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Headaches in Hodge theory

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June 24, 2021

Publisher

Chapter 1
Infinitesimal deformations

The following notes are written after many email exchanges with P. Deligne in Novermber 2018. The main objective of the notes is two folded. First, we would like rewrite the results of [Mov19, Chapter 18] using the formalism of infinitesimal schemes over a field k of characteristic zero. Second, to reproduce some of P. Deligne's questions and suggestions.

1.1 Deformation over infinitesimal schemes

Let T be a reduced and smooth scheme over a field k of chararacteristic 0, $0 \in \mathsf{T}$ be a k-rational point of T and $S \hookrightarrow \mathsf{T}$ be a possibly non-reduced sub scheme of T. For instance, $\mathsf{T} := \mathrm{Spec}(\mathsf{k}[t_1, t_2, \dots, t_s])$ and

$$S := \mathrm{Spec}\left(\mathsf{k}[t_1, t_2, \dots, t_s]/\langle t_1^{N+1}, \cdots, t_s^{N+1}, f_1, f_2, \cdots\rangle\right) = \mathrm{Spec}\left(\mathsf{k}[t]/\langle t^{N+1}, f\rangle\right) \tag{1.1}$$

where $f_1, f_2, \dots$ is a set of polynomials in $\mathsf{k}[t]$. We take a projective smooth scheme X_{T} over T and denote by X/S the induced scheme. In general we may work with

Definition 1.1 Let $\mathscr{M}$ be the maximal ideal of $\mathscr{O}_{S,0}$. We call

$$S^N := \mathrm{Spec}\left(\mathscr{O}_{S,0}\Big/\mathscr{M}^{N+1}\right) \tag{1.2}$$

the N-th infinitesimal scheme of S at 0. We also call the induced scheme X/S, the N-th order deformation of X_0.

The fiber over 0 is denote by X_0. From now on I will use X/T instead of $X_{\mathsf{T}}/\mathsf{T}$. The relative algebraic de Rham cohomology $H^m_{\mathrm{dR}}(X/\mathsf{T})$ is a free $\mathscr{O}_{\mathsf{T}}$-module of rank $\dim_{\mathsf{k}} H^m_{\mathrm{dR}}(X_0/\mathsf{k})$. We denote the pieces of its Hodge filtration by $F^i = F^i H^m_{\mathrm{dR}}(X/\mathsf{T}),\ i = 0, 1, 2, \dots, m$. We have also the Gauss-Manin connection

$$\nabla : H^m_{\mathrm{dR}}(X/\mathsf{T}) \to \Omega^1_{\mathsf{T}} \otimes_{\mathscr{O}_{\mathsf{T}}} H^m_{\mathrm{dR}}(X/\mathsf{T})$$

In an earlier draft of this text, I worte the formalisim of Gauss-Manin connection directly for S (not using the bigger space T). This was trivially false as rings like the one used in (1.1) might have elements f such that $df = 0$ but f is not constant. This resulted in the following comment.

1.1 (P. Deligne, 20 November, 2018) You need the crystalline story, which tells that for X smooth (proper is not needed) over a S such as (1.1), $H_{\mathrm{dR}}(X/S)$ depends only (up to a <u>canonical</u> isomorphism) on $X_0/\mathrm{Spec}(\mathbb{C})$. In other words, one has a canonical isomorphism

$$H_{\mathrm{dR}}(X/S) \cong H_{\mathrm{dR}}(X_0) \otimes \mathsf{R} \tag{1.3}$$

[R being the ring of S]. In your cases, you will have $S \hookrightarrow \mathsf{T}$, T smooth, and X/S is induced by $X_{\mathsf{T}}/\mathsf{T}$, proper and smooth. In that case, the Gauss-Manin connection induces a constantification of $H_{\mathrm{dR}}(X_{\mathsf{T}}/\mathsf{T})$ on a formal neighborhood of $0 \in \mathsf{T}$, and it induces (1.3) on S.

Now we can talk about the Gauss-Manin connection

$$\nabla : H^m_{\mathrm{dR}}(X/S) \to \Omega^1_S \otimes_{\mathscr{O}_S} H^m_{\mathrm{dR}}(X/S)$$

and any other object related to de Rham cohomologies, such as trace map, polarization and cup product, as it is explained in the comments above. We take a subscheme Z_0 of codiemsnion $p := \frac{m}{2}$ in X_0/k and denote its cohomology class by $\mathrm{cl}(Z_0) \in H^m(X_0/\mathsf{k})$. There is a unique section s of $H^m_{\mathrm{dR}}(X/S)$ such that $\nabla(\mathsf{s}) = 0$ and $\mathsf{s}_0 = \mathrm{cl}(Z_0)$. This is called the horizontal extension of $\mathrm{cl}(Z_0)$ or a flat section of the cohomology bundle.

Definition 1.2 Let X/S and X_0 be as above. The Hodge locus $V^{X/S}_{[Z_0]}$ is a subscheme of S given by the conditions

$$\nabla(\mathsf{s}) = 0, \tag{1.4}$$

$$\mathsf{s} \in F^{\frac{m}{2}} H^m_{\mathrm{dR}}(X/S), \tag{1.5}$$

$$\mathsf{s}_0 = \mathrm{cl}(Z_0). \tag{1.6}$$

For the reduced smooth T and X/T as above we use the notation

$$V^{\infty}_{[Z_0]} = V_{[Z_0]} := V^{X/\mathsf{T}}_{[Z_0]}$$

and for the N-th order deformation X/S^N of X_0 as in Definition 1.1 we use

$$V^N_{[Z_0]} := V^{X/S^N}_{[Z_0]}$$

and call it the N-th infinitesimal Hodge locus.

1.2 (P. Deligne, November 13, 2018) Note that we did not need to require that s is at each point $V_{[Z_0]}$ an integral cohomology class-which would be a transcendental condition-it comes free from (1.4) and (1.6).

Definition 1.3 We define $\check{\mathsf{T}} = \check{\mathsf{T}}_{Z_0}$ to be the algebraic subset of T coming from Cattani-Deligne-Kaplan theorem applied to the Hodge locus $V_{[Z_0]} = V_{[Z_0]}^{X/\mathsf{T}}$. We denote by $V_{Z_0} \subset (\mathsf{T}, 0)$ the local analytic subset of $(\mathsf{T}, 0)$ parametrizing flat deformations of Z_0. We have

$$V_{Z_0} \subset V_{[Z_0]} \subset \check{\mathsf{T}}_{Z_0}$$

i

Definition 1.4 We say that the alternative Hodge conjecture (AHC) holds for the pair (X_0, Z_0) if

$$V_{Z_0} = V_{[Z_0]}$$

as analytic varieties.

For many examples of (X_0, Z_0) such that AHC does not hold see [Mov19, §18.2].

Proposition 1.1 *If $X/S, X_0, Z_0$ are defined over* k *then $V_{[Z_0]}$ is also defined over* k.

Proof. This follows from the fact that the Gauss-Manin connection and the cohomology class of Z_0 are defined over k. □

However, note that the condition (1.4) is mixed algebraic and holomorphic condition. Whereas ∇ is algebraic, its flat section s for reduced schemes is given by holomorphic functions. After passing to infinitesimal schemes we use the truncation of of such series, and hence, get polynomials. The section s can be computed explicitly once the Gauss-Manin connection is computed. In this way the equations of the Hodge locus are hidden in (1.5) which can be written as

$$\langle \omega, \mathsf{s} \rangle = 0, \quad \forall \omega \in F^{\frac{m}{2}+1} H^m_{\rm dR}(X/S).$$

where

$$\langle \cdot, \cdot \rangle : H^m_{\rm dR}(X/S) \times H^m_{\rm dR}(X/S) \to \mathscr{O}_S(S), \ \ (\mathsf{s}, \beta) \mapsto \mathrm{Tr}(\mathsf{s} \cup \beta \cup \theta^{2n-2m}). \qquad (1.7)$$

and $\theta \in H^2(X/S)$ is the polarization. For T reduced and smooth scheme and ω a section of $H^m_{\rm dR}(X/\mathsf{T})$ around 0, we know that

$$\int_{\delta_t} \omega \in \mathsf{k}[[t]] = \text{ ring of formal power series in } t \text{ with coefficients in } \mathsf{k} \qquad (1.8)$$

where $\delta_t \in H_m(X_t, \mathbb{Z})$ is the monodromy of $\delta_0 := [Z_0]$ to nearby fibers, that is, periods are formal power series (actually convergent) in t with coefficients in k. The ideal of the Hodge locus $V_{[Z_0]}$ in this case is given by

i I am not sure whether V_{Z_0} in general exists! Duco van Straten kindly reminded me [January 14, 2019] that "the requirement of flatness of a cycle is not a reasonable condition at all. Flatness is just too special. There is the notion of families of cycles used by Barlet in analytic and by Kollar in algebraic geometry[see [Kol95], page 45-46 and [Bar75]]. It is very ugly and algebraically hard to use, but geometrically reasonable."

$$\left\langle F_i \;\middle|\; i=1,2,\ldots,b \right\rangle, \quad F_i := \int_{\delta_t} \omega_i,$$

where ω_i's are sections of $H^m_{\rm dR}(X/\mathsf{T})$ around 0 which form a basis of $F^{\frac{m}{2}+1}H^m_{\rm dR}(X/\mathsf{T})$ at each fiber near 0. For families of hypersurfaces and the Fermat variety X_0, one can give closed formulas for the coefficients of (1.10), see [Mov19, Theorem 13.2, Theorem 18.9]. In this case, instread of $[Z_0]$ we can take any Hodge cycle and k is always an abelian extension of the cyclotomic field $\mathbb{Q}(\zeta_d)$, see Deligne's lecture motes in [DMOS82].

Remark 1.1 Let X/T be a deformation of X_0 with T smooth and reduced. The tangent space of $V^{X/\mathsf{T}}_{[Z_0]}$ at 0 is just the first order infinitesimal Hodge locus $V^1_{[Z_0]} = V^{X/\mathsf{T}^1}_{[Z_0]}$.

Replacing $\pi: X \to S$ with $\pi^{-1}(V^{X/S}_{[Z_0]}) \to V^{X/S}_{[Z_0]}$ we get the following type of families:

Definition 1.5 Let X/S be a deformation of X_0 and $Z_0 \subset X_0$ be an algebraic cycle of codimension of codimension m. We say that ${\rm cl}(Z_0)$ remains Hodge in S if we have (1.4), (1.5) and (1.6).

The following question was originally posed for the number of parameters equal to 1 and $N=1$.

1.3 (P. Deligne, November 11, 2018) In what sense should we deform Z_0 into Z over S to make sure that ${\rm cl}(Z_0)$ remains Hodge over S, meaning that its horizontal extension in $H^m_{\rm dR}(X/S)$, the bundle over S of relative de Rham cohomology, remains of Hodge filtration $\frac{m}{2}$? I expect it suffices that for some E_0 of codimension ≥ 2 in Z_0, we have outside of E_0 a flat extension, noted $Z-E$, of $Z_0 - E_0$. The point is that such a $Z-E$ should give us a natural extension of ${\rm cl}(Z_0)$ in

$$F^p(H_{\rm dR}(X/S)) =$$
$$H^p(X/S, \text{ truncated relative dR complex starting at } \Omega^p \text{ put in cohomological degree 0 }).$$

1.4 (P. Deligne, November 15, 2018) I still think it might be useful to better understand, over an infinitesimal basis, in which sense (weaker than flat deformation as a subscheme) a cycle should be deformed to ensure that its class remains Hodge by some (infinitesimal, for instance first order) deformation: if the F_i are the local equations you use for the Hodge locus, and if an intersection of $dF_i = 0$ is bigger than expected, but if this can be explained by deformations in a weak sense, then it is no evidence for an Hodge locus of dimension bigger than expected.

1.5 (P. Deligne, November 20, 2018) Barlet and Argéniol have tried to define family of Chow cycles in the X_s of X/S, I do not remember what they could do, nor a family of Chow cycles in their sense give what one wants. One idea they use is that such [Chow cycle] to be defined

$$\begin{array}{ccc} Z & \hookrightarrow & X \\ & & \downarrow \\ & & S \end{array}$$

should locally, for $U \subset X$ a coordinate system and $U \to V$ a smooth projection with $\dim V = \dim(Z)$, Z finite over V, one should have a trace map

$$\mathrm{Tr}_{Z/V} : \mathscr{O}_U \to \mathscr{O}_V$$

with nice properties (such as, for some d, to be the degree of Z over V, and $(f_i)_{i\in I},\ \ |I| = d+1$)

$$\sum_{S\subset I} (-1)^{|S|}\mathrm{Tr}(\prod_{i\in S} f_i) = 0\big). \tag{1.9}$$

Long ago, I had hopes that having this for one projection gives us the same for any, but Argéniol showed I was wrong.

Note that the condition (1.9) can be re written in the following way. Taking the trace of the homogenuous pieces of

$$\prod_{i=1}^{d+1}(1-f_i)$$

and still summing them up, we get 1. We have assumed that $\mathrm{Tr}(1) = 1$ and note that Tr is not necessarily additive. For more discussion on this topic see 4.6. I do not have any intuition regarding this comment.

1.2 Main example

Let T be the parameter space of smooth hypersurfaces of degree d and dimension n. For

$$-1 \le m \le \frac{n}{2} - 2$$

let $\check{\mathsf{T}}_m$ be the parameter space of hypersurfaces X containing two linear $\mathbb{P}^{\frac{n}{2}}$ and $\check{\mathbb{P}}^{\frac{n}{2}}$ with

$$\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m, \tag{1.10}$$

that is, their intersection is of dimension m.

Proposition 1.2 *Any deformation (X_t, Z_t) of the pair $(X, \mathbb{P}^{\frac{n}{2}} + \check{\mathbb{P}}^{\frac{n}{2}})$ will give us a flat deformation of $Z_0 = \mathbb{P}^{\frac{n}{2}} + \check{\mathbb{P}}^{\frac{n}{2}}$ and $Z_t = \mathbb{P}_t^{\frac{n}{2}} + \check{\mathbb{P}}_t^{\frac{n}{2}}$ with $\mathbb{P}_t^{\frac{n}{2}} \cap \check{\mathbb{P}}_t^{\frac{n}{2}} = \mathbb{P}_t^m$.*

Proof. A pair $(X, \mathbb{P}^{\frac{n}{2}} + \check{\mathbb{P}}^{\frac{n}{2}})$ with (1.10) cannot be deformed into another pair with different m. This is because the topological intersection of these two cycles inside X is given by the formula:

$$\mathbb{P}^{\frac{n}{2}} \cdot \check{\mathbb{P}}^{\frac{n}{2}} = \frac{1-(-d+1)^{m+1}}{d}. \tag{1.11}$$

see [Mov19, Section 17.6]. □

Note that the deformation of the pair $(\mathbb{P}^{\frac{n}{2}}, \check{\mathbb{P}}^{\frac{n}{2}})$ with (1.10) is in the opposite direction: It can be deformed into another pair with an arbitrary m and the generic deformation will produce a pair with $m=-1$. It must be easy to prove the following:

Proposition 1.3 *Let $X \to T$ be a morphism of algebraic varieties over $\bar{\mathsf{k}}$ such that its generic fiber is $\mathbb{P}^N \cup_{\mathbb{P}^m} \mathbb{P}^N$ and its fiber over 0 is $\mathbb{P}^N \cup_{\mathbb{P}^{\tilde{m}}} \mathbb{P}^N$. If $\tilde{m} \neq m$ then X/S is not flat over 0.*

The conclusion is that a falt deformation (in our case smooth which is stronger) of a variety X_0, forces either an algebraic cycle $Z_0 \subset X_0$ to disappear or to be deformed flat. It might be interesting to find a counterexample to this, that is, to find a smooth X/S with an algebraic cycle Z of codimension $\frac{m}{2}$ in X/S such that the only non-flat fiber of Z is the central fiber Z_0. This is the topic of §4.6.

For some computer assisted proofs we will take

$$\mathbb{P}^{\frac{n}{2}}: \begin{cases} x_0 - \zeta_{2d}x_1 = 0, \\ x_2 - \zeta_{2d}x_3 = 0, \\ x_4 - \zeta_{2d}x_5 = 0, \\ \cdots \\ x_n - \zeta_{2d}x_{n+1} = 0. \end{cases} \qquad \check{\mathbb{P}}^{\frac{n}{2}}: \begin{cases} x_0 - \zeta_{2d}x_1 = 0, \\ \cdots \\ x_{2m} - \zeta_{2d}x_{2m+1} = 0, \\ x_{2m+2} - \zeta_{2d}^3 x_{2m+3} = 0, \\ \cdots \\ x_n - \zeta_{2d}^3 x_{n+1} = 0. \end{cases} \tag{1.12}$$

where $\zeta_{2d} := e^{\frac{2\pi\sqrt{-1}}{2d}}$. These are linear algebraic cycles in the Fermat variety $X_n^d \subset \mathbb{P}^{n+1}$ given by the homogeneous polynomial $x_0^d + x_1^d + \cdots + x_{n+1}^d = 0$, and satisfy $\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m$.

1.3 First order deformation

Recall the first order infinitesimal scheme $\mathsf{T}^1 \subset \mathsf{T}$ at 0 and the corresponding first order deformation X/T^1 of X_0. Recall also $V^1_{[Z_0]} := V^{X/\mathsf{T}^1}_{[Z_0]}$ and $V_{[Z_0]} := V^{X/\mathsf{T}}_{[Z_0]}$.

Proposition 1.4 *If $\dim_{\mathsf{k}} V^1_{[Z_0]} = \dim(V_{Z_0})$ then $V_{[Z_0]} = V_{Z_0}$, that is, AHC holds for (X_0, Z_0). Moreover, $V_{[Z_0]}$ is smooth and reduced.*

Proof. The Zariski tangent space of $V_{[Z_0]}$ at 0 is $V^1_{[Z_0]}$ and $V_{Z_0} \subset V_{[Z_0]}$. □

The following examples satisfies the hypothesis of Proposition 1.4. We have taken X/T to be the full family of smooth hypersurfaces.

1. Complete intersection algebraic cycles after [Dan14a, MV18]: Assume that $n \geq 2$ is even and $f \in \mathbb{C}[x]_d$ is of the following format:

$$f = f_1 f_{\frac{n}{2}+2} + f_2 f_{\frac{n}{2}+3} + \cdots + f_{\frac{n}{2}+1} f_{n+2}, \;\; f_i \in \mathbb{C}[x]_{d_i}, \;\; f_{\frac{n}{2}+1+i} \in \mathbb{C}[x]_{d-d_i}, \tag{1.13}$$

where $1 \leq d_i < d, \; i = 1, 2, \ldots, \frac{n}{2}+1$ is a sequence of natural numbers. Let $X_0 \subset \mathbb{P}^{n+1}$ be the hypersurface given by $f = 0$ and $Z_0 \subset X_0$ be the algebraic cycle given by $f_1 = f_2 = \cdots = f_{\frac{n}{2}+1} = 0$. We call Z_0 a complete intersection algebraic cycle in X_0.

2. Sum of two linear cycles [MV18, Mov19]: Let X_0 be a smooth hypersurface of degree d and dimension n which contains two linear $\mathbb{P}^{\frac{n}{2}}$ and $\check{\mathbb{P}}^{\frac{n}{2}}$ with $\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m$. For a generic choice of X_0 [ii] The algebraic cycle $Z_0 := r\mathbb{P}^{\frac{n}{2}} + \check{r}\check{\mathbb{P}}^{\frac{n}{2}},\ r, \check{r} \neq 0$ with (n,d,m) in the list

$$
\begin{aligned}
&(2,d,-1),\ 5 \leq d \leq 14,\\
&(4,4,-1),(4,5,-1),(4,6,-1),(4,5,0),(4,6,0),\\
&(6,3,-1),(6,4,-1),(6,4,0),\\
&(8,3,-1),(8,3,0),\\
&(10,3,-1),(10,3,0),(10,3,1),
\end{aligned}
$$

In [Vil20] this has been generalized to cases:

$$d < (d-2)(\frac{n}{2} - m) \tag{1.14}$$

For instance, for all $m \leq \frac{n}{2} - 2$ and $d \geq 5$.

1.4 Second order deformations

Remark 1.2 In the following when we talk about Hilbert schemes, we actually mean a connected open subset of it, over which the corresponding family is flat.

Let us consider an irreducible algebraic cycle Z_a in X_a and assume that in the flag Hilbert scheme $\mathrm{Hilb}(X_a, Z_a)$ it has a degeneration (X_0, Z_0) in which X_0 is still smooth but $Z_0 = P_0 + \check{P}_0$, where P_0 and $\check{P}_0$ are two irreducible algberaic cycles. It follows that the codimension of $P_0 \cap \check{P}_0$ in both P_0 and $\check{P}_0$ is one. Assume that for three algebraic cycles $C = Z_0, P_0, \check{P}_0$ the following property holds: the tangent space of $V_{[C]}$ at the point $0 \in \mathrm{Hilb}(X_0)$ corresponding to X_0 is equal to the dimension of the image of

$$\mathrm{Hilb}(X_0, C) \to \mathsf{T} := \mathrm{Hilb}(X_0).$$

This implies that the alternative Hodge conjecture (see [Mov19, Conjecture 18.2]) is true for the pairs $(X_0, Z_0),\ (X_0, P_0), (X_0, \check{P}_0)$.

Conjecture 1.1 *Assume that*

$$\dim F^{\frac{m}{2}+1} H^m_{\mathrm{dR}}(X_0) > 1. \tag{1.15}$$

The second infinitesimal Hodge locus $V^2_{r[P_0]+\check{r}[\check{P}_0]}$ for $r, \check{r} \in \mathbb{Z}, r, \check{r} \neq 0, r \neq \check{r}$ is singular at 0, and hence, the Hodge locus $V_{r[P_0]+\check{r}[\check{P}_0]}$ is singular at 0. The condition (1.15) *is*

[ii] Here, generic means in some Zariski open subset of the algebraic subset of T parametrizing those X_0.

needed, otherwise all Hodge loci would be either the whole space or of codimension one.

Note that ‘singular’ in the above statement is used in the scheme theoretic context. If the underlying analytic variety of $V_{r[P_0]+\check{r}[\check{P}_0]}$ is smooth then the above statement says that it is not reduced. Note also that in Conjecture 1.1, $r = \check{r}$ is excluded as $(X_0, Z_0),\ Z_0 = P_0 + \check{P}_0)$ has a larger deformation space in which the deformed algebraic cycle Z_t is irreducible.

Theorem 1.6 ([Mov19], Theorem 18.3, part 1) *Conjecture 1.1 is true for the Fermat hypersurface X_0 of degree d and dimension n and $Z_0 = \mathbb{P}^{\frac{n}{2}} + \check{\mathbb{P}}^{\frac{n}{2}}$ with (1.12) and in the following cases: For all $r, \check{r} \in \mathbb{Z},\ 1 \leq |r| \leq |\check{r}| \leq 10,\ r \neq \check{r}$ and (n,d) in the list*

$$(2,d),\ 5 \leq d \leq 9, \tag{1.16}$$

$$(4,4),(4,5),(6,3),(8,3) \tag{1.17}$$

One might claim that the underlying analytic variety of $V_{r[P_0]+\check{r}[\check{P}_0]},\ \ r, \check{r} \neq 0, r \neq \check{r}$ is $V_{[P_0]} \cap V_{[\check{P}_0]}$. However, this is stronger than Conjecture 1.1. The dimension of $V_{[P_0]} \cap V_{[\check{P}_0]}$ is less than the dimension of the Zariski tangent space of $V_{r[P_0]+\check{r}[\check{P}_0]}$ at 0, and when the differenc is only one, this stronger statement is true. This is the case, for instance, for Example 4.6 and Example 4.7. The second order approximation of Hodge loci is also formulated by Maclean [Mac05, Theorem 7], see also Theorem 8 for some applications in the case of quintic surfaces with two disjoint lines, that is, $(n,d,m) = (2,5,-1)$.

