $\deg(E_6) = 6$, and so we can write $P_f = P_f(x, E_4, E_6) \in \mathbb{Q}[x, E_4, E_6]$. By Theorem 2.2 and Theorem 2.3 we know that $\frac{1}{(2\pi i)^a} t_a$, $a = 2, 3$ is a modular form of weight $2a$ defined over $\mathbb{Q}$. This implies that $P_{t_a} \in (2\pi i)^{-Na}\mathbb{Q}[x, t_2, t_3]$, $a = 2, 3$. Let us consider an elliptic curve $E := \mathbb{C}/\Lambda$ defined over $\bar{\mathbb{Q}}$, that is, $t_2(\Lambda), t_3(\Lambda) \in \bar{\mathbb{Q}}$. Up to some power of $2\pi i$ factor $P_{t_a}(x, t_2(\Lambda), t_3(\Lambda)) \in \bar{\mathbb{Q}}[x]$ and this finishes the proof of the fact that $\check{E}$ is defined over $\bar{\mathbb{Q}}$.

Let $(e_1, 0)$ be a 2-torsion point of E, see Section 2.14. The function $x - e_1$ has a zero of order 2 at P and a pole of order 2 at O. Therefore, the zero and polar set of $(x - e_1) \circ f = P(x) - e_1$ are respectivey $f^{-1}(P)$ and $f^{-1}(O)$. Since $\check{E}$ is defined over $\overline{\mathbb{Q}}$, its torsion points are also defined over $\overline{\mathbb{Q}}$, and in paricular $f^{-1}(O), f^{-1}(P) \subset E(\bar{\mathbb{Q}})$. This implies that all the zeros of $P(x)$ are defined over $\bar{\mathbb{Q}}$, and hence $P(x)$ up to multiplication by a constant is defined over $\bar{\mathbb{Q}}$. Since f sends torsions of $\check{E}$ to torsions of E, evaluating $(x - e_1) \circ f$ at any torsion points of $\check{E}$, which is not in $f^{-1}(O)$ and $f^{-1}(P)$, we get the fact that P itself is defined over $\bar{\mathbb{Q}}$. In a similar way, we consider $\frac{y \circ f}{\check{y}}$ which is an even function and its zeros and poles are all on torsion points of $\check{E}$, see Figure 7.1.

Chapter 8
Congruence groups

The real purpose of mathematics is to be the means to illuminate reason and to exercise spiritual forces (August Leopold Crelle, see [BKK$^+$98b, page 32]).

8.1 Introduction

In this chapter we work with modular forms for subgroups of $\mathrm{SL}(2,\mathbb{Z})$, and in particular for congruence groups. One of the most well-known applications of such modular forms is the so-called arithmetic modularity theorem which has been one of the great achievements in mathematics of 20th century.

8.2 Congruence groups

We have seen that $\mathrm{SL}(2,\mathbb{Z})$ appears as the monodromy group of the Weierstrass familly of elliptic curves. If we take other families of elliptic curves and compute the corresponding monodromy group then we will get subgroups of $\mathrm{SL}(2,\mathbb{Z})$ of finite index. Congruence groups are the most well-known subgroups of $\mathrm{SL}(2,\mathbb{Z})$. They appear as monodromy groups of universal family of elliptic curves enhanced with torsion structures.

Let N be a positive integer number. Define

$$\Gamma(N) := \left\{ A \in \mathrm{SL}(2,\mathbb{Z}) \middle| A \equiv_N \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

not]$\Gamma(N)$, congruence group It is the kernel of the canonical homomorphism of groups $\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z})$.

Definition 8.1 A subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ is called a congruence subgroup of level N if $\Gamma(N) \subset \Gamma$.std]Congruence subgroup of level N

Our main examples are

$$\Gamma_0(N) := \left\{ A \in \mathrm{SL}(2,\mathbb{Z}) \middle| A \equiv_N \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \right\},$$

not]$\Gamma_0(N)$, congruence group

$$\Gamma_1(N) := \left\{ A \in \mathrm{SL}(2,\mathbb{Z}) \middle| A \equiv_N \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \right\},$$

not]$\Gamma_1(N)$, congruence group which are congruence groups of level N.

Exercise 8.1 For a description of a fundamental domain for the action of $\Gamma_0(p)$, p a prime number, see [Apo90, Theorem 4.2]. Write a report on this.

Definition 8.2 *std]Modular form A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a moduler form of weight k for Γ if*

1. *For all $A \in \Gamma$ we have $f|_k A = f$.*
2. *For all $A \in \mathrm{SL}(2,\mathbb{Z})$ the limit $\lim_{\mathrm{Im}(\tau)\to+\infty} f|_k A$ exists.*

8.3 Weil pairing

Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ be a lattice with $\mathrm{Im}(\frac{\omega_1}{\omega_2}) > 0$ and let $E = \mathbb{C}/\Lambda$ be the corresponding elliptic curve. Recall the set $E[N]$ of N-torsion points of E defined in Chapter 6. The following definition of Weil pairing is taken from [Sil94b, page 89] and [Sil92b, Chapter 3, Section 8].

Definition 8.3 *std]Weil pairingnot]e_N, Weil pairing Let E be an elliptic curve. The Weil pairing is*

$$\begin{aligned} &e_N \;\; E[N] \times E[N] \to \mu_N \\ &e_N \left(\frac{a\omega_1 + b\omega_2}{N}, \frac{c\omega_1 + d\omega_2}{N} \right) = e^{2\pi i \frac{ad-bc}{N}}, \end{aligned}$$

where $\mu_N := \{e^{\frac{2\pi i k}{N}} \mid k \in \mathbb{Z}\}$.

For an elliptic curve E in the Weierstrass format we consider the Weil pairing in E through the Weierstrass uniformization theorem.

Exercise 8.2 Prove that the above definition is well-defined, that is, it is independent of the choice of the basis ω_1, ω_2 for Λ. Moreover, the Weil pairing

1. is bilinear, $e_N(P_1+P_2,Q)=e_N(P_1,Q)e_N(P_2,Q))$ and alternating, $e_N(P,Q)=e_N(Q,P)^{-1}$.
2. It is non-degenerate, if $e_N(P,Q)=1$ for all Q, then $P=O$.
3. It is Galois invariant, $e_N(\sigma(P),\sigma(Q))=\sigma(e_N(P,Q)))$ for all $\sigma\in\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$.
4. It is compatible, $e_{NN'}(P,Q)=e_N(N'P,Q)$ for $P\in E[NN']$ and $Q\in E[N]$.

Theorem 8.1 *Let*

$$Y_0(N):=\Gamma_0(N)\backslash\mathbb{H},\ Y_1(N):=\Gamma_1(N)\backslash\mathbb{H},\ Y(N):=\Gamma(N)\backslash\mathbb{H}.$$

not]$Y_0(N)$, modular curvenot]$Y_1(N)$, modular curvenot]$Y(N)$, modular curve

1. *The set $Y_0(N)$ is the moduli space of pairs (E,C), where E is a complex elliptic curve and C is a cyclic subgroup of E of order N. For $\tau\in Y_0(N)$ the corresponding pair is:*

$$(E,C)=\left(\mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z}),\left\{\left[\frac{1}{N}\right],\left[\frac{2}{N}\right],\dots,\left[\frac{N-1}{N}\right]\right\}\right).$$

2. *The set $Y_1(N)$ is the moduli space of pairs (E,p), where E is a complex elliptic curve and p is a point of E of order N. For $\tau\in Y_1(N)$ the corresponding pair is:*

$$(E,C)=(\mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z}),[\frac{1}{N}]).$$

3. *The set $Y(N)$ is the moduli space of pairs $(E,(p,q))$, where E is a complex elliptic curve and (p,q) is a pair of points of E that generates the N-torsion subgroup of E with Weil pairing $e_N(p,q)=e^{\frac{2\pi i}{N}}$. For $\tau\in Y_1(N)$ the corresponding pair is:*

$$(E,C)=(\mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z}),([\frac{\tau}{N}],[\frac{1}{N}])).$$

The proof is simple and can be found in [DS05, Theorem 1.5.1, page 38].

Exercise 8.3 *Show that*

$$\begin{aligned}
&\#[\mathrm{SL}(2,\mathbb{Z}):\Gamma_1(N)]=n^2\ \Pi\ \left(1-\tfrac{1}{p^2}\right)\\
&\#[\mathrm{SL}(2,\mathbb{Z}):\Gamma(N)]=n^3\ \Pi\ \left(1-\tfrac{1}{p^2}\right)\\
&\#[\mathrm{SL}(2,\mathbb{Z}):\Gamma_0(N)]=\psi(N):=n\ \Pi\ \left(1+\tfrac{1}{p}\right).
\end{aligned}$$

Here, $\psi(N)$ is called the Dedekind ψ function.std]Dedekind ψ function

Remark 8.1 It is a natural question to ask when universal families over modular curves exist. In [MFK94, Theorem 7.9 and comments after] it is stated that if $N>6^g\cdot d\cdot\sqrt{g!}$ then the fine moduli scheme $A_{g,d,N}$ for abelian varieties of dimension g, with level N structure and polarization of degree d exists. It is a quasi-projective over Sepc($\mathbb{Z}$). Actually there is a finer result which is called Serre's Lemma and it states that $N \geq 3$ is enough for the existence of such universal families. The main reference for this discussion is [Gro62, Appendix by Serre].

8.4 Modular forms for congruence groups

We consider the moduli space of elliptic curves enhanced with torsion point structures and holomorphic 1-forms. The difference between these moduli spaces and those in Theorem 8.1 is the presence of a differential form ω together with E. Recall that by integration the pair (E,ω) is identified with a lattice $\Lambda \subset \mathbb{C}$. not]$\mathsf{L}_1(N), \mathsf{L}_0(N), \mathsf{L}(N)$, moduli of enhanced elliptic curves

$$\begin{aligned}
\mathsf{L} &:= \text{ moduli of elliptic curves } (E,\omega), \\
\mathsf{L}_1(N) &:= \text{ moduli of } (E,\omega,z),\ z \in E[N] \text{ a torsion point of order } N\ , \\
\mathsf{L}(N) &:= \text{ moduli of } (E,\omega,\{z_1,z_2\}),\ z_1,z_2 \in E[N],\ e_N(z_1,z_2) = \zeta_N, \\
\mathsf{L}_0(N) &:= \text{ moduli of } (E,\omega,C),\ C \subset E[N] \text{ cyclic group of order } N.
\end{aligned}$$

Let f be an elliptic function of weight k with poles at 0. For instance, we use $f = \wp, \wp'$ which are of weight 2 and 3, respectively. We know the following functions on these moduli spaces:

$$\begin{aligned}
f_{1,N} &: \mathsf{L}_1(N) \to \mathbb{C},\ \ f_{1,N}(E,\omega,z) = f\ (\Lambda,z), \\
f_N^i &: \mathsf{L}(N)\ \to \mathbb{C},\ \ f_N(E,\omega,z_1,z_2) = f\ (\Lambda,z_i),\ \ i = 1,2, \quad \Lambda := \int_{H_1(E,\mathbb{Z})} \omega \\
f_{0,N} &: \mathsf{L}_0(N) \to \mathbb{C},\ \ f_{0,N}(E,\omega,C) = \sum_{z\in C} f\ (\Lambda,z).
\end{aligned}$$

Note that in the last item we could $f_{0,N}$ in different ways. For instance, the sum can run 1. all generators z of C. 2. For $d \in \mathbb{N}$ dividing N, it can run on all $z \in C$ with $dz = 0$. Depending on applications, these other definitions might be useful. Any function g as above has the following functional equation:

$$g(E,a\omega,*) = a^{-k} g(E,\omega,*),\ \ \forall a \in \mathbb{C}^*, \tag{8.1}$$

where k is the weight of the elliptic function f. We consider the following maps

$$\begin{aligned}
&i : \mathbb{H} \to \mathsf{L}_1(N) && \tau \mapsto (\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}), dz, \left[\tfrac{1}{N}\right]), \\
&i : \mathbb{H} \to \mathsf{L}(N) && \tau \mapsto \left(\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}), dz, \left(\left[\tfrac{\tau}{N}\right], \left[\tfrac{1}{N}\right]\right)\right), \\
&i : \mathbb{H} \to \mathsf{L}_0(N) && \tau \mapsto \left(\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}), dz, \left\{\left[\tfrac{1}{N}\right], \left[\tfrac{2}{N}\right], \ldots, \left[\tfrac{N-1}{N}\right]\right\}\right).
\end{aligned}$$

Proposition 8.1 *Let f be an elliptic function of weigh k with poles at Λ. Then the pull-back of $f_{1,N}, f_N, f_{0,N}$ by i is a holomorphic modular form of weight k for $\Gamma_1(N), \Gamma(N)$, and $\Gamma_0(N)$ respectively.*

Proof. We only prove the $\mathsf{L}_1(N)$ case. The other cases are similar. For all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in$ $\Gamma_1(N)$ we have

$$\begin{aligned} f_{1,N}\left(\frac{a\tau+b}{c\tau+d}\right) &= f\left(\mathbb{Z}\frac{a\tau+b}{c\tau+d}+\mathbb{Z}, \frac{1}{N}\right) \\ &= (c\tau+d)^k\, f(\mathbb{Z}(a\tau+b)+\mathbb{Z}(c\tau+d), \frac{c\tau+d}{N}) \\ &= (c\tau+d)^k\, f(\mathbb{Z}\tau+\mathbb{Z}, \frac{1}{N}). \end{aligned}$$

Now, we have to show the growth condition. For this by Exercise 2.14 it is enough to assume that $f = \wp$ or $\wp'$. Note that if f has poles only at Λ then R_1, R_2 in Exercise 2.14 are polynomials in $\wp(z)$. In these cases the affirmation follows from Exercise 2.10.

Exercise 8.4 Write the proof of Proposition 8.1 for $\mathsf{L}_0(N)$ and $\mathsf{L}(N)$.

Proposition 8.2 *If f is a modular form of weight k for* $\mathrm{SL}(2,\mathbb{Z})$ *then* $g(\tau) := f(N\tau)$ *is a modular form of weight k for* $\Gamma_0(N)$.

Proof. First note that $N^{k-1} \cdot g = f|_k \begin{bmatrix} N & 0 \\ 0 & 1 \end{bmatrix}$ and so $g(\tau)$ is a priori a modular form for

$$\begin{bmatrix} N & 0 \\ 0 & 1 \end{bmatrix}^{-1} \mathrm{SL}(2,\mathbb{Z}) \begin{bmatrix} N & 0 \\ 0 & 1 \end{bmatrix}.$$

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ we have

$$\begin{bmatrix} N^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} N & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & N^{-1}b \\ N\cdot c & d \end{bmatrix},$$

If we take A's with $N|b$, this means that g is modular for $\Gamma_0(N)$.

Another important example of a modular forms comes from the following. Despite being trivial and well-known for experts, I found it in a natural way in the study of geometric quasi-modular forms in [Mov22b], see also Chapter 13.

Proposition 8.3 *The following is a modular form of weight two for* $\Gamma_0(N)$*:*

$$N \cdot E_2(N\tau) - E_2(\tau). \tag{8.2}$$

Proof. The proof is a mere computation and it follows from the functional equation of E_2 in Theorem 2.4. Let f be the holomorphic function in (8.2). For $\begin{bmatrix} a & b \\ c & s \end{bmatrix} \in \Gamma_0(N)$ we have

$$
\begin{aligned}
f\left(\frac{a\tau+b}{c\tau+d}\right) &= NE_2\left(\frac{aN\tau+Nb}{cN^{-1}N\tau+d}\right)-E_2\left(\frac{a\tau+b}{c\tau+d}\right)\\
&= N\left((c\tau+d)^2E_2(\tau)+\frac{12}{2\pi i}cN^{-1}(c\tau+d)\right)\\
&\quad -(c\tau+d)^2E_2(\tau)-\frac{12}{2\pi i}c(c\tau+d)\\
&= (c\tau+d)^2f(\tau).
\end{aligned}
$$

The following modular form appears in a natural way in the study of Picard's curious example in Theorem 13.5.

Exercise 8.5 For $N\in\mathbb{N}$ let f_N be an elliptic function with pole and zero of order N at $[0]$ and $[\frac{1}{N}]$, respectively, and with no other poles or zeros (see Exercise 2.17). The following function

$$
F_N(\tau):=\frac{1}{N}\frac{f_N'}{f_N}-\frac{1}{2}\frac{\wp'(z,\tau)+\wp'(\frac{1}{N},\tau)}{\wp(z,\tau)-\wp(\frac{1}{N},\tau)}
$$

is independent of z and it is a modular form for $\Gamma_1(N)$. Here, $'$ is derivation with respect to z. Hint: The two terms in $F_N(\tau)$ have poles of order one at $[0]$ and $[\frac{1}{N}]$ with residues 1 and -1 at these points. Therefore, the difference is a holomorphic elliptic function in z, and hence, it is constant in z. For the functional equation of $F_N(\tau)$ either prove it directly or interpret it as a function in $\mathsf{L}_1(N)$.

The motivation for the following exercise comes from Picard's moduli space in Section 13.7 and in particular Theorem 13.5.

Exercise 8.6 Let

$$
a:=\frac{\wp\left(\frac{1}{N},\tau\right)}{F_N},\quad b:=\frac{\wp'\left(\frac{1}{N},\tau\right)}{F_N},\quad c:=\frac{60G_4(\tau)}{F_N}.
$$

Show that there is a meomorphic function g on the upper half plane such that

$$
\begin{aligned}
g\cdot\frac{\partial a}{\partial\tau} &= 2c-24a^2+6ab+6b,\\
g\cdot\frac{\partial b}{\partial\tau} &= -3c+36a^2-36ab+9b^2,\\
g\cdot\frac{\partial c}{\partial\tau} &= 12ca+12cb-144a^3+36b^2.
\end{aligned}
$$

8.5 Fourier or q-expansion

Let f be a modular form of weight k for a congruence group Γ of level N. It follows that for all $A\in\mathrm{SL}(2,\mathbb{Z}),\ f|_kA$ has Fourier expansion and in this section we aim to explain this. We have

$$T^N = \begin{bmatrix} 1 & N \\ 0 & 1 \end{bmatrix} \in \Gamma(N), \quad \text{where } T := \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

and so for all $A \in \mathrm{SL}(2,\mathbb{Z})$

$$(f|_k A)|_k T^N = f|_k (AT^N A^{-1})|_k A = f|_k A$$

because $AT^N A^{-1} \in \Gamma(N) \subset \Gamma$. This implies that if we define $q_N := e^{\frac{2\pi i\tau}{N}}$ thennot]q_N, the q-expansion

$$f|_k A = \sum_{n=0}^{\infty} f_n \, q_N^n, \;\; a_n \in \mathbb{C}.$$

Exercise 8.7 For an elliptic function f such as $\wp$, compute the first few coefficients of q-expansions of modular forms $f_{1,N},\ f_N,\ f_{0,N}$ in Proposition 8.1 and F_N in Proposition 8.3.

Exercise 8.8 The quotient

$$\varphi(\tau) := \frac{\Delta(N\tau)}{\Delta(\tau)}$$

is a modular function for the group $\Gamma_0(N)$. This follows from the functional equation of $\Delta(\tau)$. It is a holomorphic in $\mathbb{H}$ and at $i\infty$:

$$\varphi(\tau) = \frac{q^N \prod_{n=1}^{\infty} (1-q^{nN})^{24}}{q \prod_{n=1}^{\infty} (1-q^n)^{24}}$$

and so it has a zero of order $N-1$ at $q=0$. There are no non-constant holomorphic functions on a compact Riemann surface, and so, φ as a function on $\overline{\Gamma_0(N)\backslash\mathbb{H}}$ is necessarily meromorphic. Compute its pole order at other cusps. For more information on φ see [Apo90, Section 4.7].

Definition 8.4 Let us now consider the equality

$$\begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix}^{-1} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix} = \begin{bmatrix} d & -cN^{-1} \\ -N\cdot b & a \end{bmatrix}$$

This together with Proposition 7.1 implies that we have a well-defined map

$$W_N : \mathrm{M}_k(\Gamma) \to \mathrm{M}_k(\Gamma), \;\; f \mapsto N^{-1}\tau^{-k} f(\frac{-1}{N\tau}), \quad \Gamma = \Gamma_0(N), \Gamma_1(N). \tag{8.3}$$

This map sends cusp forms to cusp forms and it is called the Fricke involution. std]Fricke involutionnot]W_N, Fricke involution

8.6 Transcendence degree of modular forms

Let $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ be a subgroup of finite index a and $f \in M_k(\Gamma)$. We define

$$\sum_{i=0}^{a} g_{a-i} \cdot X^i = \prod_{A \in \Gamma \backslash \mathrm{SL}(2,\mathbb{Z})} (X - f|_k A) \ , \quad g_0 := 1 \tag{8.4}$$

Theorem 8.2 *We have* $g_i \in M_{k\cdot i}(\mathrm{SL}(2,\mathbb{Z}))$.

Proof. Let $P(X)$ be the right hand side of (8.4). Then for $B \in \mathrm{SL}(2,\mathbb{Z})$ we have

$$P(X)|_k B = \prod_{A \in \Gamma \backslash \mathrm{SL}(2,\mathbb{Z})} \Big(X - (f|_k A)|_k B \Big) = P(X).$$

Therefore, the coefficients of $P(X)$ has the correct functional equation. The finite growth of g_i's follow from the finite growth of $f|_k A$'s for all $A \in \Gamma \backslash \mathrm{SL}(2,\mathbb{Z})$.

As a corrolary we get:

Proposition 8.4 *Let* $N \in \mathbb{N}$ *be a natural number and* f *be a modular form of weight* k *for* $\mathrm{SL}(2,\mathbb{Z})$*. There are modular forms* $g_i \in M_{k\cdot i}(\mathrm{SL}(2,\mathbb{Z})), \quad i = 0,1,\dots,\psi(N)$ *such that*

$$\sum_{i=0}^{\psi(N)} g_{\psi(N)-i} f(N\cdot\tau)^i = 0.$$

Proof. We know that $f(N\tau)$ is a modular form for $\Gamma_0(N)$, see Proposition 8.2, and the index of $\Gamma_0(N)$ is $\mathrm{SL}(2,\mathbb{Z})$ is $\psi(N)$, see Exercise 8.3. In this way Proposition 8.4 follows from Theorem 8.2.

We have also the following statement.

Theorem 8.3 *The field generated by modular forms for congruence groups is of transcendence degree two. More precisely, any modular form for a congruence group is in the algebraic closure of the field* $\mathbb{C}(E_4, E_6)$.

We may also try to state Proposition 8.4 for weakly holomorphic modular forms, that is, in Proposition 8.4 assume that f is a weakly holomorphic modular forms of weight k and pole order $m_1, m_2, \dots, m_s$ at the cusps of $\mathbb{H}/\Gamma$. Then g_i must be a weakly holomorphic modular form of weight $k\cdot i$ and its pole order at $i\infty$ must be computed in terms of m_i's. This might lead to a proof of some classial statements such as Exercise 13.1.

Let us consider the converse of Theorem 8.2, that is, let us consider $g_i \in M_{k\cdot i}\Big(\mathrm{SL}(2,\mathbb{Z})\Big)$, $i = 1,2,..a$, and define

$$P(X) = \sum_{i=0}^{a} g_{a-i} \cdot X^i.$$

The resultant of $P(X)$ is a homogeneous polynomial of degree $2\cdot k\cdot a$ in

$$\mathbb{Q}[g_1, g_2, \dots, g_a], \text{ weight } (g_i) = ki.$$

This is a weight $2ka$ modular form for $\mathrm{SL}(2,\mathbb{Z})$. Assume that this resultant has no zeros. Since $\mathbb{H}$ is simply connected, we can find holomorphic functions $f_1, f_2, .., f_a : \mathbb{H} \to \mathbb{C}$ such that

$$P(X) = (X - f_1)(X - f_2) \cdots (X - f_a).$$

We have the representation

$$\chi : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{GL}(a,\mathbb{Z}),$$

whose image is isomorphic to the permutation group in a elements and such that

$$\begin{pmatrix} f_1|_k A \\ f_2|_k A \\ \vdots \\ f_a|A \end{pmatrix} = \chi(A) \begin{pmatrix} f_1 \\ f_2 \\ \\ f_a \end{pmatrix}, \quad \forall A \in \mathrm{SL}(2,\mathbb{Z}). \tag{8.5}$$

Here, $\chi(A)$ is just a permutation matrix in $1, 2, \dots, a$. Let

$$\Gamma_i := \{A \in \mathrm{SL}(2,\mathbb{Z}) \Big| \chi(A)e_i = e_i\},$$

where $e_i = [0, 0, \dots, 1, 0, \dots, 0]^{\mathrm{tr}}$ and 1 is in the i-th place. We have $f_i \in M_k(\Gamma_i)$ which is a direct consequence of (8.5) and the definition of Γ_i.

Exercise 8.9 Show that $P(X)$ is irreducible over $M_k\Big(\mathrm{SL}(2,\mathbb{Z})\Big)[X]$ if and only if an orbit of χ in $\{1, 2, \dots, a\}$ is the whole set. It follows that if $P(X)$ is irreducible over $M_k\Big(\mathrm{SL}(2,\mathbb{Z})\Big)[X]$. Then

$$\{f_1, f_2, \dots, f_a\} = \{f_i|_k A, \quad A \in \Gamma\backslash\mathrm{SL}(2,\mathbb{Z})\}$$

and

$$\Gamma_i := A^{-1}\Gamma_1 A \qquad A \in \Gamma\backslash\mathrm{SL}(2,\mathbb{Z}).$$

The following question is natural to ask: under which conditions on g_i's, Γ_1 is a congruence group?

Chapter 9
Quasi-modular forms

There are five elementary arithmetical operations: addition, subtraction, multiplication, division, and modular forms, (a quote atributed to Martin Eichler).

9.1 Introduction

Like modular forms, examples of quasi-modular forms are abundant in the literature of 19 and 20 centuries. Typical examples are the Eisenstein series E_2 and logarithmic derivatives of theta series. The job of defining the algebra of quasi-modular forms, and also stamping the name, is done in [KZ95]. In this article the authors give a direct proof for a formula stated in [Dij95] which deals with counting ramified covering of elliptic curves and it is in the context of string theory. An algebraic geometric framework for quasi-modular forms using de Rham cohomology of elliptic curves has been introduced by the author in [Mov08, Mov12].

In Chapter 2 we have seen that the classical modular fotms are holomorphic functions on the Poincaré upper half plane which satisfy a functional equation with respect to the action of a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}$ and have some growth condition at infinity. Quasi-modular forms can be also presented in a similar way, despite the fact that their functional equation is longer, and so uglier. We will skip this approach and refer the reader to [MR05]. Instead, we adopt the algebraic geometric encarnation of quasi-modular forms in [Mov12], as this is open to generalizations in the context of Calabi-Yau varieties. The bridge between the two notions of algebraic and holomorphic quasi-modular forms is given by the generalized period map which is constructed by elliptic integrals.

Algebraic de Rham cohomology of algebraic varieties is introduced by A. Grothendieck in [Gro66]. Our aim in this chapter is to introduce this concept for elliptic curves without going to the details of the general theory. For a computational approach using a covering (Čech cohomology) see also [MV21]. Let E be an elliptic curve over a field k of characteristic zero. The algebraic de Rham cohomologies $H^i_{\mathrm{dR}}(E), i=0,1,2$ are k-vector spaces of dimensions respectively $1,2$ and

1, see Proposition 9.2, Proposition 9.4 and Proposition 9.5. We have $H^0_{\mathrm{dR}}(E) = \mathsf{k}$, an isomorphism $\mathrm{Tr} : H^2_{\mathrm{dR}}(E) \cong \mathsf{k}$ and a bilinear map

$$H^1_{\mathrm{dR}}(E) \times H^1_{\mathrm{dR}}(E) \to H^2_{\mathrm{dR}}(E).$$

The map Tr composed with the bilinear map gives us:

$$\langle \cdot, \cdot \rangle : H^1_{\mathrm{dR}}(E) \times H^1_{\mathrm{dR}}(E) \to \mathsf{k}$$

which is non-degenerate and anti symmetric. We call it the intersection bilinear form. We have also a natural filtration of $H^1_{\mathrm{dR}}(E)$ which is called the Hodge filtration:

$$\{0\} = F^2 \subset F^1 \subset F^0 = H^1_{\mathrm{dR}}(E)$$

Its non-trivial piece F^1 is generated by a regular differential form (a differential form of the first kind). In the present chapter we define all these in a down-to-earth manner.

9.2 De Rham cohomology of affine varieties

In this section we recall basic definitions and properties of differential forms in an algebraic geometry. For further details the reader is referred to [Har77, page 172] and [Mov21, Chapter 10].

Let A and R be commutative k-algebra and $\mathsf{R} \to A$ be a morphism of k-algebras. In all of our examples R will be a sub algebra of A. Using this morphism, A can be seen as an R-algebra. We assume that A as an R-algebra is finitely generated.

Definition 9.1 Let $\Omega_{A/\mathsf{R}}$ denote the module of relative (Kähler) differentials, that is, $\Omega_{A/\mathsf{R}}$ is the quotient of the A-module freely generated by symbols $dr,\ r \in A$, modulo its submodule generated by

$$dr,\ r \in \mathsf{R},\ d(ab) - adb - bda,\ d(a+b) - da - db,\ a, b \in A.$$

The A-module $\Omega_{A/\mathsf{R}}$ is finitely generated and it is equipped with the derivation

$$d : A \to \Omega_{A/\mathsf{R}},\ r \mapsto dr.$$

It has the universal property that for any R-linear derivation $D : A \to M$ with the A-module M, there is a unique A-linear map $\psi : \Omega_{A/\mathsf{R}} \to M$ such that $D = \psi \circ d$. Let $X = \mathrm{Sepc}(A)$ and $T = \mathrm{Sepc}(\mathsf{R})$ be the corresponding affine varieties over k and $X \to T$ be the map obtained by $\mathsf{R} \to A$. We will mainly use the Algebraic Geometry notation $\Omega^1_{X/T} := \Omega_{A/\mathsf{R}}$.

Definition 9.2 Let

$$\Omega^i_{X/T} = \bigwedge_{k=1}^{i} \Omega_{A/\mathsf{R}},$$

be the i-th wedge product of $\Omega_{X/T}$ over A, that is, $\Omega^i_{X/T}$ is the quotient of the A-module freely generated by the symbols $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_i$ modulo its submodule generated by elements which make $\wedge$ A-linear in each ω_i and

$$\omega_1 \wedge \cdots \wedge \omega_j \wedge \omega_{j+1} \wedge \cdots \wedge \omega_i = 0, \text{ for } \omega_j = \omega_{j+1}.$$

It is convenient to define

$$\Omega^0_{X/T} := A.$$

Definition 9.3 The differential operator

$$d_i : \Omega^i_{X/T} \to \Omega^{i+1}_{X/T}$$

is defined by assuming that it is R-linear and

$$d_i(a da_1 \wedge \cdots \wedge da_i) = da \wedge da_1 \wedge \cdots \wedge da_i, \;\; a, a_1, \ldots, a_i \in A.$$

Sometimes it is convenient to remember that d_i's are defined relative to R. One can verify easily that d_i is in fact well-defined and satisfy all the properties of the classical differential operator on differential forms on manifolds. From now on we simply write d instead of d_i. If $\mathsf{R} = \mathsf{k}$ is a field then we write X instead of X/T.

Exercise 9.1 1. Prove the universal property of the differential map $d : A \to \Omega_{A/\mathsf{R}}$.
2. Prove the following properties of the wedge product: For $\alpha \in \Omega^i_{X/T}$, $\beta \in \Omega^j_{X/T}$, $\gamma \in \Omega^r_{X/T}$
$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma),$$
$$\alpha \wedge \beta \wedge \gamma = (-1)^{ij+jr+ir} \gamma \wedge \beta \wedge \alpha,$$
3. Prove that $d \circ d = 0$.
4. For $\alpha \in \Omega^i_{X/T}$, $\beta \in \Omega^j_{X/T}$ we have:
$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^i \alpha \wedge (d\beta).$$

After the definition of differential forms, we get the de Rham complex of X/T, namely:

$$\Omega^0_{X/T} \to \Omega^1_{X/T} \to \cdots \Omega^i_{X/T} \to \Omega^{i+1}_{X/T} \to \cdots$$

Since $d \circ d = 0$, we can define the de Rham cohomologies

$$H^i_{\mathrm{dR}}(X/T) := \frac{\ker(\Omega^i_{X/T} \xrightarrow{d} \Omega^{i+1}_{X/T})}{\mathrm{Im}(\Omega^{i-1}_{X/T} \xrightarrow{d} \Omega^i_{X/T})}.$$

Exercise 9.2 1. Let m be the number of generators of the R-algebra A. Show that for $i \geq m+1$ we have $\Omega^i_{X/T} = 0$ and hence $H^i_{\mathrm{dR}}(X/T) = 0$.

2. Let $A = \mathsf{R}[x_1, x_2, \dots, x_n]$. In this case, we use the notation $\mathbb{A}^n_{\mathsf{R}} := \mathrm{Sepc}(A)$. The A-module $\Omega^1_{\mathbb{A}^n_{\mathsf{R}}}$ is freely generated by the elements $dx_1, dx_2, \dots, dx_n$. Prove that

$$H^i(\mathbb{A}^n_{\mathsf{R}}) = 0,\ i = 1, 2, \dots$$

This is in [Eis95, Exercise 16.15c, page 414].
3. Let us come back to the case of an arbitrary A. Let $a_1, a_2, \dots, a_m \in A$ generate the R-algebra A. Define

$$I = \{P \in \mathsf{R}[x_1, x_2, \dots, x_m] \mid P(a_1, a_2, \dots, a_m) = 0\}.$$

The set I is an ideal of $\mathsf{R}[x_1, x_2, \dots, x_m]$ and we have

$$A \cong \mathsf{R}[x_1, x_2, \dots, x_m]/I,$$

$$\Omega^i_{X/T} \cong \Omega^i_{\mathbb{A}^n_{\mathsf{R}}}/(dI \wedge \Omega^{i-1}_{\mathbb{A}^n_{\mathsf{R}}} + I\Omega^i_{\mathbb{A}^n_{\mathsf{R}}}),$$

where by $dI \wedge \Omega^{i-1}_{\mathbb{A}^n_{\mathsf{R}}} + I\Omega^i_{\mathbb{A}^n_{\mathsf{R}}}$ we mean the A-module generated by

$$dr_1 \wedge \omega_1 + r_2\omega_2,\ r_1, r_2 \in I,\ \omega_1 \in \Omega^{i-1}_{\mathbb{A}^n_{\mathsf{R}}},\ \omega_2 \in \Omega^i_{\mathbb{A}^n_{\mathsf{R}}}.$$

4. Discuss conditions on A such that $H^0(X/T) = \mathsf{R}$. For instance, show that if $\mathsf{R} = \mathsf{k}$ is an algebraically closed field of characteristic zero and X is an irreducible reduced variety over k then $H^0(X) = \mathsf{k}$.

9.3 Two incomplete elliptic curves

In this section we find an explicit basis for the de Rham cohomology of the main examples of this text, that is, affine elliptic curves in Weierstrass form. The general theory uses the notion of a Brieskorn module which is essentially the same as de Rham cohomology. Our main source for this section is [Mov21, Chapter 10] [Ked08].

Let $t_2, t_3 \in \mathsf{R}$, $P(x) = 4x^3 - t_2x - t_3 \in \mathsf{R}[x]$ and $f = y^2 - P(x)$. Define $A = \mathsf{R}[x, y]/\langle f \rangle$, where $\langle f \rangle$ is the ideal generated by f. We have

$$\Omega^1_{X/T} = \Omega^1_{\mathbb{A}^2_{\mathsf{R}}}/\langle f\Omega^1_{\mathbb{A}^2_{\mathsf{R}}} + \Omega^0_{\mathbb{A}^2_{\mathsf{R}}} df\rangle,\ \Omega^2_{X/T} = \Omega^2_{\mathbb{A}^2_{\mathsf{R}}}/\langle f\Omega^2_{\mathbb{A}^2_{\mathsf{R}}} + df \wedge \Omega^1_{\mathbb{A}^2_{\mathsf{R}}}\rangle.$$

We have to say some words about $\Omega^2_{X/T}$. We define the auxiliary R-module:

$$V := \Omega^2_{\mathbb{A}^2_{\mathsf{R}}}/df \wedge \Omega^1_{\mathbb{A}^2_{\mathsf{R}}},$$

and $\Delta := 27t_3^2 - t_2^3$. The quotient $\mathsf{R}[x, y]/\langle f_x, f_y\rangle$ is isomorphic to V by sending $P \in V$ to $P dx \wedge dy$.

Proposition 9.1 *We have*

$$\Delta \Omega^2_{X/T} = 0.$$

Proof. This is just the reformulation of Proposition 4.1.

From now on we assume that Δ is irreducible in R and we replace R with its localization on its multiplicative group generated by Δ. Therefore, Δ is invertible in R and we can talk about the pole or zero order along Δ of an element in any R-module. In this way $\Omega^2_{X/T} = 0$ and

$$H^1_{\mathrm{dR}}(X/T) \cong \Omega^1_{\mathbb{A}^2_{\mathsf{R}}} / \langle f\Omega^1_{\mathbb{A}^2_{\mathsf{R}}} + df\Omega^0_{\mathbb{A}^2_{\mathsf{R}}} + d\Omega^0_{\mathbb{A}^2_{\mathsf{R}}} \rangle.$$

There are two polynomials $A, B \in \mathsf{R}[x]$ such that $AP + BP' = \Delta$. We define

$$\omega = \frac{1}{\Delta}(Aydx + 2Bdy),$$

which satisfies:

$$dx = y\omega, dy = \frac{1}{2}P'\omega, \tag{9.1}$$

We denote by $\frac{dx}{y}$ and $\frac{xdx}{y}$ the elements ω, respectively $x\omega$. Note that these two elements have poles of order at most one along Δ.

Proposition 9.2 *The R-module $H^1_{\mathrm{dR}}(X/T)$ is freely generated by the elements $\frac{dx}{y}$ and $\frac{xdx}{y}$.*

Proof. Using the equalities (9.1) and $y^2 = P(x)$, every element of $H^1_{\mathrm{dR}}(X/T)$ can be written in the form $(C + yD)\omega$, $C, D \in \mathsf{R}[x]$. Since $Dy\omega = Ddx$ is exact, this reduces to $C\omega$. From another side the elements

$$d(x^a y) = (\frac{1}{2}P'x^a + ax^{a-1}P)\omega$$

are cohomologous to zero. This equality can be written as

$$d(x^a y) = \left((4a+6)x^{a+2} - (a+\frac{1}{2})t_2 x^a - at_3 x^{a-1}\right)\frac{dx}{y}.$$

If $\deg(C) \geq 2$, we can choose a monomial $F = x^a$ in such a way that the leading coefficient of $(\frac{1}{2}P'F + F'P)$ is equal to the leading coefficient of C. We subtract $d(Fy)$ from $C\omega$ and we get smaller degree for C. We repeat this until getting a degree one C.

The following is the same as Exercise 3.12 without integral sign.

Exercise 9.3 Show that restricted to the elliptic curve E_{t_2,t_3} we have the following equalities:

$$\begin{aligned}
\frac{x^2dx}{y} &= \frac{1}{12}t_2\frac{dx}{y}+d(\frac{y}{6}),\\
\frac{x^3dx}{y} &= \frac{1}{10}t_3\frac{dx}{y}+\frac{3}{20}t_2\frac{xdx}{y}+d(\frac{xy}{10})\\
\frac{x^4dx}{y} &= \frac{5}{336}t_2^2\frac{dx}{y}+\frac{1}{7}t_3\frac{xdx}{y}+d(\frac{1}{14}x^2y+\frac{5}{168}t_2y)\\
\frac{x^5dx}{y} &= \frac{1}{30}t_2t_3\frac{dx}{y}+\frac{7}{240}t_2^2\frac{xdx}{y}+d(\frac{1}{18}x^3y+\frac{7}{360}t_2xy+\frac{1}{36}t_3y).
\end{aligned}$$

In $H^1_{\mathrm{dR}}(X/T)$ we can further ignore the exact differential forms. The expression of exact forms will be used in Chapter 12. Hint: See Exercise 3.12.

Remark 9.1 The proof of Proposition 9.2 shows that for $n\in\mathbb{N}$ we have

$$x^n\frac{dx}{y}=a_0(t_2,t_3)\frac{dx}{y}+a_1(t_2,t_3)\frac{xdx}{y}+d(yQ(x))$$

for some homogeneous polynomials $a_0,a_1\in\mathbb{Q}[t_2,t_2],\ \deg(t_2)=2,\ \deg(t_3)=3$ of degree n and $n-1$, respectively, and a polynomial $Q\in\mathbb{Q}[x]$ of degree $\leq n-2$.

Let $P(x)\in\mathsf{R}[x]$ be as before and

$$A=\mathsf{R}[x,y,z]/\langle y^2-P(x),yz-1\rangle$$

We will simply write $\frac{1}{y}$ instead of z.

Proposition 9.3 *The* R *module* $H^1_{\mathrm{dR}}(X/T)$ *is freely generated by*

$$\frac{dx}{y},\frac{xdx}{y},\frac{dx}{y^2},\frac{xdx}{y^2},\frac{x^2dx}{y^2},$$

Proof. In this example $dy=\frac{1}{2y}P'dx$ and so every element ω of $H^1_{\mathrm{dR}}(X/T)$ can be written in the form $Cy^{-k}dx+Cy^{-k+1},\ C,D\in\mathsf{R}[x],\ k\geq 1$. We use the equality

$$d(x^ay^{-b})=ax^{a-1}y^{-b}dx+\frac{-b}{2}x^ay^{-b-2}P'dx$$

for $b=-1,-2,\dots$ and see that ω is reduced to a form with $k=1$ (each time we multiply ω with $\Delta=BP'+Ay^2$). Now, for terms $Cy^{-2}dx$ we make the division of C by P and we are thus left with the generators $\frac{dx}{y^2},\frac{xdx}{y^2},\frac{x^2dx}{y^2}$. For terms $Dy^{-1}dx$ we proceed as in Proposition 9.2 and we are left with the generators $\frac{dx}{y},\frac{xdx}{y}$.

