Leaf schemes and Hodge loci

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Abstract

We develop a theory of (singular) foliations on schemes such that the leaves are also equipped with scheme structure and they might have different dimensions for a given foliation. Natural examples of such foliations come from the study of Hodge loci. In our context we encounter non-trivial questions with foliations of dimension zero in which generic leaves are points, however, we have special leaves of different dimensions passing through the singular locus of the foliation. We also aim to formulate a local-global conjecture, which has as special cases the Katz-Grothendieck p-curvature conjecture and a conjecture of Ekedahl and Shepherd-Barron on the algebraic integrability of leaves of non-singular foliations. More importantly, we discuss its relation with two consequences of the Hodge conjecture. First, all Hodge cycles are absolute in the sense of Deligne and second, the field of definition of Hodge loci is the algebraic closure of the base field. These are the lecture notes of a course given at BIMSA and YMSC during 2023.

Contents

1 Introduction

The theory of holomorphic foliation in complex manifolds has its origin in differential equations, and in particular the second part of Hilbert's 16th problem on the uniform boundedness of number of limit cycles. However, in the last decades, it has been investigated as a branch of algebraic geometry. Despite this, considering foliations on non-reduced schemes or with non-reduced leaves might seem abstract non-sense. In the present text we develop a theory in which a holomorphic foliation is identified with its module of differential 1-forms (which may not be saturated) and so it is not just the underlying geometric object. Moreover, its leaves might be non-reduced with different codimenions for which we use the name leaf scheme. The origin of this theory goes back to [Mov17] and [Mov22, Chapter 5,6] and we follow and further develop these texts.

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For experts in holomorphic foliations the theory of holomorphic foliations in which leaves are replaced with leaf schemes might seem an abstract non-sense. This is mainly due to the lack of motivation and simple examples. We are mainly motivated by applications to Hodge loci, even though the corresponding foliations cannot be written explicitly. The main goal is to introduce tools for proving the main conjectures in [Mov23, Mov21b].

A beginner might consult the following literature on holomorphic foliations. The Jouanolou's lecture notes [Jou79] is mainly concerned with codimension one foliations. For an introduction to one dimensional holomorphic foliations in the local context, the reader is referred to Camacho and Sad's book [CS87] or Loray's book [Lor06]. Lins Neto and Scárdua's book [LNS] and Lins Neto's monograph [Net07] give a nice account of foliations in the projective spaces. Local study of foliations of arbitrary codimension is partially discussed in Medeiros' articles [dM77, dM00] and [CL16]. For a more arithmetically oriented text the reader is referred to [MP97, LPT11, LPT13, Bos01, MP23] and references therein. In the C^∞ context, Camacho and Lins Neto's book [CL85] and Godbillons' book [God91] contain many geometric statements on foliations, and depending on applications, one might try to find the corresponding scheme theoretic counterparts.

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2 Preliminaries in algebraic geometry

2.1 The base ring and field

We consider a commutative ring R with multiplicative identity element $1 \neq 0$. We also consider projective varieties defined over R. This uses a finite number of elements of R, therefore, in practice we can assume that the ring R is finitely generated, and so

$$\mathsf{R} := \frac{\mathbb{Z}[t_1, t_2, \cdots, t_r]}{I}, \text{ where } I \subset \mathbb{Z}[t_1, t_2, \cdots, t_r], \text{ is an ideal.}$$

Note that if $n \in \mathbb{N}$ is invertible in R then we reserve the variable t_1 for this purpose and assume that $nt_1 - 1 \in I$. We assume that R is an integral domain. In other words, it is without zero divisors, that is, if for some $a, b \in \mathsf{R}$, $ab = 0$ then $a = 0$ or $b = 0$, (this is the same as to say that I is a prime ideal). In particular, it is reduced, that is, it has not nilpotent elements. The characteristic of R is the smallest $p \in \mathbb{N}$ such that the sum of $1 \in \mathsf{R}$, p-times, is zero. It is either a prime number or zero. In the first case we can write $\mathsf{R} = \frac{\mathbb{F}_p[t_1, t_2, \cdots, t_r]}{I}$, where I is now an ideal in $\mathbb{F}_p[t_1, t_2, \cdots, t_r]$. In the second case, we have $\mathbb{Z} \subset \mathsf{R}$, and we define N to be the greatest positive integer which is invertible in R and we have $\mathsf{R} = \frac{\mathbb{Z}[\frac{1}{N}, t_1, t_2, \cdots, t_r]}{I}$. The primes dividing the number N are called the bad primes of the ring R. By our assumptions R is Noetherian, that is, every ideal of R is finitely generated. We denote by k the field obtained by the localization of R over $\mathsf{R} \setminus \{0\}$ and by $\bar{\mathsf{k}}$ the algebraic closure of k.

Proposition 1. *Let R be a finitely generated, commutative ring with multiplicative identity element $1 \neq 0$, of characteristic zero and without zero divisors. There exists an embedding $\mathsf{R} \hookrightarrow \mathbb{C}$ which makes R a subring of $\mathbb{C}$.*

Proof. We take the quotient field k of R and construct an embedding of fields $\mathsf{k} \hookrightarrow \mathbb{C}$. Since we have a canonical injective morphism $\mathsf{R} \hookrightarrow \mathsf{k}$ of rings this would prove the proposition. Let k be generated by $a_1, a_2, \ldots, a_r$ over $\mathbb{Q}$. The proof is by induction on the number r. The case $r = 0$ is trivial as we have a unique embedding $\mathbb{Q} \hookrightarrow \mathbb{C}$. Let us assume the proposition for $r - 1$. We define $\tilde{\mathsf{k}}$ to be the subfield of k generated by $a_1, a_2, \ldots, a_{r-1}$. By the hypothesis of induction we

have an embedding $\tilde{\mathsf{k}} \hookrightarrow \mathbb{C}$. Let

$$A := \{P \in \tilde{\mathsf{k}}[x] \mid P(a_r) = 0\}.$$

If $A = \{0\}$ then we choose a transcendental number b over $\tilde{\mathsf{k}}$ and we construct $\mathsf{k} \hookrightarrow \mathbb{C}$ by sending a_r to b. If not then A is an ideal of $\tilde{\mathsf{k}}[x]$. This ideal is generated by a unique monic polynomial $P(x)$. We take a number $b \in \mathbb{C}$ such that the minimal polynomial of b over $\tilde{\mathsf{k}}$ is $P(x)$ and send a_r to b. $\qquad\square$

When we choose a transcendental number a then we have an infinite number of options, whereas when we choose an algebraic number over $\tilde{\mathsf{k}}$ we have a finite number. For this reason, the number of embeddings $\mathsf{R} \hookrightarrow \mathbb{C}$ is infinite if the transcendendtal degree of the quotient field k of R is strictly bigger than zero. The main motivation behind Proposition 1 is the following version of Lefschetz principal. For a property P talking about schemes (and foliations on them) defined over a ring R, we may assume that $\mathsf{R} \subset \mathbb{C}$ and it has finite transcendence degree over rational numbers. Therefore, in order to prove P, we can use all the possible transcendental methods.

We will need to fix a subring $\mathfrak{R} \subset \mathsf{R}$ with the same properties as of R. In most of the cases this is going to be $\mathfrak{R} = \mathbb{Z}[\frac{1}{N}]$,   or $\mathfrak{R} := \mathbb{F}_p$. The quotient field of $\mathfrak{R}$ is denoted by $\mathfrak{k}$. The most famous rings in the present text are the following

$$(1) \qquad\qquad \mathsf{R} := \mathbb{Z}\left[t_1, t_2, t_3, \frac{1}{6(27t_3^2 - t_2^3)}\right], \quad \mathfrak{R} := \mathbb{Z}\left[\frac{1}{6}\right].$$

Note that by definition of $\mathfrak{R}$ only finite number of primes $p \in \mathbb{N}$ are invertible in T. We denote by $\tilde{\mathfrak{R}}$ the smallest subring of the quotient field $\mathfrak{k}$ of $\mathfrak{R}$ containing both $\mathfrak{R}$ and $\mathbb{Q}$. In other words, $\tilde{\mathfrak{R}} = \mathfrak{R} \otimes_{\mathbb{Z}} \mathbb{Q}$.

2.2   Schemes over rings

A basic knowledge of algebraic geometry of schemes would be enough for our purposes, see for instance the first chapters of Hartshorne's book [Har77] or Eisenbud and Harris's book [EH00].