1.5 Degeneration of algebraic cycles

Let X be a smooth projective variety and $Z = \sum_{i=1}^{r} n_i Z_i,\ \ n_i \in \mathbb{Z}$ be an algebraic cycle in X, with Z_i an irreducible subvariety of codimension $\frac{n}{2}$ in X. The following definition is done using analytic deformations and it would not be hard to state it in the algebraic context.

Definition 1.7 We say that $Z = \sum_{i=1}^{r} n_i Z_i,\ \ n_i \in \mathbb{Z}$ is semi-irreducible if there is a smooth analytic variety $\mathscr{X}$, an irreducible subvariety $\mathscr{Z} \subset \mathscr{X}$ of codimension $\frac{n}{2}$ (possibly singular), a holomorphic map $f : \mathscr{X} \to (\mathbb{C}, 0)$ such that

1. f is smooth and proper over $(\mathbb{C}, 0)$ with X as a fiber over 0. Therefore, all the fibers X_t of f are C^∞ isomorphic to X.
2. The fiber Z_t of $f|_{\mathscr{Z}}$ over $t \neq 0$ is irreducible and $Z_0 = \cup_{i=1}^{r} Z_i$.
3. The homological cycle $[Z] := \sum_{i=1}^{r} n_i [Z_i] \in H_n(X, \mathbb{Z})$ is the monodromy of $[Z_t] \in H_n(X_t, \mathbb{Z})$.

1.6 (P. Deligne 15 November, 2018) [In Definition 1.7] 3. is strangely weak: Don’t you want the fiber, as an algebraic cycle, to be $\sum n_i Z_i$? What 3. requires is weaker when the cohomology classes of the Z_i are not linearly independent.

The above comment is absolutely correct. One might regard Z_i as a scheme determined by its sheaf of ideals $\mathscr{I}_i$, and therefore, n_iZ_i has the sheaf of ideals $\mathscr{I}_i^{n_i}$. For this we assume that $n_i \in \mathbb{N}$. In this way we are talking about degeneration of schemes. Doing in this way one has to prove that

Proposition 1.5 *The cohomology class* $\mathrm{cl}(Z_t)$ *is flat, that is,* $\nabla(\mathrm{cl}(Z_t)) = 0$.

The following examples might help to define the general concept of deformation of algebraic cycles.

1.7 (P. Deligne, November 7, 2018) Take a smooth divisor Z on Y, and blow up many points of Y on Z to get X. As cycle C take the pure transform of Z. If we deform X by moving in Y the points to be blown up, the cycle is usually not able to follow. Here it matters what is to be called 'deformation': $C = (C + \text{exceptional divisors}) - (\text{exceptional divisors})$, and both can be deformed.

1.6 Equations for Hodge loci

Let $\mathsf{X} \to \mathsf{T}$ be a family of smooth projective varieties over k. Let $Z_0 \subset X_0$ be an algebraic cycle and let $\mathrm{cl}(Z_0) \in H^n_{\mathrm{dR}}(X_0/\mathsf{k})$ be its cohomology class.

1.8 (P. Deligne, November 15, 2018) It also result from the fact that Hodge cycle obtained by deforming classes of algebraic cycles, keeping them Hodge along the deformation, are absolute Hodge cycles, even "motivated cycles" in the sense of André.

The first part of this statement is in [DMOS82, Theorem 2.12, Principle B]. For motivated cycles see [And96, And04]. Assuming that both Z_0 and X_0 are defined over a field k, it turns out that there is an algebraic subset $\check{\mathsf{T}} \subset \mathsf{T}$ defined over k such that the Hodge locus $V_{[Z_0]}$ is a union of irreducible components of a small neighborhood of 0 in $\check{\mathsf{T}}$ using the analytic topology.

1.9 (P. Deligne, November 11, 2018) *Could this [Proposition 1.1] be used to guess equations for* $\check{\mathsf{T}}$*?*

The following two examples suggest that it might be possible to play with holomorphic equations of Hodge loci and get the algebraic equations. Let us consider the differential $(n+1)$-form

$$\Omega := \sum_{i=0}^{n+1} (-1)^i x_i \widehat{dx_i}$$

Example 1.8 The Fermat quartic surface has 48 lines. Let us call them $\mathbb{P}^1_i, \ \ i = 1,2,\ldots,48$. Let also $\delta_{i,t} \in H_2(X_t, \mathbb{Z}), \ \ t \in (\mathsf{T}, 0)$ be the monodromy of $[\mathbb{P}^1_i]$ to nearby fibers. We denote by $\check{\mathsf{T}} \subset \mathsf{T}$ the parameter space of quartic surfaces containg a line. It is a codimension 1 subvariety of T and hence it is the zero set of a polynomial $P \in \mathbb{C}[t]$. We have

$$\prod_{i=1}^{48}\int_{\delta_{i,t}}\text{Resi}\left(\frac{\Omega}{f}\right)=g\cdot P$$

where g is a holomorphic function in a neighborhood of $0\in\mathsf{T}$. This follows from the fact that both hand sides near $0\in\mathsf{T}$ have the same zero set. Actually, we can show that the coefficients of P are in $\mathbb{Q}(\zeta_4)$. Note that the ingredients of the product are holomorphic functions defining the Hodge locus $V_{\mathbb{P}^1_i}$.

Example 1.9 The Fermat cubic fourfold has $3\cdot 5\cdot 3^3=405$ liner cycles $\mathbb{P}^2$. Let us call them $\mathbb{P}^2_i,\ \ i=1,2,\ldots,405$. Let also $\delta_{i,t}\in H_2(X_t,\mathbb{Z}),\ \ t\in(\mathsf{T},0)$ be the monodromy of $[\mathbb{P}^2_i]$ to nearby fibers. We denote by $\check{\mathsf{T}}\subset\mathsf{T}$ the parameter space of cubic fourfolds containg a linear $\mathbb{P}^2$. It is a codimension 1 subvariety of T given by a polynomial $P\in\mathbb{Q}(\zeta_3)[t]$. We have

$$\prod_{i=1}^{405}\int_{\delta_{i,t}}\text{Resi}\left(\frac{\Omega}{f^2}\right)=g\cdot P$$

where g is a holomorphic function in a neighborhood of $0\in\mathsf{T}$.

1.7 Singularities of Hodge locus

In an attempt to describe Hodge locus as leaves of certain holomorphic foliations in [Mov20a], I arrived at a defining ideal $I\subset\mathscr{O}_{\mathbb{C}^N,0}$ of a Hodge locus with the following description. Let $\mathscr{O}_{\mathbb{C}^N,0}$ be the $\mathbb{C}$-algebra of holomorphic functions in a neighborhood of 0 in $\mathbb{C}^N$. Let also $(a_0,a_1,\cdots,a_{\frac{m}{2}-1})$ (originally Hodge numbers $h^{m,0},h^{m-1,1},\cdots,h^{\frac{m}{2}+1,\frac{m}{2}-1}$) be natural numbers. Let us consider an ideal $I\subset\mathscr{O}_{\mathbb{C}^N,0}$ generated by the entries of $f^0,f^1,\ldots,f^{\frac{m}{2}-1}$, where f^i is a $a_i\times 1$ matrix with entries in $\mathscr{O}_{\mathbb{C}^N,0}$. Moreover, assume that there are $a_i\times a_{i+1}$ matrices $\mathsf{A}^{i,i+1}$ with entries in $\Omega^1_{\mathbb{C}^N,0}$ such that

$$\begin{cases} df^0=\mathsf{A}^{01}f^1\\ df^1=\mathsf{A}^{12}f^2\\ \cdots\\ df^{\frac{m}{2}-2}=\mathsf{A}^{\frac{m}{2}-2,\frac{m}{2}-1}f^{\frac{m}{2}-1}.\end{cases} \tag{1.18}$$

This motivates us to define:

Definition 1.10 An ideal $I\subset\mathscr{O}_{\mathbb{C}^N,0}$ is differentially saturated if

$$\forall f\in I,\quad df\in I\cdot\Omega^1_{\mathbb{C}^N,0}\Rightarrow f\in I.$$

The equalities (1.18) suggest that the the ideal of a Hodge locus might be differentially saturated.

Try to prove this.

The equalities (1.18) remembers one of the Griffiths transversality, however, it seems that this kind of description cannot be done in a purely algebraic context, as in [Mov20a][Section 6.10] I had to introduce a foliation called $\mathscr{F}(2)$ with possibly all transcendental leaves, and restrict the Gauss-Manin connection to the leaves of this foliation. It might be possible to do it in an infinitesimal level.

1.8 The locus of Hodge cycles

As it was noticed in [CDK95], the total space of $F^{\frac{n}{2}}$-bundle over T is a better place to study Hodge loci. We denote it by

$$\tilde{\mathsf{T}} := F^{\frac{n}{2}} H^n_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}).$$

We have the canonical projection $\tilde{\mathsf{T}} \to \mathsf{T}$ and using [Mov17b, Section 2] we can take an affine charts in $\tilde{\mathsf{T}}$.

1.9 How it started?

The following was the starting point of the present text.

1.10 (P. Deligne, November 4, 2018) I played with the numerology of Hodge numbers $h^{p,q}$ of the primitive part of the cohomology of hypersurfaces of degree d and dimension $2n$, in $\mathbb{P}^{2n+1}$. Examples lead me to expect the following:

1. If $h^{p,q} = 1$, with $p < q$, then $h^{p+1,q-1}$ is the number of moduli: dimension $\binom{d+2n+1}{2n+1}$ of the space of equations of degree $d - (n+2)^2$.
2. If for some $p < n-1$ one has non-vanishing of $h^{p,q}$, then $h^{n-1,n+1}$ is at least the number of moduli (with equality only if this p is $n-2$, with $h^{p,q} = 1$).

If the first statement is correct, it would suggest that as for Calabi-Yau, the tangent space to the moduli is given by $\mathrm{Hom}(H^{p+1,q-1}, H^{p,q})$. If the second statement is correct, it would suggest that either only 3 $H^{p,q}$ do not vanish, as for surfaces, in which case one might have plenty of positive dimensional Hodge locus, or that we would be unable to predict by analytic means that an Hodge cycle occurs near some hypersurface. Do you know whether the statement are true? One could ask the same question for complete intersections, and for the first question, there is a priori no reason to consider only the even dimensional case.

The statements 1 and 2 follows from the description of the Hodge numbers of the Fermat hypersurface using its Jacobian ring, see [Mov19, §15.4]. Concerning "...would be unable to predict..." the following comments might be useful. If in the Fermat variety one chooses a general Hodge cycle then your argument tells us that the corresponding Hodge locus would be zero dimensional and so not useful. However, special Hodge cycles, which might exist infinite number of them, produce positive dimensional Hodge locus. For instance, a heavy computer calculations in

[Mov19, Chapter 18] shows the following. Consider the Fermat cubic tenfold X ($d = 3$ $n = 5$) and the algebraic cycle $\mathbb{P}^5_1 - \mathbb{P}^5_2$ in X, where $\mathbb{P}^5_1$ and $\mathbb{P}^5_2$ are two projective spaces of dimension 5 inside X and intersecting each other in a projective space of dimension 3 (I call such $\mathbb{P}^5$ a linear algebraic cycle). The Zariski tangent space of the Hodge locus corresponding to this Hodge cycle has codimension 32, whereas the deformation space of the triple $(X, \mathbb{P}^5_1, \mathbb{P}^5_2)$ with $\mathbb{P}^5_1$ and $\mathbb{P}^5_2$ inside X, is of codimension 36. This means that by deforming X in 4 free dimensions the homology class of $\mathbb{P}^5_1 - \mathbb{P}^5_2$ can be still a Hodge cycle. Verification of Hodge conjecture for this Hodge cycle seems to be as Hard as the Hodge conjecture itself.

1.11 (P. Deligne, November 5, 2018) For the cubic tenfold, smoothness of this Hodge locus would be extremely interesting, as it would indeed give a concrete instance where the Hodge conjecture is open. So far, the main example I know is the Weil's example of some abelian varieties with some complex multiplication (simplest case : dimension 4, complex multiplication by a quadratic imaginary field K, Lie algebra isomorphic as a K-module to K tensor $\mathbb{C}$).[For this see Weil's article [Wei77] and Mumford's example there] Presumably hypersurfaces in $\mathbb{P}^{2n+1}$ whose with only 3 Hodge number in middle cohomology would give many examples, but one can hope that in each such case one could reduce the problem to the case of divisors on a related surface.

However, my expectation is rather that the dimension of the tangent space to the Hodge locus is bigger than the dimension of that locus. I agree that if this is the case, one would like to have an explanation for why. My working hypothesis is that the reduced Hodge locus is just the codimension 36 locus where we have both $\mathbb{P}^5$, a locus where the codimension 20 locus where we have some $\mathbb{P}^5$ has two (smooth?) branches. I am puzzled by the fact that $(20+20)-36 = 36-32$, but could not use it. Are your computations of the dF_i for F_i equations of the Hodge locus only at Fermat? Do you see a way to do higher order computations?

Actually I do higher order approximations of the equations of the Hodge locus near Fermat. I was able to write down the Taylor expansion of periods at Fermat points and implement it in the computer. This is [Mov19, Theorem 18.9 in Section 18.5]. The linear part of this series encodes just the tangent space. The reducedness and smoothness of the Hodge locus boils down to identities between formal power series, and I do not know how to prove such identities in general. If such equalities happens with truncated power series up to order N, then I say it is N-smooth. In this way I can prove Theorem 18.2 and Theorem 18.3 in Section 18.1. In particular, Theorem 18.3 part 2 gives us smooth Hodge loci which are bigger than the expected one. In this list you could also put the triple $(n, d, m) = (10, 3, 3)$ which is the example in my previous example, however I had to reduce the number of parameters to get Theorem 18.3 in this case. Anyway, to be sure that such a Hodge locus exists one has to check N-smoothness for N big enough, however, I could only do computations until $N =< 4$. For now I do not know which N-smoothness implies smoothness. This N seems to be some invariant which can be derived from the Gauss-Manin connection, or the underlying geometry. If you think I have to work out other examples let me know. For me a case where the Hodge conjecture is well-known is also interesting.

For instance, for quartic Fermat fourfold with two $\mathbb{P}^2$ intersecting each other in a point the Hodge locus of the difference of two $\mathbb{P}^2$ must be bigger than the expected one, and for this apart from the computation of the tangent space I had to approximate periods (over the full moduli) up order 4. My computer ran for few days to check this.

1.10 Hodge cycles for cubic hypersurfaces

In December 2018, I started to write the article [Mov18] in order to gain more evidence to the existence of Hodge cycles predicted in my book [Mov19]. In Januray 4, 2019 P. Deligne made the following observation. Since there will be used many linear cycles, I will use the following notation:

$$\mathbb{P}^{\frac{n}{2}}_{a_1,a_2}: \begin{cases} x_0-\zeta_6 x_1=0, \\ \cdots \\ x_{n-4}-\zeta_6 x_{n-3}=0, \\ x_{n-2}-\zeta_6^{2a_1+1}x_{n-3}=0, \\ x_n-\zeta_6^{2a_2+1}x_{n+1}=0. \end{cases} \quad \text{where } 0\leq a_1,a_2\leq 2. \tag{1.19}$$

Using this notation we have

$$m=\frac{n}{2}-2,\ \ \mathbb{P}^{\frac{n}{2}}_{0,0}=\mathbb{P}^{\frac{n}{2}}, \mathbb{P}^{\frac{n}{2}}_{1,1}=\check{\mathbb{P}}^{\frac{n}{2}}$$

We can write

$$\mathbb{P}^{\frac{n}{2}}_{0,0}-\check{\mathbb{P}}^{\frac{n}{2}}_{1,1}=\left(\mathbb{P}^{\frac{n}{2}}_{0,0}+\mathbb{P}^{\frac{n}{2}}_{0,1}\right)-\left(\mathbb{P}^{\frac{n}{2}}_{1,1}+\check{\mathbb{P}}^{\frac{n}{2}}_{0,1}\right)=\check{\mathbb{P}}^{\frac{n}{2}}_{2,1}-\mathbb{P}^{\frac{n}{2}}_{0,2} \tag{1.20}$$

where the second equality is modulo $\mathbb{P}^{\frac{n}{2}+1}$ slices of the Fermat variety X_0. This way of writing was also suggested to me few months earlier by my student Roberto Villaflor. We can also do the same thing by adding and subtracting $\mathbb{P}^{\frac{n}{2}}_{1,0}$. We get apparentely three branches

$$V_Z,\ \ Z=Z_1,Z_2,Z_3=\mathbb{P}^{\frac{n}{2}}_{0,0}-\mathbb{P}^{\frac{n}{2}}_{1,1},\ \ \mathbb{P}^{\frac{n}{2}}_{2,1}-\mathbb{P}^{\frac{n}{2}}_{0,2},\ \ \mathbb{P}^{\frac{n}{2}}_{1,2}-\mathbb{P}^{\frac{n}{2}}_{2,0}, \tag{1.21}$$

of an irreducible subvariety Y of T parameterizing cubic hypersurfaces with two linear cycles of dimension $\frac{n}{2}$ and intersecting each other in dimension $m=\frac{n}{2}-2$. The variety Y parametrizes hypersurfaces given by homogeneous polynomials of the form:

$$f=f_1*+\cdots+f_{s-2}*+f_{s-1}(g_{s-1}*+g_s*)+f_s(g_{s-1}*+g_s*),\ \ s:=\frac{n}{2}+1.$$

where f_i and g_i's are degree 1 homogeneous polynomials. Since these three cycles induce the same element in the primitive homology $H_n(X_0,\mathbb{Z})_0$, the corresponding

Hodge loci $V_{[Z]}$ are the same. The analytic varieties V_{Z_i}'s are smooth and we have

$$V_{Z_1} \cup V_{Z_2} \cup V_{Z_3} \subset V_{[Z]}$$

in the scheme theoretical sense, that is, the inclusion happens in the level of ideals.

1.12 (P. Deligne, January 4, 2018) Do your computations imply that the Hodge locus is not just the union of the three branches? Is its tangent space bigger than the sum of the tangent space to the three branches? Model example : a function on $\mathbb{C}^2$ vanishing on three lines through 0 will begin at order 3, and will look smooth in a second order computation.

It is mostly the case that the tangent space of $V_{[Z]}$ is the sum of tangent spaces of $V_{Z_i}, \ \ i = 1,2,3$. This can be checked easily and at some point I have to do it. But note that in Table 1 of [Mov18], for $n = 6,8,10$ I have shown $9-$, $5-$ and $3-$ order smoothness of $V_{[Z]}$, respectively.

Proposition 1.6 *Let $V_1, V_2, \cdots, V_k \subset (\mathbb{C}^n, 0)$ be germs of smooth analytic schemes. If the union $V := V_1 \cup V_2 \cup \cdots \cup V_k$ is not equal to none of V_i's and it is N-smooth then $N < k$.*

Proof. Let $I_1, I_2, \ldots, I_k, I$ be the ideals of $V_1, V_2, \ldots, V_k, V$, respectively. By definition

$$I = I_1 \cap I_2 \cap \cdots I_k, \text{ and so } I_1 I_2 \cdots I_k \subset I.$$

We consider the coordinate system $(z_1, z_2, \ldots, z_n)$ for $(\mathbb{C}^n, 0)$. Without loss of generality we assume that the linear part of the ideal I is generated by $z_1, z_2, \cdots, z_a$. After a coordinate change in $(\mathbb{C}^n, 0)$ we further assume that $z_1, z_2, \ldots, z_a \in I$, and hence, these are in all I_i's. It is enough to prove the proposition for $z_1 = z_2 = \cdots = z_a = 0$, that is, we can assume that no element in I has non-zero linear part. By our hypothesis V_i's are proper analytic subspaces of $(\mathbb{C}^n, 0)$. For all $f_i \in I_i, \ i = 1,2,\ldots,k$ we have $f_1 f_2 \cdots f_k \in I$. If V is N-smooth then $f_1 f_2 \cdots f_k \in \mathscr{M}_{\mathbb{C}^n,0}^{N+1}$. Since V_i is smooth, we can choose f_i such that the linear part of f_i is non-zero, and hence, $f_1 f_2 \cdots f_k \notin \mathscr{M}_{\mathbb{C}^n,0}^{k+1}$ which implies that $k < N$. □

Proposition 1.7 *The inclusion of analytic schemes*

$$V_{Z_1} \cup V_{Z_2} \cup V_{Z_3} \subset V_{[Z]}$$

is strict for all cases listed in Table 1 of [Mov18].

Proof. If the equality happens then by Proposition 1.6 V can be N-smooth only for $N < 3$. This is not the case for all cases in Table 1.

Note that Proposition 1.7 does not imply that the underlying analytic variety of $V_{[Z]}$ is larger than the union of $V_{Z_1}, V_{Z_2}, V_{Z_3}$. That is, we are not yet done we the discovery of new Hodge cycles! Verifying the following conjecture might help. Do you think it is true?

Conjecture 1.2 *There is an infinite number of algebraic cycles $Z_i,\ i\in\mathbb{N}$ such that*

1. *The cohomology class of Z_i's in the primitive cohomology are equal up to multiplication by a rational number.*
2. *There is no inclusion between V_{Z_i}'s as analytic varieties.*

So far we know only three algebraic cycles Z_1,Z_2,Z_3. The moral of the story is

1. Either Conjecture 1.2 is true and we might be able to construct infinite number of algebraic cycles with small deformation spaces. In this case the union of V_{Z_i}'s is contained in $V_{[Z]}$ which is larger than all V_{Z_i}'s.
2. or it is false and there is at most k algebraic cycles $Z_i,\ i=1,2,\dots,k$ with the property of Conjecture 1.2. By Proposition 1.7, in order to predict a bigger Hodge loci we must verify N-smoothness for $N\geq k$.

Finally, the discussion for cubic fourfolds might be useful. Since for cubic fourfold we have $h^{40}=0,\ h^{31}=1$, components of any Hodge locus in this case are of codimension one. For the family of cubic fourfolds

$$X_t:\ x_0^3+x_1^3+x_2^3+x_3^3+x_4^3+x_5^3-(t_1x_2+t_2x_3)x_1x_5 \text{ the case } m=0 \tag{1.22}$$

the Hodge locus $V_{[Z]}$ is given by the zero set of a single holomorphic function

$$F(t):=\int_{\delta_t}\mathrm{Resi}\left(\frac{\Omega}{f^2}\right)=(-2z-2)t_2+\cdots$$

which has non-zero linear part, and hence, the Hodge locus in these case is smooth of codimension one, and it contains three smooth subvarities $V_{Z_i}, i=1,2,3$ which are of codimension 2.

1.11 Cubic tenfold again

1.13 (P. Deligne, January 7, 2019) *There are many more locus where the class of $P'-P''$ extends as an algebraic cycle of low complexity. One could take a P''' such that both $P'-P'''$ and $P'''-P'$ intersect in some $\mathbb{P}^{N-2}$, and write*

$$P'-P''=(P'-P''')+(P'''-P'').$$

One can, for $N=5$, take P',P''',P'' given by roots of unity

$$\begin{aligned} P' &: 0,0,0,0,0,0\\ P''' &: 0,0,0,a,1,0\\ P'' &: 0,0,0,0,1,1 \end{aligned}$$

(16 possibilities: position: $4\cdot 2$, choice of a: 2) and using the previous letter write $(L_1-L_2)+(L_3-L_4)$.