9.4 Algebraic de Rham cohomology of complete curves

Let $X = X/\mathsf{R}$ be a curve over the ring R of characteristic zero. For simplicity, we will cut $/\mathsf{R}$ from our notations. We take a covering $X = U_0 \cup U_1$ by two open set U_0 and U_1. We denote by $\Omega^i_{U_j},\ i,j = 0,1$ the set of of differential i-forms in U_j. By definition $\Omega^0_{U_j}$ is the ring of regular functions in U_j. The de Rham cohomologies of X relative to this covering are defined in the following way:

$$H^0(X/\mathsf{R}) := \mathsf{R}$$
$$H^1(X/\mathsf{R}) := \frac{\{(\omega_0,\omega_1) \in \Omega^1_{U_0} \times \Omega^1_{U_1} \mid \omega_1 - \omega_0 \in d(\Omega^0_{U_0\cap U_1})\}}{d\Omega^0_{U_0} \times d\Omega^0_{U_1}}$$
$$H^2(X/\mathsf{R}) := \frac{\Omega^1_{U_0\cap U_1}}{(\Omega^1_{U_0} + \Omega^1_{U_1} + d\Omega^0_{U_0\cap U_1})}.$$

The definition using a covering with n open sets is similar and together with the following exercise is left to the reader.

Exercise 9.4 Show that for any two covering of X, the corresponding de Rham cohomologies are isomorphic in a natural way.

As usual, our main example of an elliptic curve is written in the Weierstrass format:

$$X = E = \mathrm{Proj}(\mathsf{R}[x,y,z]/\langle zy^2 - 4x^3 + t_2xz^2 + t_3z^3\rangle)$$

which is covered by two open sets

$$U_0 = \mathrm{Sepc}(\mathsf{R}[x,y]/\langle y^2 - 4x^3 + t_2x + t_3\rangle),$$
$$U_1 = \mathrm{Sepc}(\mathsf{R}[x,z]/\langle z - 4x^3 + t_2xz^2 + t_3z^3\rangle).$$

The elliptic curve E has a closed point $O := [0;1;0]$ which is in the affine chart U_1. It is sometimes called the point at infinity. In the next section we will carry out some residue calculus at this point.

9.5 Residue calculus, trace map and cup product

We need to carry out some residue calculus near the closed smooth point O, see for instance [Tat68]. Such a machinery is usually developed for curves over a field and so it seems to be necessary to consider the elliptic curve E over the fractional field k_1 of R, that is, we use $E \otimes_{\mathsf{R}} \mathsf{k}_1$ instead of E. However, most of our calculations lead to elements in R which will be used later in the theory of quasi-modular forms.

A regular function t in a neighborhood of $O = [0;1;0]$ is called a coordinate system at O if $t(O) = 0$ and t generates the one dimensional k_1-space m_O/m_O^2, where m_O is the ring of regular functions in a neighborhood of O such that they vanish at

O, and m_O^2 is the $\mathscr{O}_{X,O}$-module generated by $ab,\ a,b\in m_O$. Recall that O is a smooth point of E. Any meromorphic function f (meromorphic 1-form ω) near O has an expansion in t:

$$f=\sum_{i=-a}^{\infty} f_i t^i,\ \text{resp. } \omega=(\sum_{i=-a}^{\infty} f_i t^i)dt,\ f_i\in \mathsf{k}_1, \tag{9.2}$$

where a is some integer. The stalk of the ring of meromorphic differential 1-forms at O is a $\mathscr{O}_{X,O}$-module generated by dt and so $\omega = f dt$ for some meromorphic function near O. Therefore, it is enough to explain the first equality. Let a be the pole order of f at O. We work with $t^a f$ and so without loosing the generality we can assume that f is regular at O. Let $f_0 = f(O)$. For some $f_1\in \mathsf{k}_1$ we have $f-f_0-f_1t\in m_O^2$. We repeat this process and get a sequence $f_0,f_1,f_2,\ldots,f_m,f_{m+1},\ldots\in\mathsf{k}_1$ such that

$$f-\sum_{i=0}^{m} f_i t^i\in m_O^{m+1}.$$

Another way of reformulating the above statement is:

$$f=\sum_{i=0}^{m} f_i t^i + O(t^{m+1}),$$

where $O(t^i)$ means a sum $\sum_{j\geq i} a_j t^j$. This is what we have written in (9.2).

The residue of ω at O is defined to be f_{-1}. It is independent of the choice of the coordinate t. In our example, we take the coordinate $t=\frac{x}{y}$ with the notation of chart U_0 (in the chart U_1 we have $t=x$). The expansions of x and y in t are of the form:

$$x=\frac{1}{4}t^{-2}+O(t^0),\ y=\frac{1}{4}t^{-3}+O(t^{-1}). \tag{9.3}$$

Exercise 9.5 1. Show that O is a smooth point of E, that is, the k_1-vector space m_O/m_O^2 is one dimensional.
2. Verify the equalities (9.3) and prove that the notion of residue does not depend on the coordinate system t.
3. Calculate the residue of $\frac{x^n dx}{y^2},\ n=0,1,2,3,4,5$ at O.
4. Calculate the first 4 coefficients of the expansion of $\frac{dx}{y}$ in the coordinate $t=\frac{x}{y}$.
5. Let us take the coordinates (x,z) in which the elliptic curve E is given by $z-4x^3-t_2xz^2-t_3z^3$ and we have $O=(0,0),\ t=x$. Consider E over the ring R. A regular function f at O can be written as $\frac{P(x,z)}{Q(x,z)}$ with $Q(0,0)\neq 0$. Show that if $Q(0,0)$ is invertible in R and $P,Q\in\mathsf{R}[x,z]$ then all the coefficients in the expansion of f belong to R (Hint: Verify this for $f=z$.)

Recall the open covering $\{U_0,U_1\}$ of E introduced in Section 9.4.

Proposition 9.4 *The canonical restriction map*

$$H^1_{dR}(E) \to H^1_{dR}(U_0),\ (\omega_0, \omega_1) \to \omega_0$$

is an isomorphism of R*-modules.*

Proof. First we check that it is injective. Let us take an element $(\omega_0, \omega_1) \in H^1_{\mathrm{dR}}(X)$ with $\omega_0 = 0$. By definition $\omega_1 = \omega_1 - \omega_0 = df,\ f \in \Omega^0_{U_0 \cap U_1}$. Since ω_1 has not poles at the closed point $O \in X$, f has not too, which implies that (ω_0, ω_1) is cohomologous to zero.

Now, we prove the surjectivity. The restriction map is R-linear and so by Proposition 9.2, it is enough to prove that $\frac{dx}{y},\ \frac{xdx}{y}$ are in the image of the restriction map. In fact, the corresponding elements in $H^1_{\mathrm{dR}}(E)$ are respectively

$$(\frac{dx}{y}, \frac{dx}{y}),\ \ (\frac{xdx}{y}, \frac{xdx}{y} - \frac{1}{2}d(\frac{y}{x})).$$

We prove this affirmation for $\frac{xdx}{y}$. We define $\tilde{U}_1 = U_1 \backslash \{x = 0\}$ and use the definition of hypercohomology with the covering $\{U_0, \tilde{U}_1\}$. We compute x and y in terms of the local coordinate $t = \frac{y}{x}$ around the point at infinity O and we have (9.3). Substituting this in $\frac{xdx}{y}$, we get the desired result.

Let U_0, U_1 be a covering of a smooth curve X. We have a well-defined map

$$\mathrm{Tr} : H^2_{\mathrm{dR}}(X) \to \mathsf{R},\ \mathrm{Tr}(\omega) = \text{ sum of the residues of } \omega_{01} \text{ around the points } X \backslash U_0,$$

where ω is represented by $\omega_{01} \in \Omega^1_{U_0 \cap U_1}$. For the elliptic curve E in the weierstrass format we take the canonical charts of E described in Section 9.4. The map Tr turns out to be an isomorphism of R-modules.

Proposition 9.5 *For the elliptic curve E in the Weierstrass format the* R*-module $H^2_{\mathrm{dR}}(X)$ is of rank one.*

Proof. According to Proposition 9.3 any element in $\Omega^1_{U_0 \cap U_1}$ modulo exact forms can be reduced to an R-linear combination of 5 elements. The classes of all these elements in $H^2_{\mathrm{dR}}(X)$ is zero, except the last one $\frac{x^2dx}{y^2}$. The first two elements are regular forms in U_0 and the next two forms are regular in U_1. We have proved that any element $\omega \in H^2_{\mathrm{dR}}(X)$ is reduced to $r\frac{x^2dx}{y^2},\ r \in \mathsf{R}$. Since $\frac{x^2dx}{y^2}$ at O has the residue $\frac{-1}{2}$ (use the local coordinate $t = \frac{x}{y}$ and the equalities (9.3)), we get the desired result.

Exercise 9.6 1. Let us take two open sets $U_1, \tilde{U}_1 \subset E$ which contain O. Show that the definition of de Rham cohomologies of E attached to the coverings $\{U_0, U_1\}$ and $\{U_0, \tilde{U}_1\}$ are canonically isomorphic.
2. By our definition of residue, it takes values in k_1, the fractional field of E. Show that the map Tr has values in R.

Now we define the cup product in the case of a curve defined over R. Let us take two elements $\omega, \alpha \in H^1_{\mathrm{dR}}(X)$. We take an arbitrary covering $X = \cup_i U_i$ of X and we assume that ω and α are given by $\{\omega_{ij}\}_{i,j \in I}$and $\{\alpha_{ij}\}_{i,j \in I}$with

$$\omega_j - \omega_i = df_{ij},\ \alpha_j - \alpha_i = dg_{ij}.$$

We define

$$\gamma := \omega \cup \alpha \in H^2_{\mathrm{dR}}(X)$$

which is given by:

$$\gamma_{ij} = g_{ij}\omega_j - f_{ij}\alpha_j + f_{ij}dg_{ij}. \tag{9.4}$$

Let us consider the elliptic curve in the Weierstrass format as in Section 9.4. In this case

$$\frac{dx}{y} \cup \frac{xdx}{y} = \{\omega_{01}\},\ \omega_{01} = \frac{-1}{2}\frac{dx}{x},$$

and

$$\langle \frac{dx}{y}, \frac{xdx}{y} \rangle = 1. \tag{9.5}$$

Exercise 9.7 1. Show that the definition of $\omega \cup \alpha$ does not depend on the covering of the curve X and that $\cup$ is non-degenerate.
2. For a curve over complex numbers show its algebraic de Rham cohomology, cup product and intersection form are essentially the same objects defined by C^∞-functions.

9.6 Eisenstein modular forms

In this section we give an application of the residue computations in Section 9.5. It can be skipped as we will not need it later. We aim to recover the Laurent expansion of the Weierstrass $\wp$ function, see Theorem 2.1, in an algebraic framework.

Let E be an elliptic curve over k and ω be a regular differential 1-form ($\omega \in F^1H^1_{\mathrm{dR}}(E)$). We take the Weierstrass coordinates (x,y) of the pair (E,ω), and so

$$E: y^2 = 4x^3 - t_2x - t_3,\ \ \omega = \frac{dx}{y},\ \ t_2,t_3 \in \mathsf{k}.$$

For our discussion we may only consider a ring $\mathsf{R} \subset \mathsf{k}$ with $t_2,t_3 \in \mathsf{R}$. Let also t be a coordinate system around the point $O = [0;1;0]$, for instance take $t = \frac{x}{y}$. We have $\frac{dx}{y} = Pdt$ for some regular function P in a neighborhood of O. Let us write the formal series of P at O and then write it as a derivation of some other formal power series $z = z(t) = \sum_{i=1}^{\infty} z_i t^i$. We have

$$\frac{dx}{y} = dz,\ z_1 = -2.$$

We call z the analytic coordinate system on E. Note that the first coefficients in the formal power series of t^3y, t^2x in t are invertible and so $z(t)$ has coefficients in R.

Proposition 9.6 *We have*

$$x = \frac{1}{z^2} + \sum_{k=1}^{\infty} g_{2k+2} z^{2k},$$

and

$$y = \frac{\partial x}{\partial z} = \frac{-2}{z^3} + \sum_{k=1}^{\infty} 2k \cdot g_{2k+2} z^{2k},\ g_{2k+2} \in \mathsf{R}.$$

Proof. We have $z(t) = -2t + O(t^2)$ and write t in terms of z, that is, $t = t(z) = \frac{-1}{2}z + O(z^2)$. Since the coefficients of $z(t)$ are in R and $z(t)$ starts with $-2t$, the coefficients of $t(z)$ are also in R. We write x in terms of z and we have: $x = \sum_{k=-2}^{\infty} g_k z^k$ for some $g_k \in \mathsf{R}$. The elliptic curve E is invariant under the involution $(x,y) \mapsto (x,-y)$. The coordinates t and z are mapped to $-t$ and $-z$, respectively, and x is invariant. This implies that $g_k = 0$ for all odd integers k. Calculating g_0, g_2 we see that $g_0 = g_2 = 0$. The expansion of y follows from the equality $dx = ydz$.

Proposition 9.7 *The mapping* $(E,\omega) \to g_{2k+2}$ *is a full modular form of weight* $2k+2$.

We denote this modular form with G_{2k+2} and we call it the (algebraic) Eisenstein modular form of weight $2k+2$.

Proof. The growth condition in the definition of a quasi-modular form follows from the fact that in the process of defining G_{2k+2}, all the coefficients are in R. For $k \in \mathbb{G}_m$, the Weierstrass coordinates system of $(E,\omega) \bullet k$ is $(\tilde{x},\tilde{y}) = (k^{-2}x, k^{-3}y)$. In this coordinates system $\tilde{t} = kt$ and $\tilde{z} = kz$ which give us the desired functional property of g_{2k+2}'s with respect to the action of $\mathbb{G}_m$.

Exercise 9.8 Show the last piece of the proof of Proposition 9.6, that is, $g_0 = g_2 = 0$. Calculate G_4 and G_6.

9.7 Ibiporanga: enhanced elliptic curves

Let N be a positive integer. In this section we use the notation of groups $\Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$ without using them, see Chapter 6 for their definitions. Their appearance in this section is for the sake of following the terminology in the literature. They are the monodromy groups of universal families of elliptic curves enhanced with certain torsion points, and hence, one usually do not see them in an algebraic context, however they are hidden there, see Section 3.3.

Definition 9.4 An enhanced elliptic curve for $\Gamma_0(N)$ is a 3-tuple (E,C,ω), where E is an elliptic curve over k, C is a cyclic subgroup of $E(\mathsf{k})$ of order N and ω is an element in $H^1_{\mathrm{dR}}(E)\backslash F^1$. An enhanced elliptic curve for $\Gamma_1(N)$ is a 3-tuple (E,Q,ω), where E,ω are as before and Q is a point of $E(\mathsf{k})$ of order N. Let us fix a primitive root of unity of order N in k, say ζ. An enhanced elliptic curve for $\Gamma(N)$ is a 3-tuple

$(E,(P,Q),\omega)$, where E,ω are as before and P and Q are a pair of points of $E(\mathsf{k})$ that generates the N-torsion subgroup $E[N]$ with Weil pairing $e_N(P,Q)=\zeta$. For the definition of Weil pairing see Chapter 8. We call C,Q or (P,Q) a torsion structure on E. The number of enhanced elliptic curves for Γ with fixed E and ω is finite and it can be shown that it is the cardinality of the quotient $\Gamma\backslash \mathrm{SL}(2,\mathbb{Z})$. For $N=1$ an enhanced elliptic curve for all $\Gamma_0(N),\Gamma_1(N)$ and $\Gamma(N)$ is the same and so we write $\Gamma=\mathrm{SL}(2,\mathbb{Z})=\Gamma_0(N)=\Gamma_1(N)=\Gamma(N)$.

The choice of $\omega\in H^1_{\mathrm{dR}}(E)\backslash F^1$ determines in a unique way a regular differential 1-form $\omega_1\in F^1$ with $\langle\omega_1,\omega\rangle=1$. This is because F^1 is a one dimensional subspace of $H^1_{\mathrm{dR}}(E)$ and any non-zero element in F^1 together with ω form a basis of $H^1_{\mathrm{dR}}(E)$, and hence, it has non-zero intersection with ω (otherwise the intersection form would be identically zero). In this way, ω_1,ω form a basis of the k-vector space $H^1_{\mathrm{dR}}(E)$. In a similar way we can define a family of enhanced elliptic curves, see [Har77, Chapter III, Section 10].

Remark 9.2 The general definition of an enhanced projective varieties given in [Mov22a, Chapter 3] includes a marked polarization $\theta\in H^2_{\mathrm{dR}}(E)$ with $\mathrm{Tr}(\theta)\in\mathbb{N}$. This is a discrete object and its presense in the enhancement can be neglected. As before, we choose $\omega_1\in F^1H_{\mathrm{dR}}(E)$ in such a way that $\langle\omega_1,\omega\rangle:=\mathrm{Tr}(\omega_1\cup\omega)=\mathrm{Tr}(\theta)\cdot\frac{\omega_1\cup\omega}{\theta}=1$. In the case of polarized abelian varieties this will produce different moduli spaces, see [Mov22a, Chapter 11].

Let $\mathsf{T}_0(N),\mathsf{T}_1(N)$ and $\mathsf{T}(N)$ be the set of enhanced elliptic curves for $\Gamma_0(N),\Gamma_1(N)$ and $\Gamma(N)$ respectively, and modulo canonical isomorphisms. In the following we will denote by T one of these moduli spaces. The additive group $\mathbb{G}_a=(\mathsf{k},+)$ and the multiplicative group $\mathbb{G}_m=(\mathsf{k}^*,\cdot)$ acts in a canonical way on T:

$$(*,*,\omega)\bullet k=(*,*,k^{-1}\omega),\ k\in\mathbb{G}_m,\ (*,*,\omega)\in\mathsf{T},$$

$$(*,*,\omega)\bullet k=(*,*,k'\omega_1+\omega),\ k'\in\mathbb{G}_a,\ (*,*,\omega)\in\mathsf{T}.$$

Both these actions can be summarized in the action of the algebraic group

$$\mathsf{G}=\left\{\begin{bmatrix}k & k'\\ 0 & k^{-1}\end{bmatrix}\ \mid\ k'\in\mathsf{k},k\in\mathsf{k}-\{0\}\right\}\cong\mathbb{G}_a\times\mathbb{G}_m \tag{9.6}$$

that is,

$$(*,*,\omega)\bullet g=(*,*,k'\omega_1+k^{-1}\omega),\ g\in G,\ (*,*,\omega)\in\mathsf{T}.$$

For $N=1$, our three moduli spaces are the same $\mathsf{T}:=\mathsf{T}_0(1)=\mathsf{T}_1(1)=\mathsf{T}(1)$, and we have:

Proposition 9.8 *For $N=1$ the moduli space T is defined over $\mathbb{Q}$ and it is the affine variety*

$$\mathsf{T}=\mathrm{Sepc}\mathbb{Q}[t_1,t_2,t_3,\frac{1}{\Delta}],\ \Delta:=27t_3^2-t_2^3$$

We have a universal family over T given by $\mathsf{E}\to\mathsf{T}$, where

$$\begin{aligned}
\mathsf{E} \ &: \ zy^2-4(x-t_1)^3+t_2z^2(x-t_1)+t_3z^3=0,\\
&\quad [x;y;z]\times(t_1,t_2,t_3)\in\mathbb{P}^2\times\mathsf{T},\\
\omega &:= \left[\frac{xdx}{y}\right]\in H^1_{\mathrm{dR}}(\mathsf{E}/\mathsf{T}), \ \textit{given in the affine coordinate } z=1.
\end{aligned}$$

The action of the algebraic group G *on* T *is given by*

$$t\bullet g := (t_1k^{-2}+k'k^{-1}, t_2k^{-4}, t_3k^{-6}), t=(t_1,t_2,t_3), g=\begin{bmatrix} k & k' \\ 0 & k^{-1}\end{bmatrix}\in\mathsf{G}.$$

Proof. We use Proposition 4.4 to write a pair (E,ω_1) of elliptic curve and holomorphic 1-form ω_1 in the Weierstrass format. We get $\omega=\frac{xdx}{y}+t_1\frac{dx}{y}$. The proof of the action of the algebraic group is as follows: Let

$$\alpha:\mathbb{A}^2_{\mathsf{k}}\to\mathbb{A}^2_{\mathsf{k}}, \ (x,y)\mapsto(k^2x-k'k, k^3y)$$

and $f=y^2-4(x-t_1)^3+t_2(x-t_1)+t_3$. We have

$$k^{-6}\alpha^*(f)=y^2-4k^{-6}(k^2x-k'k-t_1)^3+t_2k^{-6}(k^2x-k'k-t_1)+t_3k^{-6}=$$
$$y^2-4(x-k'k^{-1}-t_1k^{-2})^3+t_2k^{-4}(x-k'k^{-1}-t_1k^{-2})+t_3k^{-6}.$$

This implies that α induces an isomorphism of elliptic curves

$$\alpha:(E_{t\bullet g},\alpha^*(\frac{xdx}{y})\to(E_t,\frac{xdx}{y}).$$

Since

$$\alpha^*\frac{xdx}{y}=k\frac{xdx}{y}-k'\frac{dx}{y}$$

we get the result.

Note that from the beginning we could work with the elliptic curve E in the Weierstrass form with $t_1=0$. We have the isomorphism

$$(\{y^2=4(x-t_1)^3-t_2(x-t_1)-t_3\},\frac{xdx}{y})\cong(\{y^2=4x^3-t_2x-t_3\},\frac{xdx}{y}+t_1\frac{dx}{y}),$$
$$(x,y)\mapsto(x-t_1,y). \tag{9.7}$$

For historical reasons, we also present the following proposition.

Proposition 9.9 *The moduli space* $\mathsf{T}[2]$ *is defined over* $\mathbb{Q}$ *and it is the affine variety*

$$\mathsf{T}=\mathrm{Sepc}\mathbb{Q}[t_1,t_2,t_3,\frac{1}{t_1-t_2},\frac{1}{t_1-t_3},\frac{1}{t_2-t_3}],$$

We have a universal family over T *given by* $\mathsf{E}\to\mathsf{T}$*, where*

$$\begin{aligned}
\mathsf{E} \ &: \ zy^2 - 4(x-t_1z)(x-t_2z)(x-t_3z),\\
&[x;y;z] \times (t_1,t_2,t_3) \in \mathbb{P}^2 \times \mathsf{T},\\
&P = [t_1;0:1],\ Q = [t_2;0;1]\\
\omega &:= \left[\frac{xdx}{y}\right] \in H^1_{\mathrm{dR}}(\mathsf{E}/\mathsf{T}),\ \textit{given in the affine coordinate } z=1.
\end{aligned}$$

The action of the algebraic group G *on* T *is given by*

$$t \bullet g := (t_1k^{-2}+k'k^{-1}, t_2k^{-2}+k'k^{-1}, t_3k^{-2}+k'k^{-1}), t = (t_1,t_2,t_3), g = \begin{bmatrix} k & k' \\ 0 & k^{-1} \end{bmatrix} \in \mathsf{G}.$$

Recall that in Section 3.7 we have reinterpreted the Ramanujan (resp. Darboux-Halphen) differential equation as a vector field in the parameter space T in Proposition 9.8 (resp. Proposition 9.9).

9.8 Quasi-modular forms

The algebraic group G acts from the right on T and so it acts from the left on the space of functions on T.

Definition 9.5 A quasi-modular form f of weight m and differential order or depth n for Γ is a function $\mathsf{T} \to \mathsf{k}$ with the following properties:

1. With respect to the action of $\mathbb{G}_m$, f satisfies
$$k \bullet f = k^m f,\ \mathsf{k} \in \mathbb{G}_m. \tag{9.8}$$
2. With respect to the action of $\mathbb{G}_a$, f satisfies the following condition: there are functions $f_i : \mathsf{T} \to \mathsf{k},\ i = 0,1,2,\ldots,n$ such that
$$k' \bullet f = \sum_{i=0}^{n} \binom{n}{i} k'^i f_i,\ k' \in \mathbb{G}_a. \tag{9.9}$$
3. (Growth condition)?

We were not able to formulate a growth condition for quasi-modular forms in a purely algebraic and intrinsic way using degeneration of curves. Such a condition would correspond to the classical growth condition for holomorphic quasi-modular forms. In [Kat76b], this condition is formulated in terms of Tate curves and Eisenstein series. This does not seem to be a natural one because it assumes a priori that we know Eisenstein series. The formulation in [Hid12] allows modular forms to have poles on cusps. We are going to introduce this condition using one of its main consequences, namely, the k-algebra of quasi-modular forms for $\mathrm{SL}(2,\mathbb{Z})$ is generated by three Weierstrass coordinates. Note that combining both actions (9.8) and (9.9) we have:

$$f \bullet g = k^{-m} \sum_{i=0}^{n} \binom{n}{i} k'^i k^i f_i, \ \forall g = \begin{bmatrix} k & k' \\ 0 & k^{-1} \end{bmatrix} \in G. \tag{9.10}$$

Let us consider the case $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. We are going to describe the growth condition in this case. Using Proposition 9.8, the Weierstrass coordinate $t_i,\ i = 1,2,3$ of an enhanced elliptic curve (E,ω) satisfies the functional equations (9.8) and (9.9) with weight $m = 2i$ and differential order $n = 1$ for t_1 and $n = 0$ for t_2 and t_4. The growth condition for f in this case is that f is an element in the k-algebra

$$\mathsf{k}[t_1,t_2,t_3],\ \text{weight}(t_i) = 2i,\ i = 1,2,3.$$

It follows that it is homogeneous with $\deg(f) = m$ and $\deg_{t_1} f \leq n$. A quasi-modular form for $\mathrm{SL}(2,\mathbb{Z})$ is also called a full quasi-modular form.

Let us now describe the growth condition for arbitrary N. We only work with $\mathsf{T}[N]$. The argument for other moduli spaces is similar. Assume that k is algebraically closed and consider $f : \mathsf{T}[N] \to \mathsf{k}$ with the properties (9.8) and (9.9). We define $g_i : T \to \mathsf{k},\ i = 1,2,\ldots,a = \deg(\mathsf{T}[N] \to \mathsf{T}[1])$ satisfying (9.8) and (9.9) with weight mi and differential order ni in the following way:

$$g_i : T[1] \to \mathsf{k},$$

$$g_i(E,\omega) := \sum_{*_1,*_2,*_2,\ldots,*_i} f(E,*_1,\omega) f(E,*_2,\omega) \cdots f(E,*_i,\omega),$$

where $*_1,*_2,*_2,\ldots,*_i$ runs through i-tuples of torsion structures on E and attached to Γ. It follows that f is a root of the polynomial

$$X^a - g_1 X^{a-1} + g_2 X^{a-2} - \cdots + (-1)^{a-1} g_{a-1} X + (-1)^a g_a. \tag{9.11}$$

The growth condition for f is that the corresponding g_i's are full quasi-modular forms (the case $N = 1$). It follows that

$$g_i \in \mathsf{k}[t_1,t_2,t_3],\ g_i \text{ homogeneous},\ \deg(g_i) = mi,\ \deg_{t_1}(g_i) \leq ni.$$

For $n = 0$ we recover the definition of modular forms of weight m, see [Kat76b]. A modular form of weight m is a function from the set of enhanced elliptic curves as before but with this difference that $\omega \in F^1$ is a regular differential form and not an element in $H^1_{\mathrm{dR}}(E) \backslash F^1$. The action of $\mathbb{G}_m$ is given by $(*,*,\omega) \bullet k = (*,*,k\omega)$ and f satisfies $k \bullet f = k^{-m} f,\ \mathsf{k} \in \mathbb{G}_m$. The growth condition in this case can be also expressed using Tate curves.

We denote by $M^n_m = M^n_m(\mathsf{T})$, for T one of $\mathsf{T}(N), \mathsf{T}_0(N), \mathsf{T}_1(N)$, the set of quasi-modular forms of weight m and differential order n and we set

$$M = \sum_{m \in \mathbb{Z}, n \in \mathbb{N}_0} M^n_m.$$

If $n \leq n'$ then $M^n_m \subset M^{n'}_m$ and

$$M_m^n M_{m'}^{n'} \subset M_{m+m'}^{n+n'},\ M_m^n + M_m^{n'} = M_m^{n'}.$$

We see that M has a structure of a graded k-algebra. The k-algebra of full quasi-modular forms has also a differential structure which is given by:

$$M_m^n \to M_{m+2}^{n+1},\ t \mapsto dt(\mathsf{R}) = \sum_{i=1}^{3} \frac{\partial t}{\partial t_i} \mathsf{R}_i$$

where $\mathsf{R} = \sum_{i=1}^{3} \mathsf{R}_i \frac{\partial}{\partial t_i}$ is the Ramanujan vector field. We sometimes use $\mathsf{R} : M \to M$ to denote this differential operator.

Exercise 9.9 1. There is a canonical bijection between modular forms of weight m and quasi-modular forms of weight m and differential order 0.
2. In Definition 9.5, verify that f_i is a quasi-modular form of weight $m-2i$ and differential order $n-i$. In particular, f_n is a modular form of weight $m-2n$.
3. The algebra $M(\Gamma(2))$ is freely generated by three quasi-modular form s_1, s_2, s_3 of weight 2 and differential order 1. Show that the polynomial in (9.11) for each s_i is
$$((X-t_1)^3 - \frac{1}{4}t_2(X-t_1) - \frac{1}{4}t_3)^2.$$

Remark 9.3 It is desirable to have a coordinate free description of $\mathsf{k}[t_1,t_2,t_3]$. It turns out that $\mathsf{k}[t_1,t_2,t_3]$ is the k-algebra generated by $\mathsf{R}^i a, \quad i = 1, 2, \ldots$, where a runs through all invertible element in $\mathscr{O}_\mathsf{T}$. Note that the set of invertible elements in $\mathscr{O}_\mathsf{T}$ is a multiplicative group generated by Δ. This kind of statement does not seem to be valid in general. For example it fails for mirror quintc, see [Mov17].

9.9 Quasi-modular forms over $\mathbb{C}$

In this section we recall the definition of quasi-modular forms as holomorphic functions on the upper half plane $\mathbb{H}$ which satisfy a functional property with respect to the action of a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}$ and have some growth condition at infinity. The main references and more details can be found in [MR05, Mov08]. We show that what we have developed so far in the algebraic geometric framework is essentially the same as its complex counterpart. The bridge between two notions is the generalized period map which is constructed by elliptic integrals.

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{R}), \tau \in \mathbb{H},\ m \in \mathbb{Z}$ and f a holomorphic function in $\mathbb{H}$, recall the slash operator $f|_m A = (c\tau + d)^{-m} f(A\tau)$. Let Γ be a subgroup of $\mathrm{SL}(2,\mathbb{Z})$. For instance, take a congruence group of level N.

Definition 9.6 A quasi-modular form of weight m and differential order $n = 0$ is a classical modular form of weight m. A holomorphic function f on $\mathbb{H}$ is called a quasi-modular form of weight m and differential order n if the following two conditions are satisfied:

1. There are holomorphic functions $f_i,\ i=0,1,\dots,n$ on $\mathbb{H}$ such that

$$f|_m A = \sum_{i=0}^{n} \binom{n}{i} c^i (c\tau+d)^{-i} f_i,\ \forall A \in \Gamma. \tag{9.12}$$

2. $f_i\ |_m\ A, i=0,1,2,\dots,n$ have finite growths when $\mathrm{Im}(\tau)$ tends to $+\infty$ for all $A \in \mathrm{SL}(2,\mathbb{Z})$, that is

$$\lim_{\mathrm{Im}(\tau)\to+\infty} (f_i\ |_m\ A)(\tau) = a_{i,A} < \infty,\ a_{i,A} \in \mathbb{C}.$$

We will also denote by M_m^n the set of quasi-modular forms of weight m and differential order n and we set

$$M := \sum_{m\in\mathbb{Z}, n\in\mathbb{N}_0} M_m^n.$$

For an $f \in M_m^n$ we have $f|_m I = f_0$ and so $f_0 = f$. Note that for a quasi-modular form f the associated functions f_i are unique. If $f \in M_m^n$ with the associated functions f_i then $f_i \in M_{m-2i}^{n-i}$ with the associated functions $f_{ij} := f_{i+j}$. The set M is a differential $\mathbb{C}$-algebra:

$$\frac{\partial}{\partial\tau} : M_m^n \to M_{m+2}^{n+1}.$$

If $n \le n'$ then $M_m^n \subset M_m^{n'}$ and $M_m^n M_{m'}^{n'} \subset M_{m+m'}^{n+n'}$. It is useful to define

$$f||_m A := (\det A)^{m-n-1} \sum_{i=0}^{n} \binom{n}{i} (c_{A^{-1}})^i (c\tau+d)^{i-m} f_i(A\tau), \tag{9.13}$$

$$A \in \mathrm{GL}(2,\mathbb{R}),\ f \in M_m^n,\ \ c_{A^{-1}} = \frac{-c}{\det(A)}.$$

Exercise 9.10 Prove the following:

1. The equality (9.12) is written in the form

$$f = f||_m A, \quad \forall A \in \Gamma. \tag{9.14}$$

2. We have

$$f||_m A = f||_m (BA),\ \forall A \in \mathrm{GL}(2,\mathbb{R}),\ B \in \Gamma,\ f \in M_m^n.$$

3. The growth condition on f is required only for a finite number of cases $f_i||_m \alpha,\ \alpha \in \Gamma\backslash\mathrm{SL}(2,\mathbb{Z}),\ i=0,1,2\dots,n.$
4. The relation of $||_m$ with $\frac{\partial}{\partial\tau}$ is given by:

$$\frac{\partial(f||_m A)}{\partial\tau} = \frac{\partial f}{\partial\tau}||_{m+2} A,\ \forall A \in \mathrm{GL}(2,\mathbb{R}). \tag{9.15}$$

5. Let $A \in \mathrm{SL}(2,\mathbb{Z})$. If $f \in M_m^n(\Gamma)$ with the associated functions f_i then $f||_m A \in M_m^n(A^{-1}\Gamma A)$ with the associated functions $f_i||_m A \in M_{m-2i}^{n-i}(A^{-1}\Gamma A)$.

For a congruence group Γ of level N we have

$$T_N := \begin{bmatrix} 1 & N \\ 0 & 1 \end{bmatrix} \in \Gamma.$$

Now assume that Γ is a normal subgroup of $\mathrm{SL}(2,\mathbb{Z})$. For an $f \in M_m^n(\Gamma)$ and $A \in \mathrm{SL}(2,\mathbb{Z})$ with $[A] = \alpha \in \Gamma\backslash\mathrm{SL}(2,\mathbb{Z})$ we have $(f||_m A)|_m T_N = f$ and so we can write the Fourier expansion of $f||_m A$ at α

$$f||_m A = \sum_{n=0}^{+\infty} a_n q_N^n,\ a_n \in \mathbb{C},\ \ q_N := e^{2\pi i N\tau}.$$

We have used the growth condition on f to see that the above function in q_N is holomorphic at 0.

9.10 Generlized period domain and generalized period map

Quasi-modular forms as holomorphic functions on the upper half plane $\mathbb{H}$ are best viewed first as holomorphic functions on the generalized period domain

$$\Pi := \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \middle| x_1, x_2, x_3, x_4 \in \mathbb{C},\ x_1x_4 - x_2x_3 = 1,\ \mathrm{Im}(x_1\overline{x_3}) > 0 \right\}. \tag{9.16}$$

We let the group $\mathrm{SL}(2,\mathbb{Z})$ (resp. G in (9.6) with $\mathsf{k} = \mathbb{C}$) act from the left (resp. right) on Π by usual multiplication of matrices. The Poincaré upper half plane $\mathbb{H}$ is embedded in Π in the following way:

$$\tau \to \tilde{\tau} = \begin{bmatrix} \tau & -1 \\ 1 & 0 \end{bmatrix}.$$

We denote by $\tilde{\mathbb{H}}$ the image of $\mathbb{H}$ under this map. Note that any element of Π is equivalent to an element of $\tilde{\mathbb{H}}$ under the action of G because:

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} \frac{x_1}{x_3} & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_3 & x_4 \\ 0 & \frac{\det(x)}{x_3} \end{bmatrix}. \tag{9.17}$$

The map

$$J : \mathrm{GL}(2,\mathbb{R}) \times \mathbb{H} \to \mathsf{G},\ J(A,\tau) = \begin{bmatrix} (c\tau + d) & -c \\ 0 & (c\tau+d)^{-1}\det(A) \end{bmatrix}$$

is an automorphy factor, that is, it satisfies the functional equation:

$$J(AB,\tau) = J(A,B\tau)J(B,\tau),\ A,B \in \mathrm{GL}(2,\mathbb{R}),\ \tau \in \mathbb{H}.$$

This follows from the equality

$$A\begin{bmatrix}\tau & -1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}A\tau & -1\\ 1 & 0\end{bmatrix}J(A,\tau),\ A\in \mathrm{GL}(2,\mathbb{R}),\ \tau\in\mathbb{H}.$$

Proposition 9.10 *Quasi-modular forms $f\in M_m^n$ are in a one to one correspondence with holomorphic functions $F=\phi(f):\mathsf{\Pi}\to\mathbb{C}$ with the following properties:*

1. *The function F is Γ-invariant.*
2. *There are holomorphic functions $F_i:\mathsf{\Pi}\to\mathbb{C},\ i=0,1,\ldots,n$ such that*

$$F(x\cdot g)=k^{-m}\sum_{i=0}^{n}\binom{n}{i}k'^{i}k^{i}F_i(x),\ \forall x\in\mathsf{\Pi},\ g\in\mathsf{G},\tag{9.18}$$

3. *For all $\alpha\in\mathrm{SL}(2,\mathbb{Z})$ the restriction of F_i to $\tilde{\mathbb{H}}_\alpha$ has finite growth at infinity, where $\tilde{\mathbb{H}}_\alpha$ is the image of $\tilde{\mathbb{H}}$ under the action of α from the left on $\mathsf{\Pi}$.*

In fact we have $F_i=\phi(f_i)$. The proof is a mere calculation and can be found in [Mov08], Proposition 6.

Exercise 9.11 *1. Verify that the vector field*

$$X:=-x_2\frac{\partial}{\partial x_1}-x_4\frac{\partial}{\partial x_3}\tag{9.19}$$

is invariant under the action of $\mathrm{SL}(2,\mathbb{Z})$ and hence it induces a vector field $\tilde{X}$ in the quotient $\Gamma\backslash\mathsf{\Pi}$.

2. *Show that under the correspondence in Proposition 9.10, the differential operator on quasi-modular forms as functions on $\Gamma\backslash\mathsf{\Pi}$ is given by the vector field $\tilde{X}$. Note that X restricted to the loci $\tilde{\mathbb{H}}$ is $\frac{\partial}{\partial\tau}$.*

9.11 Generalized period map and it inverse

Recall the notations of Section 9.8 for the base field $\mathsf{k}=\mathbb{C}$ and Section 9.8. Recall also that that for $\Gamma=\mathrm{SL}(2,\mathbb{Z})$ we have

$$\mathsf{T}:=\mathsf{T}_\Gamma=\{(t_1,t_2,t_3)\in\mathbb{C}^3\mid 27t_3^2-t_2^3\neq 0\}.$$

If Γ is one of $\Gamma(N),\Gamma_1(N),\Gamma_0(N)$ then we know that the projection map $\beta:\mathsf{T}_\Gamma\to\mathsf{T}$ (neglecting the torsion point structure) is a covering of degree $\#(\Gamma\backslash\mathrm{SL}(2,\mathbb{Z}))$ (see Exercise 9.12) and so T_Γ has a natural structure of a complex manifold. We define R_Γ to be the pull-back of the Ramanujan vector field in T_Γ.

Let us fix $b\in\mathsf{T}_\Gamma$ and a basis δ_1^0,δ_2^0 of the $\mathbb{Z}$-module $H_1(E_{\beta(b)},\mathbb{Z})$ with $\langle\delta_1^0,\delta_2^0\rangle=1$. For any path γ which connects b to an arbitrary point $t\in\mathsf{T}_\Gamma$ we define $\delta_1,\delta_2\in$

$H_1(E_t,\mathbb{Z})$ to be the monodromy of δ_1^0 and δ_2^0 along the path γ. The generalized period map is defined by

$$\mathsf{P}:\mathsf{T}_\Gamma\to\Gamma\backslash\Pi,\ t\mapsto\left[\frac{1}{\sqrt{-2\pi i}}\begin{bmatrix}\int_{\delta_1}\frac{dx}{y} & \int_{\delta_1}\frac{xdx}{y}\\ \int_{\delta_2}\frac{dx}{y} & \int_{\delta_2}\frac{xdx}{y}\end{bmatrix}\right].$$

We also use P for the multi-valued map $\mathsf{T}_\Gamma\to\Pi$; being clear in the text which we mean. Brackets $[\cdot]$ means the equivalence class in the quotient $\Gamma\backslash\Pi$. It is well-defined because of Proposition 3.2, Proposition 3.5 and the following fact: different choices of the path γ lead to the action of Γ from the left on Π which is already absorbed in the quotient $\Gamma\backslash\Pi$. Different choices of b and δ_1^0,δ_2^0 lead to the composition of the generalized period map with canonical automorphisms of $\Gamma\backslash\Pi$ (see Exercise 9.12, Item 2). The factor $\frac{1}{\sqrt{-2\pi i}}$ is inserted so that the determinant of the matrix is one (Legendre relation between elliptic integrals).

Proposition 9.11 *The Gauss-Manin connection of the family of elliptic curves $y^2=4(x-t_1)^3-t_2(x-t_1)-t_3$ written in the basis $\frac{dx}{y},\ \frac{xdx}{y}$ is given as bellow:*

$$\nabla\begin{pmatrix}\frac{dx}{y}\\ \frac{xdx}{y}\end{pmatrix}=A\begin{pmatrix}\frac{dx}{y}\\ \frac{xdx}{y}\end{pmatrix}\tag{9.20}$$

where

$$A=\frac{1}{\Delta}\begin{bmatrix}-\frac{3}{2}t_1\alpha-\frac{1}{12}d\Delta & \frac{3}{2}\alpha\\ \Delta dt_1-\frac{1}{6}t_1d\Delta-(\frac{3}{2}t_1^2+\frac{1}{8}t_2)\alpha & \frac{3}{2}t_1\alpha+\frac{1}{12}d\Delta\end{bmatrix},$$

$$\Delta=27t_3^2-t_2^3,\ \alpha=3t_3dt_2-2t_2dt_3.$$

In particular, the period matrix P satisfies the following differential equation:

$$d\mathsf{P}^{\mathrm{tr}}=A\mathsf{P}^{\mathrm{tr}}.\tag{9.21}$$

Proof. This is an easy consequence of Proposition 3.4 after inserting shifting x with t_1. For this recall the isomorphism (9.7).