We will need schemes T over $\mathfrak{R}$ which are covered with a finite number of affine schemes of type $\mathrm{Spec}(\mathsf{R}_i)$, $i = 1, 2, \ldots$, where R_i is the ring in Section 2.1. In most of the cases $\mathsf{T} := \mathrm{Spec}(\mathsf{R})$. For our purpose, we make the following definition.

Definition 2. A scheme T over $\mathfrak{R}$, or $\mathfrak{R}$-scheme for simplicity, satisfies the following properties:

1. The morphism of schemes $\mathsf{T} \to \mathrm{Spec}(\mathfrak{R})$ is of finite type. This means that there is a covering of T by open affine subsets $\mathsf{T}_i := \mathrm{Spec}(\mathsf{R}_i)$ such that R_i is a finitely generated $\mathfrak{R}$-algebra.

2. T is irreducible, that is, the underlying topological space does not contain two proper disjoint nonempty open sets. In particular, T is connected, that is, it cannot be written as a disjoint union of two nonempty open sets.

3. T is reduced, that is, T is covered by open sets $\mathsf{T}_i := \mathrm{Spec}(\mathsf{R}_i)$ such that R_i is reduced.

Note that by definition an integral scheme is reduced and irreducible, and so, a $\mathfrak{R}$-scheme T is integral. The ring (resp. field) of regular (resp. rational) functions on T is denoted by $\mathfrak{k}[\mathsf{T}]$ (resp. $\mathfrak{k}(\mathsf{T})$). A morphism of R-schemes $\mathsf{T}_1 \to \mathsf{T}_2$ is a morphism of schemes such that

$$(2) \qquad\qquad \begin{array}{ccc} \mathsf{T}_1 & \to & \mathsf{T}_2 \\ & \searrow \swarrow & \\ & \mathrm{Spec}(\mathfrak{R}) & \end{array}$$

is commutative. If we have larger ring $\mathfrak{R} \subset \tilde{\mathfrak{R}}$ then we can consider T it as a scheme over $\tilde{\mathfrak{R}}$ which we denote it by $\mathsf{T}_{\tilde{\mathfrak{R}}}$:
$$\mathsf{T}_{\tilde{\mathfrak{R}}} := \mathsf{T} \times_{\mathfrak{R}} \operatorname{Spec}(\tilde{\mathfrak{R}}).$$

Definition 3. Let T be a scheme over $\mathfrak{R}$. An $\mathfrak{R}$-valued point in T is a morphism of schemes $\operatorname{Spec}(\mathfrak{R}) \to \mathsf{T}$. The set of $\mathfrak{R}$-valued points of T is denoted by $\mathsf{T}(\mathfrak{R})$.

If $\mathsf{T} = \operatorname{Spec}(\mathsf{R})$ is an affine scheme then an $\mathfrak{R}$-valued point of T correspond to an $\mathfrak{R}$-algebra homomorphism $t : \mathsf{R} \to \mathfrak{R}$. Its kernel is a prime ideal (recall that $\mathfrak{R}$ has no zero divisors). This gives a point of T in the definition of a scheme. An $\mathfrak{R}$-valued point can be identified with a prime ideal $\mathfrak{p}$ of R together with an isomorphism $\mathsf{R}/\mathfrak{p} \cong \mathfrak{R}$. If $\mathsf{R} = \mathfrak{R}[t_1, t_2, \cdots, t_r]/I$ then an $\mathfrak{R}$-valued point is identified with the image of t_i's under $\mathsf{R} \to \mathfrak{R}$. This is denoted by $P = (P_1, P_2, \ldots, P_r) \in \mathfrak{R}^r$. When there is no confusion we denote this again by t and not P. Note that $\mathfrak{R}$-valued points are not necessarily closed, except when $\mathfrak{R}$ is a field. In the prest text there will be two steps in localization: first we localize to an $\mathfrak{R}$-valued point, represented by a prime ideal $\mathfrak{p}$, and then we take a maximal ideal $\tilde{\mathfrak{p}}$ containing $\mathfrak{p}$. The first localization can be referred as geometric and the latter will be referred as arithmetic localization.

Exercise 4. Show that
$$\mathbb{P}^n_{\mathbb{Z}}(\mathbb{Z}) \cong \{(m_0, m_1, \cdots, m_n) \in \mathbb{Z}^{n+1} \mid \gcd(m_0, m_1, \cdots, m_n) = 1\}/\{\pm 1\}.$$

2.3 Differential forms on schemes

We will need the sheaf of differential 1-forms in T. It is enough to define it for the case of affine schemes $\mathsf{T} := \operatorname{Spec}(\mathsf{R})$. In this case for the definition of $\Omega_{\mathsf{R}/\mathfrak{R}} = \Omega^1_{\mathsf{T}}$ and also Ω^i_{T} see [Mov21a, Section 10.3]. Classical references are [Har77] and [Eis95].

Definition 5. The dimension of the $\mathfrak{R}$-scheme T is the number a such that $\Omega^{a+1}_{\mathsf{T}}$ is the torsion sheaf and Ω^a_{T} is not. The sheaf Ω^a_{T} is called the canonical sheaf of T.

Recall that a sheaf on T is called a torsion sheaf if any section s of this sheaf is anihilated by some non-zero section of $\mathcal{O}_{\mathsf{T}}$ in the same open set as of s. For singular T, $\Omega^{a+1}_{\mathsf{T}}$ might be non-zero and it is well-known as Milnor or Tjurina module in singularity theory, see for instance [Mov21a, Chapter 10].
Let
$$\mathfrak{m}_{\mathsf{T},t} := \left\{ f \in \mathcal{O}_{\mathsf{T},t} \mid f(t) = 0 \right\}.$$
It is a maximal ideal if $\mathfrak{R}$ is a field.

Definition 6. An $\mathfrak{R}$-valued point t is called smooth if the $\mathfrak{R}$-module $\mathfrak{m}_{\mathsf{T},t}/\mathfrak{m}^2_{\mathsf{T},t}$ is a free of rank $\dim(\mathsf{T})$. We say that T is smooth if the veriety $\mathsf{T}_{\bar{\mathfrak{k}}}$ is smooth at any $\bar{\mathfrak{k}}$-valued point.

Exercise 7. For a smooth $\mathfrak{R}$-schemes of dimension a, $\Omega^{a+1}_{\mathsf{T}}$ is the zero sheaf.

The canonical sheaf of T is an invertible sheaf, that is, in local charts it is free of rank 1. Therefore, it comes from a Cartier divisor in T, see Hartshorne's book [Har77, Propsition 6.13, page 144].

Definition 8. Let T be a $\mathfrak{R}$-scheme of dimension a and let Ω be a submodule of the $\mathcal{O}_{\mathsf{T}}$-module Ω^m_{T} for some $m \geq 1$. We have

(3)
$$\Omega \bigwedge \Omega^{a-m}_{\mathsf{T}} \subset \Omega^a_{\mathsf{T}}.$$

and so we have an ideal sheaf $\mathrm{ZeId}(\Omega) \subset \mathcal{O}_{\mathsf{T}}$, which we call it the zero ideal, such that the left hand side of (3) is equal to $\mathrm{ZeId}(\Omega) \cdot \Omega_{\mathsf{T}}^n$. The zero scheme of Ω is defined to be

$$\mathrm{ZeSc}(\Omega) := \mathrm{Spec}(\mathcal{O}_{\mathsf{T}}/\mathrm{ZeId}(\Omega)).$$

Definition 9. For an $\mathfrak{R}$-valued point t of T, we consider the ring of formal power series $\mathcal{O}_{\mathsf{T}^{\mathrm{for}},t}$ in a neighborhood of t. This is the direct limit of

$$\cdots \to \mathcal{O}_{\mathsf{T},t}/\mathfrak{m}_{\mathsf{T},t}^{n+1} \to \mathcal{O}_{\mathsf{T},t}/\mathfrak{m}_{\mathsf{T},t}^{n} \to \cdots .$$

In case $\mathfrak{R} \subset \mathbb{C}$ we are also interested in the ring of convergent power series $\mathcal{O}_{\mathsf{T}^{\mathrm{hol}},t}$ in a neighborhood of t. These are also called the germs of holomorphic functions in a neighborhood of t. We have natural inclusions

$$\mathcal{O}_{\mathsf{T},t} \subset \mathcal{O}_{\mathsf{T}^{\mathrm{hol}},t} \subset \mathcal{O}_{\mathsf{T}^{\mathrm{for}},t}$$

For which we also use the geometric notation:

$$(\mathsf{T}^{\mathrm{for}}, t) \subset (\mathsf{T}^{\mathrm{hol}}, t) \subset (\mathsf{T}, t).$$

Note that we do not define schemes $\mathsf{T}^{\mathrm{for}}$ or $\mathsf{T}^{\mathrm{hol}}$. For a definition of formal scheme the reader might consult EGA I section 10.