$L_i's$ are given by

$$\begin{array}{l} L_1 : 0\ 0\ 0\ 0\ 2\ 0 \\ L_2 : 0\ 0\ 0\ \tilde{a}\ 1\ 0 \\ L_3 : 0\ 0\ 0\ a\ 1\ 2 \\ L_4 : 0\ 0\ 0\ \tilde{a}\ 1\ 1 \end{array} \tag{1.23}$$

where $\tilde{a} = 3 - a$. Let $L = L_1 + L_2 + L_3 + L_4$ be the sum of four algebraic cycles L_i, and $V_L \subset (\mathsf{T}, 0)$ be the corresponding deformation space of L. By our consruction V_L is inside the Hodge lcus $V_{[Z]}$, and it turrns out that, V_L DOES NOT lie in the union of three branches $V_{Z_i},\ \ i = 1,2,3$ in the previous section. For this we write the most general cubic hypersurface $X : f = 0$ parametrized by V_Z. We need $4+7 = 11$ linear equations. This is the number of digits in (1.23), counted once if a digit repeats in column, and the same digit in different columns are considered different. The linear polynomials in the last columns are denoted by f_i, g_i, h_i's. The homogeneous polynomial f has the terms $g_2 f_a f_{\tilde{a}}, h_0 f_a f_{\tilde{a}}$ which makes the statement plausible. At the beginnig I made a mistake and I wrote “The loci you are describing for me lies in the intersection of three loci in your letter of January 4, and hence, it does not make N in the N-smoothness increase.”

1.14 (P. Deligne, January 7, 2019) *This gives many intersections of two branches of Y, each of codimension at most $2 \times 36 = 72$ (for N=5). This makes it for me difficult to see up to which order one should go to get evidence of smoothness at Fermat for the Hodge locus. Even first order information at some general enough cubic (not containing other $\mathbb{P}^5$ than P', P'') would be easier to interpret. Would such a computation be possible?*

All this discussion is relevant assuming that the Hodge loci is a (scheme theoretic) union of its V_{Z_i}'s. I think this assumption is wrong (example cubic fourfold case!). I do not have proof for this. I have a kind of idea how to compute N in (N-smooth implies smooth) purely from the Gauss-Manin connection. I will try to write it soon.

Regarding working in a generic cubic with two linear cycles: it is possible to do it , however it needs some patience and time to write down the algorithm and then implement it in computer. The case of Fermat already took few years of my life, and at the end no body appreciated this kind of mathematics (and even considered it trivialities) and I almost abandoned it. However, your last emails and comments now given some joy to push forward this kind of math. I will do it once I lose all my hopes that the computation around Fermat is enough.

1.12 Using cubic surfaces

Let $Y \subset \mathsf{T}$ be the set of cubic hypersurfaces containing two linear cycles $P' = \mathbb{P}^{\frac{n}{2}}$ and $P'' = \check{\mathbb{P}}^{\frac{n}{2}}$ meeting in a $\mathbb{P}^{\frac{n}{2}-2}$. For simplicity take the case $n = 10$. Let $\mathbb{P}^7 \subset \mathbb{P}^{11}$ be the projective space spanned by P' and P''. For a generic X in Y, the intersection $\mathbb{P}^7 \cap X$ is a smooth cubic 6-fold containing two $\mathbb{P}^5$ meeting in a $\mathbb{P}^3$. We consider the following non-generic case. Let $\check{Y} \subset Y$ be the set such that for a hypersurface in $\check{Y}$ such a cubic 6-fold in some coordinate system $[x_0 : x_1 : x_2 : x_3 : y_0 : y_1 : y_2 : y_3]$ of $\mathbb{P}^7$ is given by a polynomial depending only on x_i's (and hence it is singular). In geometric terms, we have a linear map $\pi : \mathbb{P}^7 \dashrightarrow \mathbb{P}^3, \quad [x : y] \mapsto [x]$ with the indeterminacy set $\mathbb{P}^3 : x = 0$ such that such a cubic 6-fold is a pull-back of a cubic surface S in $\mathbb{P}^3$. The closure of the fibers of π are $\mathbb{P}^4$'s meeting in $\mathbb{P}^3$. Any curve C in S will give us a 5-dimensional algebraic cycle $Z_C = \overline{\pi^{-1}(C)}$ in X. Of course, $\check{Y}$ is a proper subset of Y. This is the picture that I understand from your letters of [P. Deligne, January 10, 2019] and [P. Deligne, January 11, 2019]. At the end of latter letter you describe a twitted curve C in S and the resulting Z_C. It turns out that Z_C for Fermat is simple given by sum of three linear cycles:

$$Z_C = \mathbb{P}^5_{0,0,0,0,0,0,0} + \mathbb{P}^5_{0,0,0,0,2,1} + \mathbb{P}^5_{0,0,0,0,0,0,1}$$

In primitive cohomology it is the same as $\mathbb{P}^5_{0,0,0,0,0,0,0} - \mathbb{P}^5_{0,0,0,0,1,1}$. The last linear cycle intersects the other two in $\mathbb{P}^4$, and the first two intersect each other in $\mathbb{P}^3$ inside the last one. It seems that this can deformed inside the deformed cubic hypersurface, let us call it a twisted cubic (algebraic) cycle. At the end we have proved that the space of cubic hypersurfaces containing a cubic ruled cycle is a component of the Hodge locus. I am now thinking on the following problem which might be trivial.

Problem 1.1 *Let $\mathbb{P}^m_1, \mathbb{P}^m_2, \mathbb{P}^m_3$ be three linear projective subspaces of $\mathbb{P}^N$ with $\dim(\mathbb{P}^m_1 \cap \mathbb{P}^m_2) = \dim(\mathbb{P}^m_1 \cap \mathbb{P}^m_3) = m-1$ and $\mathbb{P}^m_2 \cap \mathbb{P}^m_3$ is inside $\mathbb{P}^m_1$ and of dimension $m-2$. For $m \geq 2$ show that $Z_0 = \mathbb{P}^m_1 + \mathbb{P}^m_2 + \mathbb{P}^m_3$ deforms into an irreducible algebraic subvariety Z of $\mathbb{P}^N$.*

For $m = 1$ this just the deformation into a twisted cubic curve.

1.13 Higher dimensional cubic cycles

In the first formulation of Problem 1.1, due to the confusion on the deformation spaces of $\mathbb{P}^5_{0,0,0,0,0,0,0} - \mathbb{P}^5_{0,0,0,0,1,1}$ and $\mathbb{P}^5_{0,0,0,0,0,0,0} + \mathbb{P}^5_{0,0,0,0,2,1} + \mathbb{P}^5_{0,0,0,0,0,0,1}$, I made a mistake and put the condition that Z is not a cone over a twisted cubic curve. A deformation which is a cone over a twisted cubic curve can be constructed in the following way. We choose four points in $\mathbb{P}^m_2 \backslash \mathbb{P}^m_1$, $\mathbb{P}^m_3 \backslash \mathbb{P}^m_1$, $(\mathbb{P}^m_2 \cap \mathbb{P}^m_1) \backslash \mathbb{P}^{m-2}$, $(\mathbb{P}^m_3 \cap \mathbb{P}^m_1) \backslash \mathbb{P}^{m-2}$, respectively, where $\mathbb{P}^{m-2} = \mathbb{P}^m_2 \cap \mathbb{P}^m_3$. Consider $\mathbb{P}^3$ spanned by these four points. The intersection C_0 of Z_0 with $\mathbb{P}^3$ is a twisted cubic curve which is a sum of three lines. Now, Z_0 is a cone over C_0 which can be easily deformed as cone, when C_0 deforms. Consulting Problem 1.1 with D. van Straten and for $n = 2$ I got the following answer:

1.15 (D. van Straten, January 15, 2019) The union of the three $\mathbb{P}^2$'s has a smoothing to a very simple and well known surface in $\mathbb{P}^4$, namely the 'cubic ruled surface'. It is isomorphic to $\mathbb{P}^2$ blown up in a single point, embedded by the linear system of quadrics through the point. So abstractly it is the Hirzebruch surface F_1. It is defined by the three 2×2 minors of a general 2×3 matrix of linear forms. Each cubic hypersurface containing the three $\mathbb{P}^2$'s can be lifted along the smoothing. I guess this is what is going for that example.

It turns out all the deformations of Z_0 in Problem 1.1 are cones over twisted curves. Let us consider the following equations

$$\mathbb{P}^m_1 : \cdots = f_1 = f_6 = 0$$
$$\mathbb{P}^m_2 : \cdots = f_1 = f_3 = 0$$
$$\mathbb{P}^m_3 : \cdots = f_4 = f_6 = 0$$

where f_1, f_3, f_4, f_6 are degree 1 homogeneous polynomials and $\cdots$ means the common equations between three cycles. The algebraic cycle $Z_0 := \mathbb{P}^m_1 + \mathbb{P}^m_2 + \mathbb{P}^m_3$ deforms into the following algebraic cycle:

Definition 1.11 A cubic ruled cycle in $\mathbb{P}^N$ of dimension m is given by

$$Z : g_1 = g_2 = \cdots = g_{N-m-2} = 0, \quad \text{rank} \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \\ f_5 & f_6 \end{bmatrix} \leq 1 \tag{1.24}$$

where g_i's and f_i's are homogeneous degree 1 polynomials.

The cubic ruled cycles deforms into Z_0 by setting f_2, f_5 equal to zero. The verification of the Hodge conjecture for the Hodge cycle in [Mov18][Theorem 1, part 1] reduces to the following:

Proposition 1.8 *For $n = 4, 6, 8, 10, 12$ the codimension of the locus of cubic hypersurfaces containing a cubic ruled cycle of dimension $\frac{n}{2}$ is respectively* $1, 6, 16, 32, 55$.

This will prove the following theorem.

Theorem 1.12 **($n = 4, 6, 8, 10, 12, \ d = 3$)** . *Let T be the full parameter space of smooth hypersurfaces of degree d and dimension n. Let also $\check{\mathsf{T}}$ be the subvariety of T parameterizing hypersurfaces containing a twisted cubic cycle Z of dimension $\frac{n}{2}$. There is a Zariski neighborhood U of $\check{\mathsf{T}}$ such that the any Hodge cycle deformation of the cohomology class of Z inside the deformed hypersurface is again supported in a cubic ruled cycle.*

This is namely the alternative Hodge conjecture formulated in [Mov19][section 18.2].

Proof. The tangent space of the Hodge locus $V_{[Z_0]}$, for Z_0 inside Fermat as before, has the same dimension as the branch V_{Z_0} of $\check{\mathsf{T}}$. A complete proof of this theorem without a restriction on degree and dimension (due to the usage of computer for the computation of the tangent space of $V_{[Z_0]}$ might be a headache!! In the case of sum of two linear cycles with intersection of low dimension, my student, see [Vil20] was able to remove the computer assisted part. But he got a very nasty commutative algebra computations!!

After making a proper definition of a deformation of an algebraic cycle, one might state the following corollary, which might be an easy exercise in algebraic geometry.

Corollary 1.13 *The only deformation of a pair (X,Z) of a smooth hypersurface X and a cubic ruled cycle $Z\subset X$ is again a hypersurface with a cubic ruled cycle in it.*

Once again the Hodge conjecture won. But it will not be for ever. At least, I am a kind of confident that the Taylor series of full family of periods computed in [Mov19, §18.5, Chapter 19] and its computer implementation is correct and mistake free. I will use it to form my own zoo of Hodge cycles!

1.14 Proof of Proposition 1.8

For an experimental proof of Proposition 1.8 see the last section of [Mov18]. The section is wrtten before that proof and it is messy, non-rigorous and contains many unrelated staff.

The ideal of a cubic ruled cycle of dimension m is radial. For this I have used SINGULAR, however, it must be also easy to see this by theoretical means:

```
LIB "primdec.lib";
ring r=0,(x,y,z,w),dp; //twisted cubic curve---
ideal I=x*z-y^2, y*w-z^2, x*w-y*z;
radical(I);

ring r=0,(x(1..6)),dp; //cubic ruled surface
ideal I=x(1)*x(4)-x(2)*x(3), x(1)*x(6)-x(2)*x(5), x(3)*x(6)-x(4)*x(5);
radical(I);
```

Therefore, for a hypersurface in $\mathbb{P}^{n+1}$ given by the homogeneous polynomial f and containing Z in (1.24) we have

$$f=g_1*+g_2*+\cdots+g_{\frac{n}{2}-1}*+(f_1f_4-f_2f_3)*+(f_1f_5-f_2f_6)*+(f_3f_6-f_4f_5)*. \tag{1.25}$$

where $*$ are homogeneous polynomials of proper degree and compatibel with $\deg(f)=d$. We want to compute the codimension of $\check{\mathsf{T}}\subset\mathsf{T}$ such that $X_t,\ \ t\in\check{\mathsf{T}}$ is given by the homogeneous polynomial f We also write polynomial f in (1.25) in the following way:

$$f=g_1*_1+g_2*_2+\cdots+g_{\frac{n}{2}-1}*_{\frac{n}{2}-1}+\begin{vmatrix}f_{11}&f_{12}&f_{13}\\ f_{21}&f_{22}&f_{23}\\ f_{31}&f_{32}&f_{33}\end{vmatrix} \tag{1.26}$$

where the first two columns consist of linear polynomials and the last column of degree $d-2$ homogeneous polynomials. For $d=3$ all f_{ij}'s are linear.

Proposition 1.9 *Let $n\geq 6$ and $d=3$. For generic and fixed $g_1,g_2,\ldots,g_{\frac{n}{2}-1},f_{11},f_{21},f_{31},f_{12},f_{22},f_{32}$ the space of polynomials 5.3 is a vector space of codimension:*

$$40,65,98,140\,\textit{for}\ n=6,8,10,12,\quad \textit{respectively} \tag{1.27}$$

in $\mathbb{C}[x]_d$.

Proof. We can assume that g_i's and f_{ij}'s mentioned in the proposition are among the variables $x_0, x_1, \cdots, x_{n+1}$. Then we use the following code. We only use the coefficient of t^3 in the output of the code below.

```
LIB "foliation.lib";
int n=6; //must be greater than 4.
ring r=0,(x(1..n+2)),dp;
ideal I=x(1)*x(4)-x(2)*x(3), x(1)*x(6)-x(2)*x(5), x(3)*x(6)-x(4)*x(5);
int i; for (i=1; i<=(n div 2)-1; i=i+1){I=I, x(6+i);}
I=radical(I);
I=std(I);
intvec a=hilb(I,1);

ring r=0,t,dp;
poly h;
for (i=1; i<=size(a); i=i+1){h=h+a[i]*t^(i-1); }
poly final=h*OneOver((1-t)^(n+2),std(ideal(t^10)),9);
final;
```

□

Proposition 1.10 *The space of cubic ruled cycles*

$$Z: g_1 = g_2 = \cdots = g_{\frac{n}{2}-1} = 0, \quad \text{rank} \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \\ f_5 & f_6 \end{bmatrix} \leq 1 \tag{1.28}$$

is of dimension

$$(\frac{n}{2} - 1 + 6)(n+2) - 3 \cdot 3 - 2 \cdot 2 - 6 \cdot (\frac{n}{2} - 1) - (\frac{n}{2} - 1)^2 - 1.$$

Proof (Not a rigorous proof). The first term is the dimension of the space of polynomials g_i and f_i's. The terms $3 \cdot 3$ and $2 \cdot 2$ are due to multiplication of the 3×2 matrix with respectively 3×3 and 2×2 matrices with non-zero determinant. The term $6 \cdot (\frac{n}{2} - 1)$ corresponds to adding linear combination of g_i's to f_i's. The term $(\frac{n}{2} - 1)^2$ comes from $(\frac{n}{2} - 1) \times (\frac{n}{2} - 1)$ matrices of linear changes in g_i's. Finally, ovrall multiplication of the ideal of Z gives us 1. For $n = 4, 6, 8, 10, 12$ we get:

$$34, 49, 66, 85 \tag{1.29}$$

respectively.

```
int n=4;
(n div 2+5)*(n+2)-9-4-6*(n div 2-1)-(n div 2-1)^2-1;
```

The difference between the numbers in (1.27) and (1.29) is the codimension of the space of cubic hypersurfaces containing a cubic ruled cycle. For another attempt see the tex file of this text after this sentence!

A determinantal variety

Let

$$f := \det[f_{ij}]_{d \times d} \in \mathbb{C}[x]_d$$

where f_{ij}'s are homogeneous linear polynomials in $x = (x_1, x_2, \ldots, x_m)$. We are interested in the dimension of the space of such polynomials. For this we compute its tangent space in a generic point f. Without loss of generality we can assume that the first entries of $[f_{ij}]$ are the variables $x_1, x_2, \dots, x_m$. The following code computes the dimension of the tanget space at such f.

```
//---This procedure computes the dimension of the deterninant variety of size d and with m variables.
LIB "foliation.lib";
int m=7; int d=3;
int a=(d^2-m)*m;
ring r=(0,t(1..a)),(x(1..m)),dp;
list fl;
int i; int j; poly P;
for (i=1; i<=m; i=i+1){fl=insert(fl,x(i),size(fl));}
for (i=1; i<=d^2-m; i=i+1)
          {
           P=0;
           for (j=1; j<=m; j=j+1)
               {
               P=P+x(j)*t((i-1)*m+j);
               }
          fl=insert(fl,P,size(fl));
          }
matrix fm[d][d]=fl[1..size(fl)];
ideal I=minor(fm,d-1);
I=std(I);
intvec av=hilb(I,1);
ring rr=0,t,dp;
poly h;
for (i=1; i<=size(av); i=i+1){h=h+av[i]*t^(i-1); }
poly final=h*OneOver((1-t)^m,std(ideal(t^10)),9);
final;  for (i=1; i<=3; i=i+1){final=diff(final,t);}
bigint D=binomial(m-1+3,3)-(int(subst(final,t,0)) div 6);


int n=8; int nh=n div 2;
bigint fi=(n+1)*(nh-1);
for (int i=1; i<=nh-1; i=i+1){i; fi =fi+binomial(n+2-i+2,2);}
fi=fi+D;
fi=binomial(n+1+3,3)-fi;
```

1.15 Noether-Lefschetz locus

Since 2015 I had the idea of getting an infinite number of special components for Noether-Lefschetz locus, and in 2019, I wrote the short text [Mov20b] on this, maybe the last one in defence of [Mov19], as I am a little bit tired of my own mathematics and my own computer codes.

1.16 (P. Deligne, August 02, 2019) I do not see your methods could prove that some component is not reduced. In the paper, you have only "singular or not reduced".

The computational methods introduced in [Mov19] do not tell you whether for a Hodge locus V_δ which one occures: singular or non-reduced. However, I do not know any example in which the analytic variety underlying V_δ is singular, even when δ is a Hodge cycle in the Fermat variety. I would start thinking like you:

1.17 (P. Deligne, August 02, 2019) I do not find singularities at a very symmetric point surprising.

if I am able to produce at least one singular V_δ. I do not know even what would this mean for the Hodge cycle δ.

Concerning Table 1 in [Mov20b]. The case is $d = 8$ and $N = 5$ is exceptional in the sense that it is 5-smooth for all the choices of (r_1, r_2). This means that I was supposed to analyse 6-smoothness, however, this was beyond the capacity of my computer (5-smoothness already took more than 10 days of computations).

1.18 (P. Deligne, August 02, 2019) The factors you take [in the definition of the curve C_2] are "consecutive". Does this matter?

Personally, I think this does not matter, even though one cannot get this fact using the automorphism of the Fermat variety or using Galois symmetry, as you have mentioned in your letter. I can start writting codes for other combinations of roots used in the curves C_1 and C_2 and observe that a similar phenomena happens. Assuming the main conjecture (discovery) of [Mov20b], that is, $V_{[C_1]+rC_2},\ r\in\mathbb{Q}$ are 31 codimensional smooth varieties, one can argue that such consecutive roots do not matter. This is as follows. Let $\mathsf{S}\subset\mathsf{T}_{\rm full}$ be the parametrs space of surfaces containg a line C_1 and a complete intersection curve C_2 of type $(3,3)$ with $C_1\cap C_2=\emptyset$. Since $V_{[C_1]}\cap V_{[C_2]}$ is of codimension 32, it is just a branch of S. It might not be so hard to prove that S is irreducible. If this is the case then all other possible arrangement of the roots of unity for defining C_1 and C_2 will produce different branches of S.

1.16 A fundamental proposition of [Mov20a]

(9 August 2019) How you would feel if you built up a mathematical theory based on a proposition, and after the theory is around 250 pages, you realize that you have not a rigorous proof for such a proposition? You may feel that the whole theory might be an abstract nonsense , however, you go through the theory and you see that even though the proof of the fundamental proposition lacks rigor, something beautiful is going on, and you are not allowed to judge the whole theory based on this. Nowadays, I have such feelings and I am trying to find a rigorous proof for the following statement. This is Proposition 2.4 of [Mov20a].

Proposition 1.11 *Let $X_t, t\in\mathsf{T}$ be a family of smooth projective varieties and let X, X_0 be two regular fibers of this family. We have an isomorphism*

$$(H^*_{\rm dR}(X),F^*,\cup,\theta)\overset{\alpha}{\simeq}(H^*_{\rm dR}(X_0),F^*_0,\cup,\theta_0) \tag{1.30}$$

For the proof I had written the following: It is enough to prove the isomorphism (1.30) for $X=X_t$ with t in a Zariski neighborhood of $0\in\mathsf{T}$. We can take sections $\alpha_{\mathsf{m},i}$ of the cohomology bundle $H^{\mathsf{m}}_{\rm dR}(X/\mathsf{T})$ in a Zariski open neighborhood U of 0 such that θ^i's are included in this basis, and moreover, it is compatible with the Hodge filtration and cup product for all $t\in U$. This basis will produce the required isomorphism (1.30) for X_0 and X.

I woke up from my dream of seeing the fruitful corners of [Mov20a] after the following:

1.19 (P. Deligne, May 12, 2019) I think you are overoptimistic in 2.4 page 14 [the above proposition]: while it might be true in examples you consider, I don't expect you can in general find sections compatible both with the cup-product and the Hodge filtration.

After few hours, I got the following:

1.20 (P. Deligne, May 12, 2019) On second thought, 2.14 is correct (but the proof is not convincing). It suffices to consider families over $\mathbb{C}$. Let G be the (linear algebraic) group of automorphisms of (H, cup product, polarization). The Hodge decomposition is given by the action of G_m: multiplication by z^{p-q} on $H^{p,q}$. This action is a morphism to G. One then uses that nearby morphisms from G_m to G are conjugate.

As an ignorant person in the theory of algebraic groups developed by A. Borel and many other respected mathematicians, I had to dig up the literature in order to understand why the last sentence "One then uses that nearby morphisms from G_m to G are conjugate" is true. I only found the following: Let G be a linear algebraic group over a field k (for safeness, characteristic zero and algebraically closed). All maximal tori T in G are $G(k)$-conjugate, see for instance [Con14, Proposition 1.1.19, 2]. The maximality cannot be dropped from this statement. However, the situation in Proposition 1.11 is slightly different: we have families of $G_m \to G$, that is, we have a morphism $G_m \times \mathsf{T} \to G$ of algebraic varieties such that for fixed $t \in \mathsf{T}$, the map $G_m \times \{t\} \to G$ is a morphism of algebraic groups. So far, I am able to produce such families by conjugation of a fixed morphism $G_m \to G$, and it is clear from P. Deligne's comments, there is a theorem which says that this is always the case. After a day or so, I failed to find a reference or prove by myself the following:

1.1 *Is the following true? Let G_m be the multiplicative group $(\mathbb{C}-\{0\},\cdot)$, G a linear algebraic group and T be an irreducible affine variety, all over $\mathbb{C}$. Let also $f: G_m \times \mathsf{T} \to G$ be a family of algebraic group morphisms, that is, for fixed $t \in \mathsf{T}$, the map $G_m \times \{t\} \to G$ is a morphism of algebraic groups. Then there is a map $g: \mathsf{T} \to G$ and algebraic group morphism $i: G_m \to G$ such that $f(g,t) = g(t)i(g)g(t)^{-1}$.*

I wrote back to P. Deligne: I tried to find a reference or prove by myself the "One then uses that nearby morphisms from G_m to G are conjugate" part of your email. I did not succeed and it seems that my brain is spoiled with too much heavy computer calculations!