Proposition 9.12 *We have*

1. *The generalized period map is a biholomorphism;*
2. *It satisfies*

$$\mathsf{P}(t\bullet\mathsf{g})=\mathsf{P}(t)\cdot\mathsf{g},\ t\in\mathsf{T}_\Gamma,\ \mathsf{g}\in\mathsf{G};\tag{9.22}$$

3. *The push forward of the vector field R_Γ by the generalized period map P is the vector field X in (9.19).*

Proof. It is enough to prove the Proposition for $\Gamma=\mathrm{SL}(2,\mathbb{Z})$ (Exercise 9.12, Item 4). The equality (9.22) follows from Proposition 9.8. The last statement follows from Proposition 9.11 as follows:

$$d\mathsf{P}(\mathsf{R})=\mathsf{P}(t)\cdot A^{\mathrm{tr}}(\mathsf{R})=\mathsf{P}\begin{bmatrix}0&0\\-1&0\end{bmatrix}=\begin{bmatrix}-x_2&0\\-x_4&0\end{bmatrix}.$$

We have used the notation $\mathsf{P}=\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$. Using the equality (9.21) and $\det(\mathsf{P})=1$ we have:

$$\begin{aligned} dx_1\wedge dx_3\wedge dx_2 &= A_{11}\wedge A_{12}\wedge (x_1A_3+x_2A_{22}) \\ &= \frac{1}{\Delta^3}(-\frac{1}{12}d\Delta)\wedge(\frac{3}{2}\alpha)\wedge(x_1\Delta dt_1) \\ &= \frac{3x_1}{4\Delta}dt_1\wedge dt_2\wedge dt_3, \end{aligned}$$

where $A=[A_{ij}]$ is the Gauss-Manin connection in the basis in Proposition 9.11. Using Proposition 3.2 we know that $x_1\neq 0$ and we conclude that P is a local biholomorphism. The fact the P is a global biholomorphism follows from the local case and the fact that after taking the quotient by the group G we have the inverse of the j function as in Section 3.5.

Exercise 9.12 1. For $\Gamma=\Gamma_0(N),\,\Gamma_1(N),\,\Gamma(N)$ show that the cardinality of $\Gamma\backslash\mathrm{SL}(2,\mathbb{Z})$ is the number of enhanced elliptic curves for Γ with (E,ω) fixed.
2. Show that the generalized period map P is well-defined.
3. For $A\in\Gamma\backslash\mathrm{SL}(2,\mathbb{Z})$ we have the well-defined map $F_A:\Gamma\backslash\Pi\to\Gamma\backslash\Pi,\ x\mapsto Ax$. A different choice of δ_1^0,δ_2^0 in the definition of the generalized period map leads to the composition $\mathsf{P}\circ F_A$.
4. Proposition 9.12 for $\Gamma=\mathrm{SL}(2,\mathbb{Z})$ implies the same proposition for arbitrary Γ.

Now, let us consider the case $\Gamma=\mathrm{SL}(2,\mathbb{Z})$ and

$$g=(g_1,g_2,g_3):\mathbb{H}\to\mathsf{T} \tag{9.23}$$

be the composition $\mathbb{H}\to\mathrm{SL}(2,\mathbb{Z})\backslash\Pi\overset{\mathsf{P}^{-1}}{\to}\mathsf{T}$. Here, P^{-1} is the inverse of the generalized period map. From Proposition 9.12 part 2 it follows that g_i's satisfy

$$(c\tau+d)^{-2i}g_i(\frac{a\tau+b}{c\tau+d})=g_i(\tau),\ i=2,3, \tag{9.24}$$

$$(c\tau+d)^{-2}g_1(\frac{a\tau+b}{c\tau+d})=g_1(\tau)+c(c\tau+d)^{-1},\ \tau\in\mathbb{H},\ \begin{bmatrix} a & b \\ c & d \end{bmatrix}\in\mathrm{SL}(2,\mathbb{Z}).$$

From Proposition 9.12 part 3 it follows also that g is a solution of the vector field R, that is,

$$\frac{\partial g_1}{\partial\tau}=g_1^2-\frac{1}{12}g_2,\ \frac{\partial g_2}{\partial\tau}=4g_1g_2-6g_3,\ \frac{\partial g_3}{\partial\tau}=6g_1g_3-\frac{1}{3}g_2^2 \tag{9.25}$$

Since $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\in\mathrm{SL}(2,\mathbb{Z})$, the functions g_i are invariant under $\tau\mapsto\tau+1$, and so, they can be written in terms of the new variable $q=e^{2\pi i\tau}$. In Section 9.12 we will prove that g_i's have a finite growth at infinity and hence as functions in q are holomorphic at $q=0$. This implies that up to multiplication with some constants, g_1,g_2,g_3 are E_2,E_4,E_6, respectivly. The precise equalities will be obtained in this section.

9.12 Periods and Ramanujan

In this section we consider the full modular group $\Gamma=\mathrm{SL}(2,\mathbb{Z})$ and the corresponding generalized period map. We are interested in the image L of the map g constructed in Section 9.11. This is the locus L of parameters $t\in\mathsf{T}$ such that:

$$\int_{\delta_1}\frac{xdx}{y}=-\sqrt{-2\pi i},\ \int_{\delta_2}\frac{xdx}{y}=0, \tag{9.26}$$

for some $\delta_1,\delta_2\in H_1(E_t,\mathbb{Z})$ with $\langle\delta_1,\delta_2\rangle=-1$. Using Proposition 9.12, part 2 and the equality (9.17), we know that the locus of such parameters is given by:

$$I=(I_1,I_2,I_3):=(t_1,t_2,t_3)\bullet\begin{bmatrix}(\frac{1}{\sqrt{-2\pi i}}\int_{\delta_2}\frac{dx}{y})^{-1} & -\frac{1}{\sqrt{-2\pi i}}\int_{\delta_2}\frac{xdx}{y}\\ 0 & \frac{1}{\sqrt{-2\pi i}}\int_{\delta_2}\frac{dx}{y}\end{bmatrix}=$$

$$\Big(-t_1(2\pi i)^{-1}(\int_{\delta_2}\frac{dx}{y})^2+(2\pi i)^{-1}\int_{\delta_2}\frac{xdx}{y}\int_{\delta_2}\frac{dx}{y},$$
$$t_2\cdot(2\pi i)^{-2}(\int_{\delta_2}\frac{dx}{y})^4,\ -t_3(2\pi i)^{-3}(\int_{\delta_2}\frac{dx}{y})^6\Big).$$

The mentioned locus is one dimensional and the above parametrization is by using three parameters t_1,t_2,t_3. We may restrict it to a one dimensional subspace $t=(0,12,-4\psi)$ as in Section 3.9, use the formulas of elliptic integrals in terms of hypergeometric functions (3.32) and obtain the following parametrization of L:

$$I=$$
$$\Big(a_1F(-\frac{1}{6},\frac{7}{6},1\mid\tau)F(\frac{1}{6},\frac{5}{6},1\mid z),\ a_2F(\frac{1}{6},\frac{5}{6},1\mid z)^4,\ a_3(1-2\tau)F(\frac{1}{6},\frac{5}{6},1\mid z)^6\Big),$$

where

$$(a_1,a_2,a_3)=(\frac{2\pi i}{12},12(\frac{2\pi i}{12})^2,8(\frac{2\pi i}{12})^3). \tag{9.27}$$

Since the generalized period map sends R to X, and the canonical map $\mathbb{H}\to\Pi$ sends $\frac{\partial}{\partial\tau}$ to X, we conclude that if we write I_i's in terms of the new variable

$$\tau=\frac{\int_{\delta_1}\frac{dx}{y}}{\int_{\delta_2}\frac{dx}{y}}=i\frac{F(\frac{1}{6},\frac{5}{6},1|1-z)}{F(\frac{1}{6},\frac{5}{6},1|z)},$$

that is $(I_1, I_2, I_3) = (g_1(\tau), g_2(\tau), g_3(\tau))$ then g is a solution of R. Here, g_i's are the same as in (9.23). We can write g_i's in terms of $q = e^{2\pi i\tau}$. This is $g_i := I_i(\mathsf{p}^{-1}(q))$, where $\mathsf{p} : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is the map given by (3.38). It follows that g_i as a function in q is one valued and holomorphic in the disc of radius one and center 0. Now, by (9.24) we know that g_1, g_2, g_3's satisfy the same functional equations as the E_2, E_4, E_6. This to gether with Theorem 2.5 implies that

$$E_{2i}(\tau) = a_i^{-1} g_i(\tau), \ i = 1, 2, 3. \tag{9.28}$$

Note that by Theorem 2.5 there is no modular form of weight 2 and hence we know that g_1 must be a multiple of E_2 .

Proof (Proof of Theorem 2.4). This is a direct consequence of (9.28) and the functional equation of g_1 in (9.24).

9.13 Comparision theorem

Now, we are in a position to prove that the algebraic and analytic notions of quasi-modular forms are equivalent.

Theorem 9.1 *The differential graded algebra of quasi-modular forms in the Poincaré upper half plane together with the differential operator* $\frac{\partial}{\partial\tau}$ *is isomorphic to the graded differential algebra of quasi-modular forms defined in Section 9.8 together with the differential operator* R_Γ.

Proof. According to Proposition 9.10, quasi-modular forms can be viewed as functions on $\Gamma\backslash\Pi$. Now, the generalized period map which is a biholomorphism gives us the desired isomorphism of algebras.

Our geometric approach toward quasi-modular forms and the fact that the generalized period map is a biholomorphism give also the double sum formula for the Eisenstein series E_2 in Theorem 2.13.

Proof (Proof of Theorem 2.13). Let us consider the family of elliptic curves $y^2 = 4x^3 - t_2x - t_3$ with $\alpha = \frac{dx}{y}, \ \ \omega = (x + t_1)\frac{dx}{y}$ and

$$(t_1, t_2, t_3) = (a_1E_2(\tau), a_2E_4(\tau), a_3E_6(\tau)),$$

where a_i's are given in (12.22). By Theorem 2.2 and Theorem 2.3, if we use the biholomorphism $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}) \cong E, \ z \mapsto (\sqrt{-2\pi i}^{-2}\wp(z), \sqrt{-2\pi i}^{-3}\wp'(z))$ and define δ_1, δ_2 to be cycles in E corresponding to vectors $\tau, 1 \in \mathbb{C}$ then

$$\frac{1}{\sqrt{-2\pi i}} \begin{bmatrix} \int_{\delta_1} \frac{dx}{y} & \int_{\delta_1} (x+t_1)\frac{dx}{y} \\ \int_{\delta_2} \frac{dx}{y} & \int_{\delta_2} (x+t_1)\frac{dx}{y} \end{bmatrix} = \begin{bmatrix} \tau & -1 \\ 1 & 0 \end{bmatrix}.$$

Note that in the Weierstrass uniformization the coefficients

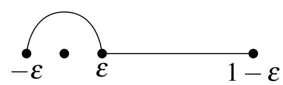
$$(g_2, g_3) = (60 \cdot 2 \cdot \zeta(4) \cdot E_4, 140 \cdot 2 \cdot \zeta(6) \cdot E_6) = ((-2\pi i)^2 a_2 E_4, (-2\pi i)^3 a_3 E_6)$$

appears. Let us consider the equality corresponding to $(1,2)$ and $(2,2)$ entries:

$$t_1 \cdot \tau = -1 - \frac{1}{\sqrt{-2\pi i}} \int_{\delta_1} \frac{xdx}{y} = -1 - \int_0^\tau \wp(\tau, z) dz \tag{9.29}$$

$$t_1 = -\frac{1}{\sqrt{-2\pi i}} \int_{\delta_2} \frac{xdx}{y} = -\int_0^1 \wp(\tau, z) dz. \tag{9.30}$$

The integration in (9.30) can be replaced with integration over the following path γ:



Recall the Weierstrass zeta function in Exercise 2.11. We continue the computation of t_1:

$$t_1 = \int_\gamma d\zeta(z) = \zeta(1-\varepsilon) - \zeta(-\varepsilon) = 2\zeta(\frac{1}{2})$$

which gives us the result.

9.14 Partial compactifications

Once a moduli space T is given, one has also the problem of its partial compactifications, that is, enlarging T in such a way that it parameterizes degenerations of the underlying objects. In our main example T in Section 9.7 after constructing coordinate system on T we get in 9.8 that $\mathsf{T} := \mathbb{C}^3 \backslash \{\Delta = 0\}$. The main goal of partial compactifications in algebraic geometry is to construct $\Delta = 0$, or part of it like $\Delta = 0$ minus its singularity which is the origin, without describing the ring of functions coordinate freely, as in general there are no methods to choose such corrdinates. In other words, it is desiable to have $\Delta = 0$ as a kind of moduli space. In this section do this.

We consider the moduli of $(\mathbb{P}^1, O, P, \alpha_1, \alpha_2)$, where $P = \{P_1, P_2\}, O, P_1, P_2 \in \mathbb{P}^1$ and P_1, P_2 are distinct different from O, and are not ordered and $\alpha_1 \in H^1_{\mathrm{dR}}(\mathbb{P}^1 - P)$ and $\alpha_2 \in H^1_{\mathrm{dR}}(\mathbb{P}^1 - P) \oplus H^1_{\mathrm{dR}}(\mathbb{P}^1, P)$ with $\langle \alpha_1, \alpha_2 \rangle = 1$.

We choose a coordinate function y on $\mathbb{P}^1$ with pole or order 1 at O. Further, we assume that $y(P_1) + y(P_2) = 0$, and hence $(y - y(P_1))(y - y(P_2)) = y^2 - b$ for some $b \in \mathbb{C}$. We consider the following generators of our one dimensional vector spaces

$$\omega_1 := \frac{dy}{y^2 - b} \in H^1_{\mathrm{dR}}(\mathbb{P}^1 - P), \quad \omega_2 := dy \in H^1_{\mathrm{dR}}(\mathbb{P}^1, P),$$

and we can verify that $\langle \omega_1, \omega_2 \rangle = 1$. The choice of y is not unique and can be replaced by any $Ay + B$, $A \in \mathbb{C}^*$, $B \in \mathbb{C}$. Since we assumed that $y(P_1) + y(P_2) = 0$ we have $B = 0$. We conclude that our moduli space is $\mathrm{Sepc}(\mathbb{C}[b, c, \frac{1}{b}])$ and we have the universal family over it given $\alpha_1 = \frac{dy}{y^2-b}$ and $\alpha_2 = dy + c\frac{dy}{y^2-b}$. We want to see this in the boundary of the moduli space T in Section 9.7. For this we have to identify the point P_1 and P_2 and get a singular curve, more precisely, we need a morphism $\mathbb{P}^1 \to \mathbb{P}^N$ such that is an embedding outside P_1 and P_2 and these poins are mapped to a single point. We can see that $y \mapsto [y^2 - b : y(y^2-b) : 1]$ is the desired map. In the affine coordinates (X,Y) for $\mathbb{P}^2$ we get

$$Y^2 = X^2(X+b),\ \alpha_1 = \frac{dX}{2Y},\ \alpha_2 = \frac{(X+c)dX}{2Y}.$$

Further transformation $(X,Y) \mapsto (X - \frac{1}{3}b, \frac{1}{2}Y)$ puts our moduli space as degeneration of T.

9.15 Hecke operators

The theory of Hecke operators for quasi-modular forms is similar to the classical case and it has been developed in [Mov15b]. In this section we review this and its main consequence:

Theorem 9.2 *Let $N \in \mathbb{N}$ be a natural number and f be a quasi-modular form of weight k for* $\mathrm{SL}(2,\mathbb{Z})$ *defined over* $\mathbb{Q}$. *There are quasi-modular forms $g_i,\ \ i = 0,1,\ldots,\psi(N)$ of weight ki for* $\mathrm{SL}(2,\mathbb{Z})$ *such that*

$$\sum_{i=0}^{\psi(N)} g_{\psi(N)-i} f(N\cdot\tau)^i = 0.$$

This same statement for modular forms is presented in Proposition 8.4.

Chapter 10
Riemann zeta function

I have told the story before, but it is ironic that being at the same university, Artin had discovered a new type of L-series and Hecke, in trying to figure out what kind of modular forms of weight one there were, said they should correspond to some kind of L-function. The L-functions Hecke sought were among those that Artin had defined, but they never made contact-it took almost forty years until this connection was guessed and ten more before it was proved, by Langlands. Hecke was older than Artin by about ten years, but I think the main reason they did not make contact was their difference in mathematical taste. Moral: Be open to all approaches to a subject, (J. Tate in [RS11] page 446).

10.1 Introduction

In this chapter we introduce the Riemann zeta function. We will follow mainly the Riemann's original article [Rie59] and the book [Edw01] which explain a historical account on Riemann's paper. Euler considered the zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for real s and Riemann introduced $\zeta(s)$ for complex s and its extension as a meromorphic function to the whole s-plane. In particular, it was known before Riemann that

$$\zeta(2) = \frac{\pi^2}{6},\ \ \zeta(4) = \frac{\pi^4}{90},\ \ \zeta(6) = \frac{\pi^6}{945},\ \ \zeta(8) = \frac{\pi^8}{9450}.$$

We will then consider Hecke's L-functions which are natural generalization of ζ in the framework of modular forms.

10.2 Riemann zeta function

In this section we are going to study the first of all zeta function, namely the Riemann zeta function:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

We mainly use the Riemman's original article [Rie59].

Proposition 10.1 *The series $\zeta(s)$ converges for all $s \in \mathbb{C}$ with* $\mathrm{Re}(s) > 1$ *and*

$$\zeta(s) = \prod_p \frac{1}{(1-p^{-s})}, \tag{10.1}$$

where p runs over all primes.

Equation (10.1) is known as Euler's product formula and [Rie59, page 671] mentions that it is a remark made by Euler.

Proof. We have $|n^{-s}| = n^{\mathrm{Re}(s)}$ and so it is enough to prove the proposition for $s \in \mathbb{R}, s > 1$. We have

$$\sum_{n=2}^{\infty} \frac{1}{n^s} < \int_1^{\infty} x^{-s} dx = \left. \frac{x^{-s+1}}{-s+1} \right|_1^{+\infty} = \frac{1}{s-1} \text{ if } s > 1.$$

In the last equality we assume that s is a real number bigger than 1. For a prime number p we have $p^{-s} < 1$ and so

$$(1-p^{-s})^{-1} = \sum_{m=0}^{\infty} p^{-ms}.$$

By unique factorization theorem

$$\prod_{p \leq N} (1-p^{-s})^{-1} = \sum_{n \leq N} n^{-s} + R_N(s).$$

Clearly

$$R_N(s) \leq \sum_{n=N+1}^{\infty} n^{-s}.$$

Since $\zeta(s)$ converges we have $R_N(s) \to 0$ as $N \to \infty$ and the result follows.

10.3 The big Oh notation

In this section for $a \in \mathbb{R} \cup \{\pm\infty\}$ we define the interval I_a to be a small one sided neighborhood of a. If $a \in \mathbb{R}$ this means that $I_a = (a, a+\varepsilon)$ or $= (a-\varepsilon, a)$ for some

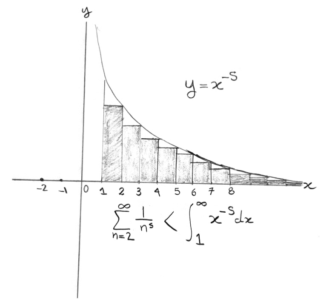


Fig. 10.1 Riemann's zeta function converges.

small ε and for $a = +\infty$ this means $I_a = (b, +\infty)$ for a big positive number b and for $a = -\infty$ this means $I_a = (-\infty, -b)$ for a big positive number b.

Definition 10.1 *Let f, g be two complex valued function in I_a we write*

$$f = O(g) \text{ or } f \sim_{x \to a} g$$

to say that $\frac{f(x)}{g(x)}$ is bounded near a, that is there exists a constant M such that

$$|f(x)| \leq M|g(x)|, \ \forall x \in I_a.$$

For three complex valued functions f, g and h in I_a we write $f(x) = h(x) + O(g(x))$ if $f(x) - h(x) = O(g(x))$.

We mainly use the following convergence criterion: Let f be a complex valued continuous function in $I_a, a \in \mathbb{R}$ and

$$f \sim_{x \to a} (x-a)^s, \ s \in \mathbb{R}.$$

For $a = \pm\infty$ we assume $f \sim_{x \to a} x^s$. We can extend this assumption to $s = \pm\infty$. For instance for $a \in \mathbb{R}$ the expression $f \sim_{x \to a} (x-a)^{+\infty}$ means

$$\forall s \in \mathbb{R}^+, \ \ f \sim_{x \to a} (x-a)^s.$$

Proposition 10.2 *Let f be a complex valued continuous function in $I_a, a \in \mathbb{R}$ and $f \sim_{x \to a} (x-a)^s, \ s \in \mathbb{R}$. For $a, s \in \mathbb{R}$ the integral $\int_{I_a} f(x)dx$ converges if $s > -1$. If $a = \pm\infty, \ s \in \mathbb{R}$ then the integral $\int_{I_a} f(x)dx$ converges if $s < -1$.*

Proof. We have

$$\left| \int_{I_a} f(x)dx \right| \leq \int_{I_a} |f(x)|\, dx \leq M \int_{I_a} |x-a|^s dx = (x-a)^{s+1} \Big|_a^{a+\varepsilon},$$

where the equality is written for $I_a=(a,a+\varepsilon)$. The last quantity is finite if $s+1>0$. The other cases are similar.

Proposition 10.2 in some instances is not enough to prove the convergence of integrals. A simple change of variable in this proposition gives us:

Proposition 10.3 *Let f be a complex valued continuous function in $I_a, a\in\mathbb{R}$ and $f\sim_{x\to a}\ln(x-a)^r(x-a)^s,\ s,r\in\mathbb{R}$. The integral $\int_{I_a}f(x)dx$ converges if $s>-1$. If $s=-1,\ s\in\mathbb{R}$ then the integral $\int_{I_a}f(x)dx$ converges if $r<-1$.*

Proof. For simplicity we assume that $I_a=(a,a+\varepsilon)$. We make the change of variable $y=-\ln(x-a)$ and we have

$$f(x)dx=-f(e^{-y}+a)e^{-y}dy\sim_{y\to+\infty}y^re^{-y(s+1)}.$$

If $s+1>0$ then

$$y^re^{-y(s+1)}\sim_{y\to+\infty}y^{-n-1},\ \forall n\in\mathbb{N}. \tag{10.2}$$

and so by Proposition 10.2 the desired integral converges. If $s=-1$ then

$$\left|\int_{I_a}f(x)dx\right|\leq\int^{+\infty}\ln(y)^rd(\ln(y))=\frac{1}{r+1}\int^{+\infty}d(\ln(y)^{r+1}),$$

and the statement follows.

Remark 10.1 We know that $\ln(x)\sim_{x\to0^+}x^{-\varepsilon}$ for all positive ε, however, there is no $a\in\mathbb{C}$ such that $\ln(x)^{-1}\sim_{x\to0^+}x^a$. This means that we cannot use Proposition 10.2 directly for integrals whose integrand contains $\ln(x)^{-1}$, that is why we have reformulated Proposition 10.3 which will be used in Section 10.9. Note also that $f\sim_{x\to a}g$ does not imply $f^s\sim_{x\to a}g^s$ for an arbitrary s, and $f\sim_{x\to a}g$ does not imply $g\sim_{x\to a}f$. In 09 June 2021 I was using $\ln(x)\sim_{x\to0^+}x^{-\varepsilon}$ implies $\ln(x)^s\sim_{x\to0^+}x^{-s\varepsilon}$ for an arbitrary s, which is trivially false, and it took a full day to find this bug in my mind!

10.4 Gamma function

The gamma function is defined by

$$\Gamma(s)=\int_0^\infty x^{s-1}e^{-x}dx,\ \ \mathrm{Re}(s)>0.$$

It converges because near 0

$$x^{s-1}e^{-x}\sim_{x\to0}x^{\mathrm{Re}(s)-1}$$

and so for $\mathrm{Re}(s)>0$ the integral near zero converges. Near infinity it is always convergent because

$$x^{s-1}e^{-x}\sim_{x\to+\infty}x^{-n-1},\ \forall n\in\mathbb{N}. \tag{10.3}$$

We have
$$\Gamma(s) = (s-1)\Gamma(s-1) \tag{10.4}$$
because
$$\Gamma(s) = -\int_0^\infty x^{s-1} de^{-x} = -x^{s-1}e^{-x}\Big|_0^\infty + \int_0^\infty e^{-x} dx^{s-1} = (s-1)\Gamma(s-1).$$
Since $\Gamma(1) = 1$ this implies that
$$\Gamma(n) = (n-1)!,\ n \in \mathbb{N}$$
and so the Γ-function is the interpolation of the factorial function.

Proposition 10.4 *The Γ-function has analytic continuation to a meromorphic function in the whole complex s-palne with poles of simple order at $s = 0, -1, -2, \ldots$. Moreover, it has no zeros.*

Proof. The equalities
$$\Gamma(s) = \frac{\Gamma(s+1)}{s} = \cdots = \frac{\Gamma(s+n+1)}{s(s+1)\cdots(s+n)},\ \ n \in \mathbb{N}$$
proves the first statement. The second statement follows from Euler's reflection formula
$$\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)} \tag{10.5}$$
There are three well-known functional equations involving the Gamma function. Two of them we have already seen in (10.4) and (10.5). The third one is the Gauss multiplication relation:
$$\frac{1}{\Gamma(m \cdot z)} \prod_{k=0}^{m-1} \Gamma(z + \frac{k}{m}) = m^{\frac{1}{2} - m \cdot z} (2\pi)^{\frac{m-1}{2}}, \tag{10.6}$$
for $z \in \mathbb{C}$. We call these three the standard relations of the Γ function. For $m = 2$ this is the Legendre duplication formula
$$\Gamma(z)\,\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z).$$

Exercise 10.1 Prove Euler's reflection formula (10.5) and Gauss multiplication relation (10.6).

Remark 10.2 Gauss introduced the notation $\Pi(s) = \Gamma(s+1)$ which is used in Riemann's original article. The notation Γ is due to Legendre, see [Edw01, page 8].

Definition 10.2 The Mellin transform of a function f defined on $\mathbb{R}^+$ is

$$\int_0^\infty x^s f(x) \frac{dx}{x}.$$

Proposition 10.5 *If $f(x)$ is locally integrable along the positive real line, and*

$$f(x)_{x\to 0+} = O(x^u) \quad \textit{and} \quad f(x)_{x\to+\infty} = O(x^v)$$

then its Mellin transform converges in the fundamental strip $[-u,-v]$.

Proof. This follows from Proposition 10.2.

By definition the Γ function is the Mellin transform of e^{-x}.

10.5 Analytic extension of Riemann's zeta function, I

Riemann in his paper [Rie59] introduces two methods to prove the analytic extension of $\zeta(s)$ to the whole $s \in \mathbb{C}$. In this section we present the first one. From the definition of Γ-function it follows:

$$\frac{\Gamma(s)}{n^s} = \int_0^\infty x^{s-1} e^{-nx} dx, \ \ \mathrm{Re}(s) > 1. \tag{10.7}$$

Taking sum for $n = 1, 2, \ldots$ we obtain

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1} dx}{e^x - 1}, \ \mathrm{Re}(s) > 1.$$

One can see that near $+\infty$ we have $\frac{1}{e^x-1} = O(x^{-\infty})$ and near 0 we have $\frac{1}{e^x-1} = O(x^{-1})$. Therefore, the convergence strip for the above integral is $\mathrm{Re}(s) \in (1,+\infty)$, see Proposition 10.5. Now, we consider the integral

$$I(s) := \int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \frac{dx}{x}.$$

Here, we have taken the branch of $(-x)^s = e^{s\ln(-x)}$ in $\mathbb{C}\backslash\mathbb{R}^+$ such that $\ln(-x)$ for negative x is a real number. The path of integration begins at $+\infty$, moves to the left down the positive real axis, circles the origin once in the clockwise direction, and returns up the positive real axis to $+\infty$, see Figure 10.2. Now, the above integral is convergent for all s and it gives an entire function in s. A simple calculation shows that it is equal to

$$I(s) = (e^{\pi i s} - e^{-\pi i s}) \int_0^\infty \frac{x^s}{e^x - 1} \frac{dx}{x}, \ \mathrm{Re}(s) > 1.$$

In particular, this shows that $I(s)$ vanishes in $s = 2, 3, \ldots$. Therefore, we get

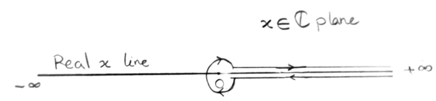


Fig. 10.2 A path of integration

$$2i\sin(\pi s)\Gamma(s)\zeta(s) = \int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x-1}\frac{dx}{x}$$

and by (10.5)

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i}\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x-1}\frac{dx}{x}. \tag{10.8}$$

Note that for $s \in \mathbb{Z}$ the integral $I(s)$ can be written as

$$I(s) = \int_{|x|=\varepsilon} \frac{(-x)^s}{e^x-1}\frac{dx}{x}$$

where the domain of integration is oriented clockwise.

Proposition 10.6 *We have*

1. *$\zeta(s)$ extends to a meromorphic function on the s-plane.*
2. *It has a unique pole at $s=1$. The point $s=1$ is a simple pole of ζ.*
3. *It vanishs at $s=-2,-4,-6,\dots$ and for $s=0,-1,-3,-5,\dots$*

$$\zeta(s) = (-1)^s\frac{B_{1-s}}{1-s},$$

where B_m's are Bernoulli numbers given by

$$\frac{x}{e^x-1} = \sum_{m=0}^{\infty} B_m\frac{x^m}{m!} = 1+\frac{\frac{-1}{2}}{1!}x+\frac{\frac{1}{6}}{2!}x^2+\frac{\frac{-1}{30}}{4!}x^4+\frac{\frac{1}{42}}{6!}x^6+\frac{\frac{1}{32}}{8!}x^8+\frac{\frac{5}{66}}{10!}x^{10}+\cdots.$$

Proof. The first item follows from the equality (10.8). The second item follows from the fact that $\Gamma(1-s)$ has a simple pole at $s=1,2,3,\dots$. Since $\zeta(s)$ has no poles at $s=2,3,\dots$ we conclude that the integral $I(s)$ in (10.8) must vanish at these points. The equality

$$\int_{|x|=\varepsilon} \frac{(-x)}{e^x-1}\frac{dx}{x} = 2\pi i \neq 0,$$

shows that the pole at $s=1$ survives. We now prove the third item. For $s \in \mathbb{Z}, s \leq 0$ we have

$$\begin{aligned}\zeta(s) &= \frac{\Gamma(1-s)}{2\pi i}\int_{|x|=\varepsilon}\left(\sum_{m=0}^{\infty}(-1)^s\frac{B_m x^{m+s-1}}{m!}\right)\frac{dx}{x}\\ &= \Gamma(1-s)(-1)^s\frac{B_{1-s}}{(1-s)!}\\ &= (-1)^s\frac{B_{1-s}}{1-s},\end{aligned}$$

where the domain of integration is oriented clockwise.

By Cauchy theorem for s with $\mathrm{Re}(s)<0$ we get

$$2\sin(\pi s)\Gamma(s)\zeta(s) = (2\pi)^s((-i)^{s-1}+i^{s-1})\sum_{n=1}^{\infty}n^{s-1}$$

see [Edw01, Section 1.6] for further details. In other words the function

$$\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}}\zeta(s)$$

remains invariant under $s \mapsto 1-s$. By analytic continuation this holds in the whole s-plane. In our way we also find the values of ζ for even positive integers:

Exercise 10.2 Prove that

$$\zeta(2n) = \frac{(2\pi)^{2n}(-1)^{n+1}B_{2n}}{2\cdot(2n)!}, \quad n \in \mathbb{N}.$$

10.6 Second proof for functional equation

In this section we present the second proof of the functional equation of $\zeta(s)$. This proof is more interesting, as in it appears a theta series and it has been the main motivation for Hecke to generalize it for a bigger class using modular forms.

In the equality (10.7) we make the change of variables $x \to n^2\pi x$ and $s \to \frac{s}{2}$ and we obtain:

$$\frac{\Gamma(\frac{s}{2})}{n^s}\pi^{-\frac{s}{2}} = \int_0^{\infty} x^{\frac{s}{2}-1}e^{-n^2\pi x}dx, \ \ \mathrm{Re}(s)>1.$$

Define

$$\psi(x) := \sum_{n=1}^{\infty} e^{-n^2\pi x}$$

and so

$$\zeta(s)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}} = \int_0^{\infty} x^{\frac{s}{2}-1}\psi(x)dx, \ \ \mathrm{Re}(s)>1. \tag{10.9}$$

Note that the sum in ψ cannot start from $n=0$ as the integral $\int_0^\infty x^{\frac{s}{2}-1}dx$ is not convergent for all $s \in \mathbb{C}$. Since ψ is a subseries of the geometric series $\sum_{n=1}^\infty (e^{-\pi x})^n$, it is convergent in the interval $(-1,1)$. For this we could also use directly the Cauchy-Hadamard theorem. We need the following functional equation of ψ. .

Proposition 10.7 *We have*

$$1+2\psi(x) = x^{-\frac{1}{2}}\left(1+2\psi(\frac{1}{x})\right). \tag{10.10}$$

We prove this in Theorem 2.11, where the main ingredient is the Poisson summation formula. We have the theta series

$$\theta_3(\tau) := \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2},\ q = e^{2\pi i\tau},\ \tau \in \mathbb{H}$$

which is related to ψ by $\theta_3(\tau) = 1+2\psi(-i\tau)$. Using (10.10) we continue computing (10.9).

$$\begin{aligned}\zeta(s)\Gamma(\frac{s}{2})\pi^{-\frac{s}{2}} &= \int_1^\infty x^{\frac{s}{2}-1}\psi(x)dx + \int_0^1 x^{\frac{s}{2}-1}\psi(x)dx \\ &= \int_1^\infty (x^{\frac{s}{2}} + x^{\frac{1-s}{2}})\psi(x)\frac{dx}{x} + \frac{1}{s(s-1)},\ \ s\in\mathbb{C}\end{aligned}$$

for which the right hand side is convergent for all $s \in \mathbb{C}$. We multiply the above equality by $\frac{s(s-1)}{2}$ and define

$$\xi(s) := \Gamma\left(\frac{s}{2}+1\right)(s-1)\pi^{-\frac{s}{2}}\zeta(s)$$

which is an entire function and satisfies $\xi(s) = \xi(1-s)$.

10.7 Hecke's L-functions

In this section we present Hecke's L-functions introduced in [Hec36]. This is mainly the imitation of the first proof of the functional equation of $\zeta(s)$ discussed in Section 10.6, however, it gives a general framework for the second proof.

Let us consider a series of the form

$$f = f_0 + f_1 q^{\frac{1}{\lambda}} + f_2 q^{\frac{2}{\lambda}} + \cdots + f_n q^{\frac{n}{\lambda}} + \cdots, \tag{10.11}$$

where $f_i \in \mathbb{C}$ and $\lambda \in \mathbb{R}^+$. We assume that f is convergent in the unit disk and hence if we set $q = e^{2\pi i\tau}$ then it defines a holomorphic function $f : \mathbb{H} \to \mathbb{C}$. We further assume that f satisfies the functional equation

$$f\left(\frac{-1}{\tau}\right) = \gamma \cdot (-i\tau)^k f(\tau) \tag{10.12}$$

for some $\gamma = \pm 1$ and $k \in \mathbb{Q}$. This format of a functional equation is useful when we restrict to the imaginary axis $\tau = ix,\ x \in \mathbb{R}^+$. The real function $g(x) := f(ix)$ satisfies

$$g(x^{-1}) = \gamma \cdot x^k g(x).$$

Note that by definition f also satisfies the functional equation

$$f(\tau + \lambda) = f(\tau).$$

Definition 10.3 The Hecke's L-function attached to f is

$$L(f,s) := \sum_{n=1}^{\infty} \frac{f_n}{n^s}$$

which converges in the region $\mathrm{Re}(s) > a$, where we have assumed that $f_n \sim n^a$.

Note that the constant term f_0 of f does not appear in the expression of L. We have

$$\frac{\Gamma(s)}{n^s} = \int_0^\infty x^{s-1} e^{-nx} dx, \ \ \mathrm{Re}(s) > 1$$

which implies

$$L(f,s)\Gamma(s) = \int_0^\infty x^{s-1} \left(\sum_{n=1}^{\infty} f_n \cdot e^{-nx} \right) dx.$$

We make the change of variables $x \to 2\pi x \cdot \lambda^{-1}$ and get

$$\frac{L(f,s)\cdot\Gamma(s)}{(2\pi\lambda^{-1})^s} = \int_0^\infty x^{s-1} \tilde{f}(ix) dx = (-i)^s \int_0^{i\infty} \tau^{s-1} \tilde{f}(\tau) d\tau \quad , \quad \tau := ix \tag{10.13}$$

where $f = f_0 + \tilde{f}$. In the following we will use the notation

$$f(i\infty) := f_0.$$

Theorem 10.1 *Let f be a holomorphic function in the upper half plane with a q-expansion of the form* (10.12) *and the functional equation* (10.12). *If $f(i\infty) = 0$ then the integral*

$$R(f,s) := (-i)^s \int_0^{i\infty} \tau^{s-1} (f(\tau) - f(i\infty)) d\tau$$

is a holomorphic function in the entire plane $s \in \mathbb{C}$ and if $f(i\infty) \neq 0$ then it is a holomorphic function in the half plane $\mathrm{Re}(s) > \max\{0,k\}$ *and it has a meromorphic extension to the whole $s \in \mathbb{C}$ with poles of order one at $s = 0, k$. In both cases it* *satisfies the functional equation*

$$R(f, k-s) = \gamma R(f,s), \quad \gamma = \pm 1. \tag{10.14}$$

Proof. This integral is always convergent at $i\infty$. The argument is as follows. Let $\tau = ix,\ x \in \mathbb{R}^+$. Using the q-series of $\tilde{f}$ in (10.11) we know that there are positive constants M and N such that

$$|\tilde{f}(ix)| \leqslant M \cdot e^{-2\pi x}, \ \ \forall x \in (N, +\infty).$$

This together with

$$\lim_{x\to+\infty} e^{-x} \cdot x^m = 0, \ \forall m \in \mathbb{N}$$

imply the convergence at $i\infty$ for all $s \in \mathbb{C}$. The convergence at 0 happens if $f_0 = 0$ and all $s \in \mathbb{C}$ or $f_0 \neq 0$ and for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \max\{1,k\}$. For this we use the functional equation (10.12) which implies the functional equation of $\tilde{f}$:

$$\tilde{f}\left(\frac{-1}{\tau}\right) = f_0 \cdot (\gamma(-i\tau)^k - 1) + \gamma(-i\tau)^k \tilde{f}(\tau).$$

or equivaelntly

$$\tilde{f}\left(i\frac{1}{x}\right) = f_0 \cdot (\gamma x^k - 1) + \gamma x^k \tilde{f}(ix).$$

We have

$$\begin{aligned}
R(f,s) &= \int_0^\infty x^{s-1}(f(ix) - f(i\infty))dx \\
&= \int_0^1 x^{s-1}\tilde{f}(ix)dx + \int_1^\infty x^{s-1}\tilde{f}(ix)dx \\
&= \int_1^\infty \left(x^{1-s}\tilde{f}(ix^{-1})x^{-2} + x^{s-1}\tilde{f}(ix)\right)dx \\
&= \int_1^\infty \left(\gamma x^{-1-s+k}\tilde{f}(ix) + x^{s-1}\tilde{f}(ix)\right)dx + f_0\int_1^\infty x^{-1-s}(\gamma x^k - 1)dx \\
&= \int_1^\infty \left(\gamma x^{-1-s+k} + x^{s-1}\right)\tilde{f}(ix)dx + f_0 \cdot \left(\frac{1}{-s} - \frac{\gamma}{-s+k}\right)
\end{aligned} \tag{10.15}$$

If $f_0 \neq 0$ then we have used $\mathrm{Re}(s) > k$ and $\mathrm{Re}(s) > 0$ in order to compute the last integral. The first integral is a holomorphic entire function in $s \in \mathbb{C}$ and so the theorem is proved. Note that for (10.14) we have used $\gamma = \pm 1$.

Remark 10.3 If we do not remove the constant term f_0 of f and then define $\int_0^\infty \tau^{s-1}f(\tau)d\tau$ then the difference of this with the previous one is $f_0\int_0^\infty \tau^{s-1}d\tau$ which is not convergent. This implies that $\int_0^\infty \tau^{s-1}f(\tau)d\tau$ is not convergent for $f_0 \neq 0$.

Example 10.1 For the theta series

$$\theta_3 = \sum_{n=-\infty}^{+\infty} q^{\frac{1}{2}n^2} = 1 + 2\sum_{n=1}^{\infty} q^{\frac{1}{2}n^2}$$

we know that

$$\theta_3\left(\frac{-1}{\tau}\right) = (-i\tau)^{\frac{1}{2}}\,\theta_3(\tau)$$

and so $k = \frac{1}{2}$, $\gamma = +1$, $\lambda = 2$ and $f_0 = 1$. In this case

$$L(\theta_3, s) = 2\sum_{n=1}^{\infty} \frac{1}{(n^2)^s} = 2\zeta(2s)$$

and so we get the functional equation and analytic continuation of $\zeta(s)$. We recover the content of Section 10.6 in this case.

Example 10.2 In this example f is the Eisenstein series $E_k,\ k \geq 4$. We have

$$\begin{aligned} L(E_k\,, s) &= b_{\frac{k}{2}} \sum_{n=1}^{\infty} \frac{\sigma_{k-1}(n)}{n^s} = b_{\frac{k}{2}} \sum_{d=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{d^{s-k+1}\, m^s} \\ &= b_{\frac{k}{2}}\, \zeta(s-k+1)\, \zeta(s), \end{aligned} \tag{10.16}$$

where $b_2 = 240, b_3 = -504$ and in general $b_k = (-1)^k \frac{4k}{B_k}$.