Definition 10. A holomorphic coordinate $z = (z_1, z_2, \cdots, z_n)$ system in (T, t) is a collection of functions $z_i \in \mathfrak{m}_{\mathsf{T},t}$, $i = 1, 2, \ldots, n$ such that they form a basis of the $\mathfrak{R}$-module $\mathfrak{m}_{\mathsf{T},t}/\mathfrak{m}_{\mathsf{T},t}^2$.

Proposition 11. *Let t be a smooth point of the $\mathfrak{R}$-scheme T of dimension n and let A be the affine scheme $\mathbb{A}_{\mathfrak{R}}^n = \mathrm{Spec}(\mathfrak{R}[x_1, x_2, \ldots, x_n])$. We have an isomorphism of holomorphic schemes:*

$$\alpha : (\mathsf{T}^{\mathrm{hol}}, t) \to (A^{\mathrm{hol}}, 0), \alpha^* x_i = z_i.$$

This proposition enables us to use the affine scheme A instead of T once we want to prove a statement of local nature for T. We remark that for the proof of this proposition we do not need to enlarge $\mathfrak{R}$. For instance, we do not need to assume that integers are invaertible in $\mathfrak{R}$.

2.4 Vector fields

Definition 12. Let T be a $\mathfrak{R}$-scheme. The sheaf of vector fields is

$$\Theta_{\mathsf{T}} := (\Omega_{\mathsf{T}}^1)^{\vee}$$

An element in $\Theta_{\mathsf{T}}(U)$ for some open set $U \subset \mathsf{T}$, is by definition an $\mathcal{O}_{\mathsf{T}}(U)$-linear map $\Omega_{\mathsf{T}}^1(U) \to \mathcal{O}_{\mathsf{T}}(U)$ and it is called a vector field in U.

If T is a smooth variety over an algebraically closed field $\mathfrak{k}$, a vector field can be also interpreted as a section of the tangent bundle of T. The $\mathcal{O}_{\mathsf{T}}$-module of vector fields Θ_{T} is isomorphic to the sheaf of derivations.

Definition 13. A map $\mathsf{v} : \mathcal{O}_{\mathsf{T}} \to \mathcal{O}_{\mathsf{T}}$ is called a derivation if it is $\mathfrak{R}$-linear and it satisfies the Leibniz rule

$$\mathsf{v}(fg) = f\mathsf{v}(g) + \mathsf{v}(f)g, \quad f, g \in \mathcal{O}_{\mathsf{T}}.$$

We denote by $\mathrm{Der}(\mathcal{O}_{\mathsf{T}})$ the $\mathcal{O}_{\mathsf{T}}$-module of derivations.

Proposition 14. *We have an isomorphism of $\mathcal{O}_{\mathsf{T}}$-modules*

$$\Theta_{\mathsf{T}} \cong \mathrm{Der}(\mathcal{O}_{\mathsf{T}}),$$
$$\mathsf{v} \mapsto (f \mapsto \mathsf{v}(df)).$$

Proof. This isomorphism maps the vector field v to the corresponding derivation $\check{\mathsf{v}}$ obtained by

(4) $$\check{\mathsf{v}}(f) = \mathsf{v}(df).$$

Since in local charts Ω^1_{T} is genereted as $\mathcal{O}_{\mathsf{T}}$-module by $d\mathcal{O}_{\mathsf{T}}$, the equality (4) also defines its inverse $\check{\mathsf{v}} \mapsto \mathsf{v}$. $\square$

Definition 15. For a vector field v on the $\mathfrak{R}$-scheme T and an $\mathfrak{R}$-valued point t of T, we say that t is a singularity of v and write $\mathsf{v}(t) = 0$ if v maps $\mathcal{O}_{\mathsf{T},t}$ to its ideal the ideal $\mathfrak{m} := \{f \in \mathcal{O}_{\mathsf{T},t} | f(t) = 0\}$.

In $\mathrm{Der}(\mathcal{O}_{\mathsf{T}})$ we have the Lie bracket $[\mathsf{v}_2, \mathsf{v}_2]$, $\mathsf{v}_1, \mathsf{v}_2 \in \mathrm{Der}(\mathcal{O}_{\mathsf{T}})$ defined by

$$[\mathsf{v}_1, \mathsf{v}_2] := \mathsf{v}_1 \circ \mathsf{v}_2 - \mathsf{v}_2 \circ \mathsf{v}_1.$$

We have to show that $[\mathsf{v}_1, \mathsf{v}_2]$ is a derivation. It is $\mathfrak{R}$-linear becuase v_1 and v_2 are. The Leibniz rule

$$[\mathsf{v}_1, \mathsf{v}_2](fg) \quad = \quad f[\mathsf{v}_1, \mathsf{v}_2](g) + g[\mathsf{v}_1, \mathsf{v}_2](f)$$

is left as an exercise to the reader.

Definition 16. Let $f : \mathsf{T}_1 \to \mathsf{T}_2$ be morphism of $\mathfrak{R}$-schemes and v_i, $i = 1, 2$ be vector fields on T_i, $i = 1, 2$. We say that f maps v_1 to v_2 if the following diagram commutes:

(5)
$$\begin{array}{ccc} \Omega^1_{\mathsf{T}_2} & \to & \Omega^1_{\mathsf{T}_1} \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathsf{T}_2} & \to & \mathcal{O}_{\mathsf{T}_1} \end{array},$$

where the down arrows are respectively v_2 and v_1.

Note that for f as above we have the induced map in the sheaf of differential 1-forms, however, we do not have a morphism $\Theta_{\mathsf{T}_1} \to \Theta_{\mathsf{T}_2}$.

Definition 17. Let $\mathsf{T}_1, \mathsf{T}_2$ be two $\mathfrak{R}$-schemes and v be a vector field in T_1. A parallel extension of v in $\mathsf{T}_1 \times \mathsf{T}_2$ is a vector field $\check{\mathsf{v}}$ in $\mathsf{T}_1 \times \mathsf{T}_2$ such that under the first projection $\mathsf{T}_1 \times \mathsf{T}_2 \to \mathsf{T}_1$ it maps to v and under the second projection $\mathsf{T}_1 \times \mathsf{T}_2 \to \mathsf{T}_2$ it maps to the zero vector field, see Figure 1.

Proposition 18. *The parallel extension of a vector field exists and it is unique.*

Proof. By definition $\mathcal{O}_{\mathsf{T}_1 \times \mathsf{T}_2} = \mathcal{O}_{\mathsf{T}_1} \otimes_{\mathfrak{R}} \mathcal{O}_{\mathsf{T}_2}$ and hence

(6) $$\Omega^1_{\mathsf{T}_1 \times \mathsf{T}_2} = \Omega_{\mathsf{T}_1} \otimes_{\mathfrak{R}} \mathcal{O}_{\mathsf{T}_2} + \mathcal{O}_{\mathsf{T}_1} \otimes_{\mathfrak{R}} \Omega^1_{\mathsf{T}_2}.$$

For a $\mathcal{O}_{\mathsf{T}_1}$-linear map $\mathsf{v} : \Omega^1_{\mathsf{T}_1} \to \mathcal{O}_{\mathsf{T}_1}$, its parallel extension $\Omega^1_{\mathsf{T}_1 \times \mathsf{T}_2} \to \mathcal{O}_{\mathsf{T}_1 \times \mathsf{T}_2}$ evaluated at the first piece (resp. second piece) in (6) is $\mathsf{v} \otimes \mathrm{Id}$ (resp. zero). $\square$

There are some other differential geometric objects that will be useful later. The definition of Lie derivative is taken from Cartan's formula

(7) $$\mathcal{L}_{\mathsf{v}} : \Omega^i_{\mathsf{T}} \to \Omega^i_{\mathsf{T}}, \quad \mathcal{L}_{\mathsf{v}} := d \circ i_{\mathsf{v}} + i_{\mathsf{v}} \circ d$$

where i_{v} is the contraction of differential forms with the vector field v. We have the identities

(8) $$\mathcal{L}_{[\mathsf{v}_1, \mathsf{v}_2]} \quad = \quad [\mathcal{L}_{\mathsf{v}_1}, \mathcal{L}_{\mathsf{v}_2}],$$