1.21 (P. Deligne, August 14, 2019) My reference is SGA 3. IX 3 uses cohomology to obtain infinitesimal statements. XI 4 proves representability of the functor M of subgroupschemes of multiplicative type. XI 5 puts it all together to prove that for an affine smooth groupscheme G/S, and M the scheme parametrizing subgroupscheme of multiplicative type, M is smooth over S and the action by conjugation of G on M gives a smooth morphism (action, Id_M) : $GxM- > MxM$.

Chapter 2
Some experiments with `SmoothReduced` of `foliation.lib`

We describe how to use the library `foliation.lib` of `Singular` in order to study the components of the Hodge loci passing through the Fermat point.

2.1 Introduction

This article is supposed to contain many computational details which are missing in the article [Mov17c]. For all definitions see this article.

2.2 Sum of two linear cycles

I have used the following code for Theorem 2 and Theorem 3 of [Mov17c]. We can change the dimension `n`, the degree `d` and the degree of truncation `tru`.

```
LIB "foliation.lib";
int n=2; int d=10; int m=n div 2-1; int tru=2; int zb=10;
intvec zarib1=1,-zb; intvec zarib2=zb,zb;
intvec mlist=d;  for (int i=1;i<=n; i=i+1){mlist=mlist,d;}
ring r=(0,z), (x(1..n+1)),dp;
poly cp=cyclotomic(2*d); int degext=deg(cp) div deg(var(1));
cp=subst(cp, x(1),z); minpoly =number(cp);
list lcycles=SumTwoLinearCycle(n,d,m,1); lcycles;
list ll=SmoothReduced(mlist,tru, lcycles, zarib1, zarib2);
ll[1];
string sss="(n,d,m,tru)=("+string(n)+","+string(d)+","+string(m)+","+string(tru)+")"+"--Smooth and Reduced";
write(":a ReducedSmoothOutputFinal", sss);
sss="Number of reduced cases=", string(size(ll[1]));
write(":a ReducedSmoothOutputFinal", sss);
write(":a ReducedSmoothOutputFinal", ll[1]);
sss="Number of noreduced cases=", string(size(ll[2]));
write(":a ReducedSmoothOutputFinal", sss);
write(":a ReducedSmoothOutputFinal", ll[2]);
write(":a ReducedSmoothOutputFinal", "**********************************************************");
```

In the following we are goint to take $\mathbb{P}_i^{\frac{n}{2}}$, $r_i \in \mathbb{Z}$, $i = 1, 2, \cdots, s$ and consider the cycle

$$\delta := \sum_{i=1}^{s} r_i [\mathbb{P}_i^{\frac{n}{2}}] \in H_n(X_n^d, \mathbb{Z}). \tag{2.1}$$

and the corresponding Hodge locus V_δ. We will assume that

$$r_1 \geq 1, \quad |r_i| \leq \texttt{zb} := 10, \; r_i \neq 0.$$

Therefore, in total we have $21^{s-1} \cdot 10$ cycles. In the codes below, $\texttt{zb} = 10$. In some cases, I had to take smaller zb in order to either make the computations faster or understand the structure of reduced and non-reduced cases.

2.3 Noether-Lefschetz locus: three lines crossing a point

We investigate the Hodge loci corresponding to three lines crossing a point in the Fermat surface.

```
 LIB "foliation.lib";
 int n=2; int d=5; int tru=3; int zb=10;
 intvec zarib1=1,-zb,-zb; intvec zarib2=zb,zb,zb;
 intvec mlist=d;  for (int i=1;i<=n; i=i+1){mlist=mlist,d;}
 ring r=(0,z), (x(1..n+1)),dp;
 poly cp=cyclotomic(2*d); int degext=deg(cp) div deg(var(1));
 cp=subst(cp, x(1),z); minpoly =number(cp);
 list  lcycles=list(intvec(0,0,0,0),intvec(0,1,2,3)),
               list(intvec(0,0,0,1),intvec(0,1,2,3)),
               list(intvec(0,0,0,2),intvec(0,1,2,3));
list ll=SmoothReduced(mlist,tru, lcycles, zarib1, zarib2);
```

1. For $d=5$ the Hodge locus V_δ is 2-reduced. It is 3-reduced only in the expected cases:

$$(r_1 = r_2 = r_3) \text{ or } (r_2 = 0 \& r_1 = r_3) \text{ or } (r_3 = 0, r_1 = r_2), \text{ or } (r_2 = r_3 = 0). \quad (2.2)$$

   Note that the first Hodge locus is actually reduced because the corresponding curve can be deformed into a curve which is a complete intersection of type $1,3$.
2. For $d=6$ the Hodge locus V_δ is NOT 2-reduced only if one of r_i is zero and the other two are non-zero and not equal to each other. It is 3-reduced only in the expected cases (2.2). The computation took few hours.
3. For $d=7$ the Hodge locus V_δ is 2-reduced only in the expcted cases (2.2).

2.4 Noether-Lefschetz locus: three lines forming U

We consider three lines $\mathbb{P}^1_i, \; i=1,2,3$ such tha $\mathbb{P}^1_1 \cdot \mathbb{P}^1_2 = \mathbb{P}^1_1 \cdot \mathbb{P}^1_3 = 1$ and all other intersections are zero.

```
 LIB "foliation.lib";
 int n=2; int d=6; int tru=2; int zb=10;
 intvec zarib1=1,-zb,-zb; intvec zarib2=zb,zb,zb;
 intvec mlist=d;  for (int i=1;i<=n; i=i+1){mlist=mlist,d;}
 ring r=(0,z), (x(1..n+1)),dp;
 poly cp=cyclotomic(2*d); int degext=deg(cp) div deg(var(1));
 cp=subst(cp, x(1),z); minpoly =number(cp);
 list lcycles=ListPeriodLinearCycle(n,d,1);
 list  lcycles=list(intvec(0,0,0,0),intvec(0,1,2,3)),
               list(intvec(0,0,0,1),intvec(0,1,2,3)),
               list(intvec(0,1,0,0),intvec(0,1,2,3));

list ll=SmoothReduced(mlist,tru, lcycles, zarib1, zarib2);
```

Note that in the code above the first line intersects the second and third line.

1. For $d=5$, all the hodge locus V_δ is 2-reduced. It is NOT 3- and 4-reduced in the cases
$$(r_2=0\&r_1\neq r_3) \text{ or } (r_3=0\&r_1\neq r_2) \tag{2.3}$$
which is $10*19+10*19$ cases. It seems that in this cases there are infininite number of components of the Hodge loci crossing the Fermat point.
2. For $d=6,7$, V_δ is 2-reduced only in the cases:
$$(r_1=r_2=r_3) \text{ or } (r_2=0,\ r_1=r_3) \text{ or } (r_3=0,\ r_1=r_2) \text{ or } (r_2=r_3=0) \tag{2.4}$$
which is $10+10+10+10$ cases. In the first case, it seems to me that the three lines can be deformed into an irreducible curve of degree 3 in a surface.

2.5 Noether-Lefschetz locus: all the lines together

In this section we take the linear combination of all the lines in the Fermat surface. It is too much. It takes an eternity!!!!

```
LIB "foliation.lib";
int n=2; int d=5; int tru=2; int zb=10;
    //-Producing the list of all lines-----
    intvec zv=0,0; intvec dm1=0,d-1; for(int i=2;i<=n div 2+1; i=i+1)
                                        {zv=zv,0,0; dm1=dm1, 0,d-1;}
    list a=aIndex(zv,dm1);  list b=bIndex(n+2); int j;
    for(i=1;i<=size(b); i=i+1){ for(j=1;j<=n+2; j=j+1){ b[i][j]=b[i][j]-1;}}
    list lcycles;
    for(j=1;j<=size(a); j=j+1)
       {for(i=1;i<=size(b); i=i+1)
           {
           lcycles=insert(lcycles, list(a[j],b[i]), size(lcycles));
           }
       }
   //-------------------------------------
int N=size(lcycles);
intvec zarib1=1; intvec zarib2=zb; for(i=2;i<=N; i=i+1){zarib1=zarib1,-zb; zarib2=zarib2,zb;}
intvec mlist=d;  for (int i=1;i<=n; i=i+1){mlist=mlist,d;}
ring r=(0,z), (x(1..n+1)),dp;
poly cp=cyclotomic(2*d); int degext=deg(cp) div deg(var(1));
cp=subst(cp, x(1),z); minpoly =number(cp);
list ll=SmoothReduced(mlist,tru, lcycles, zarib1, zarib2);
```

2.6 Noether-Lefschetz loci: The most simple tree

After the computations in §2.4 for $d=5$, it seemed reasonable to analyze the most simple three of lines, that is, $\mathbb{P}^1_i,\ i=1,2,\dots,M$ with $\mathbb{P}^1_i\cdot\mathbb{P}^1_{i+1}=1,\ \ i=1,2,\dots,M-1$, and no other intersectin point. Note that for $M=3$ the order of lines is different as in (2.4). In the code below I produce one such a tree among d^2 group of lines corresponding to $3d^2=d^2+d^2+d^2$. For instance for $M=5$ it produces the lines:

```
[1]:
   [1]:
      0,0,0,0
   [2]:
      0,1,2,3
[2]:
   [1]:
      0,0,0,1
```

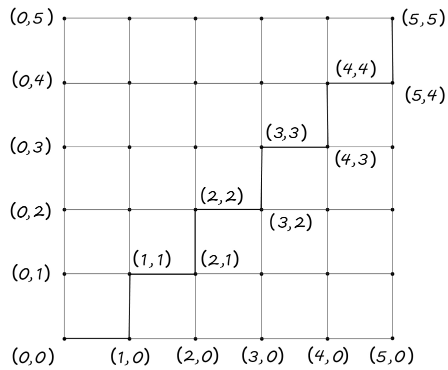

Fig. 2.1 A simple Tree of linear cycles

```
   [2]:
      0,1,2,3
[3]:
   [1]:
      0,1,0,1
   [2]:
      0,1,2,3
[4]:
   [1]:
      0,1,0,2
   [2]:
      0,1,2,3
[5]:
   [1]:
      0,2,0,2
   [2]:
      0,1,2,3
```

It could be also useful to take a tree jumping from one group of d^2 lines to another group. In the code below the number M of lines must be at most $2d+1$, see Figure 2.1.

```
LIB "foliation.lib";
int n=2; int d=6; int tru=3; int zb=4;
int M=5;   //---the size of the tree
intvec a=intvec(0,0,0,0); intvec b=intvec(0,1,2,3); list  lcycles=list(list(a,b)); int k; int i;
intvec zarib1=1; intvec zarib2=zb;
for (i=1;i<=M-1; i=i+1)
    {
    k=i div 2;  if (2*k==i){ a=intvec(0,k,0,k);}else{ a=intvec(0,k,0,k+1);}
    lcycles=insert(lcycles, list(a,b), size(lcycles));
    zarib1=zarib1,-zb; zarib2=zarib2,zb;
    }
lcycles;
intvec mlist=d;  for (int i=1;i<=n; i=i+1){mlist=mlist,d;}
ring r=(0,z), (x(1..n+1)),dp;
poly cp=cyclotomic(2*d); int degext=deg(cp) div deg(var(1));
cp=subst(cp, x(1),z); minpoly =number(cp);
list ll=SmoothReduced(mlist,tru, lcycles, zarib1, zarib2);
```

In the following the number of lines is 4, that is, $M = 4$.

1. For $d = 5$, the Hodge locus V_δ is always 2-reduced. It is NOT 3-reduced only in the cases:
$$(r_3 = r_4 = 0,\ r_1 \neq r_2,\ r_2 \neq 0)$$

which is $10*21-10-10 = 190$ cases. To be sure, I also checked that V_δ is NOT 4-reduced only in the above cases. THIS IS NOT STRANGE. Note that the first coefficients is never zero, and so, the result says that as soon as we add non-zero multiple of a one or two lines which does not intersect the first one, then the Hodge loci becomes 3-reduced. Therefore, this is not in contradiction with Voisin's result that the loci of sum of two lines in a surface is reduced if and only if either one of the coefficients is zero or two coefficients are equal. In this case it seems that we have an infinite number of components of the Hodge locu crossing Fermat.
2. The case $d=6$, I started to analyze 2-reducedness. I was not able to find a simple logical statement distinguishing the set of reduced cases from non-reduced cases. In what follows the text is hyperlinked to the corresponding data in my homepage. For $\texttt{zb}=2$ we have in total $2*5^3 = 250$ cases from which only 118 cases are 2-reduced. For $\texttt{zb}=4$ we have 844 reduced cases and for $\texttt{zb}=\texttt{10}$ we have 12430 reduced cases. This pattern seemed to me strange, and in fact it is different from, the pattern for $d=7,8$ below. I checked the 3-reducedness and get the same pattern as for higher degrees. For $\texttt{zb}=4$ there are only 144 3-reduced cases.
3. This strange pattern from $d=7,8,\cdots$ changes. For $\texttt{zb}=4$ and $d=7$, 144 cases are reduced. For $\texttt{zb}=4$ and $d=8$, 144 cases are reduced. Therefore, it seems that for $d\geq 7$ the structure of 2-reducedness is the same.

Analysing the cases $d=7,8$ I started to think on Conjecture 4.1.

2.7 Some heavy computation made in December 2017

In this section I report on more computations for sum of linear cycles as in §2.6.

1. $n=2$, $d=5$, $tru=4$, $zb=4$, $M=5$. The data of this computation can be found here. The Hodge locus in NOT reduced only when

$$r_3=r_4=r_5=0, r_2\neq 0, r_2\neq r_1.$$

   which are 28 cases. It seems that in this cases there are infininite number of components of the Hodge loci crossing the Fermat point.
2. $n=2$, $d=7$, $tru=2$, $zb=4$, $M=5$. There are 2864 2-reduced cases!. No idea what to say. Most of the cases the tree is disconnected, that is, one of the cofficients r_i is zero.
3. $n=2$, $d=6$, $tru=3$, $zb=4$, $M=5$. The same comment as above! There are 5264 which are NOT 3-reduced cases!. NO idea what to say. Have a look at the data by yourself.
4. $n=2$, $d=6$, $tru=2$, $zb=4$, $M=6$.
5. $n=2$, $d=8$, $tru=2$, $zb=4$, $M=5$. FEW DAYS Of Computing. More than 55GB of Swap+16Mem is used.

6. $n = 2$, $d = 7$, $tru = 3$, $zb = 4$, $M = 5$. 15 days of computing.
7. $n = 2$, $d = 7$, $tru = 3$, $zb = 4$, $M = 5$. 30 days of computing. 101GB swap+16GB memory used.

2.8 Noether-Lefschetz loci: four lines forming a square

Here we take four lines forming a square.

```
 LIB "foliation.lib";
 int n=2; int d=7; int tru=2; int zb=4;
 intvec zarib1=1,-zb,-zb,-zb; intvec zarib2=zb,zb,zb,zb;
 intvec mlist=d;  for (int i=1;i<=n; i=i+1){mlist=mlist,d;}
 ring r=(0,z), (x(1..n+1)),dp;
 poly cp=cyclotomic(2*d); int degext=deg(cp) div deg(var(1));
 cp=subst(cp, x(1),z); minpoly =number(cp);
 list  lcycles=list(intvec(0,0,0,0),intvec(0,1,2,3)),
               list(intvec(0,0,0,1),intvec(0,1,2,3)),
               list(intvec(0,1,0,1),intvec(0,1,2,3)),
               list(intvec(0,1,0,0),intvec(0,1,2,3));
lcycles; zarib1;zarib2;
list ll=SmoothReduced(mlist,tru, lcycles, zarib1, zarib2);
```

1. For $d = 5$ and $\texttt{zb} = 4$ we have only $56 = 28+28$ cases in which V_δ is NOT 3- and 4-reduced. These are exactly those with the coefficients zero, the corresponding lines intersect each other, and the other coefficints are non-zero and non-equal.
$$(a,b,0,0),(a,0,0,b),a,b,\neq 0,\ a\neq b.$$
2. For $d = 6,7,8$ and $\texttt{zb} = 4$ we have only $60 = 36+4+4+4+4+4+4$ 2-reduced cases.
$$(a,0,b,0),$$
$$(a,a,0,0),(a,0,0,a),(a,a,0,a),(a,a,a,0),(a,0,a,a),(a,a,a,a)$$
The novelty here is that $(1,1,1,1)$ is included in the 2-reduced cases.

2.9 Noether-Lefschetz loci: E shape arrangement

Chapter 3
Computing a PF system

These are the notes of my math conversations with P. Berglund in January 2019.

3.1 Set-up

3.1 (P. Berglund, January 31, 2019) There are two mirror CYs involved. The mirror CY with $h21 = 3$ which we call A. It has defining equation given by

$$\tilde{A} = b0*y2*y3*y4*y5*y6+b1*y3*y4*y5^4*y6^4+b2*y2^4+b3*y3^3+b4*y4^3+b5*y2^3*y3*y4*y5^7-b6*y6^7-b7*y2^3*y5^3*y6^3$$

CY A is based on a pair of reflexive polytopes in the Kreuzer-Skarke database. Everything is well-defined and there's no Laurent polynomial. We can define the algebraic coordinates $z1, z2, z3$ as I did,

$$z1 = b3b4b7/b0^3, z2 = b5b6/(b1b7), z3 = b1b2/(b0b7).$$

I can calculate the PF-operators exactly and solve for the periods explicitly. CY A has the following four PF operators in the original $z1, z2, z3$ variables:

```
L1=th1^3 - th1^2 th2 - th1^2 th3 - 6 z1 - 33 th1 z1 - 54 th1^2 z1 -
 27 th1^3 z1 - 11 th3 z1 - 36 th1 th3 z1 - 27 th1^2 th3 z1 -
 6 th3^2 z1 - 9 th1 th3^2 z1 - th3^3 z1

L2=th2^2 + th1 th2 z2 - th2^2 z2 - th1 th3 z2 + th3^2 z2

L3=-th2 th3 + th3^2 - th1 z3 - 3 th1^2 z3 + th2 z3 + 3 th1 th2 z3 +
 th3 z3 + 2 th1 th3 z3 + th2 th3 z3 + th3^2 z3

L4=9 th1^2 - 9 th1 th2 - 12 th1 th3 + 7 th2 th3 - 54 z1 - 243 th1 z1 -
 243 th1^2 z1 - 81 th3 z1 - 162 th1 th3 z1 - 27 th3^2 z1 -
 15 th2 th3 z2 + 15 th3^2 z2 + 12 th1^2 z3 - 19 th1 th2 z3 -
 12 th1 th3 z3 + 7 th2 th3 z3 - 8 th1 z2 z3 - 52 th1^2 z2 z3 +
 8 th2 z2 z3 + 52 th1 th2 z2 z3 + 8 th3 z2 z3 + 51 th1 th3 z2 z3 +
 th2 th3 z2 z3 + th3^2 z2 z3
```

However, I am not interested in CY A, but rather in CY B, which has $h21 = 2$ (and $h11 = 86$). It turns out that CY B can be obtained from CY A by taking the limit $b7 \to 0$. Rather than using the original algebraic coordinates, z_i, we can change coordinates to

$$\tilde{z}_1 = z_1 z_3, \tilde{z}_2 = z_2/z_3, \tilde{z}_3 = 1/z_3 \tag{3.1}$$

in which case $b7 \to 0$ is obtained by taking $\tilde{z}3 \to 0$. Note that both $\tilde{z}1 and \tilde{z}2$ are finite in this limit which is how we remain with the two complex structure deformations for CY B. The problem is that I do not know how to find the PF operators for the CY B model from the PF operators for CY A. I do know the defining equation which takes the form-note the Laurent monomial:

$$\tilde{B} = c0*w2*w3*w4*w5*w6 + c1*w5^4*w6^4 + c2*w2^4 + c3*w3^4 + c4*w4^4 + c5*w2^3*w3^(-1)*w4^(-1)*w5^7 - c6*w6^7$$

We can define new algebraic coordinates in terms of the above ci,

$$\hat{z}1 = c1c2c3c4/c0^4, \;\; \hat{z}2 = c0c5c6/c1^3.$$

Since the $b7 \to 0$ limit of CY A is supposed to give CY B, I believe that $\hat{z}_i = \tilde{z}_i$. In fact, I can check this by taking suitable linear combinations of the periods for CY A and then making the change of variables in the $\underline{7} \to 0$ limit, to obtain periods which I can compare with from what I can obtain directly for CY B. However, I do not know how obtain the complete set of PF operators for CY B.

By chain rule for (3.1) we have:

$$\begin{aligned}
\theta_1 &= \tilde{\theta}_1 \\
\theta_2 &= \tilde{\theta}_2 \\
\theta_3 &= \tilde{\theta}_1 - \tilde{\theta}_2 - \tilde{\theta}_3
\end{aligned}$$

Note that $\theta z^n = z^n(\theta + n)$.