Example 10.3 We can also formulate Hecke's L-function for quasi-modular forms. Let us explain this for E_2 which satisfies the functional equation

$$E_2\left(\frac{-1}{\tau}\right) = \tau^2 E_2(\tau) + \frac{12}{2\pi i}\,\tau$$

instead of (10.12), and describe the functional equation of $L(E_2, s)$. Note that similar to (10.16) we have

$$L(E_2, f) = -24\zeta(s-1)\,\zeta(s)$$

Note that $b_1 = -24$. If $E_2 = 1 + \tilde{E}_2$ then the functional equation of $\tilde{E}_2$ is

$$\tilde{E}_2\left(\frac{-1}{\tau}\right) = (-i\tau)^2 - 1 + (-i\tau)^k\, \tilde{E}_2(\tau) + \frac{12}{2\pi i}\tau$$

Similar to (10.15) this gives us

$$L(E_2, s) = \int_1^{\infty} \left(x^{1-s} + x^{s-1}\right) \tilde{f}(\tau) dx + \left(\frac{1}{-s} - \frac{1}{-s+2}\right) + \frac{6}{\pi}\frac{1}{s-1}$$

For the extra integration due to $\frac{12}{2\pi i}\tau$ we have to assume that $\mathrm{Re}(s) > 1$. Because of this we have the functional equation

$$R(E_2, s) - R(E_2, 2-s) = \frac{12}{\pi}\frac{1}{s-1}.$$

Remark 10.4 It is well-known that any quasi-modular form f of weight k and order $\leq s$ can be written as $\sum_{i=0}^{s} D^i F_i + \alpha D^{\frac{k}{2}} E_2$, where F_i is a modular form of weight $k-2i$ and $\alpha = 0$ if $s < \frac{k}{2}$, see [MR05, Proposition 4.2]. This implies that

$$L(f,s) = \sum_{i=0}^{s} L(F_i, s-i) + \alpha L(E_2, s - \frac{k}{2}).$$

10.8 L-function of cusp forms

An important class of L-functions are those attached to cusp forms, see for instance [Sil94a, pages 80-84]. Let $f = \sum\limits_{n=1}^{\infty} f_n q^n$, $f_1 = 1$, be a normalized eigenfunction of weight k. Then in Chapter 7 we have seen that

$$\begin{aligned} &f_{mn} = f_m f_n\,, \qquad (n,m) = 1, \\ &f_{p^e} \cdot f_p = f_{p^{e+1}} + p^{k-1} f_{p^{e-1}} \qquad e \geqslant 1. \end{aligned}$$

Proposition 10.8 *We have*

$$L(f,s) := \sum_{n=1}^{\infty} \frac{f_n}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - f_p p^{-s} + p^{k-1-2s}} \tag{10.17}$$

for $\mathrm{Re}(s) > \frac{k}{2} + 1$.

Proof. The convergence follows from $f_n \sim n^{\frac{k}{2}}$ (see Theorem 2.8) and the same convergence statement for Riemann zeta function. For the product formula we first observe that

$$L(f,s) = \sum f_n \cdot n^{-s} = \prod_{p \text{ prime}} \sum_{e \geqslant \circ} f_{p^e} \cdot p^{-es}.$$

Then we have

$$
\begin{aligned}
&(1 - f_p p^{-s} + p^{k-1-2s}) \left(\sum_{e \geqslant \circ} f_{p^e} \cdot p^{-es} \right) \\
&= \underbrace{\sum_{e \geqslant \circ} f_{p^e} p^{-es}}_{A} - \underbrace{\sum_{e \geqslant \circ} f_p \cdot f_{p^e} p^{-s-es}}_{B} + \underbrace{\sum_{e \geqslant \circ} f_{p^e} p^{k-1-2s-es}}_{C} \\
&= A + C - \sum_{e \geqslant 1} (f_{p^{e+1}} + p^{k-1} f_{p^{e-1}}) p^{-s-es} - f_p \cdot p^{-s} \\
&= A + C - (A - 1 - f_p p^{-s}) - C - f_p p^{-s} = 1.
\end{aligned}
$$

The product in (10.17) is also called the Euler's product formula of L-function.

Theorem 10.2 *Let f be a cusp form of weight k for* $\mathrm{SL}(2,\mathbb{Z})$*. Then*

1. *$L(f,s)$ has an analytic extension to an entire holomorphic function in $s \in \mathbb{C}$.*
2. *If we set $R(f,s) := (2\pi)^{-s}\Gamma(s)L(f,s)$ then*

$$R(f,k-s) = (-1)^{\frac{k}{2}} R(f,s).$$

Note that R is symmetric with respect to $\mathrm{Re}(s) = \frac{k}{2}$.

Proof. This is a particular case of Hecke's theorem, see Theorem 10.1. We have $\gamma = (-1)^{\frac{k}{2}},\ \lambda = 1,\ \ f(i\infty) = 0$. Note that k is even.

Remark 10.5 For the analytic continuation of L-functions attached to cusp forms for $\Gamma_0(N)$ or $\Gamma_1(N)$ we use the map W_N introduced in (8.3). For more details see [DS05, Section 5.10, page 204],

Remark 10.6 We may analyze L-functions of cusp forms from a point of view which has to do with elliptic points of the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}$. Let $\gamma = \gamma_1 - \gamma_2$, where γ_1 connects $i\infty$ to i and γ_2 connects 0 to i. Both paths are in the imaginary axis. Note that

$$\gamma_2 = S \cdot \gamma_1, \quad S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

In some sense, $R(f,\, s) = (-i)^s \int_\gamma f(\tau)\tau^s \cdot \frac{d\tau}{\tau}$ is attached to S with $S^2 = -I$ and its fixed point i. Recall that the analytic continuation of R to $\mathbb{C}$ is done through

$$L(f,\, s) = \int_{\gamma_1} f(\tau)\tau^s \cdot \frac{d\tau}{\tau} - \int_{\gamma_2} f(\tau)\tau^s \cdot \frac{d\tau}{\tau} = \int_{\gamma_1} \left(f(\tau)\tau^s \cdot \frac{d\tau}{\tau} + f(\tau)\tau^k (\frac{-1}{\tau})^s \frac{d\tau}{\tau} \right).$$

This gives us the convergency at 0 and the functional equation $R(f,k-s) = R(f,s)$. Here, we are using the fact that k is even. Now, consider

$$A = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$$

with $A^3 = -I, A\rho = \rho$, where $\rho = \frac{-1+i\sqrt{3}}{2}$. Under the iteration of A we have

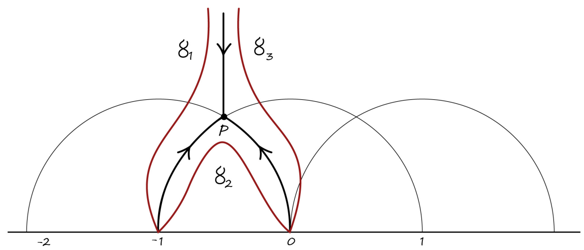


Fig. 10.3 L-function

$$i\infty \to -1 \to 0 \to i\infty$$
$$\tau \to \tfrac{\tau+1}{-\tau} \to \tfrac{-1}{\tau+1} \to \tau.$$

Furthermore, we have the functional equations

$$f\left(\frac{\tau+1}{-\tau}\right) = (-\tau)^k f(\tau) \qquad f\left(\frac{-1}{\tau+1}\right) = (\tau+1)^k f(\tau)$$

Let δ_1 be the path in the upper half plane which connects $i\infty$ to ρ. For instance, it can be in the straight line $\mathrm{Re}(\tau) = \mathrm{Re}(\rho)$. Let also $\delta_2 = A\delta_1$ and $\delta_3 = A\delta_2$, see Figure 10.3. We define

$$\begin{aligned} \gamma_1 = \delta_1 - \delta_2,\ \gamma_2 = \delta_2 - \delta_3,\ \gamma_3 = \delta_3 - \delta_1 \\ R_i := (-i)^s \int_{\gamma_i} f(\tau)\tau^{s-1}, \quad i = 1,2,3 \end{aligned} \tag{10.18}$$

We have

$$R_1 + R_2 + R_3 = 0$$

The integrals in (10.18) are convergent at $i\infty$, -1, 0 respectively. Since γ_3 connects $i\infty$ to 0, we conclude that $R_3 = R(f,s)$. By substituting the Fourier expansion of f inside these integrals one arrives at incomplete Gamma functions, and it might be interesting to investigate this further.

Remark 10.7 Let f be an eigenform of weight 2 for $\Gamma_0(N)$. Recall from (10.13) that the L-function of a modular form is basically a Mellin transform

$$\frac{\Gamma(s)}{(-2\pi i)^s}L(f,s) = \int_0^{i\infty} \tau^{s-1} f(\tau) d\tau \tag{10.19}$$

We use Mellin inversion theorem and write

$$f(\tau) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tau^{-s} \frac{\Gamma(s)}{(-2\pi)^s} L(f,s) ds \tag{10.20}$$

Now let E be an elliptic curve over $\mathbb{Q}$ and f be the corresponding modular form $L(E,s) = L(f,s)$. In the absence of f, it is tempting to insert a mirror map in (10.20) and try to recover f through this formula:

$$f\left(\frac{\int_{\delta_1}\omega}{\int_{\delta_2}\omega}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(\frac{\int_{\delta_1}\omega}{\int_{\delta_2}\omega}\right)^{-s} \frac{\Gamma(s)}{(-2\pi)^s} L(E,s) ds. \tag{10.21}$$

Note that the Left hand side is a polynomial expression in terms of elliptic integrals. One might replace E with a variety for which arithmetic modularity is not known, compute experimentally the right hand side and observe which kind of periods must pop-up in the left hand side. For instance, if the left hand side of (10.21) is a period then it must satisfy a polynomial differential equation, and one might try to compute such a differential equation directly from the right hand side (this is also true directly the formula (10.20)).

10.9 *L*-function attached to Gauss hypergeometric equation

Using the analytic continuation of the mirror map in Section 3.10 we can define a vast generalization of *L*-functions attached to linear differential equations. The main idea is that if we have a modular form $f(\tau)$ and we replace τ with the Schwarz map then we get polynomial expressions of periods. For simplicity, we will do this only in the case of Gauss hypergeometric equation with the Schwarz map in Section 3.10. This method seems to be elaborated in [Sti88] in which the author in page 229 says “A number of specific arithmetic applications will be fully discussed in our forthcoming paper ...”. The paper's name is “η, θ, ζ” which seems to be never published.

Recall the identities (3.43) in Section 3.11. Let us take for instance the identity involving E_4. We substitute the schwarz map in (10.16) and get

$$\zeta(s-3)\zeta(s) = \frac{1}{240}L(E_4,s) =$$

$$\frac{(2\pi)^s}{240\cdot\Gamma(s)}\int_0^1 \left(\frac{F(\frac16,\frac56,1|1-z)}{F(\frac16,\frac56,1|z)}\right)^{s-1}\left(F(\frac{1}{6},\frac{5}{6},1\mid z)^4-1\right) d\left(\frac{F(\frac16,\frac56,1|1-z)}{F(\frac16,\frac56,1|z)}\right).$$

Recall the differential equation (3.36) satisfied by the nominator and denominator of the Schwarz map, and the equality (3.37). We conclude that up to multiplication by a constant independent of s we have

$$\zeta(s-3)\zeta(s) =$$

$$\frac{(2\pi)^s}{\Gamma(s)}\int_0^1 \left(\frac{F(\frac{1}{6},\frac{5}{6},1|1-z)}{F(\frac{1}{6},\frac{5}{6},1|z)}\right)^{s-1}\left(F(\frac{1}{6},\frac{5}{6},1\mid z)^2-F(\frac{1}{6},\frac{5}{6},1\mid z)^{-2}\right)\left(\frac{dz}{z}+\frac{dz}{1-z}\right) \tag{10.22}$$

Let $R(E_4,s)$ be the integral in the above formula. In order to see the functional equation $R(E_4,4-s)=R(E_4,s)$, we set $\tau := \left(\frac{F(\frac{1}{6},\frac{5}{6},1|1-z)}{F(\frac{1}{6},\frac{5}{6},1|z)}\right)$, $F=F(\frac{1}{6},\frac{5}{6},1|z)$ and we write

$$\begin{aligned}
R(E_4,s) &= \int_0^1 A(s,z) = \int_0^{\frac{1}{2}} A(s,z) + \int_{\frac{1}{2}}^1 A(s,z)\\
&= \int_0^{\frac{1}{2}} A(s,z) - A(s,1-z)\\
&= \int_0^{\frac{1}{2}} (\tau^{s-1}+\tau^{3-s})(F^2-F^{-2})\left(\frac{dz}{z}+\frac{dz}{1-z}\right)+\\
&\quad \int_0^{\frac{1}{2}} (\tau^{3-s}-\tau^{-1-s})F^{-2}\left(\frac{dz}{z}+\frac{dz}{1-z}\right)
\end{aligned}$$

For the computation of the second integral, we return back to the τ coordinate and it is $-(\frac{1}{s}+\frac{1}{4-s})$. This is clearly invariant under $s\to 4-s$. This part also contains the poles $s=0, s=4$.

By Proposition 10.3 the integral 10.22 is convergent at $z=0$ for all $s\in\mathbb{C}$ because its integrand has the asymptotic

$$\text{integrand } \sim (\ln(z))^{s-1}.$$

Note that $\frac{1}{z}\left(F(\frac{1}{6},\frac{5}{6},1\mid z)^2-F(\frac{1}{6},\frac{5}{6},1\mid z)^{-2}\right)$ is holomorphic at $z=0$. By Proposition 10.3 the convergency at $z=1$ happens only for $\mathrm{Re}(s)>4$. This is because near $z=1$ we have $F(\frac{1}{6},\frac{5}{6},1|z)=$ hol $\cdot\ln(z-1)+$hol and so

$$\text{integrand } \sim_{z\to 1} (\ln(z-1))^{1-s+2}(z-1)^{-1}$$

Remark 10.8 Similar change of variable is used in [Rog13] and [KZ01, page 24] in order to compute special values of Ramanujan's L-function.

For a modular form f with the constant term f_0 we have used $f(\tau)-f_0$ in the expression of $L(f,s)$. This does not seem to be an elegant way, as we expect to make a sum of f with another modular form of the same weight and not weigh 0 modular form which is the constant f_0. This is why the L function of cusp forms are more natural. There is a very trivial way to construct cuspidal quasi-modular forms. Namely, if f is a quasi-modular form then its derivation with respect to τ is a cuspidal form and so we can directly define its R-function:

$$R(f',s) := \int_0^{i\infty} \tau^{s-1} \frac{\partial f}{\partial \tau} d\tau = \int_0^{i\infty} \tau^{s-1} df(\tau)$$

For instance, for $f = E_4$, we get

$$\zeta(s-4)\zeta(s-1) = \frac{1}{240} L(E_4', s) =$$

$$\frac{(-2\pi i)^s}{240\Gamma(s)} \int_0^1 \left(\frac{F(\frac{1}{6},\frac{5}{6},1|1-z)}{F(\frac{1}{6},\frac{5}{6},1|z)} \right)^{s-1} d\left(F(\frac{1}{6},\frac{5}{6},1 \mid z)^4 \right) \tag{10.23}$$

In order to simplify this formula further, it seems natural to define

$$x = F(\frac{1}{6},\frac{5}{6},1 \mid z)^4 - 1 : (\mathbb{C},0) \to (\mathbb{C},0) \text{ and } (0,1) \to (0,+\infty)$$

and define its inverse by $M(x)$. Let

$$F(\frac{1}{6},\frac{5}{6},1|1-z) = F(\frac{1}{6},\frac{5}{6},1|z)\ln(\frac{1}{432}z) + G(z)$$

where G is holomorphic at $z = 0$ and $G(0) = 0$. We have

$$\zeta(s-4)\zeta(s-1) = \frac{(-2\pi i)^s}{240\Gamma(s)} \int_0^{+\infty} \left(\ln(\frac{1}{432}M(x)) + \frac{G(M(x))}{(1+x)^{\frac{1}{4}}} \right)^{s-1} dx \tag{10.24}$$

Since $F(\frac{1}{6},\frac{5}{6},1 \mid z)$ is an elliptic integral of the first kind, it never vanishes in $\mathbb{C}\backslash\{0,1\}$. We have to prove that its derivative in z does not vanish in $(0,1)$ in order to be able to talk about the inverse of $G(x)$ which is defined in $(0,+\infty)$. Even if this is the case, F' might have zeros in other places. A consequence of this is that the Taylor series of $G(x)$ might have a finite radius of convergence.

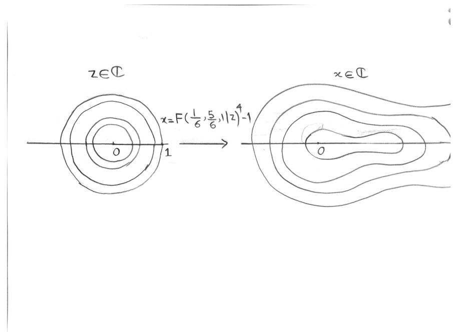


Fig. 10.4 A change of coordinate.

Chapter 11
Hasse-Weil zeta function

I should like to conclude with a brief discussion of a very interesting conjecture, due, I believe, to Hasse. As we have said, from the Kroneckerian point of view the fields of dimension 1 *are the number-fields and the function-fields of curves over finite fields; to each one of these there belongs a zeta-function, the properties of which may be said to epitomize in analytic garb some of the more important properties of the field. It is therefore reasonable to guess that similar functions can be attached to fields of higher dimension, and in the first place to the fields of dimension* 2*, i.e., to the curves over an algebraic number-field, and to the surfaces over a finite field, ([Wei52, page 99]).*

11.1 Introduction

11.2 Finite fields

A finite field, as its name indicates, is a field with finite cardinality. By definition of a field and finiteness property, the characteristic of a finite field is a prime number $p > 1$. Finite fields are completely classified as follows:

Exercise 11.1 We have the following:

1. The order of a finite field of characteristic p is p^n for some $n \in \mathbb{N}$.
2. There is a unique (up to isomorphism of fields) finite field with $p^n,\ n \in \mathbb{N}$ elements.
3. For a prime number the finite field with cardinality p is simply the quotient $\mathbb{F}_p := \frac{\mathbb{Z}}{p\mathbb{Z}}$.
4. For $q = p^n,\ n \in \mathbb{N}$ the finite field with cardinality p^n is denoted by $\mathbb{F}_q$. It is the spliting field of the polynomial $x^q - x$ over $\mathbb{F}_p$.
5. Every finite integral domain is a field and in particular, let $f(T)$ be a monic irreducible polynomial of degree n in $\mathbb{F}_p[T]$. Then the quotient $\mathbb{F}_q[T]/\langle f\rangle$ is a finite field with p^n elements.

6. Let $f(x,y) \in \mathbb{F}_p[x,y]$ be a polynomial and I be a non zero prime ideal of $R := \mathbb{F}_p[x,y]/\langle f\rangle$. Then the quotient R/I is a finite field.
7. We have
$$\overline{\mathbb{F}_p} = \bigcup_{n=1}^{\infty} \mathbb{F}_{p^n}. \tag{11.1}$$

For more on finite fields the reader is referred to [Jac85].

11.3 Zeta functions of elliptic curves over finite fields

In this section we review the zeta function of elliptic curves over finite fields which are rational functions, and hence, much simpler than L-functions. The general definition and rationality was conjectured in [Wei49] and was proved by P. Deligne (see for instance [Kat76a] for an exposition of Deligne results). In the following, p is a prime.

Definition 11.1 Let X be an affine or projective variety defined over $\mathbb{F}_p$. The zeta function of X is defined to be the formal power series in T:

$$Z(X,T) = \exp(\sum_{r=1}^{\infty} \frac{\#X(\mathbb{F}_{p^r})}{r} T^r).$$

Theorem 11.1 *Let E be an elliptic curve defined over $\mathbb{F}_p$. Then*

$$Z(E,T) = \frac{1+2a_E T + pT^2}{(1-T)(1-pT)}. \tag{11.2}$$

where a_E is an integer depending only on E. Moreover, the Riemann hypothesis holds for E, i.e. the only zeros of

$$\zeta(C,s) := Z(E,p^{-s})$$

are in the line $\mathrm{Re}(s) = \frac{1}{2}$.

Proof. For a proof see [Mil20, Theorem 9.10, page 202].

Let
$$1-2a_E T + pT^2 = (1-\alpha T)(1-\beta T)$$
and so
$$\alpha+\beta = 2a_E,\ \alpha\beta = p \tag{11.3}$$
Note that α and β are algebraic integers:

$$\alpha, \beta = a_E \pm \sqrt{a_E^2 - p}. \tag{11.4}$$

We take the logarithmic derivative of both sides of (11.2) and one easily finds the equalities

$$\#E(\mathbb{F}_{p^r}) = p^r + 1 - \alpha^r - \beta^r,\ r = 1,2,3,\dots \tag{11.5}$$

For $r = 1$ we obtain

$$\#E(\mathbb{F}_p) = p + 1 - 2a_E \tag{11.6}$$

We conclude that

Corollary 11.1 For elliptic curves over a finite field $\mathbb{F}_p$ the number of $\mathbb{F}_p$-rational points determine the number of $\mathbb{F}_{p^r}$-rational points.

Proof. This follows from the equalities (11.4), (11.5) and (11.6).

Corollary 11.2 *For an elliptic curve E over a finite field $\mathbb{F}_p$, the Riemann hypothesis holds for $Z(E, p^{-s})$ if and only if*

$$|\#E(\mathbb{F}_p) - p - 1| \leq 2\sqrt{p}. \tag{11.7}$$

Proof. By the rationality statement in Theorem 11.1 we know that if

$$|\alpha| = |\beta| = p^{\frac{1}{2}}. \tag{11.8}$$

then the Riemann hypothesis holds for $Z(E, p^{-s})$. Therefore,

$$|\#E(\mathbb{F}_p) - p - 1| = |2a_E| = |\alpha + \beta| < 2\sqrt{p}.$$

The equality cannot occur because p is prime. Conversely, if (11.7) happens then $a_E^2 - p < 0$ and so the roots of the polynomial $1 - 2a_E T + pT^2$ are complex conjugate, $\beta = \bar{\alpha}$ and since $\alpha\beta = p$, we get $|\alpha| = |\beta| = p^{\frac{1}{2}}$.

11.4 One dimensional algebraic groups

Let us consider a singular curve E of degree 3 in $\mathbb{P}^2_{\mathsf{k}}$ given by $f(x,y,z) = 0$, where $f \in \mathsf{k}[x,y,z]$ is a homogeneous polynomial. It is not hard to see that the singular point P of E is defined over k and it is unique, see Exercise 4.14. In this section, we want to remark that there is three type of singularities P. In the literature, see for instance [Mil20, Chapter II, Section 3, page 70], this is usually done using the language of algebraic groups that we mention it at the end of the this section.

For two points A and B in $\mathbb{P}^2_{\mathsf{k}}(\mathsf{k})$ let L_{AB} be the line through A and B. Let us take an arbitrary line $\mathbb{P}^1_{\mathsf{k}}$ in $\mathbb{P}^2_{\mathsf{k}}$ which does not cross P and define the map

$$f : \mathbb{P}^1_{\mathsf{k}}(\mathsf{k}) \to E(\mathsf{k}), \quad Q \mapsto \text{the third intersection point of } L_{PQ} \text{ with } E$$

1. There is exactly one point $A \in \mathbb{P}^1_{\mathsf{k}}(\mathsf{k})$ which is mapped to P. In this case P is called a cuspidal singularity.

2. There are exactly two points $A,B\in\mathbb{P}^1_{\mathsf{k}}(\mathsf{k})$ which are mapped to P (A,B are defined over k). This is called the nodal singularity with the extra information that the two tangent lines to E to two branches of E passing through P are defined over k.
3. There is $a\in\mathsf{k}$ which is not square in k and there are exactly two points $A,B\in\mathbb{P}^1_{\mathsf{k}}(\mathsf{k}(\sqrt{a}))$ which are mapped to P (A,B are defined over $\mathsf{k}(\sqrt{a})$). This is called the nodal singularity with the extra information that the two tangent lines to E to two branches of E passing through P are Galois conjugate or equivalently the product of these two lines is defined over k.

Exercise 11.2 Verify the classification above for the curve:

$$y^2+a_1xy+a_3y=x^3+a_2x^2+a_4x+a_6,\ a_1,\cdots,\ldots,a_6\in\mathsf{k},$$

or for sub familly of this (for instance take $a_1=a_3=0$).

These corresponds to the following algebraic groups which is used in [Mil20].

1. The additive group $\mathbb{G}_a:=\mathbb{A}^1:=\mathrm{Sepc}(\mathsf{k}[x])$. The group structure in the k-rational points $\mathbb{G}_a(\mathsf{k})=\mathsf{k}$ is the usual addition in k.
2. The multiplicative group $\mathbb{G}_m:=\mathbb{A}^1-\{0\}:=\mathrm{Sepc}(\mathsf{k}[x,\frac{1}{x}])$. The group structure in the k-rational points $\mathbb{G}_m(\mathsf{k})=\mathsf{k}^*$ is the usual multiplication of k.
3. Twisted multiplicative group $\mathbb{G}_m[a]:=\mathrm{Sepc}(\mathsf{k}[x,y]/\langle x^2-ay^2-1\rangle)$ for an element $a\in\mathsf{k}^*$. The group structure in the k-rational points $\mathbb{G}_m[a](\mathsf{k})$ is given by

$$(x,y)\cdot(x',y')=(xx'+ayy',xy'+x'y).$$

which is induced by the multilication in the field extension $\mathsf{k}\subset\mathsf{k}(\sqrt{a})$.

Exercise 11.3 In the above we have not given the scheme theoretical definitions of group structures, for instance, for the additive group the group structure is induced by $\mathsf{k}[x]\to\mathsf{k}[x]\otimes_{\mathsf{k}}\mathsf{k}[x],\ \ x\mapsto x\otimes 1+1\otimes x$. Give a detailed exposition of the scheme theoretical definitions of the above algebraic groups. You may consult [Mil17]

Exercise 11.4 Show that

$$\mathbb{G}_m[a]\cong\mathbb{G}_m[ac^2],\ a,c\in\mathsf{k}-0,$$

In particular, $\mathbb{G}_m[c^2]\cong\mathbb{G}_m$. Moreover,

$$\mathbb{G}_m[a](\mathbb{F}_p)=p+1.$$

By Bezout theorem a singular cubic curve E in $\mathbb{P}^2$ has a unique singular point (if there are two singularities then the line connecting that points meets the curve in 4 points counted with multiplicities). The singular point is defined over k because it is fixed under the action of the Galois group $\mathrm{G}al(\bar{\mathsf{k}}/\mathsf{k})$. Let S be the singular point of of E and

$$E^{ns}(\mathsf{k}):=E(\mathsf{k})\backslash\{S\}.$$

The same definition of group law for elliptic curves applies for E^{ns} and it turns out that E^{ns} is a group and:

Exercise 11.5 If the elliptic curve E is given by the Weierstrass form

$$y^2 = x^3 + t_4 x + t_6,\ t_2, t_3 \in \mathsf{k}, \Delta = 2(4t_4^3 + 27t_6^2) = 0.$$

then E^{ns} is isomorphic to the three one dimensional group described above:

$$E^{ns}(\mathsf{k}) \cong \mathbb{G}_m(\mathsf{k}) \text{ or } \mathbb{G}_m[c](\mathsf{k}), \text{ or } \mathbb{G}_a(\mathsf{k}).$$

Do we need $\mathrm{char}(\mathsf{k}) \neq 2, 3$? See [Mil20, page 74,75].

Exercise 11.6 For $\mathrm{char}(\mathsf{k}) = 3$ (resp. $\mathrm{char}(\mathsf{k}) = 2$) we have to consider the case (4.6) (resp. (4.4)). Discuss the reduction modulo 2 and 3 in such cases.

11.5 Reduction of elliptic curves

We take an elliptic curve in the Weierstrass form

$$y^2 = x^3 + t_4 x + t_6,\ t_2, t_3 \in \mathbb{Q}, \Delta := 2(4t_4^3 + 27t_6^2) \neq 0. \tag{11.9}$$

and by change of coordinates $(x, y) \mapsto (c^2 x, c^3 y),\ c \in \mathbb{Q}$ we assume that $|\Delta|$ is minimal.

Exercise 11.7 For p prime different from 2 and 3 we have the curve $E/\mathbb{F}_p$ and the reduction map

$$E(\mathbb{Q}) \to E(\mathbb{F}_p).$$

1. It is called a good reduction if that $E/\mathbb{F}_p$ is a (smooth) elliptic curve. This happens if p does not divide Δ
2. It is called cuspidal reduction/additive reduction if the curve $E/\mathbb{F}_p$ has a cusp as singularity, that is, its non-singular part is an additive group. This case happens if and only if $p \mid \Delta$, and $p \mid 2t_4 t_6$.
3. Nodal reduction/split multiplicative. The reduced curve $E^{ns}/\mathbb{F}_p$ is a multiplicative group. This happens if and only if $-2t_4 t_6$ is a square in $\mathbb{F}_p$.
4. Nodal reduction/nonsplit multiplicative. The reduced curve $E^{ns}/\mathbb{F}_p$ is a twisted multiplicative group. This happens if and only if $-2t_4 t_6$ is not a square in $\mathbb{F}_p$.

See [Mil20, page 78]

Exercise 11.8 Reduction modulo 3 of the elliptic curve (11.9) is singular if and only if $t_4 = 0$. In the singular case it is always a cusp. In reduction modulo 2 the elliptic curve $E/\mathbb{F}_2$ is always singular and its singular point is $S = (t_4, t_6)$. Find the four groups $E^{ns}(\mathbb{F}_2)$ corresponding to the four choice of (t_4, t_6).

Exercise 11.9 Let $E/\mathbb{Q} : y^2 + y = x^3 - x^2 + 2x - 2$. Show that 1. the primes of bad reduction for E are $p = 5$ and 7. 2. The reduction at $p = 5$ is additive, while the reduction at $p = 7$ is multiplicative.

Exercise 11.10 [Mil20, Exercise 3.4]. This exercise is taken from [Fre86].

11.6 Zeta functions of elliptic curves over $\mathbb{Q}$

We follow [Mil20, Chapter IV, Section 10, page 213].

Definition 11.2 *The non-complete zeta function of a smooth curve $E: f(x,y) = 0,\ f \in \mathbb{Z}[x,y]$ is defined to be*

$$\zeta_S(E,s) = \prod_{p \notin S} \zeta(E/\mathbb{F}_p, s),$$

where S is a finite number of prime numbers such that $E/\mathbb{F}_p$ is singular.

In the case of elliptic curves it is natural to define

$$L_S(E,s) := \prod_{p \notin S} \frac{1}{1 + (\#(E(\mathbb{F}_p)) - p - 1)p^{-s} + p^{1-2s}}$$

and call it non-complete L-function. We have

$$\zeta_S(E,s) = \frac{\zeta_S(s)\zeta_S(s-1)}{L_S(E,s)}$$

Proposition 11.1 *The product $\zeta_S(E,s)$ and hence $L_S(E,s)$ converges for* $\mathrm{Re}(s) > \frac{3}{2}$

Proof. It is direct consequence of the Riemann hypothesis for elliptic curves over finite fields, see Theorem 11.1, and the convergence of the Riemann zeta function, see Proposition 10.1.

We we want to define the complete L function by adding bad prime numbers $p \in S$. We define

$$L_p(T) = \begin{cases} 1 + (\#(E(\mathbb{F}_p)) - p - 1)T + pT^2 & \text{good reduction} \\ 1 - T & \text{split multiplicative reduction} \\ 1 + T & \text{non-split multiplicative reduction} \\ 1 & \text{additive reduction} \end{cases}$$

We have defined this in such a way that

$$L_p(p^{-1}) = \frac{\#E^{ns}(\mathbb{F}_p)}{p}$$

Now we define the L-function of an elliptic curve E over $\mathbb{Q}$:

$$L(E,s) = \prod_p \frac{1}{L_p(p^{-s})}.$$

Definition 11.3 The conductor of an elliptic curve over $\mathbb{Q}$ is defined to be

$$N_{E/\mathbb{Q}} = \prod_{p \text{ bad}} p^{f_p}$$

where $f_p = 1$ if E has multiplicative reduction at p, $f_p = 2$ if $p \not| 2, 3$ and E has additive reduction at p. For the case in which we have additive reduction modulo $p = 2, 3$ we have $f_p \geq 2,\ f_p \in \mathbb{N}$ and f_p depends on wild ramification in the action of the inertia group at of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the Tate module of E.

Exercise 11.11 Discuss the case $p = 2, 3$ in the above definition. [Mil20] is also talking about a formula of Ogg $f_p = ord_p\Delta + 1 - m_p$ using Néron models. Can you obtain some information on this.

Exercise 11.12 Show that the zeta function of the elliptic curve $y^2 = x^3 - 1$ can be expressed in terms of Dedekind's L-functions for the field $\mathbb{Q}(1^{\frac{1}{3}})$. This is taken from [Wei52, page 99]. It seems to me that A. Weil has called it Hecke's L-function and not Dedekind.

11.7 Hasse-Weil conjecture

Define

$$\Lambda(E, s) := N_{E/\mathbb{Q}}^{\frac{s}{2}} (2\pi)^{-s} \Gamma(s) L(E, s)$$

Theorem 11.2 *(Hasse-Weil conjecture for elliptic curves) The function $\Lambda(E, s)$ can be analytically continued to a meromorphic function on the whole $\mathbb{C}$ and it satisfies the functional equation*

$$\Lambda(E, s) = \pm \Lambda(E, 2 - s).$$

This theorem was first proved for *CM* elliptic curves by Deuring 1951/1952. It is proved in its generality by the works of Eichler and Shimura, Wiles, Taylor, Diamond and others, see **??**.

11.8 Birch Swinnerton-Dyer conjecture

For the functional equation of L the value $s = 1$ is in the middle, that is, it is the fixed point of $s \mapsto 2 - s$.

Conjecture 11.1 (Birch Swinnerton-Dyer conjecture(BSD)) For an elliptic curve E over $\mathbb{Q}$, the function $L(E, s)$ is holomorphic at $s = 1$ and its order of vanishing at $s = 1$ is the rank of the elliptic curve E.

A weak form of this conjecture is not also proved:

Conjecture 11.2 (Weak BSD conjecture) $L(E, 1) = 0$ if and only if E has infinitely many rational points.

For papers on BSD conjecture see [CW77, BSD63, BSD65, Tun83, Lan78b] [Ser89], [Mor69].

11.9 Congruent numbers

In this chapter we follow [Kob93b].

Definition 11.4 A natural number n is said to be congruent if it is the area of a right triangle whose sides have rational length.

In other words, for a natural number $n \in \mathbb{N}$, we are looking for the Diophantine equation:

$$C_n : x^2 + y^2 = z^2,\ n = \frac{1}{2}xy$$

in $\mathbb{Q}$, where x, y and z are the sides of a triangle. Consider the affine curve $C_n/\mathbb{Q}$ in $\mathbb{A}^3$ defined by the above equations. It intersects the projective space at infinity in 4 points:

$$[x; y; z; w] = [0; \pm 1; 1; 0],\ [\pm 1; 0; 1; 0].$$

Let

$$D_n : y^2 = x^4 - n^2,\ E_n : y^2 = x^3 - n^2 x.$$

We have morphisms

$$C_n \to D_n,\ (x, y, z) \mapsto (\frac{z}{2}, \frac{x^2 - y^2}{4})$$

and

$$D_n \to E_n,\ (x, y) \mapsto (x^2, xy)$$

defined over $\mathbb{Q}$.

Proposition 11.2 *A necessary and sufficient condition for the point $(x, y) \in E_n(\mathbb{Q})$ to be in the image of $C_n(\mathbb{Q}) \to E_n(\mathbb{Q})$ is that*

1. *x is a square and*
2. *its denominator is divisible by two and*
3. *its numerator has no common factor with n.*

The proof is simple and is left to the reader, see [Kob93b].

Exercise 11.13 Let $\bar{C}_n$ be the projectivization of C_n in $\mathbb{P}^3$. Is $\bar{C}_n$ smooth? If yes determine its genus.

Exercise 11.14 [Kob93b, Exercises 1,2,3,4, page 5].

We want to analyze the torsion points of

$$E_n : y^2 = x^3 - n^2 x$$

By definition of the group structure of E_n we know that

$$O, (0,0), (0, \pm n)$$

are 2-torsions of E_n. .

Proposition 11.3 *We have*

$$E_n(\mathbb{Q})_{\text{tors}} = \{O, (0,0), (0, \pm n)\}$$

and so $\# E_n(\mathbb{Q})_{\text{tors}} = 4$.

Proof. We follow [Kob93b, page 44, Proposition 4]. Let us first give the strategy of the proof. Let $E/\mathbb{Q}$ be an a elliptic curve in the Weierstrass form and let $p > 2$ be a prime number which does not divide the discriminant of E. By a linear change of variable $(x, y) \mapsto (a^2 x, a^3 y)$ we can assume that the ingredient coefficients of E are in $\mathbb{Z}$. Let $\bar{E}/\mathbb{F}_p$ be the elliptic curve obtained from E by considering the coefficients of E modulo p. The main ingredient of the proof is the reduction map

$$E(\mathbb{Q}) \to \bar{E}(\mathbb{F}_p),$$

which is a group homomorphism. Note that by our assumption on p, $\bar{E}/\mathbb{F}_p$ is not singular. This is an injection of $E(\mathbb{Q})_{\text{tors}}$ inside $E(\mathbb{F}_p)$ for all but finitely many p and so for such primes $m := \# E(\mathbb{Q})_{\text{tors}}$ divides $\# E(\mathbb{F}_p)$. In fact, we have not yet proved that $E(\mathbb{Q})_{\text{tors}}$ is finite (a corollary of Mordell-Weil theorem). Therefore, we take a finite subgroup G of $\# E(\mathbb{Q})_{\text{tors}}$ and prove that the reduction map restricted to G is an injection and so $m := \# G$ divides $\# E(\mathbb{F}_p)$. From another side, we prove that for $E = E_n$:

$$\# E_n(\mathbb{F}_p) = p + 1, \ \forall p \text{ prime } p \equiv -1 \bmod 4 \tag{11.10}$$

Therefore, for all but finitely many primes $p \equiv -1 \bmod 4$ we have $p \equiv -1 \bmod m$. This implies that $m = 4$. Therefore, every finite subgroup of $E(\mathbb{Q})_{\text{tors}}$ is of order 4. Since all the elements of $E(\mathbb{Q})_{\text{tors}}$ are torsion, we conclude that $\# E_n(\mathbb{Q})_{tors} = 4$.

Now let us prove that the reduction map induces an injection in a finite subgroup G of $E(\mathbb{Q})_{tors}$. Two points $P = [x; y; z], Q = [x'; y'; z'] \in E(\mathbb{Q})$ are the same after reduction if and only if

$$xy' - x'y, xz' - x'z, yz' - y'z \tag{11.11}$$

are zero modulo p. For all pairs P, Q in G, the number of numbers (11.11) is finite and so there are finitely many primes dividing at least one of them. For all other primes p, we have the injection of G in $E(\mathbb{F}_p)$ by the reduction map. The proof of (11.10) is done in the next proposition.

Proposition 11.4 *Let* $q = p^f$, $p \not| 2n$. *Suppose that* $q \equiv -1$ *mod* 4. *Then there are* $q + 1$ $\mathbb{F}_q$ *points on the elliptic curve* $E_n : y^2 = x^3 - n^2 x$.

Proof. Consider the map

$$f : \mathbb{F}_q \to \mathbb{F}_q, \ f(x) = x^3 - n^2 x$$

f is an odd function, i.e. $f(-x) = -f(x)$, and -1 is not in its image (this follows from the hypothesis on p). It follows that the index of the multiplicative group $\mathbb{F}_q^2 - \{0\}$ in $\mathbb{F}_q - \{0\}$ is two and so for all $x \in \mathbb{F}_q - \{0\}$ exactly one of x or $-x$ is square and so for all $x \in \mathbb{F}_q - \{0, n, -n\}$ exactly one of $f(x)$ or $f(-x)$ is square. Each such a pair $(x, y),\ y = f(x)$ gives us two points $(x, y),\ (x, -y) \in E_n(\mathbb{F}_q)$ and so in total we have $3 + 2\frac{q-1}{2}$ points in $E_n(\mathbb{F}_q)$.

Proposition 11.5 *The natural number n is congruent if and only if $E_n(\mathbb{Q})$ has non-zero rank.*

Proof. If n is a congruent number then by Proposition 11.2, E_n has $\mathbb{Q}$-rational point with x-coordinate in $(\mathbb{Q}^+)^2$. The x coordinates of 2-torsion points in the affine chart x, y are $0, \pm n$. The fact that n is square free and Proposition 11.3 implies that such a rational point is of infinite order.

Conversely, suppose that P is a rational point of infinite order in E_n. We use Exercise 11.15 to finish the proof.

Exercise 11.15 ([Kob93b, page 35, Exercise 2c]) If P is a point not of order 2 in $E_n(\mathbb{Q})$, then the x-coordinate of $2P$ is a square of rational number having an even denominator. By Proposition 11.2, $2P$ comes from a point in $C_n(\mathbb{Q})$ and hence n is a congruent number.

Exercise 11.16 [Kob93b, pages 49-50, Exercises 4,5,6, 7,9].