(9) $$i_{[\mathsf{v}_1, \mathsf{v}_2]} \quad = \quad [\mathcal{L}_{\mathsf{v}_1}, i_{\mathsf{v}_2}] = [i_{\mathsf{v}_1}, \mathcal{L}_{\mathsf{v}_2}].$$
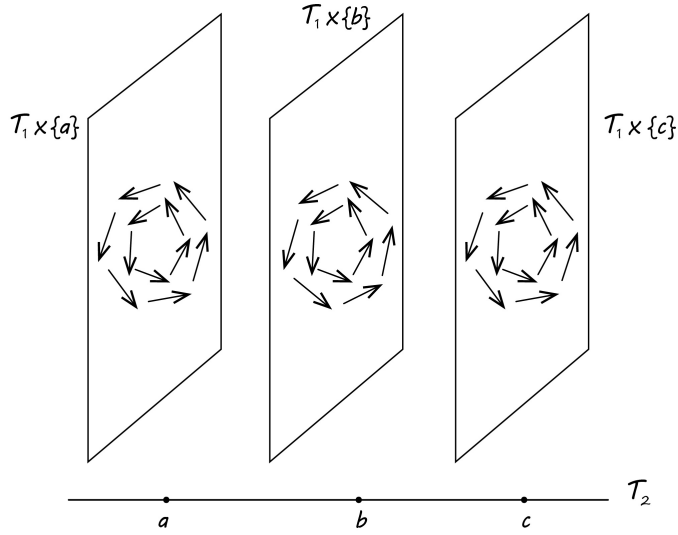
Figure 1: Parallel extension

2.5 Fundamental theorem of ODE's

In a course in ordinary differential equations one first learn the existance an uniqueness of solutions. Let $\mathbb{A}^1_{\mathfrak{R}} = \operatorname{Spec}(\mathfrak{R}[s])$ be the affine line over $\mathfrak{R}$ and $\frac{\partial}{\partial s}$ be the vector field on it such that $\frac{\partial}{\partial s}(s) = 1$. Let also $\tilde{\mathfrak{R}} := \mathfrak{R} \otimes_{\mathbb{Z}} \mathbb{Q}$. The main theorems in this section are classical in the theory of ordinary differential equations. With our algebro-geometric language we would like to point out the fact that in order to define the underlying holomorphic objects, we only need to be able to invert any natural number in the ring $\mathfrak{R}$, and that is why, we have have to use $\tilde{\mathfrak{R}}$ instead of $\mathfrak{R}$.

Theorem 19. *Let t be a smooth point of T and v be a vector field in T with $\mathsf{v}(t) \neq 0$. There is a unique holomorphic map*

$$\varphi : ((\mathbb{A}^1_{\tilde{\mathfrak{R}}})^{\mathrm{hol}}, 0) \to (\mathsf{T}, t) \quad \varphi(0) = t$$

such that φ maps the vector field $\frac{\partial}{\partial s}$ to v.

The statement that φ maps the vector field $\frac{\partial}{\partial s}$ to v is also written in the format $\frac{\partial \varphi}{\partial s} = \mathsf{v}(\varphi)$.

Proof. Let us take coordinate system z in (T, t). We denote by φ_i the pull-back of z_i by ϕ The fact that ϕ is a solution of v is translated into the following ordinary differential equation:

(10)
$$\begin{cases} \frac{\partial \varphi_1}{\partial s} = \mathsf{v}_1(\varphi(s)) \\ \frac{\partial \varphi_2}{\partial s} = \mathsf{v}_2(\varphi(s)) \\ \cdots \\ \frac{\partial \varphi_n}{\partial s} = \mathsf{v}_n(\varphi(s)) \end{cases}$$

We write $\mathsf{v} = (\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_n) = p_0 + p_1 + \cdots$, where p_i is the homogeneous piece of degree i of v. Moreover, let us write:

$$\varphi = \sum_{i=0}^{\infty} \varphi_i s^i, \ \varphi_i \in \tilde{\mathfrak{R}}^n, \ \varphi_0 := 0$$

7

and substitute all these in the above differential equation. It turns out that $i \cdot \varphi_i$ can be written in a unique way in terms of φ_j, $j < i$ with coefficients in $\mathfrak{R}$. We need to invert i, that is why in the statement of theorem we have to use $\tilde{\mathfrak{R}}$. This guaranties the existence of a unique formal power series φ. Note that if $\mathsf{v}(t) = 0$ then $\varphi_i = 0$ for all $i \geq 1$ and so φ is the constant map. It is not at all clear why φ must be convergent. For this we use Picard operator associated with the differential equation (10) and the contracting map principle. For more details see [IY08, §1.4, page 4]. $\qquad\square$

The fact that a solution of a vector field is still defined over $\mathfrak{R}$, that is we do not need to invert integers in $\mathfrak{R}$, has strong consequence.

Proposition 20. *Let T be an $\mathfrak{R}$-scheme over $\mathfrak{R}$, t be a $\mathfrak{R}$-valued point of T and v be a vector field in T with $\mathsf{v}(t) \neq 0$. If the solution φ of v through p is defined over $\mathfrak{R}$ then for any good prime v^p vanishes at t. In other words, the subscheme $\check{\mathsf{T}}$ of T_p defined by the sheaf of ideals generated by $\mathsf{v}^p \mathcal{O}_\mathsf{T}$ contains the point t.*

Proof. We know that φ maps $\frac{\partial}{\partial s}$ to v. Since φ is defined over $\mathfrak{R}$, it makes sense to have the same statement over $\mathfrak{R}_p$. We conclude that φ maps $\frac{\partial^p}{\partial s^p} = 0$ to v^p in T_p. The first vector field is zero in $\mathbb{A}^1_{\mathfrak{R}_p}$, and hence, $\mathsf{v}^p \mathcal{O}_\mathsf{T}$ vanishes at t. $\qquad\square$

The conclusion of Proposition 20 implies the hypothesis of the following conjecture which is a particular case of **??**.

Conjecture 21. *Let T be an $\mathfrak{R}$-scheme, t be a $\mathfrak{R}$-valued point of T, and v be a vector field in T with $\mathsf{v}(t) \neq 0$. If for all but a finite number of primes p, v is colinear with v^p at the point t, then the solution φ of v through t is algebraic.*

Exercise 22. The Ramanujan vector field v leaves the discriminant locus $\Delta : t_2^3 - t_3^2 = 0$ invariant and its solutions in this locus are algebraic. Morover, for examples of primes $p \neq 2, 3$, we have
$$\mathrm{Radical} \langle \mathsf{v}^p t_1, \mathsf{v}^p t_2, \mathsf{v}^p t_3 \rangle = \langle \Delta \rangle.$$
where we have considered ideals in $\mathbb{F}_p[t_1, t_2, t_3]$. Show that the only solutions of v parameterized over $\mathbb{Z}[\frac{1}{6}]$ are those inside $\Delta = 0$. Hint: For the second part use the code:

```
LIB "foliation.lib"; int pr=17;
ring r=pr, (t_1,t_2,t_3),dp;
list vecfield=1/12*(t_1^2-t_2), 1/3*(t_1*t_2-t_3), 1/2*(t_1*t_3-t_2^2);
list vf; int i; int k; int j; poly Q; int di=size(vecfield);
for (i=1; i<=di;i=i+1){vf=insert(vf, var(i),size(vf));}
     for (k=1; k<=di;k=k+1)
       {
       for (i=1; i<=pr;i=i+1)
         {
         Q=0;
         for (j=1; j<=di;j=j+1)
           {
            Q=Q+diff(vf[k], var(j))*vecfield[j];
           }
         vf[k]=Q;
         }
       }
ideal I=vf[1..size(vf)]; I=radical(I);
\\---which is alwyas Delta=t_2^3-t_3^2;
```

Theorem 23. *Let t be a smooth point of T and v be a vector field in T with $\mathsf{v}(t) \neq 0$. There is a coordinate system $(z_1, z_2, \ldots, z_n)$ in $(\mathsf{T}^{\mathrm{hol}}, t)$ such that $\mathsf{v} = \frac{\partial}{\partial z_1}$.*