3.2 Restricting non-commutative rings

In this section I will use some techniques used in [HMY17]. In particulary, I am going to use the procedure `restrictionIdeal` of `dmosapp.lib` written by Viktor Levandovskyy and Daniel Andres.

```
//|||||2019-Per-Berglund three parameter family of K3 fibered CY3|
intvec w = 0,0,1; // 1 at the place of z_3, the rest are 0. This is the limit z3=0-----------
LIB "nctools.lib";
ring r = 0, (z1,z2,z3,t1,t2,t3),dp;
def R = Weyl(); setring R;
poly th1=z1*t1; poly th2=z2*t2; poly th3=z3*t3;

poly L1 = th1^2*th3 - 6*z1*z3 - 44*z1*z3*th1 - 96*z1*z3*th1^2 -
 64*z1*z3*th1^3 + 11*z1*z3*th2 + 48*z1*z3*th1*th2 +
 48*z1*z3*th1^2*th2 - 6*z1*z3*th2^2 - 12*z1*z3*th1*th2^2 +
 z1*z3*th2^3 + 11*z1*z3*th3 + 48*z1*z3*th1*th3 + 48*z1*z3*th1^2*th3 -
 12*z1*z3*th2*th3 - 24*z1*z3*th1*th2*th3 + 3*z1*z3*th2^2*th3 -
 6*z1*z3*th3^2 - 12*z1*z3*th1*th3^2 + 3*z1*z3*th2*th3^2 + z1*z3*th3^3;

poly L2 = -z2*th1*th3 + 2*z2*th2*th3 + z2*th3^2 + z3*th2^2;

poly L3 = -th3 - 4*th1*th3 + th2*th3 + th3^2 + z3*th1^2 - 3*z3*th1*th2 +
 2*z3*th2^2 - 2*z3*th1*th3 + 3*z3*th2*th3 + z3*th3^2;

poly L4 = -8*z2*th3 - 53*z2*th1*th3 + z2*th2*th3 + z2*th3^2 - 7*z3*th2^2 +
 12*z3*th1*th3 - 7*z3*th2*th3 + 15*z2*z3*th1^2 - 45*z2*z3*th1*th2 +
 30*z2*z3*th2^2 - 30*z2*z3*th1*th3 + 45*z2*z3*th2*th3 +
 15*z2*z3*th3^2 - 3*z3^2*th1^2 + 10*z3^2*th1*th2 - 7*z3^2*th2^2 +
 12*z3^2*th1*th3 - 7*z3^2*th2*th3 - 54*z1*z3^3 - 324*z1*z3^3*th1 -
 432*z1*z3^3*th1^2 + 81*z1*z3^3*th2 + 216*z1*z3^3*th1*th2 -
 27*z1*z3^3*th2^2 + 81*z1*z3^3*th3 + 216*z1*z3^3*th1*th3 -
 54*z1*z3^3*th2*th3 - 27*z1*z3^3*th3^2;
```

```
//--new code suggested by V. Levandovskyy February 04, 2019.
ideal I=L1, L2,L3, L4;
I = slimgb(I); // computes a left Groebner basis
dim(I); // Gelfand-Kirillov dimension; 3 in this case
LIB "dmodapp.lib";
def Rz2 = restrictionIdeal(I,w);
setring Rz2;
ideal II=resIdeal;

ideal III=slimgb(II);

//--Old code suggested by V. Levandovskyy.
//-ideal I=L1, L2,L3, L4;
//-I = std(I); // computes a left Groebner basis
//-dim(I);
//-LIB "dmodapp.lib";
//-def Rz2 = restrictionModule(I,w);
//-setring Rz2;
//-print(resMod); // the ideal J you ask for
// We divide the result by gen(1) and z1.
//-module II=resMod;
//-ideal I;  for (int i=1; i<=11; i=i+1){ I=I, II[1,i];}
```

```
 Answer:
resIdeal[1]=z1^3*z2*t1^3-5*z1^2*z2^2*t1^2*t2+8*z1*z2^3*t1*t2^2-4*z2^4*t2^3-4*z1*z2^2*t1*t2^2+z2^3*t2^3+2*z1^2*z2*t1^2+6*z1*z2^2*t1*t2-14*z2^3*t2^2-4*z1*z2*t1*t2
resIdeal[2]=840*z1^3*z2^2*t1^2*t2-2331*z1^2*z2^3*t1*t2^2+1302*z1*z2^4*t2^3-1024*z1^4*t1^3+768*z1^3*z2*t1^2*t2+1152*z1^2*z2^2*t1*t2^2-320*z1*z2^3*t2^3+588*z1^3*z
resIdeal[3]=1869*z1^2*z2^3*t1^2*t2^2-5418*z1*z2^4*t1*t2^3+3360*z2^5*t2^4-1024*z1^4*t1^4+4992*z1^2*z2^2*t1^2*t2^2-3104*z1*z2^3*t1*t2^3+2232*z2^4*t2^4+4*z1^3*t1^4
resIdeal[4]=26880*z1^5*t1^4-131628*z1^2*z2^3*t1*t2^3+85281*z1*z2^4*t2^4-105*z1^4*t1^4-151552*z1^4*t1^3*t2+113664*z1^3*z2*t1^2*t2^2+1365*z1^2*z2^2*t1^2*t2^2+6297
resIdeal[5]=428064*z1^2*z2^3*t1*t2^4-279888*z1*z2^4*t2^5+557056*z1^4*t1^3*t2^2-561834*z1^2*z2^3*t1*t2^3-417792*z1^3*z2*t1^2*t2^3-960*z1^2*z2^2*t1^2*t2^3+367353*
t2^2+1053696*z1*z2*t2^3-69390*z2^2*t2^3-315*z1^2*t1^2+902784*z1^2*t1*t2-7875*z1*z2*t1*t2-8523648*z1*z2*t2^2+30240*z2^2*t2^2+3072*z1*t1*t2^2+6720*z2*t2^3+230580*
resIdeal[6]=605808315*z1*z2^5*t2^4+24980366208*z1^2*z2^3*t1*t2^3+102060*z1*z2^4*t1*t2^3-15907253376*z1*z2^4*t2^4-204120*z2^5*t2^4-552960*z1^4*t1^4+26585464832*z
z2^2*t1*t2-25962368*z1^2*t1^2*t2-17820*z1*z2*t1^2*t2-202993956864*z1*z2^2*t2^2+1718010*z2^3*t2^2+78842368*z1*z2*t1*t2^2+86143200*z2^2*t2^3-42388920*z1^2*z2*t1+3
resIdeal[7]=1944810*z1^2*z2^4*t1*t2^3-3889620*z1*z2^5*t2^4-99244992*z1^2*z2^3*t1*t2^3+62497344*z1*z2^4*t2^4-101777408*z1^4*t1^3*t2+9474003*z1^2*z2^3*t1*t2^2+763
t2-265440*z2*t2^2-5292*z1*t1+796320*z2*t2+127008*z1
resIdeal[8]=17106096*z1*z2^4*t1*t2^5-13698048*z2^5*t2^6+14090240*z1^4*t1^4*t2^2-17088*z1^2*z2^2*t1^3*t2^3-22451751*z1*z2^4*t1*t2^4-61034496*z1^2*z2^2*t1^2*t2^4+
t2^3+304560*z1*z2*t1^2*t2^3-816526512*z2^3*t2^4-295895040*z1*z2*t1*t2^4+38448*z2^2*t1*t2^4+116244480*z2^2*t2^5+13636224*z1^3*t1^3+9749880*z1^2*z2*t1^2*t2+171768
resIdeal[9]=484646652*z2^6*t2^5+40824*z1*z2^4*t1^2*t2^3+53497973424*z1*z2^4*t1*t2^4-81648*z2^5*t1*t2^4-42107786175*z2^5*t2^5-221184*z1^4*t1^5+36773003264*z1^4*t
38071886208*z1^3*t1^3+162162*z1^2*z2*t1^3+7776*z1^2*t1^4-78356454672*z1^2*z2*t1^2*t2-314118*z1*z2^2*t1^2*t2-466844768*z1^2*t1^3*t2-11880*z1*z2*t1^3*t2+116561349
```

It turns out that the above ideal is already written in the Groebner basis format.

3.3 Picard-Fuchs ideal

Geometers like the terminology of Gauss-Manin connection rather than the old Picard-Fuchs equations and systems. In this section we want to discuss the relation of both concepts to each other.

Let $X_z,\ z \in T := \mathbb{C}^h$ be a family of projective varieties and ω be a global section of the cohomology bundle $H^m_{\rm dR}(X/T)$. We will also use the notation $X/\mathbb{C}(z)$ for X as a variety defined over the function field and $\omega \in H^m_{\rm dR}(X$
$C(z))$. The case X_z a Calabi-Yau n-fold, and ω a holomorphic n-form, is of particular interest. Let $\theta_i = z_i \partial_{z_i}$.

Definition 3.1 The Picard-Fuchs ideal is a left ideal in the non-commutative ring $\mathbb{C}(z)[\theta] := \mathbb{C}(z_1,\cdots)[\theta_1,\cdots]$ given by

$$I := \left\{ P \in \mathbb{C}(z)[\theta] \middle| P \int_{\delta_z} \omega = 0 \right\}.$$

For simplicity we write $P\omega = 0$ which might be interpreted as $\nabla_\theta \omega = 0$. The following discussion is for Calabi-Yau threefolds with $h = h^{21}$, and in case of need, can be easily generalized to other varieties.

Proposition 3.1 *For $X/\mathbb{C}(z)$ a CY3 with $h^{21}=h$ assume that*

$$1,\theta_1,\theta_2,\cdots,\theta_h,\theta_1^2,\theta_2^2,\cdots,\theta_h^2,\theta_1\theta_2 \tag{3.2}$$

applied to $\omega^{3,0}$ form a basis of $H^3_{\rm dR}(X/\mathbb{C}(z))$. Then I is generated by $2h^2-h$ elements. Among them $h(h+1)$ are third order and the rest are second order differential operators.

Proof. Just apply $\theta_i,\ i=1,2,\ldots,h$ to the list of differential forms in 3.2 and write it again in term of the same. We get $h(2h+2)$ equalities. In total there are $2(h-2)+6$ trivial equalities, like $\theta_1=\theta_1$, which must be removed. □

Remark 3.1 In Proposition 3.1 for $h=2$ we do not get a second order element in the Picard-Fuchs ideal. However, I might contains such elements. For instance, when the A-model CY3 is elliptically or K3 fibered then I has a second order elements. This is also the case in Berglund example in the previous section.

Remark 3.2 It is possible to bound the degree of the denominator and nominator of the elements of the Picard-Fuchs ideal I. The denominator usually contains the discriminant, however, it might have apparent singularities. For more discussion in this direction in the context of tame polynomials see [Mov19, Theorem 10.1, §12.5].

Remark 3.3 In the paper [PERIOD INTEGRALS AND THE RIEMANN-HILBERT CORRESPONDENCE, AN HUANG, BONG H. LIAN, AND XINWEN ZHU] the authors study the space of holomorphic solutions of the ideal at any point z (even singular) and they prove that such solutions are essentially periods.

Chapter 4
Bloch's semi-regularity

I have written these notes from 2018 till 2019. It was mainly inspired by many conversations with Ananyo Dan and Roberto Villaflor. Its main aim is to review Bloch's semi-regularity map introduced in [Blo72] and some developments alterwards in [Ran93] and [DK16].

4.1 A conjecture

Our final goal is to verify the following.

Conjecture 4.1 *Let X be a smooth hypersurface of degree $d > 2 + \frac{2}{n}$ in $\mathbb{P}^{n+1}$. Let $\delta_1, \delta_2 \in H_n(X_0, \mathbb{Z})$ be two Hodge cycles such that $\delta_1 \cdot \delta_2 = 0$. If the Hodge loci $V_{\delta_i},\ \ i = 1, 2$ are smooth and reduced then the Hodge locus $V_{r_1\delta_1 + r_2\delta_2},\ r_1, r_2 \in \mathbb{Z},\ r_1, r_2 \neq 0$ is also smooth and reduced and the underlying analytic variety is $V_{\delta_1} \cap V_{\delta_2}$.*

One of the consequences of this conjecture is that if the alternative Hodge conjecture is true for both δ_1 and δ_2 then it is also true for all $r_1\delta_1 + r_2\delta_2,\ r_1, r_2 \in \mathbb{Z}$.

According to computations in [Mov17c] and Chapter 2, this conjecture must be even valid when the norm of $\delta_1 \cdot \delta_2$ is small. It seems to me that one needs something more than IVHS in order to prove this. For the formulation of Conjecture 4.1 in terms of Gorenstein rings see Conjecture **??**.

4.2 Normal bundle

Let $Z \subseteq X$ be projective varieties over complex numbers, Z being of codimension p in X, and let

$\Theta_X :=$ the sheat vector fields in X
$\Theta_{X,Z} :=$ the subsheaf of Θ_X containing vectors tangent to Z
$N_{Z\subseteq X} :=$ the normal bundle of X

We have the short exact sequence

$$0 \to \Theta_{X,Z} \to \Theta_X \to N_{Z\subseteq X} \to 0 \tag{4.1}$$

This might be taken as the definition of the normal bundle.

Proposition 4.1 *We have canonical isomorphism*

$$N_{Z\subseteq X} \cong \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{I}_{Z\subset X} \to \mathscr{O}_X).$$

$$v \mapsto (f \mapsto df(v)).$$

Proof. □

The long exact sequence of 4.1 gives us the map α and γ in

$$H^0(X, N_{Z\subseteq X}) \to H^1(X, \Theta_{X,Z}) \xrightarrow{\gamma} H^1(X, \Theta_X) \xrightarrow{\alpha} H^1(X, N_{Z\subseteq X}) \to \dots \tag{4.2}$$

Let us consider the following IVHS

$$\bar{\nabla} : H^p(X, \Omega_X^p) \to \mathrm{Hom}\left(H^1(X, \Theta_X), H^{p+1}(X, \Omega_X^{p-1})\right)$$

and $[Z]^{\mathsf{pd}} \in H^p\left(X, \Omega_X^p\right)$ be the cohomology class of Z.

Theorem 4.1 *The Bloch semi-regularity map is such that the following commutes*

$$\begin{array}{ccc} & H^1(X, N_{Z\subseteq X}) & \\ \alpha \nearrow & & \searrow \beta \\ H^1(X, \Theta_X) & \xrightarrow{\bar{\nabla}([Z]^{\mathsf{pd}})} & H^{p+1}\left(X, \Omega^{p-1}\right) \end{array} \tag{4.3}$$

For this [Dan2017] Theorem 33 and [BuchweitzFlenner2000] Theorem 4.5,5.5. From this we get:

Definition 4.2 We say that the algebraic cycle $Z \subset X$ is semi-regular if the semi-regularity map β is injective. It is called weakly semi-regular if one of the following equivalent conditions hold.

$$\mathrm{Im}(\alpha) \cap \ker(\beta) = \{0\} \quad \Leftrightarrow \quad \ker\left(\bar{\nabla}[Z]^{\mathsf{pd}}\right) = \ker(\alpha)$$

The fact that these are equivalent conditions follows from Theorem 4.1

Conjecture 4.2 *The pair (X, Z) is weakly semi-regular if and only if it satisfies the alternative Hodge conjecture.*

If the pair (X,Z) is weakly semi-regular then AHC is valid in the first order neighborhood of Z in X. According to A. Dan this follows from [Blo72, Proposition 2.6] or [Hartshorne, Deformation Theory 2010, Theorem 6.2]. For me these two references construct the scheme Z up to first order approximation. In Bloch's article Proposition 2.6, the ideal sheaf J of Z contains $J_0'^2$ which confirms this. The actual construction of Z is done at the end of Bloch's article using Artin's theorem.

4.3 Hilbert schemes

Let $\mathrm{Hilb}(X)$ and $\mathrm{Hilb}(X,Z)$ be the Hilbert scheme parametrizing deformations of X and the pair (X,Z), respectively. We have the canonical map

$$\kappa : \mathrm{Hilb}(X,Z) \to \mathrm{Hilb}(X)$$

and we denote by V_Z its image. We denote by $0 \in \mathrm{Hilb}(X)$ and $0 \in \mathrm{Hilb}(X,Z)$ the points corresponding to X and (X,Z), respectively. We would like to get some information about the tangent space of V_Z at 0. Since V_Z is given as the image of another variety, we will be able to get some information about $\mathrm{Im}(D_0\kappa)$ which might be stricktly smaller than $\mathbf{T}_0 V_Z$. Let us now consider the diagram

$$\begin{array}{ccccccc} H^0(X,N_{Z\subseteq X}) & \to & H^1(X,\Theta_{X,Z}) & \stackrel{\gamma}{\to} & H^1(X,\Theta_X) & \stackrel{\alpha}{\longrightarrow} & H^1(X,N_{Z\subseteq X}) \\ & & \uparrow KS_{X,Z} & & \uparrow KS_X & \check{\alpha} \nearrow & \\ & & \mathbf{T}_0\mathrm{Hilb}(X,Z) & \stackrel{D_0\kappa}{\to} & \mathbf{T}_0\mathrm{Hilb}(X) & & \end{array}$$

Let G be the linear reductive group acting on both $\mathrm{Hilb}(X)$ and $\mathrm{Hilb}(X,Z)$. We identify elements of $\mathrm{Lie}(\mathsf{G})$ with global vector fields in $\mathrm{Hilb}(X)$ and $\mathrm{Hilb}(X,Z)$. In this way the the differnetial $D_0\kappa$ of κ is the identity map on $\mathrm{Lie}(\mathsf{G})$. In the following proposition we need that the kernel of KS_X is given by $\mathrm{Lie}(\mathsf{G})$. For instance, this is the case for hypersurfaces.

Proposition 4.2 *If the Kodaira-Spencer map $KS_{X,Z}$ is surjective and the kernel of KS_X is given by* $\mathrm{Lie}(\mathsf{G})$ *then*

$$\mathrm{Im}(D_0\kappa) = \ker(\check{\alpha}).$$

Proof. The inclusion $\mathrm{Im}(D_0\kappa) \subset \ker(\check{\alpha})$ is trivial and does not need any hypothesis. We prove $\ker(\check{\alpha}) \subset \mathrm{Im}(D_0\kappa)$. Let $a_1 \in \ker(\check{\alpha})$ and using the first hypothesis we find

$$\begin{array}{ccccc} \to a_3 & \stackrel{\gamma}{\to} & a_2 & \stackrel{\alpha}{\longrightarrow} & 0 \\ \uparrow & & \uparrow & \nearrow & \\ a_4 & \stackrel{D_0\kappa}{\to} & a_1, a_5 & & \end{array}$$

In the final step, we find $a_5 = D_0\kappa(a_4)$. The element $a_1 - a_5$ is in the kernel of KS_X and by the second hypothesis $a_1 - a_5 \in \mathrm{Lie}(\mathsf{G})$. The map $D_0\kappa$ is identity restricted to $\mathrm{Lie}(\mathsf{G}) \subset \mathbf{T}\mathrm{Hilb}(X,Z)$ and $\mathrm{Lie}(\mathsf{G}) \subset \mathbf{T}\mathrm{Hilb}(X)$.
□

4.4 Complete intersection algebraic cycles

Consider the short exact sequence

$$0 \to N_{Z\subseteq X} \to N_{Z\subseteq \mathbb{P}^{n+1}} \to N_{X\subseteq \mathbb{P}^{n+1}}|_Z \to 0$$

all sheaves over Z, and the corresponding long exact sequence

$$H^0(Z, N_{Z\subseteq\mathbb{P}^{n+1}}) \xrightarrow{i} H^0(Z, N_{X\subseteq\mathbb{P}^{n+1}}) \to H^1(Z, N_{Z\subseteq X}) \to H^1(Z, N_{Z\subseteq\mathbb{P}^{n+1}})$$

Definition 4.3 Let f be a homogeneous polynomial of the form

$$f := \sum_{i=1}^{\frac{n}{2}+1} f_i\, f_{\frac{n}{2}+1+i} = 0.$$

We call $Z: \ f_1 = f_2 = \ldots = f_{\frac{n}{2}+1}$ a complete intersection algebraic cycle.

Proposition 4.3 *Let $Z \subseteq X$ be a complete intersection algebraic cycle. We have*

1. $N_{X\subseteq\mathbb{P}^{n+1}} \simeq \mathscr{O}_X(d)$
2. $N_{Z\subseteq\mathbb{P}^{n+1}} \simeq \bigoplus_{i=1}^{\frac{n}{2}+1} \mathscr{O}_Z(d_i)$

The map $N_{Z\subseteq\mathbb{P}^{n+1}} \to N_{X\subseteq\mathbb{P}^{n+1}}|_Z$ is given by:

$$\bigoplus_{i=1}^{\frac{n}{2}+1} \mathscr{O}_Z(d_i) \to \mathscr{O}_Z(d)$$

$$(h_1, h_2, \ldots h_{\frac{n}{2}+1}) \to \sum_{i=1}^{\frac{n}{2}+1} h_i\, f_{\frac{n}{2}+1+i}$$

Here, we have used the restriction of $f_s \in H^0(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(s)) \cong \mathbb{C}[x]_s$ for $s = \frac{n}{2} + 2, \cdots, n+2$ to Z. By Proposition 4.3 we have

$$H^1(Z, N_{Z\subseteq\mathbb{P}^{n+1}}) = \oplus_{i=1}^{\frac{n}{2}+1} H^1(Z, \mathscr{O}_Z(d_i)), \tag{4.4}$$

Therefore, in order to describe $H^1(Z, N_{Z\subseteq X})$ explicity, we need that the above cohomologies are zer and then we need to describe the map i explicity:

Proposition 4.4 *If*

$$H^1(Z,\mathscr{O}_Z(a)) = 0, \tag{4.5}$$
$$H^1(\mathbb{P}^{n+1},I_Z(a)) = 0, \quad a = d_1,d_2,\dots d_{\frac{n}{2}+1}, \tag{4.6}$$

then

$$H^1(Z,N_{Z\subseteq X}) \simeq \left(\frac{\mathbb{C}[x]}{\langle f_1,f_2,\dots f_{\frac{n}{2}+1},f_{\frac{n}{2}+2},\dots,f_{n+2}\rangle}\right)_d \tag{4.7}$$

Proof. From (4.4) and (4.5) we have

$$H^1(Z,N_{Z\subseteq X}) \simeq \frac{H^0(Z,\mathscr{O}_Z(d))}{Im\left(\bigoplus_{i=0}^{\frac{n}{2}+1} H^0(Z,\mathscr{O}_Z(d_i)) \xrightarrow{i} H^0(Z,\mathscr{O}_Z(d))\right)}$$

Let us take

$$0 \to I_Z \to \mathscr{O}_{\mathbb{P}^{n+1}} \to \mathscr{O}_Z \to 0$$

and tensor it with $\mathscr{O}_{\mathbb{P}^{n+1}}(k)$

$$0 \to I_Z(k) \to \mathscr{O}_{\mathbb{P}^{n+1}}(k) \to \mathscr{O}_Z(k) \to 0 \tag{4.8}$$

and consider the corresponding long exact sequence

$$\dots H^0(\mathbb{P}^{n+1},I_Z(k)) \to H^0(\mathbb{P}^{n+1},\mathscr{O}_{\mathbb{P}^{n+1}}(k)) \xrightarrow{\alpha} H^0(Z,\mathscr{O}_Z(k)) \to H^1(\mathbb{P}^{n+1},I_Z(k)) \to \cdots$$

Therefore, if $H^1(\mathbb{P}^{n+1},I_Z(k)) = 0$ then the restriction map α is surjective and

$$H^0(Z,\mathscr{O}_Z(k)) \cong \frac{\mathbb{C}[x]_k}{(I_Z)_k}$$

□

Proposition 4.5 *If for a complete intersection Z we have have* (4.5) *and* (4.6) *then Z is semi-regular.*

Proof. Let $I := \langle f_1,f_2,\dots f_{\frac{n}{2}+1},f_{\frac{n}{2}+2},\dots,f_{n+2}\rangle$. The diagram (4.3) becomes

$$\begin{array}{ccccc} & & \mathbb{C}[x]_d/I_d & & \\ & \alpha \nearrow & & \searrow \beta & \\ \mathbb{C}[x]_d & & \xrightarrow{\bar{\nabla}([Z]^{\mathsf{pd}})} & & \mathbb{C}[x]_{(\frac{n}{2}+2)d-n-2} \end{array}$$

Note that $\ker(\alpha) = I_d$ which is the Zariski tangent space of the Hodge locus.

Proposition 4.6 *A linear cycle* $\mathbb{P}^{\frac{n}{2}}$ *inside a smooth hypersurface of dimension n is semi-regular.*

Proof. We need to check (4.5) and (4.6).

4.5 Castelnuevo-Mumford regularity

Definition 4.4 *The Castelnuevo-Mumford regularity of a scheme $Z \subset \mathbb{P}^{n+1}$ is the smallest r such that*

$$H^i\left(\mathbb{P}^{n+1}, I_Z(r-i)\right) = 0,\ \forall i \geq 1 \tag{4.9}$$

Theorem 4.5 *The Castelnuevo-Mumford regularity of a complete intersection of type $d_1, d_2, \ldots, d_s$ is less than or equal $d_1 \cdot d_2 \cdots d_s$*

Proof. We first prove this for a hypersurface Z of degree d_1 for which we have

$$I_Z = \mathscr{O}_{\mathbb{P}^{n+1}}(-Z) \cong \mathscr{O}_{\mathbb{P}^{n+1}}(-d_1).$$

and so $I_Z(r-i) \cong \mathscr{O}_{\mathbb{P}^{n+1}}(r-i-d_1)$. Using [Har77] Theorem 5.1, we know that for all $r \geq d_1$ we have (4.9). Now assume that for a complete intersection Z_{s-1} of type $d_1, d_2, \ldots, d_{s-1}$ and all $r \geq d_1 \cdot d_2 \cdots d_{s-1}$ we have (4.9) and consider Z_s.