Let us now state the main result in Section 11.3 for the elliptic curve E_n related to the congruent numbers. The Legendre symbol is defined for integers a and positive odd primes p by

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \text{ divides } a \\ 1 & \text{for some } x \in \mathbb{Z},\ a \equiv x^2 \bmod p \\ -1 & \text{otherwise} \end{cases}$$

Exercise 11.17 In the zeta function of $E_n : y^2 = x^3 - n^2 x$ defined over $\mathbb{F}_p$, p a prime $p \not| 2n$, we have:

$$\alpha = \begin{cases} i\sqrt{p} & \text{if } p \equiv 3(\bmod 4) \text{ in this case } a_{E_n} = 0 \\ 2k + \left(\frac{n}{p}\right) + 2ki & \text{if } p \equiv 1(\bmod 4) \text{ in this case } a_{E_n} = 2k + \left(\frac{n}{p}\right) \end{cases}$$

In the second case k is determined by the fact that $\alpha\bar{\alpha} = p$

Exercise 11.18 The bad prime numbers for the elliptic curve $E_n : y^2 = x^3 - nx$ are those which divide $2n$. For $p \mid 2n,\ p \neq 2$ or $p = 2,\ 2|n$ we have an additive reduction. For $p = 2$ and $p \not| n$ we have apparently a multiplicative reduction: $y^2 = x^3 + x$. The singular point in this case is $S = (1, 0)$ and $E^{ns}(\mathbb{F}_2) = \{O, (0, 0)\}$ which is isomorphic to $(\mathbb{A}(\mathbb{F}_2), +)$ and so it is additive.The conductor of E_n is:

$$N_{E_n/\mathbb{Q}} = \begin{cases} 2^4 n^2 & \text{if } n \text{ is even} \\ 2^5 n^2 & \text{if } n \text{ is odd} \end{cases}$$

Exercise 11.19 In Theorem 11.2 the root number $\pm$ is determined in the following way:

$$\begin{cases} +1 & \text{if } n \equiv 1,2,3 \ (8) \\ -1 & \text{if } n \equiv 5,6,7 \ (8) \end{cases}$$

Exercise 11.20 Reformulating Exercise 11.17 we have we have:

$$(1-T)(1-pT)Z(E_n/\mathbb{F}_p,T) = \prod_{\mathsf{p}|\langle p\rangle}(1-(\alpha_{\mathsf{p}}T)^{\deg(\mathsf{p})})$$

where

$$\alpha_{\mathsf{p}} = \begin{cases} i\sqrt{p} & \text{if } \mathsf{p} = \langle p\rangle \\ a+ib & \text{if } p \text{ splits, where } a+ib \text{ is the unique generator of } \mathsf{p} \\ & \text{which is congruent to } (\frac{n}{p}) \bmod 2+2i. \\ 0 & p \mid 2n \end{cases}$$

The L function of E_n is

$$L(E_n,s) = \prod_{\mathsf{p}\subset\mathbb{Z}[i] \text{ prime}} (1-(\alpha_{\mathsf{p}})^{\deg(\mathsf{p})}(\mathbb{N}\mathsf{p})^{-s})^{-1}$$

Now $\mathbb{Z}[i]$ is a Dedekind domain and so we can define a unique map χ from the ideals of $\mathbb{Z}[i]$ to $\mathbb{C}$ such that $\chi_n(\mathsf{p}) = \alpha_{\mathsf{p}}^{\deg(\mathsf{p})}$. Therefore

$$L(E_n,s) = \prod_{\mathsf{p}\subset\mathbb{Z}[i] \text{ prime}} (1-\chi(\mathsf{p})(\mathbb{N}\mathsf{p})^{-s})^{-1} = \sum_{\mathfrak{a}\subset\mathbb{Z}[i]} \chi_n(\mathfrak{a})(\mathbb{N}\mathfrak{a})^{-s}$$

where the sum is taken over all non-zero ideals.

Chapter 12
Jacobi forms

Monsieur, un jeune géomètre ose vous présenter quelques découvertes faites dans la théorie des fonctions elliptiques, auxquelles il a été conduit par l'étude assidue de vos beaux écrits. C'est à vous, Monsieur, que cette partie brillante de l'analyse doit le haut degré de perfectionnement auquel elle a été portée, et ce n'est qu'en marchant sur les vestiges d'un si grand maître, que les géomètres pourront parvenir à la pousser au delà des bornes qui lui ont été prescrites jusqu'ici. C'est donc à vous que je dois offrir ce qui suit comme un juste tribut d'admiration et de reconnaissance, (in Jacobi's letter to Legendre, see [Cog14, page 533]).

12.1 Introduction

In this chapter we recover the theory of Jacobi forms in the frame work of the moduli of enhanced elliptic curves. Despite examples of Jacobi forms going back to Jacobi himself, its systematic treatment has been started in [EZ85]. The geometrization of Jacabi forms in terms of relative algebraic de Rham cohomology of elliptic curves with two marked point has been started in [CMV24] and in this chapter we follow and extend the results of this article. The relative algebraic de Rham cohomology in general is defined through the hypercohomology of a certain complex and the reader can find the missing definitions in [MV21]. We hope that the reader has become familiar with hypercohomology in Chapter 9.

12.2 Jacobi group and Jacobi forms

For any commutative ring R with unit 1, the Jacobi group Γ_R is a subgroup of the symplectic group $\mathrm{Sp}(4,R)$ consisting of matrices with fourth row of the form $[0,0,0,1]$. For our purpose we consider its conjugate with the matrix which permutes the first and second coordinates of R^4 and hence

$$\mathit{\Gamma}_R = \left\{ \begin{bmatrix} 1 & \lambda & \mu & \kappa \\ 0 & \alpha & \beta & \mu' \\ 0 & \gamma & \delta & -\lambda' \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathrm{Mat}(4,R) \mid \alpha\delta - \beta\gamma = 1, \;\; [\lambda,\mu] = [\lambda',\mu'] \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \right\}.$$

The group $\mathit{\Gamma}_{\mathbb{R}}$ acts on $\mathbb{C} \times \mathbb{H}$ by

$$g \cdot (\tau, z) = (\frac{z + \lambda\tau + \mu}{\gamma\tau + \delta}, \frac{\alpha\tau + \beta}{\gamma\tau + \delta}), \;\; g \in \mathit{\Gamma}_{\mathbb{R}}.$$

Exercise 12.1 The stabilizer of the point $(0,i)$ is the group $\mathrm{SO}(2,\mathbb{R}) \times Z_{\mathbb{R}}$, where

$$\mathrm{SO}(2,\mathbb{R}) := \left\{ \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \mid \alpha^2 + \beta^2 = 1 \right\},$$

and and $Z_{\mathbb{R}}$ is the subgroup of $\mathit{\Gamma}_{\mathbb{R}}$ generated by $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. This action induces an ismorphism

$$\mathit{\Gamma}_{\mathbb{R}} / (\mathrm{SO}(2,\mathbb{R}) \times Z_{\mathbb{R}}) \xrightarrow{\cong} \mathbb{C} \times \mathbb{H}, \;\; g \to g(i,0)$$

Hint: For further details see [BS98] or [EZ85].

Definition 12.1 *A weak Jacobi form of weight k and index m is a holomorphic function $f : \mathbb{C} \times \mathbb{H} \to \mathbb{C}$ satisfying:*

$$f(\frac{z + \lambda\tau + \mu}{\gamma\tau + \delta}, \frac{\alpha\tau + \beta}{\gamma\tau + \delta}) = (\gamma\tau + \delta)^k e^{2\pi i m [\frac{\gamma(z+\lambda\tau+\mu)^2}{\gamma\tau+\delta} - (\lambda^2\tau + 2\lambda z)]} \phi(\tau, z) \qquad (12.1)$$

for $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ and $(\lambda,\mu) \in \mathbb{Z}^2$ together with a Fourier expansion of the form

$$f(\tau, z) = \sum_{n=0}^{\infty} \sum_{r \in \mathbb{Z}} c(n,r) e^{2\pi i (n\tau + rz)}.$$

We denote the space of weak Jacobi forms of weight k and index m by $\widetilde{J}_{k,m}$. If we replace the third condition by a Fourier expansion of the form

$$f(z, \tau) = \sum_{n=0}^{\infty} \sum_{r \in \mathbb{Z}, r^2 \leq 4nm} c(n,r) e^{2\pi i (n\tau + rz)},$$

$f(z,\tau)$ is a Jacobi form of weight k and index m and the space of Jacobi forms of weight k and index m is denoted by $J_{k,m}$.

Now, consider the function F defined (2.66). By Exercise 2.39 we know that

$$F^2 \in \widetilde{J}_{-2,1}$$

is a Jacobi form of weight $k = -2$ and index $m = 1$. Moreover, by the product formula of F, we know that F^2 has a zero of order 2 at $z \in \mathbb{Z}\tau + \mathbb{Z}$.

Theorem 12.1 *Any weak Jacobi form of f of weight k and index m can be written as*

$$f = F^{2m}\left(a_0(\tau) + a_1(\tau)\wp + \left(\sum_{i=0}^{m-1} f_i(\tau)\wp^j + \wp' \sum_{i=0}^{m-2} g_i(\tau)\wp^j\right)'\right), \tag{12.2}$$

where $'$ means derivation with respect to z and a_0, a_1, f_i, g_i are modular form for $\mathrm{SL}(2,\mathbb{Z})$ *of weight $k+2m, k+2m-2, k+2m-2i-1$, $k+2m-4-2i$, respectively.*

Proof. For a Jacobi form f of weight k and index m, the quotient $\frac{f}{F^{2m}}$ is of weight $k+2m$ and index 0. Therefore, it is an elliptic function and by Exercise 2.14 it can be written as $R_1[\wp(z)] + \wp'(z)R_2[\wp(z)]$, where R_1, R_2 are rational functions in $\wp$. As $\frac{f}{F^{2m}}$ has only poles at $z = 0$, R_1, R_2 are polynomials. It is of weight $k+2m$ which implies $2\deg(R_1), 3+2\deg(R_2) \leq k+2m$. Let $\tilde{R}_2(x)$ be the integration of $R_2(x)$, that is, the derivation of $\tilde{R}_2(x)$ with respect to x is $R_2(x)$. In the geometric frame work with Weierstrass coordinates $(x,y) = (\wp, \wp')$ we have

$$\frac{f}{F^{2m}} dz = R_1(x)\frac{dx}{y} + d(\tilde{R}_2(x)) = (a_0 + a_1 x)\frac{dx}{y} + d(\tilde{R}_2(x) + yQ(x))$$

where the last equality is written resticted to the elliptic curve E_{t_2,t_3} and $Q(x)$ can be computed using Proposition 9.2 and Remark 9.1. We also have

$$\deg(\tilde{R}_2(x)) \leq m + \frac{k-1}{2},\ \deg(Q(x)) \leq m + \frac{k}{2} - 2.$$

From another side the pole order of $\frac{f}{F^{2m}}$ at $z = 0$ is $\leq 2m$. This means that

$$\deg(\tilde{R}_2(x)) \leq m - \frac{1}{2}, \deg(Q) \leq m - 2,$$

which are stronger than the previous degree condition.

Corollary 12.1 *([EZ85, Theorem 9.4]) The ring of weak Jacobi forms $\tilde{J}_{*,*}$ has the following structure:*

$$M_*(\mathrm{SL}(2,\mathbb{Z}))[a,b,c]\langle c^2 = \frac{1}{432} a(b^3 - 3E_4 a^2 b + 2E_6 a^3)\rangle$$

where

$$a = F^2 \in \tilde{J}_{-2,1},\ \ b = \frac{12}{(2\pi i)^2} F^2 \wp \in \tilde{J}_{0,1},\ \ c = \frac{-1}{(2\pi i)^3} F^4 \wp' \in \tilde{J}_{-1,2}.$$

.

Proof. We use Theorem 12.1 and we have

$$
\begin{aligned}
F^{2m}\wp'\wp^{m-2} &= cb^{m-2},\\
F^{2m}\wp''\wp^{m-2} &= F^{2m}(6\wp^2-\frac{1}{2}g_2)\wp^{m-2} = *6b^m - *\frac{1}{2}g_2a^2b^{m-2},\\
F^{2m}\wp'^2\wp^{m-3} &= F^{2m}(4\wp^3-g_2\wp-g_3)\wp^{m-3} = *4b^m - *g_2a^2b^{m-2} - *g_3a^3b^{m-3},
\end{aligned}
$$

where $*$'s are some constants.

Remark 12.1 The functional equation of the Jacobi theta function involves the exponential function, and a direct geometric realization of this on the moduli space T seems to be impossible. We need to do extra enhancement of elliptic curves with objects not related directly to de Rham cohomologies. The first suggestion is the following: consider the line bundle $\mathscr{O}(-2O)$ over the elliptic curve E. By definition its dual $\mathscr{O}(2O)$ has a global holomorphic section s with $\mathrm{div}(s) = 2O$. We can view s as $s : \mathscr{O}(-2O) \to \mathbb{C}$ which is linear at each fiber. The new enhancement is a point $Q \neq 0$ over the fiber of $\mathscr{O}(-2O)$ over P. The fiber of the line bundle $\mathscr{O}(-2O)$ at $P \in E$ is given by meromorphic differential forms ω with simple poles at O and P. In the Weierstrass coordinates this is given explicitly given by

$$
\mathscr{O}(-2O)_P := \mathbb{C}\frac{y+b}{x-a}\frac{dx}{y},\ P = [a:b:1].
$$

where $\frac{y+b}{x-a}\frac{dx}{y}$ can be interpreted as a global section of $\mathscr{O}(-2O)$ with a pole of order two at O.

12.3 Relative de Rham cohomology

Definition 12.2 Let X be a smooth variety over k and Y be a smooth subvariety of X. We consider the complex $(\Omega^\bullet_{X/\mathsf{k}}, d)$ (resp. $(\Omega^\bullet_{Y/\mathsf{k}}, d)$) of regular differential forms on X (resp. Y). The (algebraic) relative de Rham cohomology of (X,Y) is defined to be the hypercohomology of the following complex

$$
H^m_{\mathrm{dR}}((X,Y)/\mathsf{k}) := H^m(\Omega^\bullet_{(X,Y)/\mathsf{k}}, d),
$$

where

$$
\Omega^m_{(X,Y)/\mathsf{k}} := \Omega^m_{X/\mathsf{k}} \oplus \Omega^{m-1}_{Y/\mathsf{k}}
$$

and

$$
d : \Omega^m_{(X,Y)/\mathsf{k}} \to \Omega^{m+1}_{(X,Y)/\mathsf{k}},\ (\omega,\alpha) \to (d\omega, \omega|_Y - d\alpha).
$$

Consider the case in which Y consists of two points O and P. The short exact sequence

$$
0 \to \Omega^{\bullet-1}_{Y/\mathsf{k}} \to \Omega^\bullet_{(X,Y)/\mathsf{k}} \to \Omega^\bullet_{X/\mathsf{k}} \to 0
$$

induces the long exact sequence

$$\cdots \to H^0(X,\Omega^\bullet_{X/\mathsf{k}}) \to H^1(Y,\Omega^{\bullet-1}_{Y/\mathsf{k}}) \to H^1(X,\Omega^\bullet_{(X,Y)/\mathsf{k}}) \to H^1(X,\Omega^\bullet_{X/\mathsf{k}}) \to \cdots$$

which gives us the exact sequence

$$0 \to H^0_{\mathrm{dR}}(X/\mathsf{k}) \to H^0_{\mathrm{dR}}(Y/\mathsf{k}) \to H^1_{\mathrm{dR}}((X,Y)/\mathsf{k}) \to H^1_{dR}(X/\mathsf{k}) \to 0, \tag{12.3}$$

together with the isomorphisms

$$H^i_{\mathrm{dR}}((X,Y)/\mathsf{k}) \cong H^i_{\mathrm{dR}}(X/\mathsf{k}), \quad \forall i \geq 2.$$

It follows from (12.3) that $\dim_{\mathsf{k}} H^1_{\mathrm{dR}}((X,Y)/\mathsf{k}) = H^1_{\mathrm{dR}}(X/\mathsf{k}) + 1$ and

$$\text{coker }(H^0_{\mathrm{dR}}(X/\mathsf{k}) \to H^0_{\mathrm{dR}}(Y/\mathsf{k})) = \mathsf{k}\cdot f,$$

where $f : Y \to \mathsf{k}$ is given by $f(O) = 1$ and $f(P) = 0$.

Let us now X be a smooth projective curve of genus one given by the Weierstrass equation $zy^2 = 4x^3 - t_2xz^2 - t_3z^3$ in the homogeneous coordinates $[x : y : z] \in \mathbb{P}^2_{\mathsf{k}}$, and let $Y := \{O,P\}$, where $O = [0 : 1 : 0]$ is the infinity point and $P = [a : b : 1]$. Hence $t_3 = 4a^3 - t_2a - b^2$. We choose the following covering for X:

$$U_0 = X - O, \;\; U_1 = X - \{x = 0\}.$$

in order to write down the double complex of the the relative algebraic de Rham cohomology. The associated simple complex turns out to be:

$$\begin{aligned}&0 \to \Omega^0(U_0)\oplus\Omega^0(U_1) \to \Omega^1(U_0,U_0\cap Y)\oplus\Omega^1(U_1,U_1\cap Y)\oplus\Omega^0(U_0\cap U_1)\\ &\to \Omega^1(U_0\cap U_1, U_0\cap U_1\cap Y) \to 0.\end{aligned} \tag{12.4}$$

Here, for the differential of the double complex, we choose the sign rule defined in [MV21, page 29]. In particular, we have

$$H^1_{\mathrm{dR}}(X,Y) = \frac{\{((\omega_0,\alpha_0),(\omega_1,\alpha_1),f_{01}) \mid df_{01} = \omega_1|_{U_{01}} - \omega_0|_{U_{01}}\;,\; f_{01}|_Y = \alpha_1|_{U_{10}} - \alpha_0|_{U_{10}}\}}{\{((df_0, f_0|_{Y\cap U_0}),(df_1, f_1|_{Y\cap U_1}), f_1|_{U_{01}} - f_0|_{U_{01}})\}} \tag{12.5}$$

where $f_i \in \Omega^0(U_i), f_{01} \in \Omega^0(U_0\cap U_1), (\omega_i,\alpha_i) \in \Omega^1_{(X,Y)}(U_i, U_i\cap Y)$ and $U_{01} = U_0 \cap U_1$.

Proposition 12.1 *For $a \neq 0$, we can choose a basis of $H^1_{\mathrm{dR}}((X,Y)/\mathsf{k})$ as follows:*

1. $((0,0),(df,0),f|_{U_{01}})$, *where* $f = \frac{x-a}{x}$;
2. $((\frac{dx}{y}|_{U_0},0),(\frac{dx}{y}|_{U_1},0),0)$, *where* $\frac{dx}{y}$ *is a holomorphic* 1*-form on* X;
3. $((\frac{xdx}{y},0),(\frac{xdx}{y}+dg, g|_Y), g|_{U_{01}})$, *where* $g = -\frac{y}{2x}$.

Proof. For the second and third item, they form a basis of $H^1_{\mathrm{dR}}(X/\mathsf{k})$ and the details can be found in [Mov12, Proposition 2.4]. For the first one, it is enough to show that this element is not zero in $H^1_{\mathrm{dR}}((X,Y)/\mathbb{C})$. If this is not true, then we can write $((0,0),(df,0),f|_{U_{01}})$ as

$$((df_0, f_0|_{Y\cap U_0}),(df_1, f_1|_{Y\cap U_1}), f_1|_{U_{01}} - f_0|_{U_{01}}).$$

Then $f_0 = 0$ and hence $f_1 = f$. However the infinite point $O \in Y \cap U_1$ and $f(O) = 1$, which is a contradiction with $f_1|_{Y\cap U_1} = 0$.

12.4 Meromorphic forms without residues

In this section we provide another algebraic interpretation of the relative de Rham cohomology. It depends on the choice of an affine chart U containing $Y = \{O,P\}$. The advantage of this description is that the Gauss-Manin connection becomes much simpler to compute and it can be used to integrate elements of algebraic de Rham cohomologies over paths.

Proposition 12.2 *Let $U \subseteq X$ be an affine open set such that $Y \subseteq U$ and take $U_0 := X - \{O\}$, $U_1 = U$ as a covering of X. We have the isomorphism given by*

$$H^1_{\mathrm{dR}}(X,Y) \cong \frac{\omega \in \Gamma(\Omega^1_U) \text{ without residues on } X-U}{\text{exact forms } df \text{ with } f|_Y = 0}. \tag{12.6}$$

given by

$$((\omega_0, f_0),(\omega_1, f_1), f_{01}) \mapsto \omega_1 + (f_1(P) - f_1(O))df$$

where f is any regular function in U with $f(P) = 0$ and $f(O) = 1$.

Proof. The proof in [CMV24][Proposition 2.4, Proposition 2.5] is not elementary. It must be rewritten.

Under the isomorphism (12.6), we may choose the basis of the relative de Rham cohomology directly rather than using the representatives in the Čech complex. This will be useful when we compute the Gauss-Manin connection later. Depending on the coordinates of the point P, we choose the following basis on $H^1_{\mathrm{dR}}(X,Y)$:

$$d\left(\frac{x-a}{x}\right), \frac{dx}{y}, \frac{xdx}{y} - d\left(\frac{y}{2x}\right) - \frac{b}{2a} d\left(\frac{x-a}{x}\right), \quad a \neq 0, \tag{12.7}$$

$$d\left(\frac{x-a}{x-1}\right), \frac{dx}{y}, \frac{xdx}{y} - d\left(\frac{y}{2(x-1)}\right) - \frac{b}{2(a-1)} d\left(\frac{x-a}{x-1}\right), \quad a \neq 1.$$

Note that for $a \neq 0$ we are considering $U = (X - \{x = 0\}) \cup \{O\}$, while for $a \neq 1$ we take $U = (X - \{x = 1\}) \cup \{O\}$. The first differential form $\omega_1 = df$ is chosen in such a way that $f(O) = 1, f(P) = 0$. Note also that the correction of $\frac{xdx}{y}$ with an exact differential form kills its pole at O. The computations are similar as in [Mov12, Section 2.8]. We remark that when $a \neq 0,1$ both basis are equal in the right hand side of (12.6) for $U = (X - \{x(x-1)=0\}) \cup \{O\}$. For instance, the difference of the third element in both basis is an exact differential form dg with $g(P) = g(O) = \frac{b}{2a} - \frac{b}{2(a-1)}$.

For the general definition of cup product in relative algebraic de Rham cohomology see [CMV24, Section 3]. In thi section we explain what it is in the case of elliptic curves. We choose the affine open cover $\{U_0, U_1, U_{01}\}$ of the smooth projective curve and ake two elements (ω, α) and (ν, β) which are represented as

$$((\omega_0, \alpha_0), (\omega_1, \alpha_1), \omega_{01}), ((\nu_0, \beta_0), (\nu_1, \beta_1), \nu_{01}).$$

We have

$$(\omega_0 \wedge \nu_0, -\omega_0|_Y \wedge \beta_0), (\omega_1 \wedge \nu_1, -\omega_1|_Y \wedge \beta_1), (-\omega_0 \wedge \nu_{01} + \omega_{01} \wedge \nu_1, -\omega_{01}|_Y \wedge \beta_1)$$

In particular, using the basis $\omega_i, i = 1,2,3$ of $H^1_{\mathrm{dR}}((X,Y)/\mathbb{C})$ given in 12.1, we get that

$$\omega_1 \cup \omega_2 = \left((0,0), \left(df \wedge \frac{dx}{y}, 0\right), \left(\frac{(x-a)dx}{xy}, 0\right)\right), \tag{12.8}$$

$$\omega_1 \cup \omega_3 = \left((0,0), \left(df \wedge \frac{xdx}{y}, 0\right), \left(\frac{(x-a)dx}{y}, 0\right)\right), \tag{12.9}$$

$$\omega_2 \cup \omega_3 = \left((0,0), (0,0), \left(-\frac{dx}{2x}, 0\right)\right) \tag{12.10}$$

We have the composition

$$H^1_{\mathrm{dR}}(X,Y) \times H^1_{\mathrm{dR}}(X,Y) \to H^2_{\mathrm{dR}}(X,Y) \to H^2_{\mathrm{dR}}(X) \xrightarrow{\mathrm{Tr}} \mathbb{C} \tag{12.11}$$

which gives us the bilinear map

$$\langle \cdot, \cdot \rangle : H^1_{\mathrm{dR}}(X,Y) \times H^1_{\mathrm{dR}}(X,Y) \to \mathbb{C},$$

which is called the relative trace map. We find that

$$\langle \omega_2, \omega_3 \rangle = -\langle \omega_3, \omega_2 \rangle = 1,$$

and the others are zero. Here we used the fact that the trace map is the residue of ω_{01} around the infinite point, see for instance [Mov12, Page 19].

12.5 Mixed Hodge structures

The following definition is the outcome of a general definition of mixed Hodge structures.

Definition 12.3 The polarized mixed Hodge structure in $H^1(X,Y)$ consists of the following data:

1. The Hodge filtration
$$0=: F^2 \subset F^1 = \text{ Holomorphic differential 1-forms in } X \subset F^0 := H^1_{\mathrm{dR}}(X,Y)$$
2. The weight filtration
$$0=: W_{-1} \subset W_0 := \ker(H^1_{\mathrm{dR}}(X,Y) \to H^1_{\mathrm{dR}}(X)) \subset W_1 := H^1_{\mathrm{dR}}(X,Y).$$
3. The bilinear map (polarization) $H^1_{\mathrm{dR}}(X,Y) \times H^1_{\mathrm{dR}}(X,Y) \to \mathsf{k}$.

The one dimensional vector space is generated by (2) and W_0 is generated by (1). We also need the mixed Hodge structure in $H^1_{\mathrm{dR}}(X-Y)$:

1. The Hodge filtration
$$0=: F^2 \subset F^1 \subset F^0 := H^1_{\mathrm{dR}}(X-Y)$$
where F^1 is generated by regular differential 1-forms in $X-Y$ with pole order ≤ 1 along Y.
2. The weight filtration
$$0=: W_0 \subset W_1 := \mathrm{Im}(H^1_{\mathrm{dR}}(X) \to H^1_{\mathrm{dR}}(X-Y)) \subset W_2 := H^1_{\mathrm{dR}}(X,Y).$$

A basis of $H^1_{\mathrm{dR}}(X-Y)$ is given by
$$\frac{dx}{y},\ \frac{xdx}{y},\ \frac{1}{2}\frac{y+b}{x-a}\frac{dx}{y}.$$
W_1 (resp. F^1) is generated by $\frac{dx}{y},\ \frac{xdx}{y}$ (resp. $\frac{dx}{y},\ \frac{1}{2}\frac{y+b}{x-a}\frac{dx}{y}$).

We would like to have a four dimensional vector space $H^1_{\mathrm{dR}}(X-Y,Y)$ canonically attached to (X,Y) with filtrations
$$W_0 \subset W_1 \subset W_2 = H,\ 0 = F^2 \subset F^1 \subset F^0 = H^1_{\mathrm{dR}}(X-Y,Y)$$
such that $W_1/W_{-1} \cong H^1_{\mathrm{dR}}(X,Y)$ and $W_2/W_0 \cong H^1_{\mathrm{dR}}(X-Y)$ are ismorphism of mixed Hodge structures. The main issue is how to choose $\omega_3 \in W_2/W_1$ and define
$$H^1_{\mathrm{dR}}(X-Y,Y) = H^1(X,Y) \oplus \mathbb{C}\omega_3.$$
We choose $\omega_3 \in F^1 H^1_{\mathrm{dR}}(X-Y)$ and assume that it has residue $+1$ at P. We have the freedom $\omega_1 \in W_1 \cap F^1$ which is one dimensional. Therefore, we fix a parameter s and consider $\omega_3 + s\omega_1$.

Further, we would like to have a pairing $\langle \cdot,\cdot \rangle : H^1_{\mathrm{dR}}(X-Y,Y) \times H^1_{\mathrm{dR}}(X-Y,Y) \to \mathbb{C}$ which coincides with the canonical pairing of W_1 and it is symplectic. We choose a basis $\omega_0, \omega_1, \omega_2$ of $H^1_{\mathrm{dR}}(X,Y)$ with the intersection matrix

$$[\langle \omega_i, \omega_j \rangle]_{i,j=0,1,2} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Let

$$[\langle \omega_i, \omega_j \rangle]_{i,j=0,1,2,3} := \begin{pmatrix} 0 & 0 & 0 & a_0 \\ 0 & 0 & 1 & a_1 \\ 0 & -1 & 0 & a_2 \\ -a_0 & -a_1 & -a_2 & 0 \end{pmatrix}.$$

where a_i's are unknown quantities depending only on a, b, t_2, t_3. In order to find a symplectic basis, we must replace ω_3 with $\frac{\omega_3}{a_0}$ which changes the residue of ω_3 at P_3. Therefore, a_0 must be necessarily 1. Moreover, we replace ω_3 with $\omega_3 - a_1\omega_2 + a_2\omega_1$ in order to get the symplectic basis. The computations in Section 12.12 suggests that for

$$\omega_0 = \mathrm{d}\left(\frac{x-a}{x}\right), \quad \omega_1 := \frac{\mathrm{d}x}{y}, \quad \omega_2 = \frac{x\mathrm{d}x}{y} - \mathrm{d}\left(\frac{y}{2x}\right), \quad \omega_3 = \frac{1}{2}\frac{y+b}{x-a}\frac{\mathrm{d}x}{y}. \tag{12.12}$$

we must have:

$$[\langle \omega_i, \omega_j \rangle]_{i,j=0,1,2,3} := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & \frac{b}{2a} \\ -1 & 0 & -\frac{b}{2a} & 0 \end{pmatrix}.$$

Recall that ω_3 has a one dimensional freedom. If we change ω_3 to $\omega_3 + s\omega_1$, in order to have the same cup product as before, and in particular $\langle \omega_3, \omega_2 \rangle$ unchanged, we must replace ω_2 with $\omega_2 + s\omega_0$. Note that multilication of ω_3 by a constant is not allowed. We call again these elements ω_3 and ω_2 and keep in mind that it depends on a parameter s.

In the 4 dimensional space $H^1_{\mathrm{dR}}(X - Y, Y)$ the weight filtration comes from limit mixed Hodge structure, see [Sch73]. The monodromy matrix is topological and can be written as $I + N$, where I is the identity matrix and N is the nilpotent matrix with $N_{14} = 1$ and elsewhere zero ($N^2 = 0$). The nilpotent matrix N acts on H and $W_0 = \mathrm{Im}(N)$ and $W_1 = \ker(N)$.

12.6 The moduli space T

In this section we introduce a new enhancement of an elliptic curve which is a combination of enhancement with elements of the relative de Rham cohomology $H^1(X,Y)$ in [CMV24] and enhancement with differential forms with simple poles in Y which produces Picard curious example in [Mov22b, Section 10]. This has been elaborated in [CM24]

Definition 12.4 An enhanced elliptic curve with two marked points is the data

$$(X,Y),[\alpha_0,\alpha_1,\alpha_2,\alpha_4]$$

where $Y=\{O,P\},\ P\neq O$ and $\alpha_0,\alpha_1,\alpha_2,\alpha_3$ form a basis of $H^1_{\rm dR}(X-Y,Y)$ compatible with the mixed Hodge structure and with constant intersection matrix. This means that $\alpha_0\in W_0,\alpha_1\in F^1\cap W_1,\ \alpha_2\in W_1,\ \alpha_3\in W_2$ such that

$$[\langle\alpha_i,\alpha_j\rangle]_{i,j=0,1,2,3}:=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}\tag{12.13}$$

and

$$\mathrm{Resi}_P\alpha_3=+1.$$

The following algebraic group acts on T.

$$\mathsf{G}=\left\{\begin{pmatrix}1&0&v&u\\0&k&k'&vk\\0&0&k^{-1}&0\\0&0&0&1\end{pmatrix},k',v,u\in\mathbb{C},k\in\mathbb{C}^*\right\},$$

It acts on the enhancement $[\alpha_0,\alpha_1,\alpha_2,\alpha_3]$ by usual multiplication from the right:

$$[\alpha_0,\alpha_1,\alpha_2,\alpha_3]\mathsf{g}=[\alpha_0,k\alpha_1,v\alpha_0+k'\alpha_1+k^{-1}\alpha_2,w\alpha_0+kv\alpha_1+\alpha_3]$$

In order to construct the moduli space T we start with the the Weierstrass familly and the basis $\omega_0,\omega_1,\omega_2$ in Proposition 12.1. Recall that with $\tilde\omega_3:=\omega_3+a_1\omega_2-a_2\omega_1$, the intersection matrix in this basis is already of the desired format (12.13) and it is compatible with the mixed Hodge structure. The most general basis compatibel with MHS is given by $S\omega$, where

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\c-\frac{b}{2a}&t_1&1&0\\d&c-\frac{b}{2a}&0&1\end{pmatrix},$$

The zeros in this matrix are due to the compatibility with MHS and the equality of $(3,1)$ and $(4,2)$ is due to $\langle\alpha_2,\alpha_3\rangle=0$. The correction of c with $\frac{b}{2a}$ is due to the fact that the corrected c will turn out to be log derivative of Jacobi theta function. We will use the letter d and we hope that it will not be confused with the differential. We have proved:

Proposition 12.3 *We have*

$$\mathsf{T}:=\mathrm{Sepc}\ \mathsf{k}[a,b,c,d,e,t_1,t_2,\frac{1}{\Delta}]$$

and the universal family of enhanced elliptic curves over T *exists and it is given by*

$$\begin{aligned}
&X : y^2 = 4x^3 - t_2x - t_3, P = (a,b),\\
&\alpha_0 := \left[d\left(\frac{x-a}{x}\right)\right], \alpha_1 := \left[\frac{dx}{y}\right],\\
&\alpha_2 = \left[\left(c - \frac{b}{2a}\right)\alpha_0 + t_1\frac{dx}{y} + \frac{xdx}{y} - d\left(\frac{y}{2x}\right)\right],\\
&\alpha_3 := \frac{1}{2}\frac{y+b}{x-a}\frac{dx}{y} + c\alpha_1 + d\cdot\alpha_0.
\end{aligned}$$

Proposition 12.4 *The group action of* G *on the coordinates* $(a,b,c,d,t_1,t_2) \in \mathsf{T}$ *is given by*

$$(a,b,c,d,t_1,t_2) \to t \bullet \mathrm{g} = (k^{-2}a, k^{-3}b, v + k^{-1}c, d + w + u, k^{-1}k' + k^{-2}t_1, k^{-4}t_2).$$

Proof. The proof is just a mere computation. Let

$$\alpha : \mathbb{C}^2 \to \mathbb{C}^2,\ (x,y) \mapsto (k^2x, k^3y)$$

and $f = y^2 - 4x^3 + t_2x + t_3$. We have

$$k^{-6}\alpha^*(f) = y^2 - 4x^3 + t_2k^{-4}x + t_3k^{-6}.$$

This implies that α induces an isomorphism

$$(E_{t_2k^{-4},t_3k^{-6}}, \omega_0, k^{-1}\omega_1, k\omega_2, \omega_3, (k^{-2}a, k^{-3}b)) \to (E_{(t_2,t_3)}, \omega_0, \omega_1, \omega_2, \omega_4, (a,b))$$

which implies that

$$(E_{t_2,t_3}, g^{\mathrm{tr}}\alpha) = (E_{t_2,t_3}, g^{\mathrm{tr}}S(a,b,c,t_1,d)\omega) = (E_{t_2k^{-4},t_3k^{-6}}, \tilde{g}S(k^{-2}a, k^{-3}b, ck^{-1}, k^{-2}t_1, d)\omega)$$

where

$$\tilde{g} = \begin{pmatrix} 1 & 0 & v & w \\ 0 & 1 & k^{-1}k' & v \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

12.7 Self-similar homology

Let (X,O) be an elliptic curve, $P \in X$ with $P \neq O$ and $Y = \{O,P\}$. Let also

$$\phi : X \to X,\ \ \phi(Q) = Q + P$$

be the translation function (and hence $\phi(O) = P$).

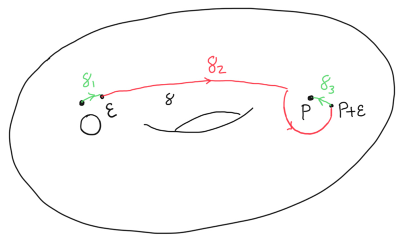


Fig. 12.1 Self-similar paths

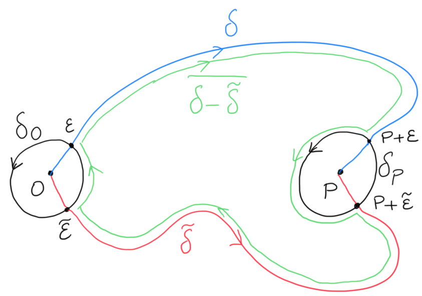


Fig. 12.2 Homology of self-similar paths

Definition 12.5 A path δ in X with boundaries at Y is called self-similar if near to O and P it behaves the same, that is, the pull-back of δ near P by ϕ is minus δ near O, see Figure 12.1.

For instance, if the self-similar path turns N-times around O anticlockwise before departing to P, then when it reaches P it must turn N-times around P clockwise before ending in P.

For simplicity, we take an anticlockwise oriented loop δ_O around O and define $\delta_P := \phi_*\delta_O$. For two self-similar paths δ and $\tilde{\delta}$ we construct a closed path $\overline{\delta-\tilde{\delta}}$ in $X-Y$ in the following way. Near to O we take the point $\tilde{\varepsilon}$ (resp. ε) of intersection between $\tilde{\delta}$ (resp. δ) and δ_O. The closed path $\overline{\delta-\tilde{\delta}}$ is as follows. It start from $\tilde{\varepsilon}$ goes along δ_O until ε, then along δ until $P+\varepsilon$, then along δ_P until the point $P+\tilde{\varepsilon}$ and returns back to $\tilde{\varepsilon}$ along $-\tilde{\delta}$. We say that δ and $\tilde{\delta}$ are homologous if the closed path $\overline{\delta-\tilde{\delta}}$ is homologous to zero in $X-Y$. Note that the pieces of $\overline{\delta-\tilde{\delta}}$ on the paths δ_O and δ_P are not mapped to each other under ϕ. Under the identification $\delta_P = \phi_*\delta_O$, these pieces forming the full loop δ_O. The homology of self-similar paths is an equivalence relation.

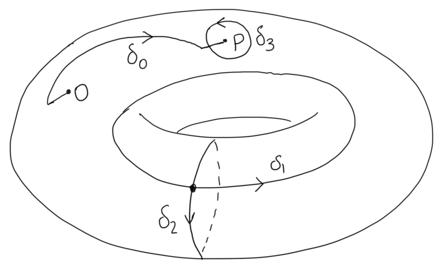


Fig. 12.3 Four paths

Definition 12.6 The self-similar homololgy is defined in the following way:

$$H_1(X-Y,Y) := \frac{\mathbb{Z}\text{-module generated by self-similar paths}}{\mathbb{Z}\text{-module of homologous to zero self-similar paths}}$$

Proposition 12.5 *The self-similar homology $H_1(X-Y,Y)$ is a free $\mathbb{Z}$-module of rank* 4.

Proof. We fix paths $\delta_0,\delta_1,\delta_2,\delta_3$ in X as follows: δ_0 is a self-similar path from O to P, δ_3 is an anticlockwise loop arround P, and δ_1,δ_2 are closed loops in $X-Y$ which from a basis of $H_1(X,\mathbb{Z})$. We prove that $H_1(X-Y,Y)$ is freely generated by these four paths. write more details.

12.8 Self-similar cohomology

Definition 12.7 Let ω be a meromorphic differential i-form ($i=0,1$) in X with order M at O and P (if M is negative $-M$ is pole order and if $M\geq 0$ this is zero order). It might have other poles. Let $N\in\mathbb{N}$. We say that ω is N-th self similar at O and P if $\phi^*\omega-\omega$ has order $\leq M+N$ at O (if M is negative then the pole order decreases by N). In case ω has poles at P ($M<0$), we say that ω is self similar if it is $-M$-th self similar, that is $\phi^*\omega-\omega$ is holomorphic at O.

For $i=0$, that is $\omega=f$ is a rational function, and $M=0$, that is f is holomorphic at O,P, and $N=1$, the N-th self similarity is equivalent to $f(O)=f(P)$. This has been used in the definition of relative de Rham cohomology. Moreover, if f is a rational function in X with pole at P of order M and homomorphic at $2P$ then ϕ^*f+f is self semilar. Anothr example of self-similar differential forms is a holomorphic differential form in X.

Definition 12.8 The self similarity cohomology is defined as follows:

$$H^1_{\mathrm{dR}}(X-Y,Y) := \frac{\{\omega \in \Omega^1(*Y) \mid \omega \text{ is residue free self similar}\}}{\{df | df \text{ is } (-M+1)\text{-th self similar}\}}.$$

where M is the order of f at Y. Here residue free, means that ω has residue zero except at O and P.

The main reason for defining self-similar differential forms is that their integration over self-similar paths is well-defined. Recall that a self similar path γ can be written as $\tilde{\gamma} = \gamma_1 + \gamma_2 + \gamma_3$, where the path γ_3 is the image of γ_1 under the translation ϕ but with the oposite direction, see Figure 12.1.

Proposition 12.6 *For a self-similar differential* 1*-forms* ω *the following limit exists:*

$$\int_\gamma \omega := \lim_{\varepsilon \to O} \int_{\gamma_2} \omega$$

Proof. The proposition has been inspired by the following not so rigorous equalities (the first integral diverges at both ends of its domain):

$$\begin{aligned}\int_\gamma \omega &= \int_{\gamma_1} \omega + \int_{\gamma_2} \omega + \int_{\gamma_3} \omega \\ &= \int_{\gamma_1} \omega + \int_{\gamma_2} \omega - \int_{\gamma_1} \phi^* \omega \\ &= \int_{\gamma_2} \omega - \int_{\gamma_1} (\phi^* \omega - \omega)\end{aligned}$$

In order to make this rigorous we observe that the last quantity is a well-defined integral and it does not depend on ε. Moreover, $\phi^*\omega - \omega$ is holomorphic in a neighborhood of O and hence $\lim_{\varepsilon \to O} \int_{\gamma_1} \omega = 0$.

Proposition 12.7 *We have the following:*

1. *A meromorphic function* f *is self-similar if and only if* df *is self-similar. For a self-similar exact* 1*-form* $\mathrm{d}f$ *we have:*

$$\int_O^P \mathrm{d}f = (\phi^* f - f)(O).$$

2. *For a meromorphic function* f *on* X*, the* 1*-form* $\frac{df}{f}$ *is self-similar if it has the same order at* O *and* P *and:*

$$\int_O^P \frac{df}{f} \in \ln\left(\frac{\phi^* f}{f}(O)\right) + 2\pi i \mathbb{Z}.$$

3. *A meromorphic* 1*-form with poles of order* 1 *at* O *and* P *(and other poles of arbitrary order in other points) is self-similar if and only if it has the same residue at* O *and* P*.*
4. *For the Weierstrass coordinates* x, y *of* X *and* $P = (a, b)$*, the following differential form is self-similar*

$$\frac{y+b}{x-a}\frac{dx}{y}-\frac{dx}{x}.$$

Proof. The first part is a direct consequence of the definition. Observe that the integration is $f(\varepsilon+P)-f(\varepsilon)=\phi^*f(\varepsilon)-f(\varepsilon)$. The proof of the second part is similar. Note that the integral is defined up to $2\pi i\mathbb{Z}$ which is due to other poles and zeros of f.