Proof. Let z be a coordinate system in (T, t). We are looking for a coordinate system F such that the push forward of the vector field $\frac{\partial}{\partial z_1}$ by F is v. This is equivalent to

$$\begin{pmatrix} \frac{\partial F_1}{\partial z_1} & \frac{\partial F_1}{\partial z_2} & \cdots & \frac{\partial F_1}{\partial z_n} \\ \frac{\partial F_2}{\partial z_1} & \frac{\partial F_2}{\partial z_2} & \cdots & \frac{\partial F_2}{\partial z_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial F_n}{\partial z_1} & \frac{\partial F_n}{\partial z_2} & \cdots & \frac{\partial F_n}{\partial z_n} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \mathsf{v}_1(F) \\ \mathsf{v}_2(F) \\ \vdots \\ \mathsf{v}_n(F) \end{pmatrix},$$

where $F = (F_1, F_2, \ldots, F_n)$. In a similar way as in Theorem 19 we have a unique solution F to the above differential equation with

$$F(0, \tilde{z}) = (0, \tilde{z}).$$

where $\tilde{z} = (z_2, \ldots, z_n)$. We have

$$\left[\frac{\partial F_i}{\partial z_j}(0)\right] = \begin{bmatrix} \mathsf{v}_1(0) & 0 \\ * & I_{(n-1)\times(n-1)} \end{bmatrix}$$

By a rotation around 0, we may assume that $\mathsf{v}_1 \neq 0$, and so F is a coordinate system. $\qquad\square$

Remark 1. In the proof of Theorem 23 observe that if v_i's does not depend on z_{i_j}, $j = 1, 2, \ldots$ then we can assume that $F_{i_j} = z_{i_j}$, that is, we do not need to change the coordinate z_{i_j}.

Remark 2. We expect that $\check{\mathsf{T}}$ is an invariant subscheme of T, and it it has an irreducible component of positive dimension containing the point t. The intuition is as follows. We assume that $\mathfrak{R} \subset \mathbb{C}$ and let $A = \mathbb{A}^1_{\mathfrak{R}}$. In $(A^{\text{hol}}, 0)$, φ is convergent, and so it makes sense to evaluate φ at a point $z \in (A^{\text{hol}}, 0)$. We consider the ring $\check{\mathfrak{R}} := \mathfrak{R}[z]$. For a dense subset of $(A^{\text{hol}}, 0)$, the prime p is not invertible in $\check{\mathfrak{R}}$, and hence, it is still a good prime for $\check{\mathfrak{R}}$. Moreover, the Taylor series of φ at z is defined over $\check{\mathfrak{R}}$. In a similar way as above, we conclude that $\varphi(z)$ is in the zero locus of $\mathsf{v}^p \mathcal{O}_{\mathsf{T}_{\check{\mathfrak{R}}_p}}$. More evidences come from Exercise 22

2.6 Module of vector fields

Proposition 24. *Let $\mathfrak{R}$ be a smooth $\mathfrak{R}$-scheme and $\Omega \subset \Omega^1_{\mathsf{T}}$ be a free subsheaf of rank r and $\Theta \in \Theta_{\mathsf{T}}$ be its dual. Ω is integrable that is, for all $\omega \in \Omega$, $d\omega \wedge \wedge^r \Omega = 0$ if and only of Θ is closed under Lie bracket.*

Proof. The theorem is of local nature and so we can take coordinate system arround a point and assume that $\mathsf{T} = (\mathbb{A}^n_{\mathfrak{R}}, 0)$. Most of the ingredients of the proof are from [CL85, Appendix 2, Theorem 2]. $\qquad\square$ <small>Proof?</small>

Definition 25. Let Θ be a submodule of the $\mathcal{O}_{\mathsf{T}}$-module Θ_{T}. Its rank is the number $a \in \mathbb{N}$ such that $\wedge^{a+1}\Theta$ is a torsion sheaf and $\wedge^a \Theta$ is not, and hence, it is an invertible sheaf. For two such modules Θ_1 and Θ_2 we define the scheme

$$\text{Sch}(\Theta_1 \subset \Theta_2) := \text{ZeSc}(\Theta_1 \wedge \bigwedge_{i=1}^{a} \Theta_2)$$

where a is the rank of Θ_2. It is also natural to define:

$$\text{Sch}(\Theta_1 = \Theta_2) := \text{Sch}(\Theta_1 \subset \Theta_2) \cap \text{Sch}(\Theta_2 \subset \Theta_1)$$

In case $\Theta_1 = \mathcal{O}_{\mathsf{T}}\mathsf{v}$ is generated by a single vector field v, we also use the notation $\text{Sch}(\mathsf{v} \in \Theta_2)$. In geometric terms if $\mathfrak{k}$ is an algebraically closed field, $\text{Sch}(\Theta_1 \subset \Theta_2)$ is the loci of points in x in T such that the vectors $\mathsf{v}(x)$ for all $\mathsf{v} \in \Theta_1$ are in the $\mathfrak{k}$-vector space generated by $\mathsf{v}(x)$, $\mathsf{v} \in \Theta_2$.

We rewrite Definition 25 in the case where Θ_1 and Θ_2 are generated by single vector fields v and w, respectively. In this case we are interested on the loci $x \in \mathsf{T}$ such that $\mathsf{v}(x)$ and $\mathsf{w}(x)$ are collinear.

Definition 26. Let v and w be two vector fields on T, all defined over $\mathfrak{R}$. The collinear or parallel scheme $\mathsf{v}\|\mathsf{w}$ of v and w is a subscheme of T given by the sheaf of ideals generated by

$$\begin{vmatrix} \mathsf{v}(P) & \mathsf{v}(Q) \\ \mathsf{w}(P) & \mathsf{w}(Q) \end{vmatrix}, \quad P, Q \in \mathcal{O}_{\mathsf{T}}.$$

Let us write [Definition 25](#) in explicit terms:

Proposition 27. *The sheaf of ideals of* $\mathrm{Sch}(\Theta_1 \subset \Theta_2)$ *is given by*

$$\begin{vmatrix} \mathsf{v}(P_1) & \mathsf{w}_1(P_1) & \mathsf{w}_2(P_1) & \cdots & \mathsf{w}_a(P_1) \\ \mathsf{v}(P_2) & \mathsf{w}_1(P_2) & \mathsf{w}_2(P_2) & \cdots & \mathsf{w}_a(P_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathsf{v}(P_a) & \mathsf{w}_1(P_a) & \mathsf{w}_2(P_a) & \cdots & \mathsf{w}_a(P_a) \end{vmatrix}, \quad P_1, P_2, \cdots, P_a \in \mathcal{O}_\mathsf{T}, \ \mathsf{v} \in \Theta_1, \ \mathsf{w}_1, \mathsf{w}_2, \ldots, \mathsf{w}_a \in \Theta_2.$$

Proof. For simplicity we proceed the proof only for $\mathsf{v}||\mathsf{w}$. Let $\mathsf{T} := \mathrm{Spec}(\mathfrak{R}[z_1, z_2, \cdots, z_n]/\mathcal{I})$. Then a vector field in T is given by

$$\mathsf{v} = \sum_{i=1}^n \mathsf{v}_i(z) \frac{\partial}{\partial z_i},$$

with

$$\sum_{i=1}^n \frac{\partial P}{\partial z_i} \mathsf{v}_i \in \mathcal{I}$$

for a set of generators P of $\mathcal{I}$ (which is equivalent to say that for all $P \in \mathcal{I}$). It is easy to verify that the sheaf of ideals of $\mathsf{v}||\mathsf{w}$ is generated by

$$\begin{vmatrix} \mathsf{v}_i & \mathsf{w}_i \\ \mathsf{v}_j & \mathsf{w}_j \end{vmatrix}, \quad i, j = 1, 2, \ldots, n, \ i \neq j.$$

For any $P, Q \in \mathfrak{R}[z_1, z_2, \ldots, z_n]$ we have

$$\begin{vmatrix} \mathsf{v}(P) & \mathsf{w}(P) \\ \mathsf{v}(Q) & \mathsf{w}(Q) \end{vmatrix} = \sum_{i,j=1}^n \frac{\partial P}{\partial z_i} \frac{\partial Q}{\partial z_j} \begin{vmatrix} \mathsf{v}_i & \mathsf{w}_i \\ \mathsf{v}_j & \mathsf{w}_j \end{vmatrix}$$

□

2.7 Foliations

Let T be an $\mathfrak{R}$-scheme.

Definition 28. Let Ω be a submodule of the $\mathcal{O}_\mathsf{T}$-module Ω^1_T. Its rank is the number $a \in \mathbb{N}$ such that $\wedge^{a+1}\Omega$ is a torsion sheaf and $\wedge^a \Omega$ is not, and hence, it is an invertible sheaf.