The following short exact sequence might be useful

$$O \to I_Z \cdot \Theta_X \to \Theta_{X,Z} \to \Theta_Z \to 0$$

4.6 Flat deformations of algebraic cycles

Let $\mathsf{X} \to \mathsf{T}$ be a family of smooth projective varieties, $0 \in \mathsf{T}$, $Z_0 \subset X_0$ and algebraic cycle of codimension $\frac{m}{2}$ and $V_{[Z_0]}$ be the corresponding Hodge locus. For simplicity, we will assume that T is the Hilbert scheme $\mathrm{Hilb}(X_0)$ (or actually a Zariski open subset of it parametrizing only smooth deformations of X_0). We denote by s the flat section of the cohomology bundle with $\mathsf{s}_0 = [Z_0]^{\mathsf{pd}}$. Even if the (rational) Hodge conjecture is true, the following statement seems to be wrong.

Property 4.1 *There is a flat family of algebraic cycles*

$$Z_t := \sum_{k=1}^{r} c_k Z_{k,t},\ Z_{k,t} \subset X_t,\ t \in V_{[Z_0]}, \tag{4.10}$$

$$\dim(Z_{k,t}) = \frac{m}{2},\ c_k \in \mathbb{Z},$$

such that $Z_{k,t}$ is irreducible for generic t and $\mathrm{cl}(Z_t) = c \cdot \mathsf{s}_t$ *for some* $c \in \mathbb{Z}$.

I do not have any counterexample to this statement. In the first versions of [Mov19] Chapter 18, I wrongly assumed that it follows from the Hodge conjecture! Two examples to investigate are the following.

Example 4.6 Consider the full family X/T of surfaces of degree 4 in $\mathbb{P}^3$ ($K3$ surfaces). Let $Z_0 = \mathbb{P}^1 + \check{\mathbb{P}}^1$ be a sum of two lines in X_0 with no intersection point (

$\mathbb{P}^1 \cap \check{\mathbb{P}}^1 = \mathbb{P}^{-1}$). In this case $(h^{20}, h^{11}, h^{02}) = (1, 20, 1)$, and so we only need one equation to define the Hodge locus. Therefore, all Hodge loci are of codimension 1. The codimension of the deformation space of X with two such lines is <u>two</u>. I do not know how to verify Lefschetz $(1,1)$ theorem for points of $V_{[Z_0]}$ in a concrete fashion.

Example 4.7 Consider the full family X/T of cubic fourfolds in $\mathbb{P}^5$. Let $Z_0 = \mathbb{P}^2 + \check{\mathbb{P}}^2$ be a sum of two linear cycles in X_0 with either no intersection point or intersecting each other at a single point ($\mathbb{P}^2 \cap \check{\mathbb{P}}^2 = \mathbb{P}^{-1}$ or $\mathbb{P}^2 \cap \check{\mathbb{P}}^2 = \mathbb{P}^0$). In this case $(h^{20}, h^{11}, h^{02}) = (1, 21, 1)$, and so we only need one equation to define the Hodge locus.

In direction of Property 4.1, we can prove the following:

Proposition 4.7 *Assume that $V_{[Z_0]}$ is irreducible. If the (rational) Hodge conjecture is true then there is a proper analytic subvariety $W \subset V_{[Z_0]}$ such that Property 4.1 is true for an étale covering of $V_{[Z_0]} \backslash W$.*

Proof. We consider the flag Hilbert scheme of pairs $\mathrm{Hilb}(X, \check{Z})$ of projective varieties $\check{Z} \subset X$, where $\check{Z}$ is of codimension $\frac{m}{2}$ in X and X is necessarily taken from $\mathrm{Hilb}(X_0) = \mathrm{Hilb}(X)$. We consider all such Hilbert schemes corresponding to different choices of $\check{Z}$. This is an enumerable set. We consider projections

$$\mathrm{Hilb}(X, Z) \to \mathsf{T} := \mathrm{Hilb}(X) \tag{4.11}$$

such that its image does not contain $V_{[Z_0]}$. The intersection of such images with $V_{[Z_0]}$ is a union A of enumerable proper analytic subvarieties of $V_{[Z_0]}$. Therefore, we can pick a point $a \in V_{[Z_0]}$ in the complement of A. We now apply the Hodge conjecture for s_a and get $Z_a := \sum_{k=1}^r c_k Z_{k,a}$ and $\mathrm{cl}(Z_a) = c \cdot \mathsf{s}_a$ for some $c \in \mathbb{N}$. By our construction the image of the map (4.11) for $(X_a, Z_{k,a})$ contains $V_{[Z_0]}$. In particular, it contains $0 \in V_{[Z_0]}$. From now on we can assume that $a = 0$ and write $Z_k = Z_{k,a}$. Note that $c \cdot [Z_0] = \sum_{k=1}^r c_k [Z_k]$ in primitive homology.

We choose subvarieties V_k in a small neighborhood of (X_0, Z_k) in $\mathrm{Hilb}(X_0, Z_k)$ such that $\dim(V_k) = \dim(V_{[Z_0]})$ and the the projection $\alpha_k : V_k \to V_{[Z_0]}$ is surjective. If we cut the ramification points of α_k from $V_{[Z_0]}$ (which lies in a codimension one subvariety W_k) then α_k becomes an étale covering. The product V of all these étale coverings (if necessary we cut all W_k's form $V_{[Z_0]}$) is still an étale covering, and hence, we get families $Z_t := \sum_{k=1}^r c_k Z_{k,t}$, $t \in V$. □

Chapter 5
Field of definition of algebraic cycles

The present chapter contains all my thoughts regarding the project started in colaboration with E. Sertöz in [MS18]. In the following I will use the notation of this text.

5.1 The most amazing

The most amzing application of the computations in [MS18] would be a computation of a Hodge cycle $\delta \in H_m(X,\mathbb{Z})$ with X defined over $\mathbb{Q}$, such that a period $p_\omega(\delta)$ is probably a transcendental number. All these might be done in a conjectural level. This would imply that δ is not abosolute, and hence, we will get a counterexample to the Hodge conjecture. In other directions which we expect that Hodge conjecture is true we will get algebraic values of special functions such as the Gauss hypergeometric function. For an example see [MR06]. My student Jorge Duque is currently working on further generalizations of this.

The main obstacle for finding new Hodge cycles is that the number of equations defining a Hodge cycle is usually much bigger than the dimension of the deformation space of X, and hence, the probability to find a Hodge cycle is almost zero. Hypersurfaces with few monomials or with large symmetry group seem to have larger space of Hodge cycles and this might be the starting point. For the first draft of the paper [MS18] we got the following comments:

5.1 (P. Deligne, October 31, 2018) In Conjecture 5, you speak of integral linear combination. In Theorem 6 (1), you speak of rational linear combination, hence you cannot say it confirms the conjecture 5. In fact, Remark 9 shows that Conjecture 5 is false integrally [take the class of a point in a real conic with no real point]. It seems clear that the conjecture 5 holds with rational coefficients : take the mean of Galois conjugates of a cycle over a bigger than wanted field.

5.2 Field of definition

Proposition 5.1 *Let X be a smooth projective variety over $\mathsf{k} \subset \mathbb{C}$ and $\delta \in H_m(X^{\mathrm{an}}, \mathbb{Z})$ be an algebraic cycle. Then there are algebraic cycles $Z_1, Z_2, \ldots, Z_s \subset X$ of pure dimension $\frac{m}{2}$ defined over k_δ and integers $a_0, \ldots, a_s$ with $a_0 \in \mathbb{N}$ such that*

$$a_0 \cdot \delta = a_1 \cdot [Z_1] + a_2 \cdot [Z_2] + \cdots + a_s \cdot [Z_s]. \tag{5.1}$$

Proof. "It seems clear that Proposition (5.1) holds with rational coefficients : take the mean of Galois conjugates of a cycle over a bigger than wanted field", (P. Deligne, personal communication, October 31). This is the main content of the proof. Let us assume that $\delta = [Z]$ and Z is defined over a field extension $\check{\mathsf{k}}$ of k_δ. If the field extension $\check{\mathsf{k}}$ is not finite then we get a family of algebraic cycles $Z_t, t \in V$, V an affine variety defined over $\bar{\mathsf{k}}$ with $\delta = [Z_t]$, see [DMOS82] Lemma 1.7. We take $t \in V(\bar{\mathsf{k}})$ and in this way we assume that $\check{\mathsf{k}}$ is a finite field extension of k_δ. We may define

$$\check{Z} := \frac{1}{\left|\mathrm{Gal}(\check{\mathsf{k}}/\mathsf{k}_\delta)\right|} \sum_{\sigma \in \mathrm{Gal}(\check{\mathsf{k}}/\mathsf{k}_\delta)} \sigma(Z)$$

which is obviously defined over k_δ. Further, for all $\sigma \in \mathrm{Gal}(\check{\mathsf{k}}/\mathsf{k}_\delta)$ we have $[\sigma(Z)] = [Z]$ in $H^{2n-m}_{\mathrm{dR}}(X/\mathsf{k})$, and hence, $[\check{Z}] = [Z]$ in $H^{2n-m}_{\mathrm{dR}}(X/\mathsf{k})$. □

Integral version of Proposition 5.1 is still a great challenge.

Theorem 5.1 (Lefschetz $(1,1)$ theorem) *Let X be a smooth projective variety over $\mathsf{k} \subset \mathbb{C}$. For any $\delta \in H^{1,1} \cap H^2(X^{\mathrm{an}}, \mathbb{Z})$ we have:*

1. *There is a divisor Z in X^{an} such that $\delta = [Z]^{\mathsf{pd}}$.*
2. *Assume that $H^1(X, \mathscr{O}_X) = 0$ and let d_1 be the smallest positive integer such that $d_1 \cdot H^2(X^{\mathrm{an}}, \mathbb{Z})_{\mathrm{tor}} = 0$. If for a field extension $\mathsf{k}_\delta \subset \check{\mathsf{k}}$ of degree d_2, X has a $\check{\mathsf{k}}$-rational point then $d_1 d_2 \cdot Z$ is linearly equivalent to a divisor $\check{Z}$ defined over $\check{\mathsf{k}}$, and hence, $d_1 d_2 \cdot \delta = [\check{Z}]^{\mathsf{pd}}$.*

Proof. The first part of the theorem is just the usual Lefschetz $(1,1)$ theorem. A topological argument for Theorem 5.1.2 can be given as follows. First, we can assume that $Z_1 := Z$ is defined over a Galois field extension $\check{\mathsf{k}}$ of k_δ. Let $Z_1, Z_2, \ldots, Z_a$ be the orbit of Z_1 under the action of the Galois group $G := \mathrm{Gal}(\check{\mathsf{k}}/\mathsf{k}_\delta)$. For $\sigma \in G$ and $\omega \in H^{2n-2}_{\mathrm{dR}}(X/\mathsf{k})$, we have

$$p_\omega(Z_i) = \sigma\left(p_\omega(Z_i)\right) = p_\omega(\sigma(Z_i))$$

We thus conclude that for any i, the difference $[Z_1] - [Z_i]$ is a torsion element of $H_{2n-2}(X^{\mathrm{an}}, \mathbb{Z})$ and so $d_1[Z_1] = d_1[Z_i]$. In particular, since $H^1(X, \mathscr{O}_X) = 0$, the isomorphism class of the line bundle corresponding to $d_1 Z_1$ is $\mathrm{Gal}(\tilde{\mathsf{k}}/\mathsf{k}_Z)$ invariant. We may now proceed as in the rest of the proof of the main theorem [MS18].

Corollary 5.2 *If $H^1(X, \mathscr{O}_X) = 0$ and $H^2(X^{\mathrm{an}}, \mathbb{Z})$ is torsion free and X has a k-rational point, then Z is linearly equivalent to a divisor $\check{Z}$ defined over k_Z.*

5.2 (P. Deligne, October 31, 2018) I do not think that X having a rational point can be of much help. To make a counterexample, I would try to take a bad Y, with no rational point, embed it in a projective space P, and take for X the blown up of P along Y. This X should inherit the badness of Y, but has rational points.

This comment is made for Theorem 5.1, 2. According to this theorem the field of definition of divisors (codimension 1 algebraic cycles) has to do with rational points (dimension 0 algebraic cycles) and so for higher codimensional algebraic cycles we might use higher dimensional algebraic cycles. For instance, the field of definition for codimension m algebraic cycles might be k_δ if X has an irreducible m dimensional divisor defined over k_δ.

5.3 Finding equations

The following has been written in June 19, 2019 and after email exchanges with Emre, and after his excitement of finding equations for algebraic cycles. It is another evidence to the fact that one has to know so much trivialities so that non-trivial things become trivial!

The idea of using Artinian-Gorenstein rings in Hodge theory goes back to the early 1980's in the work of P. Griffiths and his coauthors. Chapter 11 of my joint book [Mov20c] with Roberto Villaflor discusses this and many important trivialities which are not written explicitly in the literature. From now on I will use the notations introduced in this book.

To any Hodge cycle $\delta \in H_n(X_0,\mathbb{Z})_0$ we attach an Artinian-Gorenstein ring $I_\delta \subset \mathbb{C}[x]$. If $\delta = [Z]$ then the ideal defining Z is inside I_δ, see [Mov20c, §11.3]. Let

$$m := \dim((I_\delta)_1) = \text{ dimension of the degree 1 part of } I_\delta.$$

Proposition 5.2 *If δ is supported in a complete intersection algebraic cycle then $m \leq \frac{n}{2}+1$ and the algebraic cycle is of type $(1^m, d_{m+1}, \cdots, d_{\frac{n}{2}+1})$. In particular, if δ is supported in a linear cycle $\mathbb{P}^{\frac{n}{2}}$ then a basis of $(I_\delta)_1$ will give us equations for $\mathbb{P}^{\frac{n}{2}}$.*

Proof. Let $f = f_1 f_{\frac{n}{2}+2} + \cdots + f_{\frac{n}{2}+1} f_{n+2}$ and $Z = \mathrm{Zero}\langle f_1, f_2, \cdots, f_{\frac{n}{2}+1}\rangle$. By results of [Dan14b, MV18] we know that

$$I_\delta = \langle f_1, f_2, \cdots, f_{n+1}\rangle$$

and so their degree one part is the same. This implies that the number of degree 1, f_i's is m. The polynomials $f_i, f_{\frac{n}{2}+1+i}$ cannot be degree 1 in the same time, otherwise, $\deg(f) = 2$ which is not the case. $\square$.

Another interesting case is when we have $m = 1$. We compute the linear polynomial in I_δ. This will give us a hyperplance section of X_0. In the case of surfaces $n = 2$, this might intersect X_0 in a union of irreducible curves and one of these might be the algebraic curve we are looking for.

5.4 Twisted curves

(July 1, 2019) I would dream of an algorithm or a closed formula which computes the periods of algebraic cycles, once the integrand is an explicit differential form (for instance Griffiths basis of the de Rham cohomology of a hypersurface) and the algebraic cycle is given by an explicit ideal. Unfortunately, and until the present moment, it seems that I have failed to explain the experts in Hodge theory the importance of such a computation. This feeling might have produced because the referees of the article [MV18] did not appreciate the explicit and beautiful integral formula in Theorem 1 of this article. My former student Roberto in [Vil20] generalized this to complete intersection algebraic cycles, however, it is not clear how the most general integration formula must look like. It must be a purely commutative algebra algorithm/formula.

In this section I would like to discuss the possibility of computing periods of cubic scroll algebraic cycles. We follow [Mov18, §5]. A cubic scroll algebraic variety in $\mathbb{P}^{n+1}$ is a natural generalization of twisted curves ($n=2$) cubic ruled surface/Hirzebruch surface F_1 ($n=4$) and it is given by

$$Z: g_1 = g_2 = \cdots = g_{\frac{n}{2}-1} = 0, \quad \text{rank} \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \\ f_{31} & f_{32} \end{bmatrix} \leq 1, \tag{5.2}$$

where g_i's and f_{ij}'s are homogeneous linear polynomials in $\mathsf{k}[x] := \mathsf{k}[x_0, x_1, \cdots, x_{n+1}]$. Note that for $n=2$ the classical presentation of twisted cubic is done by setting $f_{12} = f_{21}$ and $f_{22} = f_{31}$ and this form can be obtained after row and column operations in the above 3×2 matrix.

We consider a hypersurface in $\mathbb{P}^{n+1}$ given by:

$$f = g_1 *_1 + g_2 *_2 + \cdots + g_{\frac{n}{2}-1} *_{\frac{n}{2}-1} + \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}, \tag{5.3}$$

where $*_i$'s are homogeneous polynomials of degree $d - \deg(g_i)$ and the first two columns consist of linear homogeneous polynomials and the last column of degree $d-2$ homogeneous polynomials. The cycle Z deforms to a sum Z_0 of three linear cycles by setting f_{12}, f_{31} equal to zero. Therefore, we can compute the periods of Z_0 using the formula in [Vil20], in the case of lienar cycles inside Fermat see also [MV18]. In particular, we can compute I_{Z_0} explicitly, whereas we do not know yet how to compute I_Z. Note that for simplicity we write $I_Z = I_{[Z]}$ which is an idea in $\mathsf{k}[x]$.

Conjecture 5.1 *The homogeneous pieces of the Artinian-Gorenstein ideals I_Z and I_{Z_0} have the same dimensions (at least for generic (X,Z) and (X_0,Z_0) in the underlying parameter spaces).*

Proof (Evidence to the statement). In [Mov18] for several examples of (n,d), I have computed $\dim(I_{Z_0})_d$ which turns out to be the dimension of the loci L of hypersurfaces (5.3). This implies that L is a component of the Hodge loci and it is smooth (and reduced) at the point 0 (parameterizing X_0 which contains Z_0). It follows that the tangent space of L at a generic point, and hence $(I_Z)_d$ is of dimension $\dim(I_{Z_0})_d$.

Note that we strongly use the geometric interpretation of the degree d piece of I_Z

Chapter 6
Griffiths-Harris conjecture

The following notes are written after a brief discussion with A. Dan in May 2019, see his article [Dan15] on this topic. Let $\mathsf{X} \to \mathsf{T}$ be the full family of smooth hypersurfaces of degree d and dimension n. Let $\Delta \subset H_n(X_0, \mathbb{Z})$ be a rank r lattice of Hodge cycles containing the polarization/hyperplane section algebraic cycle $[Z_\infty]$.

Definition 6.1 The (analytic) r-Hodge locus passing through 0 and corresponding to Δ is the analytic variety

$$V_\Delta := \left\{ t \in (\mathsf{T}, 0) \;\middle|\; \int_{\delta_t} \omega_1 = \int_{\delta_t} \omega_2 = \cdots = \int_{\delta_t} \omega_a = 0, \quad \forall \delta \in \Delta \right\}. \tag{6.1}$$

We consider it as an analytic scheme with

$$\mathscr{O}_{V_\Delta} := \mathscr{O}_{\mathsf{T},0} \Big/ \left\langle \int_{\delta_{i,t}} \omega_1, \int_{\delta_{i,t}} \omega_2, \cdots, \int_{\delta_{i,t}} \omega_a, \quad i = 1, 2, \ldots, r \right\rangle. \tag{6.2}$$

where we have considered a basis $\delta_i, \;\; i = 1, 2, \ldots, r$ of Δ

By definition we have the following:

$$V_\Delta = \cap_{i=1}^r V_{\delta_{i,0}}$$

Note that we have considered scheme theoretic intersection. I am not sure, but it might be possible to generalize Cattani-Deligne-Kaplan theorem and show that V_Δ is a branch of an algebraic subvariety of T.

Conjecture 6.1 (Griffiths-Harris) *For surfaces of degree d, that is $n = 2$, and for $2 < r < d$ we have*

$$\mathrm{codim}(V_\Delta) \geq (r-1)(d-3) - \frac{(r-3)(r-4)}{2}. \tag{6.3}$$

for all possible choice of X_0 and Δ.

This conjecture can be analyzed for the Fermat variety X_0 with only the knowledge of linear algebra. The description of the Hodge lattice inside $H_0^{\frac{n}{2},\frac{n}{2}}$ is done in [Mov19, Section 18.6]. Actually, in this book we have explicitly described the sublattice of the Hodge lattice generated by linear cycles $\mathbb{P}^{\frac{n}{2}}$. The computations in [AMV19] suggest that if $\gcd(d,(n+1)!)=1$ or d is prime or $d=4$ then this is the whole lattice. By works of T. Shioda and Z. Ran [Shi79] we know that this is true over rational numbers.

The inequality (6.3) follows from the following linear algebra problem.

Problem 6.1 Recall the notation of a period vector $\mathsf{p} := (\mathsf{p}_i,\ \ i \in I_{(\frac{n}{2}+1)d-n-2})$ and the matrix $[\mathsf{p}_{i+j}]$ introduced in [Mov19, Problem 20.3]. Consider $r-1$ copies of such vectors $\mathsf{p}^k,\ \ k=1,2,\dots,r-1$. Show that if there is no $\mathbb{Z}$-linear dependence between p^k's then

$$\operatorname{rank}\Big(\text{ horizontal concatenation of } [\mathsf{p}^k_{i+j}],\ \ k=1,2,\dots,r-1\Big) \geq (r-1)(d-3) - \frac{(r-3)(r-4)}{2}$$

For $r=2$ this is [Mov19, Problem 20.3] which says that if $\mathsf{p}\neq 0$ then we have the inequality

$$\operatorname{rank}([\mathsf{p}_{i+j}]) \geq d-3$$

An elementary proof of this is given in [Mov17b, Proposition 7]. Generalizing this proof might help in understanding Griffiths-Harris conjecture.

From now one we use the notation introduced in [Mov19, Section 18.6]. Note that originally we have the problem

$$\operatorname{codim} \cap_{k=1}^{r-1} \ker([\mathsf{p}_{i+j}(\delta_k)]) \geq (r-1)(d-3) - \frac{(r-3)(r-4)}{2}, \tag{6.4}$$

where δ_k form a basis of the primitive part Δ_0 of Δ. One may conjecture that the left hand side of this is $\operatorname{codim}(\ker([\mathsf{p}_{i+j}(\delta)]))$ for a generic $\delta \in \Delta_0$ (or maybe tensored with $\mathbb{C}$).

6.1 The equality

It is well known that at least for $n=2$ in the Griffiths-Harris conjecture the equality happens if δ is a sum of $r-1$ lines passing through a point. In this section we would like to write this as a problem in linear algebra.

Let $\mathbb{P}^{\frac{n}{2}}_{\xi_i},\ i=0,2,\dots,d-1$ be linear cycles obtained by the intersection of

$$\mathbb{P}^{\frac{n}{2}+1} : x_2 - \zeta_{2d}x_3 = 0, \cdots, x_n - \zeta_{2d}x_{n+1} = 0$$

with the Fermat variety X^d_n. Here, ξ_i's are all d-th roots of unity. The linear cycle $\mathbb{P}^{\frac{n}{2}}_{\xi_i}$ is given by the same equations of $\mathbb{P}^{\frac{n}{2}+1}$ plus $x_0 - \zeta_{2d}\xi_i x_1 = 0$. Let us take $r-1$ linear cycles $\mathbb{P}^{\frac{n}{2}}_{\xi_i}$, $i=1,\dots,r-1$, where we have taken $r-1$ different roots $\xi_1,\xi_2,\cdots,\xi_{r-1}$ of unity, and define

$$\delta=\sum_{k=1}^{r-1} r_i\mathbb{P}^{\frac{n}{2}}_{\xi_i}, \quad r_k\in\mathbb{Z}$$

to be their sum with unknown coefficients. By [MV18] we know that

$$\mathsf{p}_i(\delta):=\frac{1}{(2\pi\sqrt{-1})^{\frac{n}{2}}}\int_\delta \omega_i=$$

$$\begin{cases}\frac{1}{d^{\frac{n}{2}+1}\cdot\frac{n}{2}!}{\zeta_{2d}}^{i_0+1+i_2+1+\cdots+i_n+1}(\sum_{k=0}^{r-1} r_k\xi_k^{i_0+1}) \text{ if } & i_{2e-2}+i_{2e-1}=d-2, \quad \forall e=1,...,\frac{n}{2}+1, \\ 0 & \text{otherwise.}\end{cases}$$

It might be helpful to solve the following linear algebra problem:

Problem 6.2 *For generic r_k's we have*

$$\operatorname{rank}([\mathsf{p}_{i+j}(\delta)])=(r-1)(d-3)-\frac{(r-3)(r-4)}{2}$$

Chapter 7
Daniel's thesis

The action of a monodromy group on a single vanishing cycles plays an important role in the classification of foliations with a center, and this is the main topic of the Ph.D thesis of my student Daniel Lopez.