Remark 12.2 The topic of the present section might be considered as a generalization of Cauchy principal value. Note that the way we would like to have $\infty-\infty=0$ highly depends on self-similar differential forms and choice of points ε and $\varepsilon+P$ which converge to O and P, repectively. We could also take the morphism $X\to X,\ Q\mapsto 2Q$ for a computation of an integral from P to $2P$. All these choices might result in different values. The following easy examples taken from Wikipedia explain this in an elementary way:

$$\lim_{a\to\infty}\int_{-a}^{a}\frac{2xdx}{x^2+1}=0,\quad \lim_{a\to\infty}\int_{-2a}^{a}\frac{2xdx}{x^2+1}=-\ln 4.$$

Thanks to Y. Nikdelan, R. Villaflor and IGADPEs team for the remark.

12.9 Gauss-Manin connection of self-similar integrals

The Gauss-Manin connection of differential forms with poles and residues at O and P is closely related to Picard's curious example which has been formulated in [Mov22b, Section 10]. We reproduce the computation in this reference with the data of exact forms so that we can glue the Gauss-Manin connection of the open and relative cohomologies $H^1_{\mathrm{dR}}(X-Y)$ and $H^1_{\mathrm{dR}}(X,Y)$. Let

$$\omega_0:=\mathrm{d}\left(\frac{x-a}{x}\right),\ \ \omega_1:=\frac{\mathrm{d}x}{y},\ \ \omega_2:=\frac{x\mathrm{d}x}{y}-\mathrm{d}\left(\frac{y}{2x}\right),\ \omega_3:=\frac{1}{2}\frac{y+b}{x-a}\frac{\mathrm{d}x}{y}. \tag{12.14}$$

The differential form ω_3 has a simple pole at O and P with residues -1 and $+1$, respectively.

Definition 12.9 Even though the integration of differential 1-form ω_3 over a path from O to P is divergent, we define

$$\int_O^P\omega_3:=\int_{\delta_0}\left(\omega_3-\frac{\mathrm{d}x}{x}\right)+\ln(a). \tag{12.15}$$

Note that $\omega_3-\frac{\mathrm{d}x}{x}$ has simple poles at O and P with residues both equal to $+1$. The path of integration $\delta_0:[0,1]\to X,\ \delta_0(0):=O,\delta_0(1)=P$ is self-similar, that is, it is taken in such a way that for ε a positive number near to 0 we have $\delta_0(\varepsilon)+P=\delta_0(1-\varepsilon)$, see Figure 12.4. Note that the first plus sign refers to the group structure of the elliptic curve. This choice of δ_0 implies that the integration of $\omega_3-\frac{\mathrm{d}x}{x}$ over

δ_0 is convergent. The correction $\ln(a)$ is actually $\int_O^P \frac{\mathrm{d}x}{x} = \ln(a) - \infty$, and we have removed the evaluation of ln at O which is infinity. We choose a in a neighborhood of a point in $\mathbb{R}^+$ and $\ln(a)$ takes real values for a positive real number.

Proposition 12.8 *For the family of elliptic curve*

$$X : y^2 = 4(x^3 - a^3) - t_2(x-a) + b^2, \tag{12.16}$$

we have

$$\mathrm{d}\left(\int_O^P \left(\frac{1}{2}\frac{y+b}{x-a}\frac{\mathrm{d}x}{y}\right)\right) = B_{30} + B_{31}\cdot\int_O^P \left(\frac{x\mathrm{d}x}{y} - \mathrm{d}(\frac{y}{2x})\right) + B_{32}\cdot\int_O^P \frac{\mathrm{d}x}{y}, \tag{12.17}$$

where d *is the differential of holomorphic functions in* $(a,b,t_2) \in \mathbb{C}^3$ *and* $B_{3i},\ i = 0,1,2,3$ *are given by*

$$\begin{aligned}
2ab\Delta B_{30} &= (24t_2^2a^2b - 360t_2a^4b + 63t_2ab^3 + 864a^6b - 324a^3b^3 + 27b^5)\mathrm{d}a \\
&\quad + (2t_2^3a - 54t_2^2a^3 + 2t_2^2b^2 + 432t_2a^5 - 90t_2a^2b^2 - 864a^7 + 360a^4b^2 - 36ab^4)\mathrm{d}b \\
&\quad - \frac{1}{2}(2t_2^2ab - 30t_2a^3b + 3t_2b^3 + 72a^5b - 18a^2b^3)\mathrm{d}t_2 \\
\Delta B_{31} &= \frac{-1}{2}(3t_2^2b + 54t_2a^2b - 216a^4b + 54ab^3)\mathrm{d}a \\
&\quad - (2t_2^2a - 30t_2a^3 + 3t_2b^2 + 72a^5 - 18a^2b^2)\mathrm{d}b + \frac{9t_2ab - 36a^3b + 9b^3}{4}\mathrm{d}t_2 \\
\Delta B_{32} &= (45t_2ab - 108a^3b + 27b^3)\mathrm{d}a + (2t_2^2 - 30t_2a^2 + 72a^4 - 18ab^2)\mathrm{d}b - \frac{3t_2b}{2}\mathrm{d}t_2
\end{aligned}$$

Proof. We first do the following easy hand computation:

$$\begin{aligned}
\frac{\partial}{\partial a}\int_O^P \frac{1}{2}\frac{y+b}{x-a}\frac{\mathrm{d}x}{y} &= \int_O^P \frac{-3b(x+a)\mathrm{d}x}{y^3} \\
\frac{\partial}{\partial b}\int_O^P \frac{1}{2}\frac{y+b}{x-a}\frac{\mathrm{d}x}{y} &= \int_O^P \frac{(2(x^2+ax+a^2) - \frac{1}{2}t_2)\mathrm{d}x}{y^3} \\
\frac{\partial}{\partial t_2}\int_O^P \frac{1}{2}\frac{y+b}{x-a}\frac{\mathrm{d}x}{y} &= \int_O^P \frac{\frac{1}{4}b\ \mathrm{d}x}{y^3}
\end{aligned} \tag{12.18}$$

For instance, we explain the first equality which has also the contribution from the domain of integration. We skip the integral sign which is from $\delta_0(\varepsilon)$ to $P + \delta_0(\varepsilon) = \delta_0(1-\varepsilon)$ and the exact terms $\mathrm{d}f$ means $f(P + \delta_0(\varepsilon)) - f(\delta_0(\varepsilon))$. At the end of computation we see that the limit under $\varepsilon \to 0$ exists.

$$
\begin{aligned}
\frac{\partial}{\partial a}\int \frac{y+b}{x-a}\frac{\mathrm{d}x}{y} &= \frac{\partial}{\partial a}\int \left(\frac{y+b}{x-a}\frac{\mathrm{d}x}{y}-\frac{\mathrm{d}x}{x}\right)+\frac{1}{a} \\
&= \mathrm{d}(\frac{y+b}{x-a}\frac{1}{y}-\frac{1}{x})+\frac{\partial}{\partial a}\left(\frac{1}{x-a}\frac{y+b}{y}\right)\mathrm{d}x+\frac{1}{a} \\
&= \mathrm{d}(\frac{y+b}{x-a}\frac{1}{y})+\frac{1}{(x-a)^2}(1+\frac{b}{y})\mathrm{d}x+\frac{1}{x-a}\frac{-b}{2}\frac{\frac{\partial p}{\partial a}}{y^3}\mathrm{d}x \\
&= \frac{b}{x-a}\mathrm{d}(\frac{1}{y})+\frac{1}{x-a}\frac{-b}{2}\frac{\frac{\partial p}{\partial a}}{y^3}\mathrm{d}x \\
&= \frac{-1}{2}\frac{b}{x-a}\left(\frac{\frac{\partial p}{\partial x}+\frac{\partial p}{\partial a}}{y^3}\right)\mathrm{d}x
\end{aligned}
$$

Knowing that $\frac{\partial p}{\partial x}+\frac{\partial p}{\partial a}=12(x-a)(x+a)$ we get the result. From the second equality on, we have to add the evaluation of $\frac{y+b}{x-a}\frac{1}{y}-\frac{1}{x}$ at $\delta_0(\varepsilon)$. Since under $\varepsilon\to 0$ it is zero, we have not written this term. The evaluation of $\frac{y+b}{x-a}\frac{1}{y}-\frac{1}{x}$ at O is zero, that is why we have written differential of this function instead of its evaluation at $P+\delta_0(\varepsilon)$. The integration of $-\mathrm{d}(\frac{1}{x})$ over the path δ_0 cancels with $\frac{1}{a}$. For derivations with respect to $s:=t_2,b$ we have $\frac{\partial \ln a}{\partial s}=0$, and in these cases there are no exact terms.

Exercise 12.2 Prove the the last two equalities in (12.18).

In the next all the equalities are written restricted to the elliptic curve. We have

$$
\mathrm{d}(x^a y)=\left((4a+6)x^{a+2}-(a+\frac{1}{2})t_2x^a-at_3x^{a-1}\right)\frac{\mathrm{d}x}{y}.
$$

and write

$$
\begin{aligned}
\frac{x^2\mathrm{d}x}{y} &= \frac{1}{12}t_2\frac{\mathrm{d}x}{y}+\mathrm{d}(\frac{y}{6}), \\
\frac{x^3\mathrm{d}x}{y} &= \frac{1}{10}t_3\frac{\mathrm{d}x}{y}+\frac{3}{20}t_2\frac{x\mathrm{d}x}{y}+\mathrm{d}(\frac{xy}{10}) \\
\frac{x^4\mathrm{d}x}{y} &= \frac{5}{336}t_2^2\frac{\mathrm{d}x}{y}+\frac{1}{7}t_3\frac{x\mathrm{d}x}{y}+\mathrm{d}(\frac{1}{14}x^2y+\frac{5}{168}t_2y) \\
\frac{x^5\mathrm{d}x}{y} &= \frac{1}{30}t_2t_3\frac{\mathrm{d}x}{y}+\frac{7}{240}t_2^2\frac{x\mathrm{d}x}{y}+\mathrm{d}(\frac{1}{18}x^3y+\frac{7}{360}t_2xy+\frac{1}{36}t_3y).
\end{aligned}
$$

This is formulated in Proposition 9.2 and Exercise 9.3. Moreover, for a polynomial $A\in\mathbb{C}[x]$ we have

$$
\begin{aligned}
\frac{A\mathrm{d}x}{y^3} &= \frac{1}{\Delta}\frac{A(-p'a_1+pa_2)\mathrm{d}x}{py} = \frac{1}{\Delta}\left(a_2A\frac{\mathrm{d}x}{y} - \frac{Aa_1}{y}\frac{\mathrm{d}p}{p}\right) = \frac{1}{\Delta}(a_2A\frac{\mathrm{d}x}{y} + 2Aa_1\mathrm{d}(\frac{1}{y})) \\
&= \frac{1}{\Delta}\left(a_2A - 2\frac{\partial}{\partial x}(Aa_1)\right)\frac{\mathrm{d}x}{y} + \frac{1}{\Delta}\mathrm{d}\left(\frac{2Aa_1}{y}\right) \\
&= \frac{1}{\Delta}\left(A_1\frac{\mathrm{d}x}{y} + A_2\frac{x\mathrm{d}x}{y}\right) + \frac{1}{\Delta}\mathrm{d}(yB) + \frac{1}{\Delta}\mathrm{d}(\frac{2Aa_1}{y}) \\
&= \frac{1}{\Delta}\left(A_1\frac{\mathrm{d}x}{y} + A_2\left(\frac{x\mathrm{d}x}{y} - \mathrm{d}(\frac{y}{2x})\right)\right) + \frac{1}{\Delta}\left(\mathrm{d}\left(\frac{2xpB+4xAa_1+A_2p}{2yx}\right)\right)
\end{aligned}
$$

This is formulated in Proposition 3.7. As it is expected, the polynomial

$$C := 2xpB + 4xAa_1 + A_2p$$

for our three examples of A are of degree ≤ 2 and so $\frac{C}{2yx}$ evaluated at O is 0. We summarize our computations as follows:

$$A = -3b(x+a),\ A_1 = \frac{-1}{2}(3t_2^2b + 54t_2a^2b - 216a^4b + 54ab^3),\ A_2 = (45t_2ab - 108a^3b + 27b^3)$$

$$
\begin{aligned}
C \ =\ & -(6t_2^2b + 108t_2a^2b - 432a^4b + 108ab^3)\cdot x^2 - (15t_2^2ab - 36t_2a^3b + 9t_2b^3)\cdot x \\
& +(45t_2^2a^2b - 288t_2a^4b + 72t_2ab^3 + 432a^6b - 216a^3b^3 + 27b^5)
\end{aligned}
$$

$$A = \frac{1}{2}(4(x^2+ax+a^2)-t_2),\ A_1 = -(2t_2^2a - 30t_2a^3 + 3t_2b^2 + 72a^5 - 18a^2b^2),\ A_2 := (2t_2^2 - 30t_2a^2 + 72a^4 - 18ab^2)$$

$$
\begin{aligned}
C \ =\ & -(8t_2^2a - 120t_2a^3 + 12t_2b^2 + 288a^5 - 72a^2b^2)\cdot x^2 - (8t_2^2a^2 - 120t_2a^4 + 30t_2ab^2 + 288a^6 - 144a^3b^2 + 18b^4)\cdot x \\
& +(2t_2^3a - 38t_2^2a^3 + 2t_2^2b^2 + 192t_2a^5 - 48t_2a^2b^2 - 288a^7 + 144a^4b^2 - 18ab^4)
\end{aligned}
$$

$$A = \frac{b}{4},\ A_1 = \frac{9t_2ab - 36a^3b + 9b^3}{4},\ A_2 := -\frac{3t_2b}{2}$$

$$C = (9t_2ab - 36a^3b + 9b^3)\cdot x^2 + \frac{t_2^2b}{2}\cdot x - \frac{3t_2^2ab - 12t_2a^3b + 3t_2b^3}{2}$$

12.10 Gauss-Manin connection in relative cohomology

In order to compute the Gauss-Manin connection in relative cohomology we first recover the exact terms produced in the computation of the usual Gauss-Manin connection in Proposition 3.4.

Proposition 12.9 *We have*

$$
\begin{pmatrix} d\left(\int_O^P \frac{dx}{y}\right) \\ d\left(\int_O^P \left(\frac{xdx}{y} - d(\frac{y}{2x})\right)\right) \end{pmatrix} = \begin{pmatrix} -\frac{1}{12}\frac{d\Delta}{\Delta}, & \frac{3}{2}\frac{\alpha}{\Delta} \\ -\frac{1}{8}t_2\frac{\alpha}{\Delta}, & \frac{1}{12}\frac{d\Delta}{\Delta} \end{pmatrix} \begin{pmatrix} \int_O^P \frac{dx}{y} \\ \int_O^P \left(\frac{xdx}{y} - d(\frac{y}{2x})\right) \end{pmatrix} - \begin{pmatrix} B_{21} \\ B_{31} \end{pmatrix}, \tag{12.19}
$$

where

$$B_{21} = \frac{1}{\Delta}(g_1 dt_2 + g_2 dt_3) - \frac{da}{b}, \tag{12.20}$$

$$B_{31} = \frac{1}{\Delta}(g_3 dt_2 + g_1 dt_3) - \frac{ada}{b} + d\left(\frac{b}{2a}\right)$$

$$g_1 = \frac{-2a^2t_2^2 + 3at_2t_3 + 9t_3^2}{4ab}, \tag{12.21}$$

$$g_2 = \frac{18a^2t_3 - at_2^2 - 3t_2t_3}{2ab},$$

$$g_3 = \frac{6a^2t_2t_3 + (18t_3^2 - t_2^3)a - t_2^2t_3}{8ab}.$$

Proof. First, we note the following equalities:

$$\frac{\partial}{\partial a}\int_O^P \frac{x^i \mathrm{d}x}{y} = \int_O^P \mathrm{d}(\frac{x^i}{y}) + \int_O^P \frac{\frac{-1}{2}\frac{\partial p}{\partial a}x^i}{y^3}\mathrm{d}x$$

$$\frac{\partial}{\partial s}\int_O^P \frac{x^i \mathrm{d}x}{y} = \int_O^P \frac{\frac{-1}{2}\frac{\partial p}{\partial s}x^i}{y^3}\mathrm{d}x, \quad s = t_2, b.$$

We only consider $i = 0, 1$ for which $\frac{x^i}{y}(O) = 0$. Moreover,

$$\frac{\partial}{\partial a}\int_O^P \left(\frac{x\mathrm{d}x}{y} - \mathrm{d}(\frac{y}{2x})\right) = \int_O^P \left(\mathrm{d}(\frac{x}{y}) - \mathrm{d}(\frac{\frac{\partial y}{\partial x}x - y}{2x^2})\right) + \int_O^P \left(\frac{\frac{-1}{2}\frac{\partial p}{\partial a}x}{y^3}\mathrm{d}x - \mathrm{d}(\frac{\frac{1}{2}\frac{\partial p}{\partial a}}{2xy})\right)$$

$$= \int_O^P \mathrm{d}\left(\frac{D}{2x^2y}\right) + \int_O^P \left(\frac{\frac{-1}{2}\frac{\partial p}{\partial a}x}{y^3}\mathrm{d}x - \mathrm{d}(\frac{\frac{1}{2}\frac{\partial p}{\partial a}}{2xy})\right)$$

where

$$D := 2x^3 - \frac{1}{2}\frac{\partial p}{\partial x}x + p = -\frac{t_2}{2}x + t_2 a - 4a^3 + b^2$$

Now we use Proposition 3.7 in order to finish the computation.

12.11 Gluing Gauss-Manin connections

The Gauss-Manin connection matrix in the basis (12.14) is given by

$$\nabla \begin{bmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ B_{10} & B_{11} & B_{12} & 0 \\ B_{20} & B_{21} & B_{22} & 0 \\ B_{30} & B_{31} & B_{32} & 0 \end{bmatrix} \begin{bmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

where the submatrix $\{0,1,2\} \times \{0,1,2\}$ corresponds to the Gauss-Manin in $\omega_0, \omega_1, \omega_2 \in H^1_{\mathrm{dR}}(X,Y)$ and the submatrix $\{1,2,3\} \times \{1,2,3\}$ corresponds to Gauss-Manin in

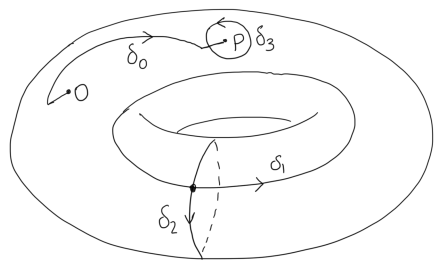


Fig. 12.4 Four paths

$\omega_1, \omega_2, \omega_3 \in H^1_{\mathrm{dR}}(X-Y)$. Recall the definition of α_i's in (12.24). The Gauss-Manin connection matrix in this basis is given by A, where the first and last column of A are zero and its last row is given by:

$$\begin{aligned}
\mathsf{A}_{30} &= (-B_{30} + c\mathsf{A}_{10} - (c - \frac{b}{2a})B_{32} + \mathrm{d}(d)), \\
\mathsf{A}_{31} &= B_{31} - B_{32}t_1 + \mathrm{d}c + c\mathsf{A}_{11}, \\
\mathsf{A}_{32} &= (B_{32} + c\mathsf{A}_{12})\alpha_2.
\end{aligned}$$

This follows from $\nabla(\omega_3) = -B_{30}\omega_0 + B_{31}\omega_1 + B_{32}\omega_2$ and

$$\begin{aligned}
\nabla(\alpha_3) &= \nabla(\omega_3 + c\alpha_1 + d \cdot \alpha_0) \\
&= (-B_{30} - (c - \frac{b}{2a})B_{32} + c\mathsf{A}_{10} + \mathrm{d}(d))\alpha_0 + (B_{31} - B_{32}t_1 + \mathrm{d}c + c\mathsf{A}_{11})\alpha_1 \\
&\quad + (B_{32} + c\mathsf{A}_{12})\alpha_2
\end{aligned}$$

Note that $\alpha_2 = (c - \frac{b}{2a})\alpha_0 + t_1 \frac{\mathrm{d}x}{y} + \omega_2$. We note that A is integrable, that is,

$$d\mathsf{A} = \mathsf{A} \wedge \mathsf{A}.$$

```
//--the procedure for writing P(x)dx/y in terms of dx/y and xdx/y with exact terms--
proc linearw(poly A, number t(2), number t(3))
   {
   int i=deg(A) div deg(var(1)); poly B=A; poly exact;  poly LM; list Bl;
   while (i>1)
        {LM=lasthomo(B);
            B=B-LM+leadcoef(LM)*((i-2+1/2)/number(4*(i-2)+6)*t(2)*var(1)^(i-2));
            if (i>2){B=B+leadcoef(LM)*(i-2)/number(4*(i-2)+6)*t(3)*var(1)^(i-2-1);}
            exact=exact+leadcoef(LM)*(1/number(4*(i-2)+6)*var(1)^(i-2));
            i=deg(B) div deg(var(1));
        }
    Bl=subst(B,var(1),0), subst(B,var(1),1)-subst(B,var(1),0);
    return(list(Bl, exact));
    }
//--Simplifying the integrand after derivation--
 proc hum(poly A, poly a1, poly a2)
    {
    A=A*a2-2*diff(A*a1,x);
    return(A);
    }
//---Computing the 4*4 Gauss-Manin connection matrix--------------------------
LIB "foliation.lib";
```

```
ring r=(0,t(1),t(2),a,b,c,d), (x, dt(1), dt(2), da,db,dc, dd),dp;
number t(3)=4*a^ 3-t(2)*a-b^2; poly P=4*x^3-t(2)*x-t(3);
poly a1=-36*x^4+(15*t(2))*x^2-t(2)^2; poly a2=(-108)*x^3+(27*t(2))*x+(-27*t(3));
number Delta=(-t(2)^3+27*t(3)^2); -diff(P,x)*a1+P*a2-Delta;
matrix B[4][4]; matrix A[4][4]; //Gauss-Manin connection matrices--
//---These are obtained after derivation of (y+b)/(2y(x-a))--
poly Aa=-6*b*(x+a)*(1/2);
poly Ab=(4*(x^2+a*x+a^2)-t(2))*(1/2);
poly At2=(b/2)*(1/2);
list la=linearw(hum(Aa,a1,a2), t(2),t(3));
list lb=linearw(hum(Ab,a1,a2), t(2),t(3));
list lt2=linearw(hum(At2,a1,a2), t(2),t(3));
poly Ca=2*var(1)*(P*la[2]+2*Aa*a1)+P*la[1][2];
poly Cb=2*var(1)*(P*lb[2]+2*Ab*a1)+P*lb[1][2];
poly Ct2=2*var(1)*(P*lt2[2]+2*At2*a1)+P*lt2[1][2];
B[4,1]=subst(Ca,var(1),a)*da+subst(Cb,var(1),a)*db+subst(Ct2,var(1),a)*dt(2);
B[4,2]=la[1][1]*da+lb[1][1]*db+lt2[1][1]*dt(2);
B[4,3]=la[1][2]*da+lb[1][2]*db+lt2[1][2]*dt(2);
B[4,1]=(1/Delta)*(1/(2*a*b))*B[4,1];
B[4,2]=(1/Delta)*B[4,2]; B[4,3]=(1/Delta)*B[4,3];
//------B[4,1]=B[4,1]+db/b;
B[4,1]=-B[4,1];
//--The Gauss-Manin connection of the relative cohomology--
 Aa=(-1/2)*diffpar(P,a);
 Ab=(-1/2)*diffpar(P,b);
 At2=(-1/2)*diffpar(P,t(2));
 la=linearw(hum(Aa,a1,a2), t(2),t(3));
 lb=linearw(hum(Ab,a1,a2), t(2),t(3));
 lt2=linearw(hum(At2,a1,a2), t(2),t(3));
 Ca=Delta*2*var(1)+  2*var(1)*(P*la[2]+2*Aa*a1)+P*la[1][2];
 Cb=                 2*var(1)*(P*lb[2]+2*Ab*a1)+P*lb[1][2];
 Ct2=                2*var(1)*(P*lt2[2]+2*At2*a1)+P*lt2[1][2];
 B[2,1]=subst(Ca,var(1),a)*da+subst(Cb,var(1),a)*db+subst(Ct2,var(1),a)*dt(2);
 B[2,2]=la[1][1]*da+lb[1][1]*db+lt2[1][1]*dt(2);
 B[2,3]=la[1][2]*da+lb[1][2]*db+lt2[1][2]*dt(2);
 B[2,1]=(1/Delta)*(1/(2*a*b))*B[2,1]; B[2,2]=(1/Delta)*B[2,2]; B[2,3]=(1/Delta)*B[2,3];
 B[2,1]=-B[2,1];
 //---------
 Aa=(-1/2)*diffpar(P,a)*var(1);
 Ab=(-1/2)*diffpar(P,b)*var(1);
 At2=(-1/2)*diffpar(P,t(2))*var(1);
 la=linearw(hum(Aa,a1,a2), t(2),t(3));
 lb=linearw(hum(Ab,a1,a2), t(2),t(3));
 lt2=linearw(hum(At2,a1,a2), t(2),t(3));
 poly D=2*var(1)^3-(1/2)*diff(P,var(1))*var(1)+P;
 Ca= -(1/2)*diffpar(P,a)*Delta+   2*var(1)*(P*la[2]+2*Aa*a1)+P*la[1][2];
 Cb= -(1/2)*diffpar(P,b)*Delta+   2*var(1)*(P*lb[2]+2*Ab*a1)+P*lb[1][2];
 Ct2=-(1/2)*diffpar(P,t(2))*Delta+2*var(1)*(P*lt2[2]+2*At2*a1)+P*lt2[1][2];
 B[3,1]=(subst(Ca,var(1),a))*da+subst(Cb,var(1),a)*db+subst(Ct2,var(1),a)*dt(2);
 B[3,1]=(1/(Delta*2*a*b))*B[3,1]+(subst(D,var(1),a)/(2*a^2*b))*da;
 B[3,1]=-B[3,1];
 B[3,2]=la[1][1]*da+lb[1][1]*db+lt2[1][1]*dt(2);
 B[3,3]=la[1][2]*da+lb[1][2]*db+lt2[1][2]*dt(2);
 B[3,2]=(1/Delta)*B[3,2]; B[3,3]=(1/Delta)*B[3,3];
 //--Computing the Gauss-Manin connection matrix B--
matrix S[4][4]=1,0,0,0,0,1,0,0,c-b/(2*a),t(1),1,0,d,c,0,1; matrix Si=inverse(S);
matrix A[4][4]=diffpar(S,d)*Si*dd+ diffpar(S,t(1))*Si*dt(1)+ diffpar(S,c)*Si*dc+
diffpar(S,a)*Si*da+diffpar(S,b)*Si*db+S*B*Si;
//----Checking the integrability of A--
list ll=diff(A,da), diff(A,db), diff(A,dc), diff(A,dd), diff(A,dt(1)), diff(A,dt(2));
list lv=a,b,c,d,t(1),t(2); int i; int j;
for (i=1; i<=6; i=i+1)
    {for (j=i+1; j<=6; j=j+1)
         {
         i,j;
         print(diffpar(ll[j],lv[i])-diffpar(ll[i],lv[j])-ll[i]*ll[j]+ll[j]*ll[i]);
         }
    }
//--Computing modular vector field--
matrix Cz[4][4]=0,0,0,0,-1,0,0,0,0,0,0,0,0,0,1,0; print(Cz);
matrix Ct[4][4]=0,0,0,0,0,0,-1,0,0,0,0,0,0,0,0,0; print(Ct);
ideal I=A-Cz; I=std(I); list lz;
lz=reduce(da,I),reduce(db,I),reduce(dc,I),reduce(dd,I),reduce(dt(1),I),reduce(dt(2),I);lz;
ideal I=A-Ct; I=std(I); list lt;
lt=reduce(da,I),reduce(db,I),reduce(dc,I),reduce(dd,I),reduce(dt(1),I),reduce(dt(2),I);lt;
//----Check local embedding of the period map--
matrix lobi[3][3]=-2*a,-3*b, -4*t(2),lz[1],lz[2],lz[6],lt[1],lt[2],lt[6];
det(lobi)-2/3*Delta;
//----Other vector fields four vector fields--
matrix C1[4][4]=0,0,0,0,0,1,0,0,0,0,-1,0,0,0,0,0; print(C1); ideal I=A-C1;   I=std(I);
reduce(da, I),reduce(db, I), reduce(dc, I), reduce(dd, I), reduce(dt(1), I), reduce(dt(2), I);
matrix C2[4][4]=0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0; print(C2); ideal I=A-C2;   I=std(I);
reduce(da, I),reduce(db, I), reduce(dc, I), reduce(dd, I), reduce(dt(1), I), reduce(dt(2), I);
matrix C3[4][4]=0,0,0,0,0,0,0,0,1,0,0,0,0,1,0,0; print(C3); ideal I=A-C3;   I=std(I);
reduce(da, I),reduce(db, I), reduce(dc, I), reduce(dd, I), reduce(dt(1), I), reduce(dt(2), I);
matrix C4[4][4]=0,0,0,0,0,0,0,0,0,0,0,1,0,0,0; print(C4); ideal I=A-C4;   I=std(I);
reduce(da, I),reduce(db, I), reduce(dc, I), reduce(dd, I), reduce(dt(1), I), reduce(dt(2), I);
//-----------relation between periods---------------------
ring r=(0,s),(x(1..4),z(1..4),inf),dp;
```

```
matrix Ih[4][4]=0,0,0,-1,0,0,-1,0,0,1,0,0,1,0,0,0; print(Ih);
matrix P[4][4]=-1,z(1),z(2),inf,0,x(1),x(2),z(3),0,x(3),x(4),z(4),0,0,0,-1; print(P);
matrix Ic[4][4]=0,0,0,1,0,0,-1,0,0,1,0,0,-1,0,0,0; print(Ic);
Ic-transpose(P)*transpose(inverse(Ih))*P;
//-----modular vector fields in the fake period domain----
LIB "foliation.lib";    ring r=(0,x(1..4),z(1..2), W), (dx(1..4),dz(1..2), dW)   ,dp;
matrix P[4][4]=-1,z(1),z(2),W,0,x(1),x(2),x(2)*z(1)-x(1)*z(2),0,x(3),x(4),x(4)*z(1)-x(3)*z(2),0,0,0,-1; print(P);
matrix dP[4][4]=0,dz(1),dz(2),dW,0,dx(1),dx(2),dx(2)*z(1)-dx(1)*z(2)+x(2)*dz(1)-x(1)*dz(2),
0,dx(3),dx(4), dx(4)*z(1)-dx(3)*z(2)+x(4)*dz(1)-x(3)*dz(2),0,0,0,0; print(dP);
matrix A=transpose(inverse(P)*dP);
A=substpar(A, x(4), (1+x(2)*x(3))/x(1));
A=subst(A, dx(4), (x(1)*x(2)*dx(3)+x(1)*x(3)*dx(2)-(1+x(2)*x(3))*dx(1))/x(1)^2);
matrix Cz[4][4]=0,0,0,0,-1,0,0,0,0,0,0,0,0,0,1,0; print(Cz);
matrix Ct[4][4]=0,0,0,0,0,0,-1,0,0,0,0,0,0,0,0,0; print(Ct);
ideal I=A-Cz;   I=std(I);
list lz=reduce(dx(1), I),reduce(dx(2), I), reduce(dx(3), I), reduce(dz(1), I), reduce(dz(2), I), reduce(dW, I); lz;
ideal I=A-Ct; I=std(I);
list lt=reduce(dx(1), I),reduce(dx(2), I), reduce(dx(3), I), reduce(dz(1), I), reduce(dz(2), I), reduce(dW, I); lt;
matrix C1[4][4]=0,0,0,0,0,1,0,0,0,0,-1,0,0,0,0,0; print(C1); ideal I=A-C1;   I=std(I);
reduce(dx(1), I),reduce(dx(2), I), reduce(dx(3), I), reduce(dz(1), I), reduce(dz(2), I), reduce(dW, I);
matrix C2[4][4]=0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0; print(C2); ideal I=A-C2;   I=std(I);
reduce(dx(1), I),reduce(dx(2), I), reduce(dx(3), I), reduce(dz(1), I), reduce(dz(2), I), reduce(dW, I);
matrix C3[4][4]=0,0,0,0,0,0,0,0,1,0,0,0,0,1,0,0; print(C3); ideal I=A-C3;   I=std(I);
reduce(dx(1), I),reduce(dx(2), I), reduce(dx(3), I), reduce(dz(1), I), reduce(dz(2), I), reduce(dW, I);
matrix C4[4][4]=0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0; print(C4); ideal I=A-C4;   I=std(I);
reduce(dx(1), I),reduce(dx(2), I), reduce(dx(3), I), reduce(dz(1), I), reduce(dz(2), I), reduce(dW, I);
//--------Fake generalized period domain is invariant under Jacobi group action-----------
LIB "foliation.lib";    ring r=(0,x(1..4),z(1..2), W,a,b,c,d,mu, la,ka), x   ,dp;
matrix P[4][4]=-1,z(1),z(2),W,0,x(1),x(2),x(2)*z(1)-x(1)*z(2),0,x(3),x(4),x(4)*z(1)-x(3)*z(2),0,0,0,-1; print(P);
number lap=la*d-mu*c; number mup=-la*b+mu*a;
matrix A[4][4]=1, la, mu, ka, 0,a,b,mup,0,c,d,-lap,0,0,0,1;  print(A);
matrix B=A*P;
poly p1=B[2,3]*B[1,2]-B[2,2]*B[1,3]-B[2,4]; subst(p1, x(4),(1+x(2)*x(3))/x(1));
poly p2=B[3,3]*B[1,2]-B[3,2]*B[1,3]-B[3,4]; subst(p2, x(4),(1+x(2)*x(3))/x(1));
//---------Reducing to tau locus------
LIB "foliation.lib";    ring r=(0,x(1..4),z(1..2), W), x   ,dp;
matrix P[4][4]=-1,z(1),z(2),W,0,x(1),x(2),x(2)*z(1)-x(1)*z(2),0,x(3),x(4),x(4)*z(1)-x(3)*z(2),0,0,0,-1; print(P);
matrix g[4][4]=1,0,z(2)*x(3)-z(1)*x(4),W+z(1)*(z(2)-z(1)*x(3)^(-1)*x(4)),0,1/x(3),-x(4),z(2)-z(1)*x(3)^(-1)*x(4),0,0,x(3),0,0,0,0,1;
print(g); print(substpar(P*g,x(4),(1+x(2)*x(3))/x(1) ));
```

12.12 Computation of a period matrix

In this section we explain the computation of period matrix using Weierstrass uniformization of elliptic curves. Let us consider the family of elliptic curves $y^2 = 4x^3 - t_2x - t_3$ with $\alpha = \frac{\mathrm{d}x}{y}$, $\omega = (x+t_1)\frac{\mathrm{d}x}{y}$ and

$$(t_1,t_2,t_3) = (a_1E_2(\tau), a_2E_4(\tau), a_3E_6(\tau)), \ \ (a_1,a_2,a_3) = (\frac{2\pi i}{12}, 12(\frac{2\pi i}{12})^2, 8(\frac{2\pi i}{12})^3). \tag{12.22}$$

By Weierstrass uniformization theorem if we use the biholomorphism

$$\mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z}) \cong E, \ z \mapsto (\sqrt{-2\pi i}^{-2}\wp(z), \sqrt{-2\pi i}^{-3}\wp'(z))$$

and define δ_1, δ_2 to be cycles in E corresponding to vectors $\tau, 1 \in \mathbb{C}$ then

$$\frac{1}{\sqrt{-2\pi i}}\begin{bmatrix} \int_{\delta_1} \frac{dx}{y} & \int_{\delta_1}(x+t_1)\frac{dx}{y} \\ \int_{\delta_2} \frac{dx}{y} & \int_{\delta_2}(x+t_1)\frac{dx}{y} \end{bmatrix} = \begin{bmatrix} \tau & -1 \\ 1 & 0 \end{bmatrix}. \tag{12.23}$$

see Section 9.13. We also define δ_0 (resp. δ_3) to be the image of the straight paths from 0 to z_0 in E (resp. an anticlockwise oriented closed path around P). In order to be able to integrate differential forms α_i over δ_i's we have to modify them according

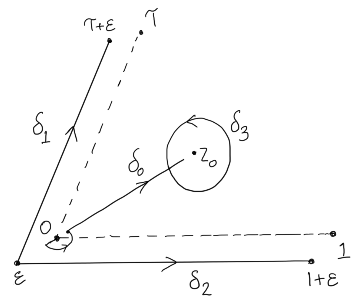


Fig. 12.5 Four paths

to Figure 12.5. In this figure ε is small complex number near to 0 with negative real and imaginary parts.

Proposition 12.10 *Let us consider the differential forms*

$$\begin{aligned}
\alpha_0 &= \mathrm{d}\left(\frac{x-a}{x}\right),\ \alpha_1 := \frac{\mathrm{d}x}{y}, \\
\alpha_2 &= \left(c-\frac{b}{2a}\right)\alpha_0 + t_1\frac{\mathrm{d}x}{y} + \frac{x\mathrm{d}x}{y} - \mathrm{d}\left(\frac{y}{2x}\right), \\
\alpha_3 &= \frac{1}{2}\frac{y+b}{x-a}\frac{\mathrm{d}x}{y} + c\alpha_1 + d\cdot\alpha_0
\end{aligned} \tag{12.24}$$

where

$$\begin{gathered}
(a,b,c,d,t_1,t_2,t_3) = \\
\left(k^2\wp, k^3\wp', -k\ln(F)', -2\ln(kF), -\frac{k^{-2}}{12}E_2, \frac{k^{-4}}{12}E_4, \frac{-k^{-6}}{6^3}E_6\right),
\end{gathered} \tag{12.25}$$

where $k = (-2\pi i)^{-\frac{1}{2}}$ *and* $'$ *is derivation with respect to* z*. The period matrix is of the format:*

$$\begin{bmatrix}
\int_{\delta_0}\alpha_0 & k\int_{\delta_0}\alpha_1 & k\int_{\delta_0}\alpha_2 & k^2\int_{\delta_0}\alpha_3 \\
\int_{\delta_1}\alpha_0 & k\int_{\delta_1}\alpha_1 & k\int_{\delta_1}\alpha_2 & k^2\int_{\delta_1}\alpha_3 \\
\int_{\delta_2}\alpha_0 & k\int_{\delta_2}\alpha_1 & k\int_{\delta_2}\alpha_2 & k^2\int_{\delta_2}\alpha_3 \\
\int_{\delta_3}\alpha_0 & k\int_{\delta_3}\alpha_1 & k\int_{\delta_3}\alpha_2 & k^2\int_{\delta_3}\alpha_3
\end{bmatrix}
=
\begin{bmatrix}
-1 & z_0 & 0 & \mathbb{Z} \\
0 & \tau & -1 & z_0-1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & -1
\end{bmatrix}.$$

Proof. The computation of the first column and the last row is trivial. The submatrix $\{1,2\}\times\{1,2\}$ is already mentioned in (12.23) and $k\int_{\delta_0}\alpha_1 = z_0$ is also easy. The proof of $\int_{\delta_0}\alpha_2 = 0$ is:

$$\begin{aligned}
(-2\pi i)^{\frac{1}{2}}\int_{\delta_0}\left(\frac{x\mathrm{d}x}{y}-\mathrm{d}(\frac{y}{2x})\right) &= \int_0^{z_0}\wp(z)\mathrm{d}z-\mathrm{d}(\frac{\wp'(z)}{2\wp(z)}) \\
&= -\int_0^{z_0}\mathrm{d}(\zeta(z)+\frac{\wp'(z)}{2\wp(z)}) = -\zeta(z_0)-\frac{\wp'(z_0)}{2\wp(z_0)} \\
&= -J_1-(-2\pi i)^{\frac{1}{2}}t_1\int_{\delta_0}\frac{\mathrm{d}x}{y}-(-2\pi i)^{\frac{1}{2}}\frac{b}{2a} \\
&= (-2\pi i)^{\frac{1}{2}}\left((c-\frac{b}{2a})-t_1\int_{\delta_0}\frac{\mathrm{d}x}{y}\right)
\end{aligned}$$

We have used $J_1=\frac{\partial}{\partial z}\ln F$ and

$$J_1(z_0)=\zeta(z_0)+\frac{(2\pi i)^2}{12}E_2 z_0, \tag{12.26}$$

see Exercise 2.40. Therefore, $J_1=\zeta-(-2\pi i)^{\frac{1}{2}}t_1\int_{\delta_0}\frac{\mathrm{d}x}{y}$.