Definition 29. A codimension a foliation $\mathcal{F}$ in T is given by an $\mathcal{O}_\mathsf{T}$-module $\Omega \subset \Omega^1_\mathsf{T}$ of rank a and with the integrability condition (or sometimes it is called Frobenius condition). We say that Ω is integrable if for all $\omega \in \Omega$, $d\omega \wedge \wedge^a \Omega$ is a torsion sheaf.

Our examples of foliations in the present text are actually algebraically integrable.

Definition 30. We say that $\mathcal{F}(\Omega)$ is algebraically integrable of $d\Omega \subset \Omega^1_\mathsf{T} \wedge \Omega$. We also say that Ω is geometrically integrable if for all $\omega \in \Omega$ there is $f \in \mathcal{O}_\mathsf{T}$, $f \neq 0$ (depending on ω) such that

$$(11) \qquad\qquad\qquad f \cdot d\omega \in \Omega^1_\mathsf{T} \wedge \Omega.$$

By definition it is clear that algebraical integrability implies the geometric integrability and this implies the general definition of integrability in [Definition 29](#). We do not assume that Ω is saturated.
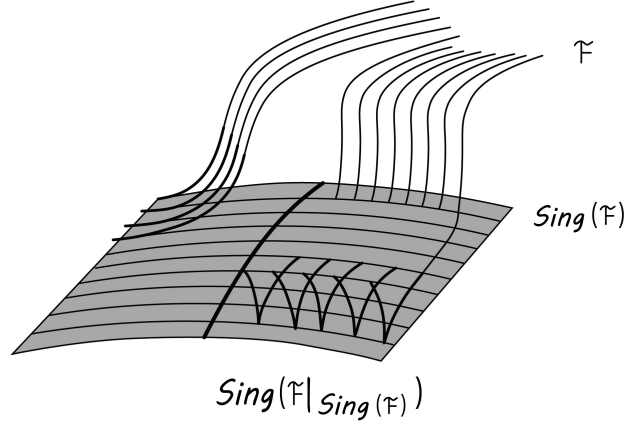
Figure 2: Singular set of a foliation

Definition 31. we sya that Ω is sturated if for some $0 \neq f \in \mathcal{O}_{\mathsf{T}}$ and $\omega \in \Omega_{\mathsf{T}}$ we have $f\omega \in \Omega$ then $\omega \in \Omega$.

The notion of leaf scheme highly depends on Ω itself and not the underlying geometric foliation. In the most of the literature when talka about a foliation, then it is usually non-singular, and so, one has reserved the term singular foliation for those with singularities. In the present text a foliation might have singularities and actually we are going to find many leaves with different codimensions inside the singular locus.

Definition 32. The codimension of a foliation $\mathcal{F}(\Omega)$ is the rank of the module Ω, that is, the number $c \in \mathbb{C}$ such that $\wedge^{c+1}\Omega \subset \Omega_{\mathsf{T}}^{c+1}$ is a torsion sheaf but $\wedge^c \Omega$ is not.

Definition 33. Let T be am $\mathfrak{R}$-scheme and $\mathcal{F}(\Omega)$ be a foliation in T of codimension c. We define

$$\mathrm{Sing}(\mathcal{F}(\Omega)) := \mathrm{ZeSc}(\wedge^c \Omega),$$

where the zero scheme ZeSc is defined in Definition 8, see Figure 2.

If Ω is an $\mathcal{O}_{\mathsf{T}}$-module generated by a global sectio ω of Ω_{T}^1 then the integrability in this case is just the equality $d\omega \wedge \omega = 0$. This is a non-trivial condition when $\dim(\mathsf{T}) \geq 3$. In this case we write $\mathcal{F}(\Omega) = \mathcal{F}(\omega)$ and call it codimension one foliation.

Exercise 34. Show that if $\mathsf{T} := \mathbb{A}_{\mathfrak{R}}^n = \mathrm{Spec}(\mathfrak{R}[z])$, $z = (z_1, z_2, \ldots, z_n)$ and Ω is generated by a single differential 1-form $\omega = P_1 dz_1 + P_2 z_2 + \cdots + P_n dz_n$, $P_i \in \mathfrak{R}[z]$ then

$$\mathrm{Sing}(\mathcal{F}(\omega)) = \mathrm{Spec}(\mathfrak{R}[z]/\langle P_1, P_2, \ldots, P_n \rangle).$$

For a vector field v in T we consider its dual in Ω_{T}^1:

$$\Omega := \left\{ \omega \in \Omega_{\mathsf{T}}^1 \mid \omega(\mathsf{v}) = 0 \right\}$$

For T smooth this is a free $\mathcal{O}_{\mathsf{T}}$-module of rank $n-1$, where $n = \dim(\mathsf{T})$. In this case, the integrability condition is trivial and hence we get a foliation $\mathcal{F}(\Omega)$ which we call it foliation by curves or dimension one foliation.

Exercise 35. Let φ be as in Theorem 19. Show that the subscheme L of T given by the ideal

$$\mathcal{I} := \{ f \in \mathcal{O}_{\mathsf{T}^{\mathrm{hol}},t} \mid f(\varphi) = 0 \}$$

is a leaf of $\mathcal{F}(\Omega)$ (recall Definition 39).

3   Local theory

In this section we only consider $\mathsf{T} := (\mathbb{C}^n, 0)$, a small neighborhood of 0 in $\mathbb{C}^n$. In this way, $\mathcal{O}_{\mathsf{T}}$ is just the ring of convergent fuctions in z with coefficients in $\mathbb{C}$. For a $\mathfrak{R}$-scheme T and a $\mathfrak{R}$-valued smooth point t of T, we will apply our results for the local ring $\mathcal{O}_{\mathsf{T}^{\mathrm{hol}},t} \subset \mathcal{O}_{\mathsf{T}^{\mathrm{for}},t}$.

3.1   Frobenius theorem

Theorem 36 (Frobenius theorem). *Let $\mathcal{F}(\Omega)$ be a foliation in $\mathsf{T} = (\mathbb{C}^n, 0)$ and assume that Ω is freely generated by r differntial 1-forms and 0 is not a singular point of $\mathcal{F}$. There is a coordinate system $(z_1, z_2, \ldots, z_n)$ in T such that Ω is freely generated as $\mathcal{O}_{\mathsf{T}}$-module by $dz_1, dz_2, \cdots, dz_r$.*

The proof of Theorem 36 gives also the following algebraic statement. Recall that $\tilde{\mathfrak{R}} := \mathfrak{R} \otimes_{\mathbb{Z}} \mathbb{Q}$ is the smallest subring of $\mathbb{C}$ containing both $\mathfrak{R}$ and $\mathbb{Q}$.

Theorem 37. *Let T be an $\mathfrak{R}$-scheme, $\mathcal{F}(\Omega)$ be a foliation on T and P be a $\mathfrak{R}$-valued smooth point of T. Assume that the the restriction of Ω in $\Omega^1_{\mathsf{T},P}$ is freely generated by r differntial 1-forms and P is not a singular point of $\mathcal{F}$. There are formal power series $z_1, z_2, \ldots, z_r \in \mathcal{O}_{\mathsf{T}^{\mathrm{for}}_{\tilde{\mathfrak{R}}},P}$ such that Ω is freely generated as $\mathcal{O}_{\mathsf{T}}$-module by $dz_1, dz_2, \cdots, dz_r$ and $dz_1 \wedge dz_2 \wedge \cdots \wedge dz_r(P) \neq 0$. Moreover, if $\mathfrak{R} \subset \mathbb{C}$ then $z_1, z_2, \ldots, z_r \in \mathcal{O}^{\mathrm{hol}}_{\mathsf{T},P}$, that is, z_i's are convergent.*

Note that we claim that the Taylor series of z_i have coefficients in the $\tilde{\mathfrak{R}}$, that is, in order to define them we must able to invert integers. The Frobenius theorem is no more true for points in $\mathrm{Sing}(\mathcal{F})$. For a study of foliations in $(\mathbb{C}^2, 0)$ with an isolated singularity at 0 the reader might consult [CS87] and [Lor06].