7.1 Motivation

Let $F(d)$ be the space of degree d foliations in $\mathbb{C}^2$ and let $H(d)$ be its subset containing foliations of the form $\mathscr{F}(df)$, where $f \in \mathbb{C}[x,y]$ is a degree $d+1$ polynomial such that 1. f has only isolated singularities 2. $f=0$ intersects infinity transversally. It follows that the sum of Milnor number of critical points of f is d^2. Let also $M(d) \subset F(d)$ be the set of foliations with at least one center singularity. It is well-known that $M(d)$ is an algebraic set. We want to study the components of

$$M(d) \cap H(d).$$

Definition 7.1 Let $1 \leq n \leq d^2$ and $d = d_1 + d_2 + \cdots + d_n$ and write $\underline{d} = (d_1, d_2, \cdots, d_n)$. Let also $H(\underline{d}, n) \subset H(d)$ be the closure of the set of polynomials f with exactly n critical values with d_i crtical point on top of each.

Proposition 7.1 *$H(\underline{d}, n)$ is an algebraic set.*

Proof. See the discussion on the discriminant in [Mov19, Chapter 10]. The proof does not imply that $H(d,n)$ is irreducible. It might have many components. □

Question 7.1 *Is a component of $H(\underline{d}, n)$ necessarily an intersection of a component of $M(d)$ with $H(d)$?*

For explicit d and n the equations of $H(d,n)$ and its decompostion can be computed by computer. It might be worth to do this for $d=2$ and $d=3$. The discriminant $\Delta(s)$ of $f-s$ is a polynomial of degree $(d+1)^2$ in s.

7.2 Two Dynkin diagrams produce two different classification of monodromy action

Definition 7.2 A vanishing cycle is called simple if its path of vanishing does not self intersect!

For example for fibarations defined over $\mathbb{R}$, real vanishing cycles are simple.

7.3 computing the discriminant

```
LIB "foliation.lib";
ring r=(0,t(0..2), s(0..2),s),(x,y),dp;
poly f=x^3+t(2)*x^2+t(1)*x+y^3+s(2)*y^2+s(1)*y-s;
number disc=discriminant(f-s);
poly disc2=substpar(disc,s,x);  coef(disc2, x);
```

```
help mulmat;
example mulmat;
mulmat(f,f);
```

```
LIB "foliation.lib";
ring r=(0,t(1..3), s(1..3),s),(x,y),dp;
poly f=x^4+t(3)*x^3+t(2)*x^2+t(1)*x+ y^4+s(3)*y^3+s(2)*y^2+s(1)*y-s;
number disc=discriminant(f-s);
poly disc2=substpar(disc,s,x);  coef(disc2, x);
```

7.4 Center conditions

27 January 2020 I visited C. Christopher at Plymouth and he showed me his article with P. Mardeic (The Monodromy Problem and the Tangential Center Problem). Following some ideas from the zero dimensional case in [GM07], they prove that the action of monodromy on a single vanishing cycle in the hyperelliptic case $f := y^2 -$

$P(x)$ either generates the whole homology or P is a pull-back polynomial, that is, $P = P_1 \circ P_2$ with degree $\neq 1$ polynomials P_1 and P_2. In the latter case, the monodromy action does not generate the whole homology. This suggest that

Theorem 7.3 *In the space $\mathscr{F}(2d)$ of holomorphic foliations of degree $2d$ in $\mathbb{C}^2$ with* $\deg(y) = d,\ \deg(x) = 2$ *the space $M(2d)$ of holomorphic foliations with a center singularity has only two irreducible components. 1. The Hamiltonian foliations. 2. The pull-back foliations.*

C. Christpher told me that any differential equation in this case can be reduced to Lienard equation and the above theorem in this case has been proved by L.A. Cherkas in [Conditions for a Liénard equation to have a center] or another article after this (I have to check this).

The case $f := y^N + P(x)$ for $N \geq 3$ turns out to be a difficult question. Let T be the parameter space of $P(x)$ and $C_{d=d_1+d_2+\cdots+d_s}$ be the space of polynomials P with d_i critical points having the same critical value. This space is independent of N. Therefore, for each N a component of C might correspond to some strange center component. Note that for $N = 2$ only one component C matters as for other components the action of the monodromy is the full homology.

7.5 Action of monodromy

[05 August 2020] I asked Daniel Lopez to study the action of monodromy in the case of mirror quintic (see [Mov17a]) as in this case the monodromy group is generated by two well-known matrices M_0, M_1. The result was the paper [Lop20] in which Daniel studies this action making mod primes. It turns out that

$$M_0^p \equiv_p I,\ p \neq 2,3,\ \ M_0^4 \equiv_2 I,\ M_0^9 \equiv_3 I,$$

$$M_1^p \equiv_p I \forall \text{ prime } p$$

and so the action of monodromy modulo primes reduces to the action of a finite number of $M_0^i M_1^j$ on a finite set until such a finite set becomes stable. Using the following one can apply this idea in many other cases.

Proposition 7.2 *Let h be a monodromy map around any singular fiber for a fibration over a curve. There are numbers $a, b \in \mathbb{N}$ such that*

$$h^{a \cdot p^{[\log_p(b)]+1}} \equiv_p I$$

Proof. This follows from a theorem of Landman in [Lan73, page 110] which says that the monodromy map h satisfies $(h^a - I)^b = 0$ for some $a \in \mathbb{N}$, where I is the identity map and $b - 1$ is the complex dimension of the fibers of f. □

For a smooth hypersurface X and a smooth hyperplane section Y, we can take a basis of vanishing cycles $\delta_\beta,\ \beta \in I$ for $H_n(X \backslash Y, \mathbb{Z})$, write down the intersection formula between them (Dynkin diagram), see F. Pham formula in [Mov19, Proposition 7.7] apply Picard-Lefschetz formula, and use Proposition 7.2 and study the fowllowing problem.

Problem 7.1 *Determine the subset of $H_n(X,\mathbb{Z})$ whose elements are supported in diffeomorphict-n-sphere submanifolds of X.*

There is something which must be fixed in this approach. This is as follows. Let $\mathbb{P}^N$ be the projective space parametrizing hypersurfaces of degree d in $\mathbb{P}^{n+1}$. Let also $\Delta \subset \mathbb{P}^N$ be the discriminant locus, that is, its points parameterize (scheme theoretically) singular hypersurfaces (for us $x_0^d = 0$ is a singular hypersurface). Let also P be a line in $\mathbb{P}^N$ which intersects Δ transversely at points $C = \{c_1, c_2, \dots, c_r\}$. It is well-known that

$$\pi_1(P\backslash C, b) \to \pi_1(\mathbb{P}^N\backslash\Delta, b)$$

induced by inclusion, is a surjective map, and hence, in order to study the action of monodromy in the higher dimensional parameter space $\mathbb{P}^N\backslash\Delta$ we need to restrict our attention to the one dimensional case $P\backslash C$. After an identification $P = \mathbb{P}^1$ with the coordinate system $t \in \mathbb{P}^1$, we need to consider a pencil of hypersurfaces of the form

$$X_t : \frac{F}{G} = t, \quad t \in \mathbb{P}^1 \tag{7.1}$$

I do not know any literature describing the Dynkyn diagram of this´. The action of monodromy is partially studied in my Ph.D thesis [Mov00]. Note that the tame polynomial case corresponds to $G = x_0^d$, and the corresponding line P does not intersects Δ transversely at $\infty \in P$. This means that using the generic pencil (7.1) we might get more vanishing cycles in $H_n(X_b,\mathbb{Z})$ and with $\frac{F}{z^d}$ we might get less.

Chapter 8
Jorge Duque's thesis

(22.05.2019) My student Jorge Duque is supposed to generalize my first article with S. Reiter in [MR06]. The idea of the problem is as follows. When one computes the periods of many one parameter family of projective varieties, one gets the Gauss hypergeometric function. Now, if the projective variety is even dimensional then we have the notion of Hodge cycles. If the Hodge conjecture is true for such varieties then according to an observation of Deligne in [DMOS82], integration over a Hodge cycle is algebraic up to a proper factor of $2\pi i$. This must produce special values of the Gauss hypergeometric function!!

8.1 Picard-Fuchs equation

```
LIB "foliation.lib";
int d=3;
ring r=(0,t), (x(1..2), y),wp(3*d,6, 2*d);
poly f=x(1)^2+x(2)^d+y*(1-y)*(t-y);
   PFeq(f,1,t);

   vector ve=[1];
  PFequ(f, poly(1),ve);
```

Chapter 9
Walter Gaviria's thesis

(20/05/2019) Walter's thesis might have the title: Gauss-Manin connection in disguise: K3 surfaces. I have written many articles with the same title but with different subtitle.

Let us consider any moduli of lattice polarized K3 surfaces for which we have a universal family $\mathsf{X} \to \mathsf{S}$ of triples (X, i, ω), where i is the lattice polarization and ω is a holomorphic 2-forms. This includes many families which Clingher and Doran has studied, see [CD12, CD11]. My favorite one is a rank 17 polarization for which I have computed the Gauss-Manin connection, see [DMWH16]. Using Griffiths-Steenbrink theorem, see for instance [Mov19] one can take a basis of the de Ram cohomology compatible with the Hodge filtration for all K3 surfaces in the family. Such a basis in an affine chart is formulated in [Mov19, Chapter 12] under the name Gauss-Manin system.

Remark 9.1 I first asked Walter to check this property for the full family of K3 surfaces given by degree 4 polynomials in four variables. It turned out that in this case Griffiths basis might work only in a Zarsiki open subset of the parameter space. A better strategy would be to reduce the number of parameters considering the $GL(4)$-action and then apply the Griffiths basis. One might try to reduce the $GL(4)$-action into a G_m action similar to families in [CD12, CD11].

Thinking twice on the above remark and recalling the history of tame polynomials in my book I get the following strange question!!!

Question 9.1 *Is this true? There is a curve C of degree 4 in $\mathbb{P}^2$ (and hence genus 3) such that any smooth degree 4 surface in $\mathbb{P}^3$ has a hyperplane section isomorphic to C.*

One may ask. Let X be a smooth hypersurface of degree d in $\mathbb{P}^{n+1}$. Hyperplane sections of X gives us a morphism from Grassmanian of hyperplanes in $\mathbb{P}^{n+1}$ (or dual projective space $\check{\mathbb{P}}^{n+1}$) to the moduli of smooth hypersurfaces of dimension $n-1$ and degree d. When this morphisim is surjective? Is it surjective for $(n,d) = (2,4)$? or maybe for all Calabi-Yau hypersurfaces $d = n+2$!!? A simple dimension computations shows that the answer is negative even in the case $(n,d) = (2,4)$.

Even if we get a common hyperplane section for all smooth hypersurfaces, we have to analyse the proof of [Mov19, Proposition 11.8] in order to see whether new components are not multiplied with the discriminant, see the proof of this proposition. Note that a basis of the Brieskorn module in this works for all smooth members of $f = 0$, see [Mov19, Chapter 10].

Problem 9.1 For the families in [CD12, CD11] analyse the proof of [Mov19, Proposition 11.8] and its computer implementation in `foliation.lib` and compute all c's in this proof!

9.1 Constructing T and modular vector fields

Remark 9.2 It would be nice to collect all families in the literature given by a tame polynomial in the sense of [Mov19]. Clingher and Doran also studied a rank 16 family. In a diffrent direction see also [HLTY18] and references therein.

The advantage of having $\mathsf{X} \to \mathsf{S}$ is that the construction of T is just adding new parameters (S-matrix in my articles) and for computing modular vector fields one only nees to compute two things: Cup product in the Griffiths basis and the Gauss-Manin connection. Even though we do not compute these objects it is possible to describe modular vector fields in terms of Cup product and Gauss-Manin connection. The idea is to intoduce variables for quantities whose computation might produce huge polynomials (like Δ for descriminant) and compute modular vector fields in terms of these variables!!

9.2 A basis of Gauss-Manin system

(24.05.2019) There was a mitake in the procedure `multdF` of `folition.lib` and hence in `dbeta`.

```
LIB "foliation.lib";
ring r=(0,a,b,c,d,s),(x,y,w),wp(8,9,6);
poly f=y^2*w-4*x^3+3*a*x*w^2+b*w^3+c*x*w-(1/2)*(d*w^2+w^4);
 number disc=discriminant(f-s);
 list a1=dbeta(f-s,1);
 list a2=dbeta(f-s,2);
```

Chapter 10
Equations for Humbert surfaces

During 10-12 October 2019 I visited Adrian Clingher at St. Louis, and here is a summary of our discussion. Let $f := f_{a,b,c,d} = 0$ be the 4 parameter family of $K3$ surfaces introduced in [CD12] and written in the affine chart $z = 1$. In [DMWH16] I had to add a new parameter s and compute the Gauss-Manin connection of the family $f - s$. In order to work with moduli spaces and not merely with parameter space, this turned to lie inside a rank 12 polarized family of K3 surfaces. One might try to describe the Picard group of the generic fiber, the corresponding Hermitian symmetric domains, invert the period map etc. The study of Noether-Lefschetz loci in this case will help us to understand better the Noether-Lefschetz loci in the original 4-parameter family. These are in one to one correspondence with Humbert surfaces in the moduli of principally polarized abelian surfaces. For this need we have to fix a $K3$ surface with computable periods. For this I take

$$X_0: \quad g = 1$$

where g is the last homogeneous piece of f. This is not in the original family of Clingher-Doran family. Let us denote by 0 the point in the parameter space corresponding to this point. In [Mov19, §13.9] I give a closed formula for the Taylor series of periods of differential forms over topological cycles and in terms of the periods of X_0. This gives us closed formulas for the analytic equation of Noether-Lefschetz loci:

$$\int_{\delta_t} \omega^{2,0} = 0.$$

One might try to play with this equation and get the algebraic equations of Humbert surfaces. This is a nice concrete example to analyse a question asked by Deligne in §1.6.

Chapter 11
Relative algebraic de Rham cohomology

(20/11/2019) This is a summary of my thoughts on a geometrization of Jacobi forms and their differential equations interpreted as vector fields in a moduli of enhanced elliptic curves. It is inspired by many conversations with E. Scheidegger and J. Walcher (April and November 2019) on the possible theory of CY modular forms which include the generating functions of disc countings/open Gromov-Witten invariants in string theory. The main ingredients in all this, is the theory of relative algebraic de Rham cohomology. Note that the term relative refers to (co)homology of pairs of varieties and there is another term relative when we have families of varieties. In the combined context we will use the term double relative (a suggestion after discussion with Emanuel Scheidegger). After few days of doing some elementary combinatorics of coverings etc., I realized that my intuition on Hode filtration of relative de Rham cohomologies was wrong. The wrong statements are not removed from the text and marked with ~~this~~.

In 03 December 2020, I talked with Murad Alim regarding the content of this project and he showed me few interesting articles related to this topic and his old note writtem in 2013, see §11.7.

11.1 Elliptic curves with a marked point

An elliptic curve X by definition comes with a marked point O, and in this section we consider another marked point P. Therefore, we consider a curve of genus two with two marked points O and P. Let $Y := \{O,P\}$ and consider the long exact sequence of the pair (X,Y):

$$\begin{array}{ccccccccc} 0 \to & H_1(X) & \to & H_1(X,Y) & \to & H_0(Y) & \to & H_0(X) & \to 0 \\ & \| & & \| & & \| & & \| & \\ & \mathbb{Z}^2 & & \mathbb{Z}^3 & & \mathbb{Z}^2 & & \mathbb{Z} & \end{array}$$

There is a natural bilinear map $H_1(X,Y) \times H_1(X,Y) \to \mathbb{Z}$ and we can choose a basis $\delta_1, \delta_2, \delta_3$ of $H_1(X,Y)$ with the following intersection matrix:

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We would like to reproduce the above discussion using relative algebraic de Rham cohomology. Even though we have not defined relative algebraic de Rham cohomology we expect the following: If X,Y are both defined over a field k of characteristic zero then we have the exact sequence

$$0 \to H^0(X) \to H^0_{\mathrm{dR}}(Y) \to H^1_{\mathrm{dR}}(X,Y) \to H^1_{\mathrm{dR}}(X) \to 0$$

We have a bilinear map

$$H^1_{\mathrm{dR}}(X,Y) \times H^1_{\mathrm{dR}}(X,Y) \to \mathsf{k}$$

which is degenerate, and the Hodge filtration

$$0 = F^3 \subset F^2 \subset F^1 \subset F^0 = H^1_d R(X,Y)$$

~~with $\dim(F^1) = 2$. Moreover the kernel of the map $H^1_{\mathrm{dR}}(X,Y) \to H^1_{\mathrm{dR}}(X)$ is $H^1_{\mathrm{dR}}(X,Y)^{\perp}$ which is one dimensional ($\perp$ with respect to the pairing above).~~

11.2 The moduli of enhanced elliptic curves

~~Let T be the moduli of triple $(X,P,\alpha_0,\alpha_1,\alpha_2)$, where E is an elliptic curve (and hence it comes with a point $O \in X$), $P \in X$ and $\alpha_0,\alpha_1,\alpha_2$ is a basis of $H^1_{\mathrm{dR}}(X,Y)$, $Y := \{O,P\}$ such that~~

1. $\alpha_0 \in H_d R^1(X,Y)^{\perp}$, $\alpha_1 \in F^1 H^1_d R(X,Y)$;
2. We have

$$[\langle \alpha_i, \alpha_j \rangle] = \Phi := \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

~~The dimension of the moduli space T must be $7 = 1+1+1+2+2$=dimension of the moduli space of elliptic curves+of the space of the point P+of the choice of α_0+of the choice of α_1 and α_2 in $H_d R^1(X,Y)/H_d R^1(X,Y)^{\perp}$+of translation of α_1 and α_2 by α_0. The following algebraic group acts on T by base change~~

$$\mathsf{G} := \left\{ g := \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ 0 & g_{22} & g_{23} \\ 0 & 0 & g_{33} \end{bmatrix} \middle| g\Phi g^{\mathsf{tr}} = g \right\}$$

~~with $\dim(\mathsf{G}) = 5$. The quotient T/G is two dimensional. It has two multiplicative group $\mathbb{G}_m$ and three additive group $\mathbb{G}_a$.~~

11.3 Relative algebraic de Rham cohomology

After a web search of the title of the present section one lands on a link in mathoverflow which offer two definitions of relative de Rham cohomology. In the following we present the hypercohomology version (and hence algebraic) of this. These are inspired by the corresponding definitions in the C^∞ context.

Let X be a variety and $Y \subset X$ be a subvariety. We define $\Omega^m_{X,Y}$ the sheaf of differential m-forms in X whose restriction to Y is zero. We have complex $(\Omega^\bullet_{X,Y}, d)$ and define

$$H^m_{\mathrm{dR}}(X,Y) := \mathbb{H}^m(\Omega^\bullet_{X,Y}, d) \tag{11.1}$$

see [Godbillon, Elements de topologie algébrique] Chapter XII. We call this Godbillon relative algebraic de Rham cohomology. We may also define

$$\Omega^m_{X,Y} := (\Omega^m_X, \Omega^{m-1}_Y),$$

$$d : \Omega^m_{X,Y} \to \Omega^{m+1}_{X,Y},\ d(\omega,\alpha) := (d\omega, \omega|_Y - d\alpha)$$

and define $H^m_{\mathrm{dR}}(X,Y)$ to be the hypercohomology of this complex. We call this Bott-Tu relative algebraic de Rham cohomology.

Theorem 11.1 *For varieties X and Y (possibly singular) with Y closed in X, the map $\omega \mapsto (\omega, 0)$ from the Godbillon complex to Bott-Tu complex is a quasi-isomorphism and hence it induces an isomorphism between Godbillon and Bott-Tu relative de Rham cohomologies.*

Proof. This basically follows from the fact that if X is affine and Y is a closed sub affine variety of X then by definition any differential form in Y is a restriction of a differential form in X. Surjectivity: Let (ω,α) be a closed element in the Bott-Tu sense. This means that $d\omega = 0, \Omega|_Y = d\alpha$. we extend α to X and call it $\tilde{\alpha}$, $\tilde{\alpha}|_Y = \alpha$. Now (ω,α) is equivalent to $(\omega - d\tilde{\alpha}, 0)$. Injectivity: Assume that $(\omega, 0)$ is zero in the Bott-Tu case. This means that $(\omega, 0) = (d\tilde{\omega}, \tilde{\omega}|_Y - d\tilde{\alpha})$. We extend $\tilde{\alpha}$ to X and denote it by the same letter $\tilde{\alpha}$. Now, $(\omega, 0)$ is the the differential of $(\tilde{\omega} - d\tilde{\alpha}, 0)$ and $\tilde{\omega} - d\tilde{\alpha}$ restricted to Y is zero. . □

11.4 Hodge filtration of relative de Rham cohomology

Similar to the case of algebraic de Rham cohomology of projective varieties, it is natural to define the Hodge filtration in the relative case by truncating the underlying complex.

$$F^q = F^q H^m_{\mathrm{dR}}(X,Y) = \mathrm{Im}\left(\mathbb{H}^m(X, \Omega^{\bullet \geq q}_{X,Y}) \to \mathbb{H}^m(X, \Omega^{\bullet}_{X,Y})\right).$$

$$0 = F^{m+1} \subset F^m \subset \cdots \subset F^1 \subset F^0 = H^m_{\mathrm{dR}}(X),$$

11.5 A basis of relative algebraic de Rham cohomology for elliptic curves

Let us return back to the notation of (X,Y) of §11.1. See [Mov12] and Scheidegger's notes. We use the covering in [Mov12]:

$$U_{i_1} = X - \{O\}, \ \ U_{i_2} := X - \{y = 0\},$$

and in particular in $U_{i_1} \cap U_{i_2}$ there is only one point P of Y. We have

$$H^1_{\mathrm{dR}}(X,Y) := \frac{\left\{(\omega^0_{i_1 i_2}, (\omega^1_{i_1}, \alpha^0_{i_1}), (\omega^1_{i_2}, \alpha^0_{i_2}))\ \middle|\ d\omega^0_{i_1 i_2} = \omega^1_{i_2} - \omega^1_{i_1}, \ \ \omega^0_{i_1 i_2}|_Y = \alpha^0_{i_2} - \alpha^0_{i_1}\right\}}{\left\{(\omega^0_{i_2} - \omega^0_{i_1}, (d\omega^0_{i_1}, \omega^0_{i_1}|_Y), (d\omega^0_{i_2}, \omega^0_{i_2}|_Y)\right\}}$$

Proposition 11.1 *A basis of $H^1_{\mathrm{dR}}(X,Y)$ is given by the following:*

1.

$$(0, (\omega|_{U_{i_1}}, 0), (\omega|_{U_{i_2}}, 0)),$$

where ω is a holomorphic 1-form in X. This is also a basis of $F^1 H^1_{\mathrm{dR}}(X,Y)$.

2.

$$(f, (\frac{xdx}{y}, 0), (\frac{xdx}{y} + df, f|_Y)), \quad f := -\frac{1}{2}\frac{y}{x}$$

3.

$$(f, (0,0), (df, 0))$$

where f is a regular function in U_{i_2} such that $f(P) \neq 0$ and $f(O) = 0$, for instance $f := \frac{1}{y}$. This is mapped to zero under the canonical map $H^1_{\mathrm{dR}}(X,Y) \to H^1_{\mathrm{dR}}(X)$.