Next, we prove the equalities corresponding to $(1,4),(2,4)$ and $(3,4)$ entries. we are going to use the equality

$$\frac{1}{2}\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}=\zeta(u+v)-\zeta(u)-\zeta(v).$$

see Exercise 2.12. Using (12.26) we have

$$\begin{aligned}
\frac{1}{2}\frac{\wp'(z)-\wp'(-z_0)}{\wp(z)-\wp(-z_0)} &= (J_1(z-z_0)-J_1(z)+J_1(z_0)) \\
&= \frac{\partial}{\partial z}\left(\ln\left(\frac{F(z-z_0)}{F(z)}\right)+zJ_1(z_0))\right)
\end{aligned}$$

The proof of $\int_{\delta_0}\alpha_3\in 2\pi i\mathbb{Z}$ is as follows:

$$\begin{aligned}
\int_{\delta_0}\omega_3-\ln(a) &:= \int_{\delta_0}(\omega_3-\frac{\mathrm{d}x}{x}) \\
&= \int_{\varepsilon}^{P+\varepsilon}(\omega_3-\frac{\mathrm{d}x}{x}) \\
&= \left(\ln\frac{F(z-z_0)}{F(z)\wp(z)}+J_1(z_0)z\right)\Big|_{\varepsilon}^{z_0+\varepsilon} \\
&=_* \ln\frac{F(\varepsilon)F(\varepsilon)\wp(\varepsilon)}{F(z_0+\varepsilon)\wp(z_0+\varepsilon)F(\varepsilon-z_0)}+J_1(z_0)z_0 \\
&=_* \ln\frac{(2\pi)^2}{F(z_0)\wp(z_0)F(-z_0)}+J_1(z_0)z_0 \\
&=_* \int_{\delta_0}\left(-d\cdot\mathrm{d}\left(\frac{x-a}{x}\right)-c\frac{\mathrm{d}x}{y}\right)-\ln(a).
\end{aligned}$$

where $=_*$ means modulo $2\pi i\mathbb{Z}$. Here, we take $\lim_{\varepsilon\to 0}$. We have used the fact that $\wp(z)=z^{-2}+\cdots$ and $F(z)=(-2\pi)z+\cdots$ and $F(-z)=-F(z)$. For the other equalities we will use

$$\begin{aligned} F(z+1,\tau) &= -F(z,\tau) \\ F(z+\tau,\tau) &= -e^{-\pi i(\tau+2z)}F(z,\tau) \end{aligned}$$

These implies for $g(z):=\frac{F(z-z_0)}{F(z)}$ we have

$$g(z+1)=g(z),\ g(z+\tau)=e^{2\pi i z_0}g(z).$$

We take ε near to zero, between the vectors -1 and $-\tau$. Moreover, we take $-z_0$ in the parallelogram formed by 1 and τ For the equalities below see Figure 12.6:

$$\begin{aligned} \int_{\delta_2}\alpha_3 &= \int_{\delta_2}\frac{1}{2}\frac{y+b}{x-a}\frac{\mathrm{d}x}{y}+c\frac{\mathrm{d}x}{y}=\int_{\varepsilon}^{1+\varepsilon} d\left(\ln\frac{F(z-z_0)}{F(z)}\right) \\ &= \int_{\varepsilon}^{1+\varepsilon}(J_1(z-z_0)-J_1(z))dz=\int_{\varepsilon-z_0}^{1+\varepsilon-z_0}J_1(z)dz-\int_{\varepsilon}^{1+\varepsilon}J_1(z)dz \\ &= \int_a^{a+1}J_1(z)dz-\int_{\varepsilon}^{1+\varepsilon}J_1(z)dz=-2\pi i\mathrm{Resi}(J_1(z),z=0) \\ &= -2\pi i \end{aligned} \tag{12.27}$$

We have used $J_1(z+1)=J_1(z)$. Next we compute:

$$\begin{aligned} \int_{\delta_1}\alpha_3 &= \int_{\delta_2}\frac{1}{2}\frac{y+b}{x-a}\frac{\mathrm{d}x}{y}+c\frac{\mathrm{d}x}{y}=\int_{\varepsilon}^{\tau+\varepsilon} d\left(\ln\frac{F(z-z_0)}{F(z)}\right) \\ &= \int_{\varepsilon}^{\tau+\varepsilon}(J_1(z-z_0)-J_1(z))dz=\int_{\varepsilon-z_0}^{\tau+\varepsilon-z_0}J_1(z)dz-\int_{\varepsilon}^{\tau+\varepsilon}J_1(z)dz \\ &= \int_a^{a+\tau}J_1(z)dz+2\pi i(\varepsilon-z_0-a)-\int_{\varepsilon}^{\tau+\varepsilon}J_1(z)dz \qquad (12.28) \\ &= 2\pi i(\varepsilon-z_0-a)-2\pi i(a-\varepsilon)+2\pi i\mathrm{Resi}(J_1(z),z=0) \\ &= 2\pi i(-z_0+1) \qquad (12.29) \end{aligned}$$

We have used $J_1(z+\tau)=J_1(\tau)-2\pi i$. The residue of J_1 at $z=0$ after the integration on the parallelograms in Figure 12.6.

Remark 12.3 We could also compute the integrals (12.27) and (12.29) by Mathematica, see the latex file of the present text. Let $\check{\theta}_i(z,q),\ i=1,2,3,4$ be the theta sereis in Mathematica. These are related to our notation by

$$\check{\theta}_1(\pi z,e^{2\pi i\tau})=-\theta_{\frac{1}{2},\frac{1}{2}},\ \check{\theta}_2(\pi z,e^{2\pi i\tau})=\theta_{\frac{1}{2},0},\ \check{\theta}_3(\pi z,e^{2\pi i\tau})=\theta_{0,0},\ \check{\theta}_4(\pi z,e^{2\pi i\tau})=\theta_{0,\frac{1}{2}}$$

Note that $F=\frac{\theta_{\frac{1}{2},\frac{1}{2}}}{\eta^3}$.

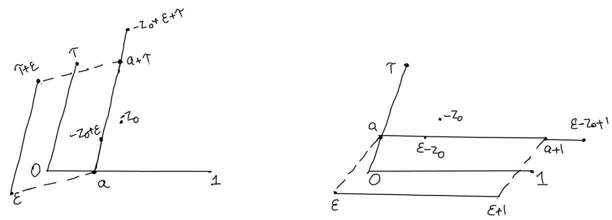


Fig. 12.6 Integration of α_3

Remark 12.4 The $2\pi in,\ n \in \mathbb{Z}$ in the $(4,1)$ entry of the period matrix can be absorbed in $d := -2\ln(kF)$ and so we can assume that this entry is zero provided that in Proposition 12.10 we say that there is a branch of ln such that the period matrix is of the desired format.

12.13 Quasi-Jacobi forms

Remark 12.5 In [Lib11] and [Obe18], both of the authors propose a definition of quasi-Jacobi forms. A quasi-Jacobi form is one of the follwoing equivalent definitions:

1. Any polynomial in F with coefficients in $\mathbb{C}[a,b,c,t_1,t_2]$ which is holomorphic at $z=0$.
2. Any $F^m P(a,b,c,t_1,t_2)$ hololorphic at $z=0$.

Note that if P is a homogeneous polynomial of degree m then $F^m P(a,b,c,t_1,t_2)$ is automatically a quasi-Jacobi form.

Chapter 13
Modular curves and differential equations

The applications of modular curves and modular functions to number theory are especially exciting: you use GL_2 to study GL_1, so to speak! There is clearly a lot more to come from that direction ... may be even a proof of the Riemann Hypothesis some day!, (J. P. Serre in an interview by [CTC01]).

13.1 Introduction

A classical model for modular curves is given as a curve in the affine variety $\mathbb{A}^2$ which is the zero set of the polynomial relation between $j(\tau)$ and $j(d\tau)$. This is highy singular and in order to have less singular models we need more functions and modular forms. In this chapters we are interested in those models which are solutions of ordinary differential equations, or in a more geometric language, they are leaves of holomorphic foliations. The main example of this returns back to [Pic89, pages 298-299] in the case of $X_1(d)$. We mainly follow a new treatment of this in [Mov22b] which is based on a moduli space interpretation of the ambient space. We also give models for $X_0(d)$ for which we first describe its three dimensional version embedded in the double copy $\mathsf{T}\times\mathsf{T}$ of the moduli T of enhance elliptic curves discussed in Chapter 9. We take a quotient of this six dimensional space and get a four dimensional weighted projective space containing a singular model of all modular curves $X_0(d)$ defined over $\mathbb{Q}$. All these curves are the only algebraic leaves of a vector field which is constructed from a double copy of the Ramanujan vector field. Recall the Dedekind ψ function:

$$\psi(d) := d\prod_{p|d}(1+\frac{1}{p})$$

where p runs through primes p dividing d. The first singular model of the modular curve $X_0(d)$ is given by the following:

Exercise 13.1 For any $d \in \mathbb{N}$, there is a polynomial $P_d(x,y) \in \mathbb{Q}[x,y]$ of degree $\psi(d)$ in both variables x,y and symmetric in x,y, that is $P_d(x,y) = P_d(y,x)$, such that

$$P_d(j(\tau), j(d\tau)) = 0,$$

where $j(\tau)$ is the j-function. Hint: We may try to deduce this statement from the assertion of Theorem 13.1 for $E_4.E_6$, as if $E_4(d\tau)$ and $E_6(d\tau)$ are algebraic over the field $\mathbb{C}(E_4(\tau), E_6(\tau))$ then any rational function of them is also algebraic over the same field. One has to then discuss the degree of algebraicity as it is announced in Exercise 13.1. We may also directly generalize Proposition 8.4 to weakly holomorphic modular forms. The treatment of the polynomial P_d is old, and for instance, we can find it in [Fri22, page 348] [Fri11, page 136].

Exercise 13.2 Show that the affine curve $P_d(x,y) = 0$ is not smooth. Can you characterize a singular point of this curve interms of the isogeny of the underlying elliptic curves.

13.2 A consequence of Hecke operators

Recall the following rescaling of Eisenstein series:

$$g_i(\tau) = a_i E_{2i}(q) := a_i \left(1 + b_i \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{2i-1}\right) q^n\right), \tag{13.1}$$

$$i = 1,2,3, \; q = e^{2\pi i \tau}, \; \mathrm{Im}(\tau) > 0$$

with

$$(b_1, b_2, b_3) = (-24, 240, -504), \;\; (a_1, a_2, a_3) = (2\pi i, (2\pi i)^2, (2\pi i)^3).$$

Theorem 13.1 For $i = 1,2,3$ and $d \in \mathbb{N}$, there is a unique homogeneous polynomial $I_{d,i}$ of degree $i \cdot \psi(d)$, where $\psi(d) := d \prod_p (1 + \frac{1}{p})$ is the Dedekind ψ function and p runs through primes p dividing d, in the weighted ring

$$\mathbb{Q}[t_i, s_1, s_2, s_3], \; \mathrm{weight}(t_i) = i, \; \mathrm{weight}(s_j) = j, \; j = 1,2,3 \tag{13.2}$$

and monic in the variable t_i such that $t_i(\tau) := d^{2i} \cdot g_i(d \cdot \tau), s_1(\tau) := g_1(\tau), s_2(\tau) := g_2(\tau), s_3(\tau) := g_3(\tau)$ satisfy the algebraic relation:

$$I_{d,i}(t_i, s_1, s_2, s_3) = 0.$$

Moreover, for $i = 2,3$ the polynomial $I_{d,i}$ does not depend on s_1.

Proof. For the statement for g_2 and g_3 we only need the theory of Hecke operators for modular forms. We apply Proposition 8.4 to $f(\tau) := g_k(\tau),\ k = 2,3$ and find modular forms $g_{\psi(N)-i}(\tau)$ of weight $2k\cdot(\psi(N)-i)$ such that $\sum_{i=0}^{\psi(N)} g_{\psi(N)-i}(\tau) f(N\tau)^i = 0$. The reader must be care that the letter g is used for two different purposes. Next, we use Theorem 2.5 in order to write $g_{\psi(N)-i}$ as polynomials in g_2, g_3. For g_1 we have to use the theory of Hecke operators for quasi-modular forms. The statement follows from Theorem 9.2 and **??**.

13.3 A model of modular curve in dimension four

Recall Proposition 8.2 and Proposition 8.3. Despite the fact that we have not constructed a basis of modular forms for $\Gamma_0(d)$, they give us an interesting model for $X_0(d)$. In the following we denote the coordinate system in the weighted projective space $\mathbb{P}^{4,6,4,6,2}$ by $[x_2 : x_3 : y_2 : y_3 : y_1]$.

Proposition 13.1 *The map given by*

$$\begin{aligned} &X_0(d) \to \mathbb{P}^{4,6,4,6,2} \backslash \{27x_3^2 - x_2^3 = 0\} \cup \{27y_3^2 - y_2^3 = 0\}, \\ &\tau \mapsto \left[g_2(\tau) : g_3(\tau) : d^2 g_2(d\tau) : d^3 g_3(d\tau) : d g_1(d\tau) - g_1(\tau)\right]. \end{aligned} \tag{13.3}$$

is a biholomorphism between $X_0(d)$ and its image $S_0(d)$. Moreover, $S_0(d)$ is a smmoth complete intersection curve in $\mathbb{P}^{4,6,4,6,2} \backslash \{27x_3^2 - x_2^3 = 0\} \cup \{27y_3^2 - y_2^3 = 0\}$ defined over $\mathbb{Q}$.

Proof. The image of the map (13.3) is an algebraic curve in $\mathbb{P}^{4,6,4,6,2}$ over $\mathbb{Q}$ and outside the discriminant locus $\{27x_3^2 - x_2^3 = 0\} \cup \{27y_3^2 - y_2^3 = 0\}$ it is a complete intersection $Q_{d,1} = Q_{d,2} = Q_{d,3} = 0$, where $Q_{d,1}(y_1, x_2, x_3)$, $Q_{d,2}(y_2, x_2, x_3)$, $Q_{d,3}(y_3, x_2, x_3)$ are homogeneous polynomials of degrees respectively $\psi(d), 2\psi(d),\ 3\psi(d)$ in the ring $\mathbb{Q}[x_2, x_3, y_1, y_2, y_3]$, $\deg(x_i) = \deg(y_i) := i$ such that

$$Q_{d,1}(d\cdot g_1(d\cdot\tau) - g_1(\tau), g_2(\tau), g_3(\tau)) = 0, \tag{13.4}$$
$$Q_{d,2}(d^2\cdot g_2(d\cdot\tau), g_2(\tau), g_3(\tau)) = 0, \tag{13.5}$$
$$Q_{d,3}(d^3\cdot g_3(d\cdot\tau), g_2(\tau), g_3(\tau)) = 0. \tag{13.6}$$

This follows from Theorem 8.2 and the fact that $\Gamma_0(d)$ has index $\psi(d)$ in $\mathrm{SL}(2,\mathbb{Z})$. Now, we prove that the complete intersection

$$\mathbb{P}\{Q_{d,1} = Q_{d,2} = Q_{d,3} = 0\} \subset \mathbb{P}^{4,6,4,6,2} \backslash \{27x_3^2 - x_2^3 = 0\} \cup \{27y_3^2 - y_2^3 = 0\}$$

is the image of the image of the map (13.3). details

We consider the affine chart $\mathbb{C}^4 \subset \mathbb{P}^{2,3,2,3,1}$ given $y_1 = 1$. The map 13.3 in this affine chart is given by

$$\tau \mapsto \left(\frac{g_2(\tau)}{(g_1(\tau) - d \cdot g_1(d \cdot \tau))^2}, \ \frac{g_3(\tau)}{(g_1(\tau) - d \cdot g_1(d \cdot \tau))^3}, \right. \tag{13.7}$$
$$\left. \frac{d^2 \cdot g_2(d \cdot \tau)}{(d \cdot g_1(d \cdot \tau) - g_1(\tau))^2}, \ \frac{d^3 \cdot g_3(d \cdot \tau)}{(d \cdot g_1(d \cdot \tau) - g_1(\tau))^3} \right).$$

We consider the following vector field in $(x_2, x_3, y_2, y_3) \in \mathbb{C}^4$:

$$\mathrm{v} := \left(2x_2 - 6x_3 + \frac{1}{6}(x_2 - y_2)x_2\right) \frac{\partial}{\partial x_2} + \left(3x_3 - \frac{1}{3}x_2^2 + \frac{1}{4}(x_2 - y_2)x_3\right) \frac{\partial}{\partial x_3} \tag{13.8}$$
$$- \left(2y_2 - 6y_3 + \frac{1}{6}(y_2 - x_2)y_2\right) \frac{\partial}{\partial y_2} - \left(3y_3 - \frac{1}{3}y_2^2 + \frac{1}{4}(y_2 - x_2)y_3\right) \frac{\partial}{\partial y_3}.$$

It is obtained through the proposition.

Proposition 13.2 *Let* (x_2, x_3, y_2, y_3) *be the coordinates of the map* (13.7). *We have*

$$\frac{\partial x_2}{\partial \tau} = (g_1(\tau) - d \cdot g_1(d \cdot \tau)) \left(2x_2 - 6x_3 + \frac{1}{6}(x_2 - y_2)x_2\right)$$
$$\frac{\partial x_3}{\partial \tau} = (g_1(\tau) - d \cdot g_1(d \cdot \tau)) \left(3x_3 - \frac{1}{3}x_2^2 + \frac{1}{4}(x_2 - y_2)x_3\right)$$
$$\frac{\partial y_2}{\partial \tau} = (g_1(\tau) - d \cdot g_1(d \cdot \tau)) \left(-2y_2 + 6y_3 - \frac{1}{6}(y_2 - x_2)y_2\right)$$
$$\frac{\partial y_3}{\partial \tau} = (g_1(\tau) - d \cdot g_1(d \cdot \tau)) \left(-3y_3 + \frac{1}{3}y_2^2 - \frac{1}{4}(y_2 - x_2)y_3\right)$$

Proof. The proof is a mere computation.

As a corollary we get:

Proposition 13.3 *The curve* $S_0(d)$ *in the affine chart* $y_1 = 1$ *is smooth.*

Proof. The singular set of the foliation the vector field v in the weighted projective space $\mathbb{P}^{2,3,2,3,1}$ consists of an isolated point and a rational curve:

$$\{[0:0:0:0:1]\} \cup \left\{ 27x_3^2 - x_2^3 = 27y_3^2 - y_2^3 = x_3^{\frac{1}{3}} + y_3^{\frac{1}{3}} - 2y_1 = 0 \right\}. \tag{13.9}$$

Exercise 13.3 Show that the vector field v in the homogeneous coordinates $[x_2 : x_3 : y_2 : y_3 : y_1]$ is given by

$$\frac{2x_2y_1 - 6x_3}{y_1} \frac{\partial}{\partial x_2} + \frac{3x_3y_1 - \frac{1}{3}x_2^2}{y_1} \frac{\partial}{\partial x_3} + \tag{13.10}$$
$$\frac{2y_2y_1 - 6y_3}{y_1} \frac{\partial}{\partial y_2} + \frac{3y_3y_1 - \frac{1}{3}y_2^2}{y_1} \frac{\partial}{\partial y_3} + \frac{1}{12} \frac{y_2 - x_2}{y_1} \frac{\partial}{\partial y_1}$$

In particular, $\mathrm{Sing}(\mathrm{v})$ intersects the projective space $\mathbb{P}^{2,3,2,3}$ at infinity at the point

$$[3:-1:3:1:0].$$

13.4 Moduli space I

In this section we give a moduli space interpretation for the weighted projective space $\mathbb{P}^{2,3,2,3,1}$ used in Section 13.3.

Theorem 13.2 *Let us consider the moduli space of triples (E_1,E_2,f), where E_1 and E_2 are two elliptic curves and $f: H^1_{\mathrm{dR}}(E_1) \xrightarrow{\sim} H^1_{\mathrm{dR}}(E_2)$ is an isomorphism which sends $F^1H^1_{\mathrm{dR}}(E_1)$ to $F^1H^1_{\mathrm{dR}}(E_2)$ and it respects the intersection form in $H^1_{\mathrm{dR}}(E_i),\ i=1,2$. This as a coarse moduli space exists and it is*

$$\mathbb{P}^{2,3,2,3,1}\backslash\left(\{27x_3^2-x_2^3=0\}\cup\{27y_3^2-y_2^3=0\}\right), \tag{13.11}$$

where we have considered the projective coordinates $[x_2:x_3:y_2:y_3:y_1]\in\mathbb{P}^{4,6,4,6,2}$.

Proof. For a triple (E_1,E_2,f) as above we write both elliptic curves E_1 and E_2 in the Weierstrass format and so it becomes of the format $(E_{x_2,x_3},E_{y_2,y_3},\ f)$. We also write f in the Weierstrass basis $\alpha_i,\ \omega_i$ of $H^1_{\mathrm{dR}}(E_i),\ i=1,2$:

$$[f(\alpha_1),\ f(\omega_1)]=[\alpha_2,\ \omega_2]\,g_f,\quad g_f\in\mathsf{G},$$

where G is the algebraic group (9.6). Let us take a different choise of the Weierstrass format $(E_{k^{-4}x_2,k^{-6}x_3},E_{\tilde{k}^{-4}y_2,\tilde{k}^{-6}y_3},\ \check{f})$ isomorphic to $(E_{x_2,x_3},E_{y_2,y_3},\ f)$. We have

$$\alpha: E_{k^{-4}x_2,k^{-6}x_3}\cong E_{x_2,x_3},\alpha^*\omega_1=k\omega_1$$

which uses similar computation as in the proof of Proposition 9.8. This together with the same statement for (y_2,y_3) implies that

$$g_{\check{f}}=\begin{bmatrix}\tilde{k}^{-1} & 0\\ 0 & \tilde{k}\end{bmatrix}g_f\begin{bmatrix}k & 0\\ 0 & k^{-1}\end{bmatrix},$$

By the action of $k\in\mathbb{G}_m$ or $\tilde{k}\in\mathbb{G}_m$ we can assume that the $(1,1)$ and $(2,2)$ entries of g_f and $g_{\check{f}}$. The above equality is equivalent to $k=\tilde{k}$ and

$$\text{the }(1,2)\text{ entry of }g_{\check{f}}=k^2\cdot\text{ the }(1,2)\text{ entry of }g_f.$$

If we denote by y_1 the $(1,2)$ entry of g_f and identify f with g_f we get

$$(E_{k^{-4}x_2,k^{-6}x_3},E_{\tilde{k}^{-4}y_2,\tilde{k}^{-6}y_3},\ \begin{bmatrix}1 & k^{-2}y_1\\ 0 & 1\end{bmatrix})\cong(E_{x_2,x_3},E_{y_2,y_3},\ \begin{bmatrix}1 & y_1\\ 0 & 1\end{bmatrix})$$

and so, we get a unique point in $\mathbb{P}^{4,6,4,6,2}$ minus $\{27x_3^2-x_2^3=0\}\cup\{27y_3^2-y_2^3=0\}$. Conversely, for a point in (13.11) we consider the triple $(E_{x_2,x_3},E_{y_2,y_3},\ f)$, where f

is uniquely determined by

$$[f(\alpha_1), f(\omega_1)] = [\alpha_1, \ \ \omega_2] \begin{bmatrix} 1 & y_1 \\ 0 & 1 \end{bmatrix}.$$

From the proof of Theorem 13.2 we know that over the affine chart $y_1 \neq 0$ of the moduli space (13.11) we have the universal family of triples (E_1, E_2, f): For $(x_2, x_3, y_2, y_3) \in \mathbb{C}^4$ we have $(E_{x_2,x_3}, E_{y_2,y_3}, \ f)$ and f is uniquely determined by

$$f(\alpha_1) = \alpha_2, \ \ f(\omega_1) = \omega_2 + \alpha_2. \tag{13.12}$$

Over this affine chart we have the two dimensional cohomology bundles $H^1_{\mathrm{dR}}(E_{x_2,x_3})$ and $H^1_{\mathrm{dR}}(E_{y_2,y_3})$, and hence, the four dimensional bundle H given by

$$H_{x_2,x_3,y_2,y_3} := H^1_{\mathrm{dR}}(E_{x_2,x_3})^\vee \otimes_{\mathbb{C}} H^1_{\mathrm{dR}}(E_{y_2,y_3})$$

which has a global section given by f. Recall that for two vector spaces A, B we have

$$A^\vee \otimes B \cong \mathrm{Hom}(A,B), \ \sum a_i^\vee \otimes b_i \mapsto (a \mapsto \sum a_i^\vee(a) b_i).$$

The cohomology bundle in (x_2, x_3) has global sections represented by the Weierstrass basis α_1, ω_1 for $H^1_{\mathrm{dR}}(E_{x_2,x_3})$. The dual bundle has also global sections $\alpha_1^\vee := \langle \alpha_1, \cdot \rangle$ and $\omega_1^\vee := \langle \omega_1, \cdot \rangle$, where $\langle \cdot, \cdot \rangle$ is the intersection bilinear form in $H^1_{\mathrm{dR}}(E_{x_2,x_3})$. The conclusion is that f is represented by the following global section of H:

$$\alpha_1^\vee \otimes \alpha_2 + \alpha_1^\vee \otimes \omega_2 - \omega_1^\vee \otimes \alpha_2 + 0\omega_1^\vee \otimes \omega_2.$$

However, this is not the section which we need in Equation (13.8). We will need the section

$$f(\alpha_1) = -\alpha_2, \ \ f(\omega_1) = \omega_2 + \alpha_2. \tag{13.13}$$

or equivalently $\alpha_1^\vee \otimes \alpha_2 + \alpha_1^\vee \otimes \omega_2 + \omega_1^\vee \otimes \alpha_2$. The Gauss-Manin connection matrix ∇ can be transported to H in a natural way.

Theorem 13.3 *There is a unique vector field* v *in* T *such that*

$$\nabla_{\mathrm{v}}(\alpha_1^\vee \otimes \alpha_2 + \alpha_1^\vee \otimes \omega_2 + \omega_1^\vee \otimes \alpha_2) = 0,$$

$$\nabla_{\mathrm{v}}(\alpha_1^\vee \otimes \alpha_2) = \frac{y_2 - x_2}{12} \alpha_1^\vee \otimes \alpha_2 + \alpha_1^\vee \otimes \omega_2 - \omega_1^\vee \otimes \alpha_2.$$

This is is given by (13.8).

Proof. The proof is purely computational. Let A_1 and A_2 be the Gauss-Manin connection matrices (3.23) in the coordinates system (x_2, x_3) and (y_2, y_3) instead of (t_2, t_3). For a connection $\nabla : H \to \Omega^1_{\mathsf{T}} \otimes H$ on a vector bundle H on T, the dual connection $\nabla^\vee : H^\vee \to \Omega^1_{\mathsf{T}} \otimes H^\vee$ is defined through the equality:

$$\nabla^\vee(a)(b) = -a(\nabla(b))$$

for sections a and b of H and $H^\vee$ respectively. The minus sign is necessary in order to make the dual connection intgerable. Note that the evaluation of a in a section of $\Omega^1_{\mathsf{T}} \otimes H$ is Ω^1_{T}-linear. The dual Gauss-Manin connection matrix in the dual basis $[\alpha^\vee, \omega^\vee]^{\mathrm{tr}}$ is given by

$$\begin{bmatrix} -(\mathsf{A}_1)_{22} & (\mathsf{A}_1)_{12} \\ (\mathsf{A}_1)_{21} & -(\mathsf{A}_1)_{11} \end{bmatrix} = \mathsf{A}_1,$$

The last equality follows from the fact that the sum of the $(1,1)$ and $(2,2)$ entries of the Gauss-Manin connection matrix (3.23) is zero. Therefore, the Gauss-Manin connection matrix in the bundle H and the basis

$$[\alpha_1^\vee \otimes \alpha_2, \alpha_1^\vee \otimes \omega_2, \omega_1^\vee \otimes \alpha_2, \omega_1^\vee \otimes \omega_2]^{\mathrm{tr}} \tag{13.14}$$

is given by

$$\mathsf{A} = \begin{bmatrix} (\mathsf{A}_1)_{11} & 0 & (\mathsf{A}_1)_{12} & 0 \\ 0 & (\mathsf{A}_1)_{11} & 0 & (\mathsf{A}_1)_{12} \\ (\mathsf{A}_1)_{21} & 0 & (\mathsf{A}_1)_{22} & 0 \\ 0 & (\mathsf{A}_1)_{21} & 0 & (\mathsf{A}_1)_{22} \end{bmatrix} + \begin{bmatrix} (\mathsf{A}_2)_{11} & (\mathsf{A}_2)_{12} & 0 & 0 \\ (\mathsf{A}_2)_{21} & (\mathsf{A}_2)_{22} & 0 & 0 \\ 0 & 0 & (\mathsf{A}_2)_{11} & (\mathsf{A}_2)_{12} \\ 0 & 0 & (\mathsf{A}_2)_{21} & (\mathsf{A}_2)_{22} \end{bmatrix}.$$

We have

$$\mathsf{A}_{\mathsf{v}} := \begin{bmatrix} -\frac{x_2-y_2}{12} & 1 & -1 & 0 \\ -\frac{y_2}{12} & -1 & 0 & -1 \\ \frac{x_2}{12} & 0 & 1 & 1 \\ 0 & \frac{x_2}{12} & -\frac{y_2}{12} & \frac{x_2-y_2}{12} \end{bmatrix}$$

and so the result follwos.

Remark 13.1 The global section f in (13.12) written in the basis (13.14) has the coefficients $C = [1,1,-1,0]$, that is, f is C times the matrix (13.14). However, in Equation (13.8) we have used $C = [1,1,1,0]$.

13.5 Proof of Theorem 13.5

13.6 Dynamics

We consider the foliation in $\mathbb{C}^4$ given by the vector field v. The reader is not supposed to know the theory of holomorphic foliations and we only want to transmit a feeling of this concept using the vector field v. Roughly speaking, by this foliation we mean the images of all solutions of v, that is all holomorphic maps $a : (\mathbb{C},0) \to \mathbb{C}^4$ such that $a'(t) = \mathrm{v}(a(t)), \quad t \in (\mathbb{C},0)$. It can be points or one dimensional transcendental or algebraic curves. The singular set $\mathrm{Sing}(\mathrm{v})$ of v is the set of points p such that $\mathrm{v}(p) = 0$. This is the zero set of the coefficients of $\frac{\partial}{\partial a}$, $a = x_2, x_3, y_2, y_3$ in v. The singular set of the foliation $\mathbb{F}(\mathrm{v})$ in the weighted projective space $\mathbb{P}^w := \mathbb{P}^{2,3,2,3,1}$ with the coordinate system $[x_2 : x_3 : y_2 : y_3 : y_1]$ consists of an isolated point and a rational curve:

$$\mathrm{Sing}(\mathrm{v}) = \{0\} \cup \left\{ 27x_3^2 - x_2^3 = 27y_3^2 - y_2^3 = x_3^{\frac{1}{3}} + y_3^{\frac{1}{3}} - 2y_1 = 0 \right\}. \tag{13.15}$$

The call the second component of $\mathrm{Sing}(\mathrm{v})$ the curve singularity and parametrize it by

$$g : \mathbb{P}^1 \to \mathbb{P}^w, \ \ [t:s] \mapsto \left[3t^2 : t^3 : 3s^2 : s^3 : \frac{1}{2}(s+t) \right]. \tag{13.16}$$

It can be easily checked that

Exercise 13.4 The following union of two hypersurfaces

$$\Delta := \Delta_1 \cup \Delta_2, \tag{13.17}$$

$$\Delta_1 := \{27x_3^2 - x_2^3 = 0\}, \ \Delta_2 := \{27y_3^2 - y_2^3 = 0\} \tag{13.18}$$

is tangent to v, that is, $\frac{d\Delta_i}{\Delta_i}(\mathrm{v}) \in \mathbb{C}[x_2, x_3, y_2, y_3]$.

Theorem 13.4 ([Mov22b], Theorem 1) *The algebraic curves $S_0(d) \subset \mathbb{P}^w, d \in \mathbb{N}$ are not contained in Δ and they are tangent to the vector field* v *in* (13.8). *They are the only algebraic curves in $\mathbb{P}^w \backslash \Delta$ with this property. The curve $S_0(d)$ intersects Δ only at the points $g([a:-b]), \ d = ab, \ a, b \in \mathbb{N}$ in the curve singularity of* v.

13.7 Picard's curious example

In this section we give a new moduli space interpretation of E. Picard's "équation différentielle curieuse" in [Pic89, pages 298-299], see also article [Maz01]. Our presentation follows [Mov22b, Section 10].

Definition 13.1 Let T be the moduli space of triples (E, P, ω), where E is an elliptic curve over k and by definition it comes together with a point $O, P \in E(\mathsf{k}), \ P \neq O$ is another point and ω is a meromorphic differential 1-form in E with poles only at O and P and with the order of pole equal to one at both points. Moreover, the residue of ω at P and O is respectively $+1$ and -1. We call T the Picard moduli space.

As it is the case with many other moduli spaces in this text, T turns out to be affine and there is a universal family over a Zariski open subset of T.

Proposition 13.4 *We have*

$$\mathsf{T} \simeq \mathbb{P}_{\mathsf{k}}^{1,2,3,4} \backslash \{[s:a:b:c] \in \mathbb{P}_{\mathsf{k}}^{1,2,3,4} | \Delta = 0\}, \tag{13.19}$$

where $\Delta := 27(-b^2 + 4a^3 - ca)^2 - c^3$, and the family over T is given by the following family of elliptic curves written in the affine coordinates $(x, y) \in \mathbb{A}_{\mathsf{k}}^2$:

$$
\begin{aligned}
E = E_{a,b,c} \quad &: \quad y^2 = 4x^3 - cx + b^2 - 4a^3 + ca, \\
\omega = \omega_{s,a,b} &:= \frac{1}{2}\frac{y+b}{x-a}\frac{dx}{y} + s\frac{dx}{y}, \quad P = (a,b).
\end{aligned} \tag{13.20}
$$

Note that we do not have universal family over T. We have such a universal family over the open subset $\{s \neq 0\}$ of T.

Proof. Let (E, ω, P) be an element of the moduli space T. We choose Weierstrass coordinates x, y on E. These are rational functions on E with pole of order 2 and 3 at O, respectively. In this way we can write E in the Weierstrass format $E_{c,\check{c}} : y^2 = 4x^3 - cx - \check{c}$ with $\Delta := 27\check{c}^2 - c^3 \neq 0$. In these coordinates we write $P = (a,b)$ and therefore $b^2 = 4a^3 - cx - \check{c}$. From this we compute $\check{c}$ and repalce it in the expression of the elliptic curve $E_{c,\check{c}}$ and we get $E_{a,b,c}$. The differential form $\frac{1}{2}\frac{y+b}{x-a}\frac{dx}{y}$ has the same properties as ω, therefore, their difference is a holomorphic 1-form in $E_{a,b,c}$ and hence it is $s\frac{dx}{y}$ for some $s \in \mathsf{k}$. It follows that any triple of the moduli space T is isomorphic to a triple in (13.20) for some $(s,a,b,c) \in \mathsf{k}^4$. Note that $\check{c} = 4a^3 - ca - b^2$ and so it can be discarded. For $k \in \mathsf{k}^*$ we have

$$
\begin{aligned}
&f : E_{k^{-4}c, k^{-6}\check{c}} \simeq E_{c,\check{c}}, \\
&f(x,y) = (k^2x, k^3y), \\
&f_*\omega_{k^{-1}s, k^{-2}a, k^{-3}b} = \omega_{s,a,b}.
\end{aligned}
$$

and so (s,a,b,c) and $(k^{-1}s, k^{-2}a, k^{-3}b, k^{-4}c)$ represents the same point in T. Any isomorphism of elliptic curves in the Weierstrass format comes from the k^* action above, and so, any two points in the right hand side of (13.19) are not isomorphic.

Let us consider the affine chart $s = 1$ for the moduli space T. We are going to talk about integration of differential forms and so we have to work over complex numbers, that is, $\mathsf{k} = \mathbb{C}$. Let $\delta \in H_1(E_{a,b,c}, \mathbb{Z})$ be a continuous family of cycles. A more algebro-geometric version of the statement bellow similar to the Gauss-Manin connection of families of elliptic curves, see Proposition 3.4, might be necessary.

Proposition 13.5 *We have*

$$
d\int_\delta \left(\frac{1}{2}\frac{y+b}{x-a}\frac{dx}{y} + \frac{dx}{y}\right) = \frac{\alpha_1}{\Delta} \cdot \int_\delta \frac{xdx}{y} + \frac{\alpha_2}{\Delta} \cdot \int_\delta \frac{dx}{y}, \tag{13.21}
$$

where d is the differential of holomorphic functions in $(a,b,c) \in \mathbb{C}^3$, $\alpha_i = \alpha_{1i}da + \alpha_{2i}db + \alpha_{3i}dc$, $i = 1,2$ and α_{ij}'s are given in

$$
\alpha := \begin{pmatrix}
3c^2 - 36ca^2 + 45cab - 108a^3b + 27b^3 & -\frac{1}{2}\left(9c^2a + 3c^2b - 144ca^3 + 54ca^2b + 9cb^2 + 432a^5 - 216a^4b - 108a^2b^2 + 54ab^3\right) \\
(2c^2 - 30ca^2 + 6cb + 72a^4 - 18ab^2) & -(2c^2a - 30ca^3 + 9cab + 3cb^2 + 72a^5 - 36a^3b - 18a^2b^2 + 9b^3) \\
-\frac{1}{2}\left(3ca + 3cb - 36a^3 + 9b^2\right) & \frac{1}{4}\left(c^2 - 18ca^2 + 9cab + 72a^4 - 36a^3b - 18ab^2 + 9b^3\right)
\end{pmatrix}.
$$

Proof. We know already that derivation of the integrant in the left hand side of (13.21) has no more residues around O and P, therefore, it must be an element of $H^1_{\mathrm{dR}}(E_{a,b,c})$, and hence, it must be a linear combination of $\frac{x^i dx}{y}$, $i = 0,1$ with coefficients which we are now going to compute. This is a simple, but long, calculus computation.

Proposition 13.6 *The loci of points in* T *such that the integral* $\int_\delta \left(\frac{1}{2}\frac{y+b}{x-a}\frac{dx}{y}+\frac{dx}{y}\right)$ *is constant for all choices of* $\delta \in H_1(E_{a,b,c},\mathbb{Z})$ *are the solutions of the vector field*

$$\mathrm{v} := (2c-24a^2+6ab+6b)\frac{\partial}{\partial a}-(3c-36a^2+36ab-9b^2)\frac{\partial}{\partial b} \tag{13.22}$$

$$+(12ca+12cb-144a^3+36b^2)\frac{\partial}{\partial c}$$

Proof. Such a loci is given by $\alpha_1=0,\alpha_2=0$. The matrix α has rank two and the vector field v is in the kernel of α_1 and α_2 and generates it. This implies that along the solutions of the vector field v in T, the integral in the left hand side of (13.21) is constant for all continuous family of cycles.

Theorem 13.5 *The foliation induced by* v *in* T *has infinite number of algebraic leaves* $S_1(N), N=2,3,\cdots$. *The leaf* $S_1(N)$ *parameterizes the triples* $(E,\frac{1}{N}\frac{df_N}{f_N},P)$, *where* P *is a torsion point of order* N *and* f_N *is a rational function in* E *with* $\mathrm{div}(f_N)=N\cdot(P-O)$. *It is given by the image of the holomorphic map*

$$\Gamma_1(N)\backslash\mathbb{H}^* \to \mathbb{P}^{1,2,3,4},\ \tau\mapsto \left[F_N(\tau):\wp\left(\frac{1}{N},\tau\right):\wp'\left(\frac{1}{N},\tau\right):60G_4(\tau)\right]. \tag{13.23}$$

It is quite natural to give a purely complex analysis proof to the fact that the image of the map (13.23) is tangent to the vector field v (similar to the case in Proposition 13.2). This has been formulated in Exercise 8.6.

Proof. We write the triple $(E,\frac{1}{N}\frac{df_N}{f_N},P)$ in the Weierstrass coordinates (x,y) and by Proposition 13.4, we get a unique point $[s:a:b:c]\in\mathsf{T}$. In this way, f_N is a rational function in x,y with coefficients in k and

$$\frac{1}{N}\frac{df_N}{f_N}-\left(\frac{1}{2}\frac{y+b}{x-a}+s\right)\frac{dx}{y} \tag{13.24}$$

restricted to $E_{a,b,c}$ is identically zero. The periods $\int_\delta \frac{df_N}{N},\quad \delta\in H_1(E_{a,b,c}-\{O,P\})$ are all in $2\pi i\mathbb{Z}$, and hence, they are constants independent of a,b,c. By Proposition 13.6 this implies that the curve $S_1(N)$ is tangent to v.

Next, we describe a parametrization of $S_1(N)$ by modular forms. We consider the complex torus $E:=\frac{\mathbb{C}}{\mathbb{Z}\tau+\mathbb{Z}}$ and its embedding in $\mathbb{P}^2$ using $z\mapsto[\wp(z,\tau):\wp'(z,\tau):1]$, where $\wp(z,\tau)$ is the Weierstrass $\wp$ function and its derivation means with respect to z. We also consider the torsion point $P=\frac{1}{N}$ in E. The following function

$$F_N(\tau):=\frac{1}{N}\frac{f_N'}{f_N}-\frac{1}{2}\frac{\wp'(z,\tau)+\wp'(\frac{1}{N},\tau)}{\wp(z,\tau)-\wp(\frac{1}{N},\tau)}$$

is holomorphic on the torus and hence it is independent of z. Here, $f_N(z)$ is a double periodic function in z with period 1 and τ and it has a zero (resp. pole) of order N at $z=\frac{1}{N}$ (resp. $z=0$) and $'$ means derivation with respect to z. The compactification of the curve $S_1(N)$ in $\mathbb{P}^{1,2,3,4}$ is birational to the modular curve $X_1(N) := \Gamma_1(N)\backslash\mathbb{H}^*$, where

$$\Gamma_1(N) := \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in \mathrm{SL}2\mathbb{Z} \middle| a_3 \equiv 0 \; a_1 \equiv a_4 \equiv 1, \pmod{N} \right\}.$$

Such a birational map is given by

$$\Gamma_1(N)\backslash\mathbb{H}^* \to \mathbb{P}^{1,2,3,4}, \; \tau \mapsto \left[F_N(\tau) : \wp\left(\frac{1}{N}, \tau\right) : \wp'\left(\frac{1}{N}, \tau\right) : 60G_4(\tau) \right].$$

The four functions involved in the above parameterization are modular forms for $\Gamma_1(N)$. The precise comparision of our Picard's differential equation given by v in (13.22) and the Picard's differential equation in [Pic89] pages 298-299 is left to the reader. For the line bundle $L := \mathscr{O}(P-O)$ on $E = E_{a,b,c}$ with its global meromorphic section s such that $\mathrm{div}(s) = P - O$, we can associate the holomorphic connection $\nabla : L \to \Omega_E \otimes L, \; s \mapsto \omega_{s,a,b} \otimes s$. Isomonodromic deformations of (E, ∇) is the same as deformations with constant integrals in the left hand side of (13.21). This and [Lor16] Corollary 2 and 3 have been the starting point of our reformulation of Picard's example.

Part II
Calabi-Yau varieties

Chapter 14
Calabi-Yau threefolds

Honestly speaking, the main goal of writing the present book has been to generalize the whole theory of elliptic curves (Calabi-Yau varieties of dimension one) and elliptic modular forms to the framework of Calabi-Yau threefolds. The author's attempts to do this has been successful in some directions which are explained in [Mov11, Mov15a, Mov17, AMSY16, Mov22a] and has failed in many other directions. As the history of modular forms and elliptic curves is almost two centuries of mathematical research, the development of similar topics for Calabi-Yau varieties might take even more time. In this chapter I would like to report on what tiny things are already done and the tremendous amount of work which must be done in the future.

For a smooth projective variety X over $\mathbb{C}$ we assume that the reader is familiar with the algebraic de Rham cohomology, see [Mov21], and singular homology, see [] and the references therein.

14.1 Hypersurfaces with a finite group action

The parameter space of smooth hypersurfaces of degree d and dimension n in $\mathbb{P}^{n+1}$ is of large dimension and it is desirable to have families of hypersurfaces with some symmetry and depending on few parameters. In this section we try to elaborate this idea and justify the definition of mirror quintic from a purely mathematical perspective. This section is a further elaboration of [Mov22a, Section 12.7].