Proof. The proof in Camacho and Lins Neto's book [CL85, Appendix 2], can be reproduced in the holomorphic context and we do it here. We proceed the proof with the notation of Theorem 37. As t is a smooth point of T, the $\mathcal{O}_{\mathsf{T},t}$-module $\Omega^1_{\mathsf{T},t}$ is free of rank $n := \dim(\mathsf{T})$. This implies thatthe restriction of Ω in $\Omega^1_{\mathsf{T},t}$ is also free of rank, let us say $n - r$. The same statements are also valid for $\Theta_{\mathsf{T},t}$ and so $\Theta := \{ \mathsf{v} \in \Theta_{\mathsf{T},t} \mid \mathsf{v}(\Omega) = 0 \}$ is free generated by r vector fields v_i, $i = 1, 2, \ldots, r$. By Proposition 24 the integrability of Ω is equivalent to Θ is closed under Lie bracket. We choose local coordinates $(z_1, z_2, \ldots, z_n)$ in (T, t) and write

$$\mathsf{v}_i := \sum_{j=1}^n \mathsf{v}_{ij} \frac{\partial}{\partial z_j}, \quad i = 1, 2, \ldots, r.$$

Since the matrix formed by v_{ij} has rank r, after a $\mathcal{O}_{\mathsf{T},t}$-linear change of v_i, we can assume that

$$\mathsf{v}_i := \frac{\partial}{\partial z_i} + \sum_{j=r+1}^n \mathsf{v}_{ij} \frac{\partial}{\partial z_j}, \quad i = 1, 2, \ldots, r.$$

This format of v_i's imply that $[\mathsf{v}_i, \mathsf{v}_j] = 0$. There is a new holomorphic corrdinate system $z = (z_1, z_2, \ldots, z_n)$ in (T, t) such that $\mathsf{v}_i = \frac{\partial}{\partial z_i}$. For this we apply Theorem 23 for v_1 and so we assume that $\mathsf{v}_1 = \frac{\partial}{\partial z_1}$. The equality $[\mathsf{v}_1, \mathsf{v}_2] = 0$ implies that if $\mathsf{v}_2 := \sum_{j=1}^n \mathsf{v}_j \frac{\partial}{\partial z_j}$ then v_j does not depend on z_1. By Remark 1 we can find a new coordinate system such that z_1 is unchanged and $\mathsf{v}_2 = \frac{\partial}{\partial z_2}$. We continue this process and get the result.
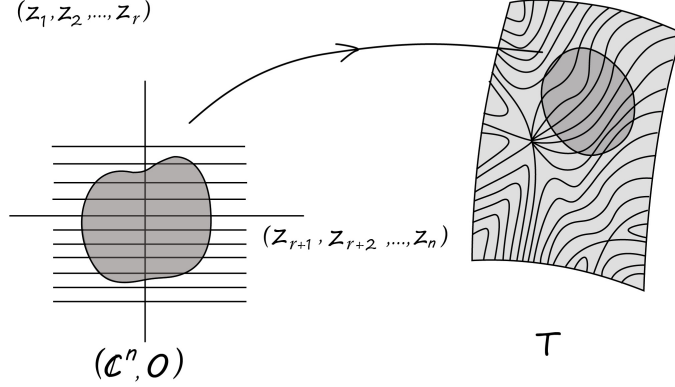
$\square$

Figure 3: Local charts of a foliation

3.2 Leaf scheme

The notion of leaf scheme is independent of the integrability condition above. Actually, all the statements proved in [Mov22, Chapter 4] do not use this condition. A deep understanding of foliations with leaf schemes at some point must use it. In Theorem 36, the set

$$L : z_1 = \mathrm{const}_1, \ z_2 = \mathrm{const}_2, \cdots, z_r = \mathrm{const}_r$$

is a smooth and reduced and it is the classical definition of leaf of $\mathcal{F}$. We enlarge the notion of a leaf into a leaf scheme.

Definition 38. Let L be a subscheme of T passing through 0, that is we have the ideal $\mathcal{I} \subset \mathcal{O}_{\mathsf{T}}$ which vanishs at 0 and $\mathcal{O}_L := \mathcal{O}_{\mathsf{T}}/\mathcal{I}$ (which might have nilpotent elements). Let also $\mathcal{F} = \mathcal{F}(\Omega)$ be a foliation in T. We say that L is a leaf scheme of $\mathcal{F}$ if Ω and $\mathcal{O}_{\mathsf{T}} \cdot d\mathcal{I}$ projected to $\Omega^1_{\mathsf{T}}/\mathcal{I}\Omega^1_{\mathsf{T}}$ and regarded as $\mathcal{O}_{\mathsf{T}}/\mathcal{I}$-modules are equal. In other words, Ω and $\mathcal{O}_{\mathsf{T}}d\mathcal{I}$ are equal modulo $\mathcal{I}\Omega^1_{\mathsf{T}}$. We say that L is a separatrix of $\mathcal{F}(\Omega)$ if we have only the inclusion $\Omega \subset \mathcal{O}_{\mathsf{T}}d\mathcal{I} + \mathcal{I}\Omega^1_{\mathsf{T}}$ and $\mathcal{O}_{\mathsf{T}}d\mathcal{I} \not\subset \Omega + \mathcal{I}\Omega^1_{\mathsf{T}}$.

Definition 39. Let T be an $\mathfrak{R}$-scheme, $\mathcal{F}(\Omega)$ be a foliation on T, t be a $\mathfrak{R}$-valued point of T and L be a leaf of $\mathcal{F}(\Omega)$ through t. If $\mathcal{I} \subset \mathcal{O}^{\mathrm{hol}}_{t,t}$ (resp. $\mathcal{I} \subset \mathcal{O}_{\mathsf{T}^{\mathrm{for}},t}$) then we say that L is a holomorphic leaf (resp. formal leaf) of $\mathcal{F}$ defined over $\mathfrak{R}$ and write it L^{hol} (resp. L^{for}) if it is necessary to emphasize its property of being holomorphic or fornal. We say that L is parametrized over $\mathfrak{R}$ if there is a morphism of $\mathfrak{R}$-schemes

$$\varphi : ((\mathbb{A}^r_{\mathfrak{R}})^{\mathrm{hol}}, 0) \to (\mathsf{T}^{\mathrm{hol}}, t),$$

where r is the dimension of L, such that $\varphi^*\mathcal{I} = 0$ and the induced map $\mathcal{O}_{\mathsf{T},t}/\mathcal{I} \to \mathcal{O}_{(\mathbb{A}^r_{\mathfrak{R}})^{\mathrm{hol}},0}$ is an isomorphism.

Note that by Theorem 37 the leaves given by Frobenius theorem are defined in $\mathfrak{R} \otimes_{\mathbb{Z}} \mathbb{Q}$. Moreover, the parametrization of a leaf of a vector field in Theorem 19 is also defined over $\mathfrak{R} \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let us proceed this section in the geometric framework $\mathsf{T} := (\mathbb{C}^n, 0)$. The following concept seems to be fundamental and non-trivial for singular leaves.

Definition 40. For the ideal $\mathcal{I}$ as above we define its integral to be

$$\mathrm{Int}(\mathcal{I}) := \left\{ f \in \mathcal{O}_\mathsf{T} \;\middle|\; f(0) = 0, \;\; df \in \mathcal{I} \cdot \Omega^1_\mathsf{T} \right\}.$$

It can be checked that $\mathrm{Int}(\mathcal{I})$ is an algebra and it is not necessarily an ideal. However, the intersection $\mathrm{Int}(\mathcal{I}) \cap \mathcal{I}$ is an ideal. The definition of $\mathrm{Int}(\mathcal{I})$ is motivated by the following proposition.

Proposition 41. *Let L be a leaf of $\mathcal{F}(\Omega)$ and $\mathcal{I}$ be its ideal. We have the leaf $\check{L}$ whose ideal is generated by $\mathcal{I}$ and $\mathrm{Int}(\mathcal{I})$.*

Proof. We know that $\Omega = \mathcal{O}_\mathsf{T} d\mathcal{I}$ modulo $\mathcal{I}\Omega^1_\mathsf{T}$. The inclusion $\Omega \subset \mathcal{O}_\mathsf{T} d\check{\mathcal{I}}$ modulo $\check{\mathcal{I}}\Omega^1_\mathsf{T}$ follows directly from this. From another side if we have an element f of $d\check{\mathcal{I}}$, it is of the form $f = d(\sum f_i g_i)$ with $f_i \in \mathcal{O}_\mathsf{T}$ and $g_i \in \mathcal{I}$ or $dg_i \in \mathcal{I}\Omega^1_\mathsf{T}$. In the first case we use $d\mathcal{I} \subset \Omega$ modulo $\mathcal{I}\Omega^1_\mathsf{T}$ to conclude that $f \in \Omega$ modulo $\check{\mathcal{I}}\Omega^1_\mathsf{T}$, and in the second case this is direct. . $\qquad\square$

Let $L_0 := L$ and $L_1 := \check{L}$. We can repeat the construction in [Proposition 41](#) and get leaves $\cdots \subset L_2 \subset L_1 \subset L_0$. Since $\mathcal{O}_\mathsf{T}$ is a Noethering ring, this stabilizes at some point, that is, for some $k \in \mathbb{N}$ we have $L_k = L_{k+1} = \cdots$ which means that L_k is differentially saturated, that is,

Definition 42. For a subscheme L of T given by the ideal $\mathcal{I}$, we say that it is differentially saturated if

$$\forall f \in \mathcal{O}_\mathsf{T}, \;\; \text{with } f(0) = 0, \text{ if } df \in \mathcal{I} \cdot \Omega^1_\mathsf{T} \text{ then } f \in \mathcal{I}.$$

In other words, $\mathrm{Int}(\mathcal{I}) \subset \mathcal{I}$.