We take a finite group G acting on $\mathbb{P}^{n+1}_{\mathfrak{k}}$. In practice, this will be a subgroup of the automorphism group of the classical Fermat variety

$$X_0 = X_n^d : \ \mathbb{P}\{x_0^d + x_1^d + \cdots + x_{n+1}^d = 0\} \subset \mathbb{P}^{n+1}_{\mathfrak{k}}$$

of dimension n and degree d. The group S_{n+2} of all permutations in $n+2$ elements $\{0, 1, \ldots, n+1\}$ acts on X_n^d in a natural way. An element in $b \in \mathsf{S}_{n+2}$ acts on X_n^d by permuting the coordinates:

$$(x_0, x_1, \ldots, x_{n+1}) \mapsto (x_{b_0}, x_{b_1}, \ldots, x_{b_{n+1}}).$$

Multiplication of the coordinates by d-th roots of unity provides other automorphisms of the Fermat variety. Let

$$\mu_d^{n+2}/\mu_d := \underbrace{\mu_d \times \mu_d \times \cdots \times \mu_d}_{(n+2)-\text{ times}}/\mathrm{diag}(\mu_d), \tag{14.1}$$

where

$$\mu_d := \{1, \zeta_d, \ldots, \zeta_d^{d-1}\} \tag{14.2}$$

is the group of d-th roots of unity and $\mathrm{diag}(\mu_d)$ is the image of the diagonal map

$$\mu_d \to \mu_d^{n+2}, \ \ \zeta \mapsto (\zeta, \zeta, \cdots, \zeta).$$

The group μ_d^{n+2}/μ_d acts on X_n^d by multiplication of coordinates:

$$(\zeta_0, \zeta_1, \ldots, \zeta_{n+1}), (x_0, x_1, \ldots, x_{n+1}) \mapsto (\zeta_0 x_0, \zeta_1 x_1, \ldots, \zeta_{n+1} x_{n+1}). \tag{14.3}$$

Let us define the free product group

$$G_n^d := \left(\mu_d^{n+2}/\mu_d\right) * \mathsf{S}_{n+2}, \tag{14.4}$$

which is a subgroup of the automorphism group of the Fermat variety X_n^d.

Exercise 14.1 Is the automorphism group of the Fermat variety either as a complex manifold or an algebraic variety over $\mathbb{C}$ is equal to G_n^d?

Let $G \subset G_n^d$ be any finite subgroup. The group G acts on the space V_n^d of smooth hypersurfaces in a canonical way and we define

$$V_G := \left\{ t \in V_n^d \middle| g \cdot t = t, \right\} \tag{14.5}$$

that is, V_G parametrizes hypersurfaces X with $G \subset \mathrm{Aut}(X)$. By definition the Fermat point $0 \in V_n^d$ is in V_G. Our main examples for G are

Example 14.1 Let

$$G := \left\{ \zeta \in \mu_d^{n+2}/\mu_d \middle| \ \zeta_0 \zeta_1 \ldots \zeta_{n+1} = 1 \right\}, \tag{14.6}$$

for the case $d = n+2$. The corresponding family of hypersurfaces is give by

$$X_t: \ \ t_0 x_0^{n+2} + t_1 x_1^{n+2} + \ldots + t_{n+1} x_{n+1}^{n+2} - (n+2) t_{n+2} x_0 x_1 \ldots x_n = 0, \tag{14.7}$$

which is called the Dwork family. It is smooth if and only if

$$\Delta := t_0 t_1 \cdots t_{n+1} (t_{n+2}^{n+2} - t_0 t_1 \cdots t_{n+1}) \neq 0.$$

For $\Delta = 0$ and t_i's different from *zero*, X_t has only isolated singularities. This might help to discribe the homology of X_t as explicit as possible using vanishing cycles.

Example 14.2 For the permutation group $G = \mathsf{S}_{n+2}$ we will consider the case $d = 3$. The corresponding family of hypersurfaces is given by

$$X_t: \quad t_0(x_0^3 + \cdots) + t_1(x_0^2 x_1 + \cdots) + t_3(x_0 x_1 x_2 + \cdots) = 0, \tag{14.8}$$

where $\cdots$ means the sum of all possible monomials obtained from the leading monomial by permuting the variables. We call X_t the Deligne's family, as working with hypersurfaces with large automorphism group is proposed by P. Deligne, (personal communication, February 20, 2016).

Definition 14.1 An automorphism of a smooth projective variety leaves the Hodge filtration invariant and hence it is natural to consider the invariant part of $H^n_{\mathrm{dR}}(X)$

$$H^n_{\mathrm{dR}}(X)^G := \left\{ \omega \in H^n_{\mathrm{dR}}(X) \;\middle|\; \sigma^* \omega = \omega \quad \forall \sigma \in G \right\} \tag{14.9}$$

and the induced Hodge filtration. This is also called the invariant cohomology of X. In a similar way, we define the invariant holology:

$$H_n(X, \mathbb{Z})^G := \left\{ \delta \in H_n(X, \mathbb{Z}) \;\middle|\; \sigma_* \delta = \delta \quad \forall \sigma \in G \right\}.$$

The fact the invariant homology is dual to invariant cohomology follows from the same statement for usual (co)homologies.

Exercise 14.2 Let $V_{\mathbb{C}}$ be a $\mathbb{C}$-vector space and $V_{\mathbb{Z}}$ be a free $\mathbb{Z}$-module with a bilinear map $V_{\mathbb{Z}} \times V_{\mathbb{C}} \to \mathbb{C}$, $(\delta, \omega) \mapsto \int_\delta \omega$ which makes $V_{\mathbb{Z}}$ dual $V_{\mathbb{C}}$, that is, the natural map $V_{\mathbb{Z}} \to V_{\mathbb{C}}^\vee$, $\delta \mapsto (\omega \mapsto \int_\delta \omega)$ is injective and $V_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{C} \cong V_{\mathbb{C}}^\vee$. Let G be a finite group acting on both $V_{\mathbb{Z}}$ and $V_{\mathbb{C}}$ such that $\int_\delta g\omega = \int_{g\delta} \omega$ for all $g \in G$, $\omega \in V_{\mathbb{C}}$, $\delta \in V_{\mathbb{Z}}$. Show that $V_{\mathbb{C}}^G$ is dual to $V_{\mathbb{Z}}^G$.

Proposition 14.1 *A basis of* $H^n_{\mathrm{dR}}(X)^G$ *for a member* $X = X_t$ *of the Dwork and Deligne families and for t in a neighborhood of the Fermat point are given by*

$$\frac{(x_0 x_1 \cdots x_{n+1})^{k-1} \Omega}{f^k}, \quad k = 1, 2, \ldots, n+1,$$
$$\frac{(x_0 x_1 \cdots x_{3k-n-2} + \cdots) \Omega}{f^k}, \quad k = 1, 2, \ldots, n+1,$$

respectively, where $\cdots$ *means the sum of all possible monomials obtained from the leading monimial by permuting the variables. It is compatible with the Hodge filtration. For the Dwork family* $\dim(H^n_{\mathrm{dR}}(X)^G) = n+1$ *and the Hodge numbers are*

$$\underbrace{1, 1, 1, \ldots, 1}_{n+1-\ \textit{times}},$$

and for Deligne family $\dim(H^n_{\mathrm{dR}}(X)^G) = n+1-2\left[\frac{n+1}{3}\right]$ *and the Hodge numbers are*

$$\underbrace{0,0,\cdots,0}_{\left[\frac{n+1}{3}\right]-\ times}, \quad \underbrace{1,1,\cdots,1}_{n+1-2\left[\frac{n+1}{3}\right]-\ times}, \quad \underbrace{0,0,\cdots,0}_{\left[\frac{n+1}{3}\right]-\ times}.$$

Proof. This follows from Griffiths theorem on the cohomology of hypersurfaces in [Gri69], see also [MV21].

Note that using Exercise 14.2 and Proposition 14.1 we can cobclude that for Dwork family $H_n(X,\mathbb{Z})_G$ is free of rank $n+1$ and for Deligne family it is free of rank $n+1-2\left[\frac{n+1}{3}\right]$.

14.2 Eisenstein series

Despite being natural, I did not find in the literature the way of writing Eisenstein series and Weierstrass $\wp$ function as in Proposition 3.3 and Remark 3.1. The main goal is to see whether in this way we can generalize these convergent series to other periods, and in particular, periods of Calabi-Yau threefolds. One can formulate similar formal power series, however, in general the main obstacle is whether they are convergent.

Exercise 14.3 Let X be a Calabi-Yau n-fold and $\delta_1 \in H_n(X,\mathbb{Z})$ be a primitive element, that is, it is not divisible by an integer. Is the following sum convergent

$$\sum_{\delta \text{ a monodromy of } \delta_1} \left(\int_\delta \omega^{n,0}\right)^{-k}, \ k \in \mathbb{N}, \tag{14.10}$$

where the sum runs in all monodromies $\delta \in H_n(X,\mathbb{Z})$ of δ_1. Note that in the case of elliptic curves $n=1$, if we find δ_2 such that δ_1, δ_2 form a basis of $H_n(X,\mathbb{Z})$ then the action of monodromy produces $c\delta_1 + d\delta_2$ for all coprime integers c and d and the above sum is the Eisenstein series E_k (up to multiplication by a constant). If we use the Eisenstein summation as in Section 2.18 then the above series is even convergent even for $k=1,2$. It might be useful that in the case of mirror quintic to write a precise description of all 4-tuples $(a_1,a_2,a_3,a_4) \in \mathbb{Z}^4$ such that $a_1\delta_1 + \cdots + a_4\delta_4$ is obtained by the mondromy of a fixed δ_1, see Exercise 2.6. We might first to ellaborate the case $n=2$, in which the moduli space of Calabi-Yau two folds ($K3$ surfaces) is a Hermitian symmetric domain and we have a well developed theory of automorphic functions in this case. For this we may take the Dwork family [MN21] or lattice polarized K3 surfaces [CD12].

My feeling is that in some way one has to use periods of forms over the homology path between rational curves inside Calabi-Yau threefolds. For instance,

$$\sum \left(\int_{\mathbb{P}^1_0}^{\mathbb{P}^1} \omega^{3,0} \right)^{-k}$$

which does not seem to be convergent. Here, $\mathbb{P}^1_0$ is a fixed rational curve of degree one, $\mathbb{P}^1$ is a rational curve of degree d such that $d\mathbb{P}^1_0$ is homologous to $\mathbb{P}^1$, and the sum runs through all $\mathbb{P}^1$'s and such homology paths. Before going to Calabi-Yau threefols, one intermediate challenge may help us. This is namely defining Eisenstein series for differential Siegel modular forms develped in [Mov22a, Chapter 11] and [CMY21].

References

AMSY16. M. Alim, H. Movasati, E. Scheidegger, and S.-T. Yau. Gauss-Manin connection in disguise: Calabi-Yau threefolds. *Comm. Math. Phys.*, 334(3):889–914, 2016.

Apo90. Tom M. Apostol. *Modular functions and Dirichlet series in number theory*, volume 41 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990.

Bac12. H. Bachmann. *Multiple Zeta-Werte und die Verbindung zu Modulformen durch Multiple Eisensteinreihen*. 2012. Thesis (Master)–Hamburg University.

BCDT01. Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor. On the modularity of elliptic curves over $\mathbf{Q}$: wild 3-adic exercises. *J. Amer. Math. Soc.*, 14(4):843–939 (electronic), 2001.

BKK$^+$98a. H. G. W. Begehr, H. Koch, J. Kramer, N. Schappacher, and E.-J. Thiele, editors. *Mathematics in Berlin*. Birkhäuser Verlag, Berlin, 1998.

BKK$^+$98b. H. G. W. Begehr, H. Koch, J. Kramer, N. Schappacher, and E.-J. Thiele, editors. *Mathematics in Berlin*. Birkhäuser Verlag, Berlin, 1998.

BS98. Rolf Berndt and Ralf Schmidt. *Elements of the representation theory of the Jacobi group*. Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 1998. [2011 reprint of the 1998 original] [MR1634977].

BSD63. B. J. Birch and H. P. F. Swinnerton-Dyer. Notes on elliptic curves. I. *J. Reine Angew. Math.*, 212:7–25, 1963.

BSD65. B. J. Birch and H. P. F. Swinnerton-Dyer. Notes on elliptic curves. II. *J. Reine Angew. Math.*, 218:79–108, 1965.

Cas66. J. W. S. Cassels. Diophantine equations with special reference to elliptic curves. *J. London Math. Soc.*, 41:193–291, 1966.

Cas90. J. W. S. Cassels. Trygve Nagell. *Acta Arith.*, 55(2):109–112, 1990.

CD12. Adrian Clingher and Charles F. Doran. Lattice polarized K3 surfaces and Siegel modular forms. *Adv. Math.*, 231(1):172–212, 2012.

CdlOGP91. Philip Candelas, Xenia C. de la Ossa, Paul S. Green, and Linda Parkes. A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory. *Nuclear Phys. B*, 359(1):21–74, 1991.

Cha85. K. Chandrasekharan. *Elliptic functions*, volume 281 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1985.

Cha14. Ching-Li Chai. Riemann's theta function. *Preprint*, 2014.

CM24. J. Cao and H. Movasati. On the Lie algebra of Jacobi group. *Preprint*, 2024.

CMV24. J. Cao, H. Movasati, and R. Villaflor. Gauss-Manin connection in disguise: Index zero Jacobi forms. *Int. Math. Res. Not. IMRN* , 2024.

CMY21. Jin Cao, Hossein Movasati, and Shing-Tung Yau. Gauss-Manin connection in disguise: genus two curves. *Adv. Math.*, 383:37, 2021. Id/No 107684.

Cog14. Alberto Cogliati. On Jacobi's transformation theory of elliptic functions. *Arch. Hist. Exact Sci.*, 68(4):529–545, 2014.

Coo09. Shaun Cooper. Inversion formulas for elliptic functions. *Proc. Lond. Math. Soc. (3)*, 99(2):461–483, 2009.

CS99. J. H. Conway and N. J. A. Sloane. *Sphere packings, lattices and groups. With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov.*, volume 290. New York, NY: Springer, 1999.

CTC01. Y. K. Leong C. T. Chong. An interview with jean-pierre serre. In *Mathematical Conversations*. Springer, New York, NY, 2001.

CW77. J. Coates and A. Wiles. On the conjecture of Birch and Swinnerton-Dyer. *Invent. Math.*, 39(3):223–251, 1977.

Del71. Pierre Deligne. Formes modulaires et représentations ℓ-adiques. Sémin. Bourbaki 1968/69, No. 355, Lect. Notes Math. 179, 139-172 (1971)., 1971.

Del73. Pierre Deligne. La conjecture de Weil. I. *Publ. Math., Inst. Hautes Étud. Sci.*, 43:273–307, 1973.

Del80. Pierre Deligne. La conjecture de Weil. II. *Publ. Math., Inst. Hautes Étud. Sci.*, 52:137–252, 1980.

DGMS13. C. Doran, T. Gannon, H. Movasati, and K. Shokri. Automorphic forms for triangle groups. *Communications in Number Theory and Physics*, 7(4):689–737, 2013.

Die72. J. Dieudonné. The historical development of algebraic geometry. *Amer. Math. Monthly*, 79:827–866, 1972.

Dij95. Robbert Dijkgraaf. Mirror symmetry and elliptic curves. In *The moduli space of curves (Texel Island, 1994)*, volume 129 of *Progr. Math.*, pages 149–163. Birkhäuser Boston, Boston, MA, 1995.

DLMF. *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.1.2 of 2021-06-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

DS05. Fred Diamond and Jerry Shurman. *A first course in modular forms*, volume 228 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

Ebe94. Wolfgang Ebeling. *Lattices and codes. A course partially based on lectures by F. Hirzebruch*. Braunschweig: Vieweg, 1994.

Edw01. H. M. Edwards. *Riemann's zeta function*. Dover Publications Inc., Mineola, NY, 2001. Reprint of the 1974 original [Academic Press, New York; MR0466039 (57 #5922)].

Eis47. G. Eisenstein. Beiträge zu Theorie der elliptischen Functionen. *J. Reine Angew. Math.*, 35:137–184, 1847.

Eis95. David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

Elk07. Noam D. Elkies. Three lectures on elliptic surfaces and curves of high rank. *arXiv e-prints*, page arXiv:0709.2908, September 2007.

EZ85. Martin Eichler and Don Zagier. *The theory of Jacobi forms*, volume 55. Birkhäuser/Springer, Basel, 1985.

Fre86. Gerhard Frey. Links between stable elliptic curves and certain diophantine equations. Ann. Univ. Sarav., Ser. Math. 1, No. 1, 40 p. (1986)., 1986.

Fri16. Robert Fricke. Die elliptischen Funktionen und ihre Anwendungen. Erster Teil: Die funktionentheoretischen und analytischen Grundlagen. Leipzig und Berlin: B. G. Teubner, X u. 500 S. 8° (1916)., 1916.

Fri22. Robert Fricke. Die elliptischen Funktionen und ihre Anwendungen. Zweiter Teil: Die algebraischen Ausführungen. Leipzig: B. G. Teubner, VIII u. 546 S. 8° (1922)., 1922.

Fri11. Robert Fricke. *Die elliptischen Funktionen und ihre Anwendungen. Dritter Teil: Anwendungen*. Berlin: Springer, 2011.

Gan06. Terry Gannon. *Moonshine beyond the Monster*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 2006. The bridge connecting algebra, modular forms and physics.

Gri69. Phillip A. Griffiths. On the periods of certain rational integrals. I, II. *Ann. of Math. (2) 90 (1969), 460-495; ibid. (2)*, 90:496–541, 1969.

Gro62. Alexander Grothendieck. Techniques de construction en géométrie analytique. VII: Étude locale des morphismes; éléments de calcul infinitesimal. VIII: Rapport sur les théorèmes de finitude de Grauert et Remmert; IX: Quelques problèmes de modules; X: Construction de l'espace de Teichmüller. Familles d'Espaces Complexes et Fondements de la Géom. Anal., Sém. H. Cartan 13 (1960/61), Fasc. 2, No. 14, 27 p. (1962); No. 15, 10 p. (1962); No. 16, 20 p. (1962); No. 17, 20 p. (1962)., 1962.

Gro66. Alexander Grothendieck. On the de Rham cohomology of algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, (29):95–103, 1966.

Gro78. Benedict H. Gross. On the periods of abelian integrals and a formula of Chowla and Selberg. *Invent. Math.*, 45(2):193–211, 1978. With an appendix by David E. Rohrlich.

Gro79. Benedict H. Gross. On an identity of Chowla and Selberg. *J. Number Theory*, 11(3 S. Chowla Anniversary Issue):344–348, 1979.

Gun90. Robert C. Gunning. *Introduction to holomorphic functions of several variables. Volume I: Function theory. Volume II: Local theory. Volume III: Homological theory. (Rev. version and complete rewriting of: Analytic functions of several complex variables by Hugo Rossi and the author).* Florence, KY: Wadsworth &— Brooks/Cole Advanced Books &— Software, 1990.

Har77. Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

Hec36. E. Hecke. Über die Bestimmung Dirichletscher Reihen durch ihre Funktionalgleichung. *Math. Ann.*, 112(1):664–699, 1936.

Hec37. E. Hecke. Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung. I. *Math. Ann.*, 114:1–28, 1937.

Hid12. Haruzo Hida. *Geometric modular forms and elliptic curves*. Singapore: World Scientific, 2012.

Hus04. D. Husemöller. *Elliptic curves*, volume 111 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2004. With appendices by Otto Forster, Ruth Lawrence and Stefan Theisen.

IR90. Kenneth Ireland and Michael Rosen. *A classical introduction to modern number theory.*, volume 84. New York etc.: Springer-Verlag, 1990.

Jac29. C. G. J. Jacobi. Fundamenta nova theoriae functionum ellipticarum. Königsberg, 1829.

Jac85. Nathan Jacobson. *Basic algebra. I.* W. H. Freeman and Company, New York, second edition, 1985.

Kam92. S. Kamienny. Torsion points on elliptic curves and q-coefficients of modular forms. *Invent. Math.*, 109(2):221–229, 1992.

Kat76a. N. M. Katz. An overview of Deligne's proof of the Riemann hypothesis for varieties over finite fields. In *Mathematical developments arising from Hilbert problems (Proc. Sympos. Pure Math., Vol. XXVIII, Northern Illinois Univ., De Kalb, Ill., 1974)*, pages 275–305. Amer. Math. Soc., Providence, R.I., 1976.

Kat76b. N. M. Katz. p-adic interpolation of real analytic Eisenstein series. *Ann. of Math. (2)*, 104(3):459–571, 1976.

Ked08. K. S. Kedlaya. p-adic cohomology: from theory to practice. In *p-adic geometry*, volume 45 of *Univ. Lecture Ser.*, pages 175–203. Amer. Math. Soc., Providence, RI, 2008.

KF17a. Felix Klein and Robert Fricke. *Lectures on the theory of elliptic modular functions. Vol. 1. Translated from the German by Arthur M. DuPre*, volume 1. Beijing: Higher Education Press, 2017.

KF17b. Felix Klein and Robert Fricke. *Lectures on the theory of elliptic modular functions. Vol. 2. Translated from the German by Arthur M. DuPre*, volume 2. Beijing: Higher Education Press, 2017.

Kob93a. Neal Koblitz. *Introduction to elliptic curves and modular forms*, volume 97 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1993.

Kob93b. Neal Koblitz. *Introduction to elliptic curves and modular forms*, volume 97 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1993.

KZ95. Masanobu Kaneko and Don Zagier. A generalized Jacobi theta function and quasi-modular forms. In *The moduli space of curves (Texel Island, 1994)*, volume 129 of *Progr. Math.*, pages 165–172. Birkhäuser Boston, Boston, MA, 1995.

KZ01. Maxim Kontsevich and Don Zagier. Periods. In *Mathematics unlimited—2001 and beyond*, pages 771–808. Springer, Berlin, 2001.

Lan78a. Serge Lang. *Elliptic curves: Diophantine analysis*, volume 231 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1978.

Lan78b. Serge Lang. Sur la conjecture de Birch-Swinnerton-Dyer (d'après J. Coates et A. Wiles). In *Séminaire Bourbaki, 29e année (1976/77)*, volume 677 of *Lecture Notes in Math.*, pages Exp. 503, pp. 189–200. Springer, Berlin, 1978.

Lib11. Anatoly Libgober. Elliptic genera, real algebraic varieties and quasi-Jacobi forms. In *Topology of stratified spaces*, volume 58 of *Math. Sci. Res. Inst. Publ.*, pages 95–120. Cambridge Univ. Press, Cambridge, 2011.

LMF13. The LMFDB Collaboration. The l-functions and modular forms database. `http://www.lmfdb.org`, 2013.

Lor16. F. Loray. Isomonodromic deformation of Lamé connections, Painlevé VI equation and Okamoto symmetry. *Izv. Math.*, 80(1):113–166, 2016.

Lut37. Elisabeth Lutz. Sur l'équation $y^2 = x^3 - Ax - B$ dans les corps p-adiques. *J. Reine Angew. Math.*, 177:238–247, 1937.

Man63. Yu.I. Manin. Rational points on algebraic curves over function fields. *Izv. Akad. Nauk SSSR Ser. Mat.*, 27:1395–1440, 1963.

Man64. Yu.I. Manin. Algebraic curves over fields with differentiation. *Transl., Ser. 2, Am. Math. Soc.*, 37:59–78, 1964.

Maz77. B. Mazur. Modular curves and the Eisenstein ideal. *Inst. Hautes Études Sci. Publ. Math.*, (47):33–186 (1978), 1977.

Maz78. B. Mazur. Rational isogenies of prime degree (with an appendix by D. Goldfeld). *Invent. Math.*, 44(2):129–162, 1978.

Maz91. B. Mazur. Number theory as gadfly. *Am. Math. Mon.*, 98(7):593–610, 1991.

Maz01. Marta Mazzocco. Picard and Chazy solutions to the Painlevé VI equation. *Math. Ann.*, 321(1):157–195, 2001.

Mer96. Loïc Merel. Bornes pour la torsion des courbes elliptiques sur les corps de nombres. *Invent. Math.*, 124(1-3):437–449, 1996.

Mey05. Christian Meyer. *Modular Calabi-Yau threefolds*, volume 22 of *Fields Inst. Monogr.* Providence, RI: American Mathematical Society (AMS), 2005.

MFK94. D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2)*. Springer-Verlag, Berlin, third edition, 1994.

Mil04. Victor S. Miller. The Weil pairing, and its efficient calculation. *J. Cryptology*, 17(4):235–261, 2004.

Mil17. J. S. Milne. *Algebraic groups. The theory of group schemes of finite type over a field*, volume 170. Cambridge: Cambridge University Press, 2017.

Mil20. James S. Milne. *Elliptic curves*. Hackensack, NJ: World Scientific, 2020.

MN20. Hossein Movasati and Younes Nikdelan. Product formulas for weight two newforms. *Cadernos de IME, No. 14*, pages 1–7, March 2020.

MN21. H. Movasati and Y. Nikdelan. Gauss-Manin connection in disguise: Dwork family. *J. Differ. Geom.*, 119(1):73–98, 2021.

MO97. Yves Martin and Ken Ono. Eta-quotients and elliptic curves. *Proc. Amer. Math. Soc.*, 125(11):3169–3176, 1997.

Mor22. L. J. Mordell. On the rational solutions of the indeterminate equations of the third and fourth degrees. *Proc. Camb. Philos. Soc.*, 21:179–192, 1922.

Mor69. L. J. Mordell. *Diophantine equations*. Pure and Applied Mathematics, Vol. 30. Academic Press, London, 1969.

Mov08. H. Movasati. On differential modular forms and some analytic relations between Eisenstein series. *Ramanujan J.*, 17(1):53–76, 2008.

Mov11. H. Movasati. Eisenstein type series for Calabi-Yau varieties. *Nuclear Phys. B*, 847(2):460–484, 2011.

Mov12. H. Movasati. Quasi-modular forms attached to elliptic curves, I. *Ann. Math. Blaise Pascal*, 19(2):307–377, 2012.

Mov15a. H. Movasati. Modular-type functions attached to mirror quintic Calabi-Yau varieties. *Math. Zeit.*, 281, Issue 3, pp. 907-929(3):907–929, 2015.

Mov15b. H. Movasati. Quasi-modular forms attached to elliptic curves: Hecke operators. *J. Number Theory*, 157:424–441, 2015.

Mov17. H. Movasati. Gauss-Manin connection in disguise: Calabi-Yau modular forms. *Surveys of Modern Mathematics, Int. Press, Boston.*, 2017.

Mov21. H. Movasati. *A course in Hodge theory. With emphasis on multiple integrals*. Somerville, MA: International Press, 2021.

Mov22a. H. Movasati. *Modular and automorphic forms & beyond*, volume 9 of *Monogr. Number Theory*. Singapore: World Scientific, 2022.

Mov22b. Hossein Movasati. On elliptic modular foliations. II. *Mosc. Math. J.*, 22(1):103–120, 2022.

MR05. Francois Martin and Emmanuel Royer. Formes modulaires et périodes. In *Formes modulaires et transcendance*, volume 12 of *Sémin. Congr.*, pages 1–117. Soc. Math. France, Paris, 2005.

Mum91. David Mumford. *Tata lectures on theta. I, II, III. With the collaboration of C. Musili, M. Nori, E. Previato, and M. Stillman*. Basel: Birkhäuser, 1983, 1984, 1991.

MV21. Hossein Movasati and Roberto Villaflor. *A course in Hodge theory: Periods of algebraic cycles. 33º Colóquio Brasileiro de Matemática, IMPA, Rio de Janeiro, Brazil, 2021*. Rio de Janeiro: Instituto Nacional de Matemática Pura e Aplicada (IMPA), 2021.

Nag35. Trygve Nagell. Solution de quelques problèmes dans la théorie arithmétique des cubiques planes du premier genre. Technical report, Skr. Norske Vid.-Akad., Oslo 1935, No.1, 1-25 , 1935.

Nag52. Trygve Nagell. Problems in the theory of exceptional points on plane cubics of genus one. In *Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949*, pages 71–76. Johan Grundt Tanums Forlag, Oslo, 1952.

Obe18. Georg Oberdieck. Gromov-Witten invariants of the Hilbert schemes of points of a $K3$ surface. *Geom. Topol.*, 22(1):323–437, 2018.

Ogg71. A. P. Ogg. Rational points of finite order on elliptic curves. *Invent. Math.*, 12:105–111, 1971.

Pha79. Frédéric Pham. Singularités des systèmes différentiels de Gauss-Manin. Avec des contributions de Lo Kam Chan, Philippe Maisonobe et Jean-Etienne Rombaldi. Progress in Mathematics. 2. Boston-Basel-Stuttgart: Birkhäuser. V, 339 p. SFr. 34.00 (1979)., 1979.

Pic89. E. Picard. Mémoire sur la théorie des fonctions algébriques de deux variables. *Journ. de Math. (4)*, 5:135–319, 1889.

Ram16. S. Ramanujan. On certain arithmetical functions. *Trans. Cambridge Philos. Soc.*, 22:159–184, 1916.

Ram00. Srinivasa Ramanujan. *Collected papers of Srinivasa Ramanujan. Edited by G. H. Hardy, P. V. Seshu Aiyar and B. M. Wilson. With a new preface and commentary by Bruce C. Berndt*. Providence, RI: AMS Chelsea Publishing, 2000.

Rie59. B. Riemann. Ueber die anzahl der primzahlen unter einer gegebenen grösse. *Monat. der Königl. Preuss. akad. der Wissen. zu Berlin (also Gesmmelte math. Werke, 1892, 145-155)*, pages 671–680, 1859.

Rog13. Mathew Rogers. Identities for the Ramanujan zeta function. *Adv. Appl. Math.*, 51(2):266–275, 2013.

RS11. Martin Raussen and Christian Skau. Interview with Abel Laureate John Tate. *Notices Am. Math. Soc.*, 58(3):444–452, 2011.

Sai01. Kyoji Saito. Primitive automorphic forms. In *Mathematics unlimited—2001 and beyond*, pages 1003–1018. Springer, Berlin, 2001.

Sas74. T. Sasai. Monodromy representations of homology of certain elliptic surfaces. *J. Math. Soc. Japan*, 26:296–305, 1974.

Sch73. Wilfried Schmid. Variation of Hodge structure: the singularities of the period mapping. *Invent. Math.*, 22:211–319, 1973.

Sch13. Matthias Schütt. Two lectures on the arithmetic of $K3$ surfaces. In *Arithmetic and geometry of $K3$ surfaces and Calabi-Yau threefolds. Proceedings of the workshop, Toronto, Canada, August 16–25, 2011*, pages 71–99. New York, NY: Springer, 2013.

Sel51. Ernst S. Selmer. The Diophantine equation $ax^3+by^3+cz^3=0$. *Acta Math.*, 85:203–362 (1 plate), 1951.

Ser69. Jean-Pierre Serre. Une interpretation des congruences rélatives à la fonction de Ramanujan. Semin. Delange-Pisot-Poitou 9 (1967/68), Theorie Nombres, No. 14, 17p. (1969)., 1969.

Ser78. Jean-Pierre Serre. A course in arithmetic. Translation of "Cours d'arithmetique". 2nd corr. print. Graduate Texts in Mathematics, Vol. 7. New York, Heidelberg, Berlin: Springer-Verlag. IX, 115 p. $ 18.70; DM 34.00 (1978)., 1978.

Ser89. Jean-Pierre Serre. *Lectures on the Mordell-Weil theorem*. Aspects of Mathematics, E15. Friedr. Vieweg & Sohn, Braunschweig, 1989. Translated from the French and edited by Martin Brown from notes by Michel Waldschmidt.

Ser97. Jean-Pierre Serre. *Lectures on the Mordell-Weil theorem. Transl. and ed. by Martin Brown from notes by Michel Waldschmidt. 3rd ed.* Wiesbaden: Vieweg, 3rd ed. edition, 1997.

Shi86. Tetsuji Shioda. An explicit algorithm for computing the Picard number of certain algebraic surfaces. *Am. J. Math.*, 108:415–432, 1986.

Shi92. Tetsuji Shioda. Some remarks on elliptic curves over function fields. In *Journées arithmétiques. Exposés présentés aux dix-septièmes congrès à Genève, Suisse, 9-13 septembre 1991*, pages 99–114. Paris: Société Mathématique de France, 1992.

Sil92a. Joseph H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1992. Corrected reprint of the 1986 original.

Sil92b. Joseph H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1992. Corrected reprint of the 1986 original.

Sil94a. Joseph H. Silverman. *Advanced topics in the arithmetic of elliptic curves*, volume 151 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.

Sil94b. Joseph H. Silverman. *Advanced topics in the arithmetic of elliptic curves*, volume 151 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.

SS96. Norbert Schappacher and René Schoof. Beppo Levi and the arithmetic of elliptic curves. *Math. Intelligencer*, 18(1):57–69, 1996.

SS03. Elias M. Stein and Rami Shakarchi. *Complex analysis*. Princeton Lectures in Analysis, II. Princeton University Press, Princeton, NJ, 2003.

ST92. Joseph H. Silverman and John Tate. *Rational points on elliptic curves*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1992.

Sti88. P. F. Stiller. Classical automorphic forms and hypergeometric functions. *J. Number Theory*, 28(2):219–232, 1988.

Tat68. John Tate. Residues of differentials on curves. *Ann. Sci. École Norm. Sup. (4)*, 1:149–159, 1968.

Tun83. J. B. Tunnell. A classical Diophantine problem and modular forms of weight $3/2$. *Invent. Math.*, 72(2):323–334, 1983.

Wal06. Michel Waldschmidt. Transcendence of periods: the state of the art. *Pure Appl. Math. Q.*, 2(2):435–463, 2006.

Wei29. A. Weil. L'arithmétique sur les courbes algébriques. *Acta Math.*, 52:281–315, 1929.

Wei49. André Weil. Numbers of solutions of equations in finite fields. *Bull. Amer. Math. Soc.*, 55:497–508, 1949.

Wei52. André Weil. Number-theory and algebraic geometry. Proc. Internat. Congr. Math. (Cambridge, Mass., Aug. 30-Sept. 6, 1950) 2, 90-100 (1952)., 1952.

Wei99. André Weil. *Elliptic functions according to Eisenstein and Kronecker. Reprint of the 1976 edition.* Berlin: Springer, reprint of the 1976 edition edition, 1999.

Wil95. Andrew Wiles. Modular elliptic curves and Fermat's last theorem. *Ann. of Math. (2)*, 141(3):443–551, 1995.

Yui13. N. Yui. Modularity of Calabi-Yau varieties: 2011 and beyond. In *Arithmetic and geometry of K3 surfaces and Calabi-Yau threefolds*, volume 67 of *Fields Inst. Commun.*, pages 101–139. Springer, New York, 2013.

Zag08. Don Zagier. Elliptic modular forms and their applications. In *The 1-2-3 of modular forms*, Universitext, pages 1–103. Springer, Berlin, 2008.

Zar52. Oscar Zariski. The fundamental ideas of abstract algebraic geometry. Proc. Internat. Congr. Math. (Cambridge, Mass., Aug. 30-Sept. 6, 1950) 2, 77-89 (1952)., 1952.

Index

A, Gauss-Manin connection matrix, 60
B_k, Bernoulli numbers, 24
B_m, Bernoulli numbers, 165
E, elliptic curve, 49
$E(\mathbb{Q})_{\text{free}}$, free part of $E(\mathbb{Q})$, 99
$E(\mathbb{Q})_{\text{tors}}$, torsion part of $E(\mathbb{Q})$, 99
$E[m]$, m-torsions of E, 108
E_2, Eisenstein series, 27
E_k, Eisenstein series, 26
G_n, Eisenstein series, 19, 22
$H^1(X/\mathsf{R})$, De Rha cohomology, 139
$H^n_{\text{dR}}(X)^G$, invariant cohomology, 233
$H_n(X,\mathbb{Z})^G$, invariant homology, 233
$H^i_{\text{dR}}(X/T)$, De Rham cohomology, 135
$L(f,s)$, Hecke's L-function, 168
$L_S(E,s)$, non-complete L-function, 184
$M(\mathrm{SL}(2,\mathbb{Z}))$, algebra of modular forms, 28
N, level of modular curves, degree of an isogeny, 110
$N_{E/\mathbb{Q}}$, conductor, 185
$R(f,s)$, a modification of $L(f,s)$, 168
S, an element of the modular group, 16
$SL(2,\mathbb{R})$, a group, 15
T, an element of the modular group, 16
$[N]$, multiplication by N, 110
$\mathbb{A}^n$, affine space, 78
$\mathbb{C}^*$, multiplicative group of $\mathbb{C}$, 12
Δ, discriminant, 30
$\mathrm{Div}(C/\mathsf{k})$, divisors of C, 91
$\mathbb{F}_p$, Finite field, 179
$\mathbb{F}_p$, finite field with p elements, 77
$\mathbb{G}_a$, additive group, 182
$\mathbb{G}_m$, multiplicative group, 182
$\mathbb{G}_m[a]$, twisted multiplicative group, 182
$\Gamma(s)$, Gamma function, 162
L, Period domain, space of lattices, 57
L, space of lattices, 12
Λ, lattice, 11
Λ, a lattice, 12
$\mathrm{M}_k(\mathrm{SL}(2,\mathbb{Z}))$, modular forms of weight k, 116
$\Omega^i_{X/T}$, differential i-forms, 135
$\mathbb{P}^n$, affine space, 78
$\mathrm{Pic}(C), \mathrm{Pic}_0(C)$, Picard group, 92
$\mathrm{SL}(2,Z)$, $\mathrm{PSL}(2,\mathbb{Z})$ Modular group, 12
$\mathsf{T}_0(N), \mathsf{T}_1(N), \mathsf{T}(N)$, the set of enhanced elliptic, 144
$\mathrm{div}(f)$, divisor of a rational function, 92
$\eta(\tau)$, Dedekind eta function, 31
G, a finite group, 231
μ_d, group of d-th roots of unity, 232
k, field, 77
M^n_m, quasi-modular forms of weight m and differential order n, 149
M, quasi-modular forms, 149
Π, generlized period domain, 150
P, period matrix, 60
S_{n+2}, permutations in $n+2$ elements, 231
σ, Weierstrass sigma function, 20
$\sigma_a(n)$, sum of divisors, 24
τ, a coordinate or point in the upper half plane, 12
$\tau(n)$, Ramanujan's τ function, 30
$\theta(z,\tau)$, Jacobi's theta function, 36
$\theta_{a,b}$, theta constant, 40
$\wp$, Weierstrass $\wp$ function, 18
ζ, Weierstrass zeta function, 20
$\zeta(s)$, Riemann's zeta function, 24, 160
$\zeta_S(E,s)$, non-complete zeta function, 184
d, differential, 134
$f(i\infty)$, evaluation at $i\infty$, 28
$f\,|_k\,A$, slash operator, 115
$g_i(\tau)$, a rescaling of Eisenstein series, 218
j, j or modular function, 32
[, 123–126, 129

Additive group, 182
Additive reduction, 183
Affine space, 78
Algebra of modular forms, 28
Automorphic form, 41
Automorphy factor, 41

Bernoulli numbers, 24, 165
Birch Swinnerton-Dyer conjecture, 185
BSD conjecture, 185

Canonical height, 104
Chern class, 92
Classical fundamental domain, 15
CM, complex multiplication, 112
compactification, 78
Complex multiplication, 112
Conductor, 185
Congrunet number, 186
cusp form, 28
Cuspidal reduction, 183

Darboux-Halphen differential equation, 45
Darboux-Halphen vector field, 64
De Rham cohomology, 135, 139
Dedekind eta function, 4, 31
Deligne's family, 233
Descent theorem, 100
differential, 134
differential *i*-forms, 135
Discriminant idela, 84
distinguished set of paths, 50
divisor, 91
Dual isogeny, 110
Dwork family, 111, 232

Eigenform, 120
Eisenstein summation, 43
Elliptic curve, 49
Elliptic function, 14
Elliptic integral, 47
Enhanced elliptic curve, 108, 143
Euler's product formula, 160, 172
Euler's reflection formula, 163

Fendamental domain, 15
Fermat curve, 78
Fibonacci sequence, 1
Finite field, 179
Fourier expansion, 23
Fourire transform, 42

Gamma function, 162
Gauss hypergemetric function, 67
Gauss multiplication relation, 163
Gauss-Manin connection, 60
Generlized period domain, 150
Golden ration/number, 2
Good reduction, 183

Heat equation, 40
Hecke operator, 116
Hecke's L-function, 168
Height function, 100
Holomorhic modular form, 22
Holomorphic at infinity, 22

Intersection form, 51
Invariant cohomology, 233
Invariant homology, 233
inversion formulas, 71
Isogeny, 110

Jacobi family, 65, 66
Jacobi's identity, 40
Jacobi's theta function, 36

Lattice, 11
Lattice of elliptic integrals, 52
Lefschetz principle, 108
Legendre relations between elliptic integrals, 61

Mellin inversion theorem, 174
Mellin transform, 164
mermorphic modular form, 23
Modular form, 14, 22
Modular function, 23, 32
Modular group, 12
Monodromy, 53
Monodromy action, 52
Monodromy group, 53
Mordell-Weil theorem, 99
Multiplicative group, 182

Neron-Tate height, 104
Nodal reduction, 183
Non-complete L-function, 184
Non-complete zeta function, 184
Nonsplit multiplicative reduction, 183
Normalized eigenform, 120

Period domain, 57
Period map, 57
Period matrix, 60
Picard group, 41, 92
Picard moduli space, 224
Picard-Fuchs equation, 65

Picard-Lefschetz theory, 52
Poincaré series, 41
Point at infinity, 49
Poisson summation formula, 42
Projective space, 78
Projective space at infinity, 78

Quasi-modular form, 148

Ramanujan relations between eisenstein series, 29
Ramanujan vector field, 64
Ramanujan's τ function, 30
Rank of an elliptic curve, 99
Riemann's zeta function, 24, 160

Schwarz map, 69
Schwarzian derivative, 33
Serre derivative, 29
Singular point, 81
Singularity, 81
Slash operator, 115
Split multiplicative reduction, 183

Theta function, 36
Torsion point, 108
Torus, 12
Trivial automorphy factor, 41
Twisted multiplicative group, 182

upper half plane, 15

Vanishing cycle, 53

Weak BSD conjecture, 186
Weakly holomorphic modular form, 23
Weierstrass $\wp$ function, 18
Weierstrass coordinates, 94
Weierstrass form, 49
Weierstrass format, 52
Weierstrass sigma function, 20
Weierstrass uniformization theorem, 55
Weierstrass zeta function, 20