From now on by a leaf we mean the differentially saturated one, and so

$$\mathcal{I} \cdot \mathcal{I} \subset \mathrm{Int}(\mathcal{I}) \subset \mathcal{I}.$$

For a differentially saturated ideal $\mathcal{I}$, it turns out that that $\mathrm{Int}(\mathcal{I})$ is also an ideal. Recall that this is not true for an arbitrary $\mathcal{I}$.

Proposition 43. *For a non-zero ideal $\mathcal{I}$, $\mathcal{I}\backslash\mathrm{Int}(\mathcal{I})$ is non-empty. If $\mathcal{I}$ is differentially saturated this means that the inclusion $\mathrm{Int}(\mathcal{I}) \subset \mathcal{I}$ is strict.*

Proof. For $f \in \mathcal{O}_\mathsf{T}$, we write the Taylor series of f at 0, $f = f_a + f_{a+1} + \cdots + \cdots, f_a \neq 0$, where f_i is a homogeneous polynomial of degree i in z and coefficients in $\mathbb{C}$. We call f_a the leading polynomial of f and $\deg(f) := a$.

Take $f \in \mathcal{I}$ with minimial degree. We cannot have $df \in \mathcal{I}\Omega^1_\mathsf{T}$ which is verified by examining the leading polynomial equality derived from this one. $\qquad\square$

Remark 3. We follows the notation in [Mov22, page 75]. Let L be a leaf of $\mathcal{F}(\Omega)$. We write $\omega = Pdf$ and $df = \check{P}\omega$ modulo $\mathcal{I}\Omega^1_\mathsf{T}$. Therefore, $\omega = d(Pf)$ and $df = \check{P}d(Pf) = d(\check{P}Pf)$. This implies that $(I_{s\times s} - \check{P}P)f$ has entries in $\mathrm{Int}(\mathcal{I})$. The existance of two matrices P and $\check{P}$ with the menioned condition is enough to construct the foliation $\mathcal{F}$ with $\omega = Pdf + \sum f_i\alpha_i, \; \alpha_i \in \Omega^1_\mathsf{T}$.

3.3 Conormal sequence

Take $\mathcal{O}_{\mathsf{T}}$ the ring of formal power series in n variables $z_1, z_2, \ldots, z_n$ and with coefficients in a ring $\mathfrak{R}$. Originally we are interested in the case $\mathfrak{R} \subset \mathbb{C}$ and the ring of convergent sereis. We will also use the coefficient in $\mathfrak{R}/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal. Take also an ideal $\mathcal{I}$ of $\mathcal{O}_{\mathsf{T},0}$ and set $L = \mathcal{O}_{\mathsf{T},0}/L$. In geometric terms, L is an analytic subscheme of T. We have the sequence

$$\frac{\mathcal{I}}{\mathcal{I} \cdot \mathcal{I}} \xrightarrow{d} \frac{\Omega^1_{\mathsf{T}}}{\mathcal{I}\Omega^1_{\mathsf{T}}} \to \Omega^1_L \to 0,$$

which is called the conormal sequence. With our notation we have $\ker(d) = \frac{\mathrm{Int}(\mathcal{I})}{\mathcal{I}\cdot\mathcal{I}}$. Therefore, Proposition 43 is equivalent to say that d in the conormal sequence is not identically zero. By [Eis95, Proposition 16.3] we know that this is an exact sequence of $\mathcal{O}_L$-modules. By [Eis95, Proposition 16.12], the first map is an split injection if and only if there is a map of $\mathfrak{R}$-algebras $\mathcal{O}_{\mathsf{T},0}/\mathcal{I} \to \mathcal{O}_{\mathsf{T},0}/\mathcal{I}^2$ splitting the projection map $\mathcal{O}_{\mathsf{T},0}/\mathcal{I}^2 \to \mathcal{O}_{\mathsf{T},0}/\mathcal{I}$. By [Eis95, Exercise 12.17], the first map in the conormal sequence is injection for radical complete intersections, without being split injection. Andre-Quillin homology theory explain the rest of the conormal sequence, for a survey on this see [Iye07]. The study of leaves of codiemsnion one foliations reduces to the following purely commutative algebra problem:

Problem 44. *Give an example of ideal $\mathcal{I} \neq 0$ which is not generated by one element, but the $\mathcal{O}_{\mathsf{T},0}/\mathcal{I}$-module $\left(\frac{\mathcal{I}}{\mathcal{I}\cdot\mathcal{I}}\right)/\ker(d)$ is generated by one element. A more general problem is to classify these ideals.*

If d is an injection, then it can be easily shown that there is no such an ideal, see the last part of the proof of **??**. This problem is not yet our final issue. If $\mathcal{I} = \langle f_1, f_2, \ldots, f_s \rangle$ such that f_1 generates $\left(\frac{\mathcal{I}}{\mathcal{I}\cdot\mathcal{I}}\right)/\ker(d)$ then we are interested on those ideals such that $\omega = df_1 + \sum_{i=2}^{s} f_i\alpha_i$, $\alpha_i \in \Omega^1_{\mathsf{T}}$ is integrable, that is, $\omega \wedge d\omega = 0$.

3.4 Smooth and quasi-smooth leaves

Definition 45. We say that L is smooth (and reduced of codimension s) if $\mathcal{I} = \langle f_1, f_2, \ldots, f_s \rangle$, and the linear part of f_i's are linearly independent. By holomorphic implicit function theorem this is equivalent to say that in some holomorphic coordinate system $(z_1, z_2, \ldots, z_n)$ in $(\mathbb{C}^n, 0)$ we have $\mathcal{I} = \langle z_1, z_2, \ldots, z_s \rangle$.

Another reformulation of the above definition is to say that L is a smooth local complete intersection.

Proposition 46. *A smooth leaf is differentially saturated. Moreover,*

$$\mathrm{Int}(L) = \mathcal{I} \cdot \mathcal{I}.$$

Proof. The inclusion $\supset$ is trivial and holds for any ideal $\mathcal{I}$. Let us prove $\mathrm{Int}(L) \subset \mathcal{I} \cdot \mathcal{I}$. For $f \in \mathrm{Int}(L)$ we have $\frac{\partial f}{\partial z_i} \in \mathcal{I}$ for all $i = 1, 2, \cdots, n$. Since $f(0) = 0$, we can write

$$f = \sum_{i=1}^{s} z_i g_i + \sum_{i=s+1}^{n} z_i g_i$$

for which we can assume that g_i does not depend on $z_1, z_2, \cdots, z_s$. We write this as $f = f_1 + f_2$. The fact $\frac{\partial f}{\partial z_i} = \frac{\partial f_1}{\partial z_i} \in \mathcal{I}$, $i = 1, 2, \cdots, s$ imply that $g_i \in \mathcal{I}$, $i = 1, 2, \ldots, s$ and so $f_1 \in \mathcal{I}\mathcal{I}$. Now, we use $\frac{\partial f}{\partial z_i} \in \mathcal{I}$, $i = 1, 2, \cdots, s$ and conclude that $\frac{\partial f_2}{\partial z_i} \in \mathcal{I}$. However, f_2 does not depend on $z_1, , z_2, \cdots, z_s$ and it vanishes at $z = 0$, therefore, it must be identically zero. $\square$

The definition of differentially saturated ideals is inspired from modular foliations, see [Mov22, Section 6.10].

Definition 47. A leaf L is called quasi-smooth if $\text{Int}(L) = \mathcal{I} \cdot \mathcal{I}$. Equivalently, in the conormal sequence d is injective.

We have seen that smooth leaves are quasi-smooth.

Proposition 48. *If the ideal $\mathcal{I}$ is generated by one irreducible element then the leaf L is quasi-smooth.*

Proof. This follows from [Eis95, Exercise 12.17] which says that the first map in the conormal sequence is injection for radical complete intersections. □

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