Proof. First and second item: By definition we must have $\omega^0_{i_1 i_2} = 0$ and so $\omega^1_{i_1}$ and $\omega^1_{i_2}$ glue to each other to give a holomorphic 1-form in X. For further details see [Mov12]

Third item. we show that this element is not zero in $H^1_{\mathrm{dR}}(X,Y)$. If this is the case then $(d\omega^0_{i_1}, \omega_{i_1}|_Y) = (0,0)$ implies that $\omega^0_{i_1} = 0$ and hence $f = \omega^0_{i_2}$. Since $\omega^0_{i_2}$ over $Y = \{O, P\} \subset U_{i_2}$ is not constant, we get a contradiction.

11.6 A basis of relative algebraic de Rham cohomology for mirror quintic

Let X be the mirror quintic CY3 as in [Mov17a]. We also pick to homologous lines C_+ and C_- in X:

$$C_\pm: \ \ x_1+x_2=x_3+x_4=x_5^2\pm\sqrt{5\psi x_1x_3}=0$$

and set $Y:=C_+\cup C_-$. Let us write down the long exact sequence of the pair (X,Y) is homology:

$$\begin{array}{ccccccccccc} 0\to & H_3(X) & \to & H_3(X,Y) & \to & H_2(Y) & \to & H_2(X) & \to & H_2(X,Y) & \to 0 \\ & \| & & \| & & \| & & \| & & \| & \\ & \mathbb{Z}^3 & & \mathbb{Z}^5 & & \mathbb{Z}^2 & & \mathbb{Z}^{101} & & \mathbb{Z}^{100} & \end{array}$$

One might still argue that there are no torsions in the above picture. Acording to Lefschetz duality we have

$$H_3(X-Y)\cong H^3(X,Y)$$

see for instance page 54 of Green's lecture notes in [GMV94]. Since in $H_3(X-Y)$ we have the intersection pairing (which is degenerate), we must

11.1. Construct a natural bilinear map in the relative algebraic de Rham cohomology $H^3_{\rm dR}(X,Y)$.

We would like to construct an explicit basis for $H^3_{\rm dR}(X,Y)$. Recall that an element of this cohomology is of the form

$$(\omega^0_3,0)+(\omega^1_2,\alpha^0_2)+(\omega^2_1,\alpha^1_1)+(\omega^3_0,0).$$

....

11.7 Some references from Physics literature

The following is a summary of my conversation with Murad Alim in 03/12/2019. He showed me his note related to the relative de Rham cohomology of elliptic curves with two marked points. He takes the family of elliptic curves given in an affine chart by $f:=1+\hat{z}x_1-zx_2+(x_1x_2)^{-1}=0$ and two marked points given by $x_1=-1$. Using toric geometry techniques, he constructs three elements $\mathscr{L}_i\in\mathbb{C}[z,\hat{z},\theta,\hat{\theta}],\ i=1,2,3$ such that they annihilate the holomorphic $(1,0)$-form ω. Two urgent tasks for the construction of the corresponding moduli of enhanced elliptic curves are:

1. It is claimed that $\omega,\theta\omega,\ \hat{\theta}\omega$ form a basis of $H^1_{\rm dR}(X,Y)$. If this is true one must be able to write $\theta\hat{\theta}\omega,\theta^2\omega,\ \hat{\theta}^2\omega$ as a linear combination of $\omega,\theta\omega,\ \hat{\theta}\omega$. These

equations must be in the left ideal generated by $\mathscr{L}_i,\ \ i=1,2,3$, and they give us the full Gauss-Manin connection matrix in the mentioned basis.

2. The computation of the cup product in the mentioned basis. Still for me it not clear how to define the cup product in the algebraic de Rham cohomology.

Murad also computes the Taylor expansion of the periods of ω over a basis of $H_1(X,Y)$ which are essential to do any kind of Fourier expansion. His joint article [Hints for Off-Shell Mirror Symmetry in type II/F-theory Compactifications] does similar computations in higher dimensions of X and Y a divisor.

Another interesting article which Murad showed me is [Matrix Factorizations And Mirror Symmetry: The Cubic Curve] by Brunner-Herbst-Lerche-Walcher. These authors take the Dwork family of elliptic curves. I was not able to figure out which two points they take. The corresponding special loci in the generalized period domain is

$$\begin{bmatrix} 1 & \tau & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \tau \in \mathbb{H}, \ \ z \in \mathbb{C}.$$

A special frame of the relative de Rham cohomology must be taken in such a way that the Gauss-Manin connection matrix becomes

$$\nabla_{\frac{\partial}{\partial \tau}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\nabla_{\frac{\partial}{\partial z}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

11.8 Computing the Gauss-Manin connection in relative cohomology

I wrote these notes in 25/03/2021. Deuring this time I was discussing with Jin Cao and Roberto Vilaflor regarding the main objective of this chapter: to get a geometric framework for Jacobi modular forms. For this we have to compute the Gauss-Manin connection of elliptic curves together with the exact forms produces in our way. For this reason I took the procedure 'gaussmanin' from my library 'foliation.lib' and wrote 'GMExact'. Using the code

```
 LIB "foliation.lib";
 ring r=(0,t(2..3)), (x,y),wp(2,3);
 poly f=y^2-4*x^3+t(2)*x+t(3);
    list l=t(2..3);
    GMExact(f,l,1);
    GMExact(f,l,x);


 >     GMExact(f,l,1);
[1]:
-27/(t(2)^3-27*t(3)^2)
[2]:
   [1]:
      _[1,1]=(t(3))/6
      _[1,2]=(t(2)^2)/108
```

```
   [2]:
      _[1,1]=0
      _[1,2]=-1/6*x^2*y+(t(2))/36*y
      _[2,1]=0
      _[2,2]=0
[3]:
   [1]:
      _[1,1]=(-t(2))/9
      _[1,2]=(-t(3))/6
   [2]:
      _[1,1]=0
      _[1,2]=-1/6*x*y
      _[2,1]=0
      _[2,2]=0
>     GMExact(f,1,x);
[1]:
-27/(t(2)^3-27*t(3)^2)
[2]:
   [1]:
      _[1,1]=(-t(2)^2)/108
      _[1,2]=(-t(2)*t(3))/72
   [2]:
      _[1,1]=0
      _[1,2]=-1/3*x^3*y+1/24*y^3+(5*t(2))/72*x*y
      _[2,1]=0
      _[2,2]=0
[3]:
   [1]:
      _[1,1]=(t(3))/6
      _[1,2]=(t(2)^2)/108
   [2]:
      _[1,1]=0
      _[1,2]=-1/6*x^2*y+(t(2))/36*y
      _[2,1]=0
      _[2,2]=0
```

We take a family of elliptic curves $y^2 = 4x^3 - t_2x - t_3$ with a marked point $P = (a,b)$, the above computation in the latexed and compact format is:

$$\begin{pmatrix} d\left(\int_O^P \frac{dx}{y}\right) \\ d\left(\int_O^P \frac{xdx}{y}\right) \end{pmatrix} = \begin{pmatrix} -\frac{1}{12}\frac{d\Delta}{\Delta}, & \frac{3}{2}\frac{\alpha}{\Delta} \\ -\frac{1}{8}t_2\frac{\alpha}{\Delta}, & \frac{1}{12}\frac{d\Delta}{\Delta} \end{pmatrix} \begin{pmatrix} \int_O^P \frac{dx}{y} \\ \int_O^P \frac{xdx}{y} \end{pmatrix} + \begin{pmatrix} A_1(\check{x},\check{y}) + \frac{da}{b} \\ A_2(\check{x},\check{y}) + \frac{ada}{b} \end{pmatrix} \qquad (11.2)$$

where

$$\Delta := 27t_3^2 - t_2^3,\ \alpha := 3t_3dt_2 - 2t_2dt_3.$$

$$\begin{aligned} A_1 &= (9x^2y - \frac{3}{2}t_2y)dt_2 + (9xy)dt_3 \\ A_2 &= (18x^3y - \frac{9}{4}y^3 - \frac{15}{4}t_2xy)dt_2 + (9x^2y - \frac{3}{2}t_2y)dt_3 \end{aligned}$$

THE ABOV UNDERSTANDING OF THE ABOVE COMPUTATION of GUASS-MANIN TOGETHER WITH PRODUCED EXACT FORMS IS WRONG. First of all among the integrals $\int_O^P \frac{x^i dx}{y}$ only the integral with $i = 0$ is convergent. Moreover, $\int_O^P d($ polynomial$)$ is also divrgent near O. Note that the produced exact forms are all polynomials in x, y.

Remark 11.1 Whatever is the Gauss-Manin connection in the relative case, it must satisfy Proposition 13.1 in my book [Mov19]. We have the variation $\tilde{\delta}_t$ of the boundary of the cycle δ_t, and this is part of the boundary of D_t. In order to generalize this it seems that we have to assume that ω restricted to $\tilde{\delta}_t$ is identiclly zero. In the case of elliptic curve which we are going to discuss, $\tilde{\delta}_t$ has fixed x and so $\frac{dx}{y}$ restricted to it is identically zero.

In order to compute the Gauss-Manin connection in the case of elliptic curves we claim that

$$H^1_{\mathrm{dR}}(X,Y) = \frac{\text{differential forms in X with poles away from Y}}{\text{exact forms } df \text{ with } f(O)=f(P)}.$$

where $Y=\{O,P\}$. We can choose the following basis of $H^1_{\mathrm{dR}}(X,Y)$.

$$d(\frac{b}{a}\frac{x}{y}),\ \frac{dx}{y}, \frac{xdx}{y} - \frac{1}{2}d(\frac{y}{x}).$$

In the computation of GM we have to use $\frac{xdx}{y} - \frac{1}{2}d(\frac{y}{x})$ the resulting exact form will be holomorphic at O and hence THEY CAN BE INTEGRATED OVER THE PATH FROM O TO P. The meromorhic function $f := \frac{b}{a}\frac{x}{y}$ satifies $f(O)=0$ and $f(P)=1$. All the integrals of df over OP is an integer and so it is a flat section of the Gauss-Manin connection.

(27/04/2021) Let us try to compute the relative Gauss-Manin connecton in the basis?

$$\omega_1 = d(\frac{b}{y}),\ \omega_2 = \frac{dx}{y},\ \omega_3 = \frac{xdx}{y} - d(\frac{y}{2x}).$$

In the following I assume that we know the following computation for the family $y^2=f(x)$:

$$\begin{pmatrix} d\left(\int \frac{dx}{y}\right) \\ d\left(\int \frac{xdx}{y}\right)\end{pmatrix} = \begin{pmatrix} -\frac{1}{12}\frac{d\Delta}{\Delta}, & \frac{3}{2}\frac{\alpha}{\Delta} \\ -\frac{1}{8}t_2\frac{\alpha}{\Delta}, & \frac{1}{12}\frac{d\Delta}{\Delta}\end{pmatrix} \begin{pmatrix} \int \frac{dx}{y} \\ \int \frac{xdx}{y}\end{pmatrix} + \begin{pmatrix} \left(\int dg_1\right) dt_2 + \left(\int dg_2\right) d_3 \\ \left(\int dg_3\right) dt_2 + \left(\int dg_4\right) dt_3 \end{pmatrix} \tag{11.3}$$

where d is the differential with respect to t_2,t_3 and the integration is any path in E minus O. We write this equality with the correction of $\frac{xdx}{y}$, taking differential with respect to parameters t_2,t_3,a. After this we can take the path of integration from O to P.

$$\begin{pmatrix} d\left(\int \frac{dx}{y}\right) \\ d\left(\int \left(\frac{xdx}{y} - d(\frac{y}{2x})\right)\right)\end{pmatrix} = \begin{pmatrix} -\frac{1}{12}\frac{d\Delta}{\Delta}, & \frac{3}{2}\frac{\alpha}{\Delta} \\ -\frac{1}{8}t_2\frac{\alpha}{\Delta}, & \frac{1}{12}\frac{d\Delta}{\Delta}\end{pmatrix} \begin{pmatrix} \int \frac{dx}{y} \\ \int \left(\frac{xdx}{y} - d(\frac{y}{2x})\right)\end{pmatrix} \tag{11.4}$$

$$+\begin{pmatrix} \left(\int dg_1\right) dt_2 + \left(\int dg_2\right) dt_3 + \left(\int d(\frac{y}{2x})\right)\frac{3\alpha}{2\Delta} + \frac{da}{b} \\ \left(\int d(g_3+\frac{1}{4y})\right) dt_2 + \left(\int d(g_4+\frac{1}{4xy})\right) dt_3 + \left(\int d(\frac{y}{2x})\right)\frac{d\Delta}{12\Delta} + \frac{ada}{b} - d(\frac{b}{2a}) - \frac{dt_3+adt_2}{4ab}. \end{pmatrix} \tag{11.5}$$

Let us denote by X,Y the entries of the last matrix. We now write this in the basis $\omega_1,\omega_2,\omega_3$:

$$\begin{pmatrix} \nabla(\omega_1) \\ \\ \nabla(\omega_2) \\ \\ \nabla(\omega_3) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \\ X & -\frac{1}{12}\frac{d\Delta}{\Delta}, & \frac{3}{2}\frac{\alpha}{\Delta} \\ \\ Y, & -\frac{1}{8}t_2\frac{\alpha}{\Delta}, & \frac{1}{12}\frac{d\Delta}{\Delta} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \\ \omega_2 \\ \\ \omega_3 \end{pmatrix} \tag{11.6}$$

Finally, it is natural to expect:

Proposition 11.2 *For a smooth curve X and $Y \subset X$ a finite number of points we have a canonical isomorphism*

$$H^1_{\mathrm{dR}}(X,Y) \cong \frac{\textit{differential forms in X with poles away from Y and without residues}}{\textit{exact forms } df \textit{ with } f|_Y = \textit{constant}}. \tag{11.7}$$

Proof. **Proof is not complete.** We define the map A from the right hand side of (11.7) to the left hand side. For a meromorphic differential form ω in X with $\mathrm{pole}(\omega)\cap Y = \emptyset$ and without residues, we choose any other mermorphic differential form ω_0 with $\mathrm{pole}(\omega_0)\cap Y = \mathrm{pole}(\omega_0)\cap\mathrm{pole}(\omega_1) = \emptyset$, $\omega - \omega_0 = df$ and consider the covering of X by $\#Y+2$ open sets:

$$U_1 = X\backslash\mathrm{pole}(\omega),\ U_0 = X\backslash\mathrm{pole}(\omega_0),\ \ U_p = (X\backslash(\mathrm{pole}(\omega)\cup Y))\cup\{p\},\ \ p\in Y$$

We define $A(\omega)$ to be

$$(\omega, f|_Y), (\omega_0, 0), (\omega, f(p)),\ \ p\in Y.$$

If $\omega = dg$ and g is the constant c restricted to Y then we have $A(\omega)=0$ as we can choose $\omega_0 = 0$. We have to show that the map does not depend on the choise of ω_0 and so it induces a map from the left hand side of (11.7) to its right hand side. The fact that it is an isomorphism is easy and is left to the reader.

Chapter 12
The transcedental degree of periods of CY threefolds

In the articles [YY04, AL07] the authors have assumed a certain algebraic independence of periods of CY threefolds in order to break the anti-holomorphic derivation into holomorphic derivations. This assumption must be proved rigorously. Let me explain this in the case of CY threefolds with middle Hodge numbers $1,1,1,1$. In this case one can prove that $16 = 4 \times 4$ periods satisfies 6 polynomial relations (coming from the cup product) and these are the only algebraic relations, see [Mov17a, §4.11, Proposition 11]. For the proof one has used the fact that the Picard-Fuchs equation of the corresponding Picard-Fuchs equation is $\mathrm{Sp}(4,\mathbb{C})$ and the monodromy group has maximal unipotent matrix. Both conditions are satisfied by the mirror quintic and 14 families of CY3's with hypergeometric periods. In this case one needs the algebraic independence of 7 period expressions, see the Proof of Theorem 6 in [Mov17a, §4.11], in order to break the anti-holomorphic derivation.

Conjecture 12.1 *If for a moduli of CY3's the Yukawa couplings C_{ijk} (either as functions in the moduli space of enhanced CY3's or as functions in terms of periods) are algebraically independent then the only algebraic relations among $(2\mathsf{h}^{21}+2)^2$ periods are those coming from the cup product.*

Chapter 13
$\mathfrak{sl}_2$ and AMSY Lie algebras

(10/12/2019) In 2015 when I was doing my sabbatical at Harvard, I talked with Murad Alim about the $\mathfrak{sl}_2$ sub Lie algebras of the Lie algebras we introduced in [AMSY16]. None of us took the job serious until few years ago, when I talked this topic with my former student Y. Nikdelan and after he wrote the paper [Nik17] on this topic. During the conference "Mirror symmetry in higher genus" which I organized it in November 2019, I talked with Younes Nikdelan, Emanuel Scheidegger and Murad Alim regarding a possible applications of these $\mathfrak{sl}_2$ Lie algebras. The following is the main content of my conversation.

In the following I will work with with the moduli T of enhanced mirror quintic CY3's and call it the ibiporanga[i] for mirror quintic. In this case the AMSY Lie algebra is generated by seven vector fields $\mathsf{R}_i, \;\; i=0,1,\ldots,6$ in T. Let $\mathscr{O}_{\mathsf{T}}$ be the ring of global regular functions in T and hence by definition the AMSY-Lie algebra is the $\mathscr{O}_{\mathsf{T}}$-sub module of the module of global vector fields in T, generated by R_i's. Let us take one of the $\mathfrak{sl}_2$ Lie algebras described in [Nik17]. For instance let us take:

$$
\begin{aligned}
e = {} & (t_3 - t_1 t_2)\frac{\partial}{\partial t_1} + \frac{t_3^3 t_4 - 5^4 t_2^2 (t_1^5 - t_5)}{5^4(t_1^5 - t_5)}\frac{\partial}{\partial t_2} + \frac{t_3^3 t_6 - 3\times 5^4 t_2 t_3 (t_1^5 - t_5)}{5^4(t_1^5 - t_5)}\frac{\partial}{\partial t_3} \\
& + (-t_2 t_4 - t_7)\frac{\partial}{\partial t_4} + (-5 t_2 t_5)\frac{\partial}{\partial t_5} + (-t_2 t_6 - 2 t_3 t_4 + 5^5 t_1^3)\frac{\partial}{\partial t_6} + (-5^4 t_1 t_3 - t_2 t_7)\frac{\partial}{\partial t_7}, \\
h = {} & t_1\frac{\partial}{\partial t_1} + 2t_2\frac{\partial}{\partial t_2} + 3t_3\frac{\partial}{\partial t_3} + 5t_5\frac{\partial}{\partial t_5} + t_6\frac{\partial}{\partial t_6} + 2t_7\frac{\partial}{\partial t_7}, \\
f = {} & \frac{\partial}{\partial t_2} - t_4\frac{\partial}{\partial t_7}.
\end{aligned}
$$

(using the t_i notation of [MN18]). Any sub Lie algebra of AMSY gives us a foliation in T and so we get a three dimensional foliation $\mathscr{F}(e,f,h)$ in T.

Conjecture 13.1 *The foliation $\mathscr{F}(e,f,h)$ has no algebraic leaf.*

---
[i] For this naming see [Mov20a, §1.4]

The following heuristic argument in favor of this conjecture is as follows. The vector field $\mathsf{R} := e$ is the Modular vector field which gives us the classical q-expansion of t_i's as in [Mov17a]. Since such q-expansions are expected to have exponential growth for coefficients, see [Mov17a, §11.2], there are no polynomial operations relating t_i's to classical modular forms. If the foliation $\mathscr{F}(e,f,h)$ has an algebraic leaf L, then it seems to me that L must be isomorphic to an ibiporanga constructed from elliptic curves and hence the restriction of t_i's to L must be related in some way to classical modular forms.

Let us replace e with

$$e := (t_3 - t_1t_2)\frac{\partial}{\partial t_1} + (-t_2^2)\frac{\partial}{\partial t_2} + (-3t_2t_3)\frac{\partial}{\partial t_3} + (-t_2t_4 - t_7)\frac{\partial}{\partial t_4}$$
$$+ (-5t_2t_5)\frac{\partial}{\partial t_5} + (-t_2t_6 - 2t_3t_4 + 5^5t_1^3)\frac{\partial}{\partial t_6} + (-5^4t_1t_3 - t_2t_7)\frac{\partial}{\partial t_7},$$

Nikdlelan in [Nik17] finds that e,f,h form another copy of $\mathfrak{sl}_2$ Lie algebra. I would be very happy if $\mathscr{F}(e,f,h)$ has an algebraic leaf. If this is the case then we can find a relation between the q-expansion of t_i's with some classical modular forms. This is as follows. Let $p = (t_{1,0}, t_{2,0}, \cdots, t_{7,0})$ be the first term of the q-expansion of $(t_1, t_2, \cdots, t_7)$. I expect that we have still an algebraic leaf of $\mathscr{F}(e,f,h)$ passing through p. Let us write the $\tilde{q}$-expansion of t_i's using this new e. For the coefficients of $\tilde{q}^1$ we might have some constant ambiguity that we can fix it by some integrality ansatz. Such $\tilde{q}$-expansions are expected to be modular forms. The transcendental degree of the field generated by $t_i(\tilde{q})$ is 3......(still thinking!!!).

Chapter 14
Gauss-Manin connection in disguise: genus two curves

(11/12/2019) During June and July 2018, I held a series of lecture on my new book [Mov20a] (thanks Prof. Yau for the invitation). Jin Cao was in the audience and he showed his interest to work on a related topic. In an intensive one month work (February 2019) at CMSA Harvard, we (me+Jin Cao+Prof. Yau) were able to write down the details of a genral theory developed in [Mov20a] in another specific example (Thanks Prof. Yau again to put us together at Harvard). This is namely to write down the theory of differential Siegel modular forms starting from the moduli of genus two curves. I am not still happy with the outcome [CMY19] as it is necessary to recover many works in Siegel modular forms, and in particular Igusa's work, using the geometric language of the moduli space of enhanced curves. However, it is clear that if one writes all details, even this particular case would turn into a voluminous book. In this note I would like to highlight the most important things to do after [CMY19].

14.1 Taylor expansion at $\tau_3 = 0$

In order to get families of genus two curves $y^2 = f(x),\ deg(f) = 5$ with $\tau_3 \to 0$, it is not enough to take two roots of f becoming the same (double root). It seems that we need a triple root as follows. We consider a genus two curve $E : y^2 = f(x),\ deg(f) = 5$ with a symplectic basis $e_1, e_2, e_3, e_4 \in H_1(E, \mathbb{Z})$, $e_1 \cdot e_3 = 1,\ e_2 \cdot e_4 = 1$. We let e_4 vanish and f converge to a polynomial with a double root. We assume that the corresponding singular curve is given by:

$$E_{a,b,\lambda} : y^2 = (x-\lambda)^2(x-a)(x-b)(x+a+b+2\lambda). \tag{14.1}$$

Hence we get the following expressions for t_i's:

$$
\begin{aligned}
t_2 &= -(a^2+ab+2a\lambda+b^2+2b\lambda+3\lambda^2)\\
&= -(3(\lambda-a)^2+8a(\lambda-a)+2b(\lambda-a)+6a^2+3ab+b^2),\\
t_3 &= (a^2b+2a^2\lambda+ab^2+4ab\lambda+4a\lambda^2+2b^2\lambda+4b\lambda^2+2\lambda^3)\\
&= 2(\lambda-a)^3+10a(\lambda-a)^2+4b(\lambda-a)^2+16a^2(\lambda-a)+12ab(\lambda-a)\\
&+ 2b^2(\lambda-a)+8a^3+9a^2b+3ab^2,\\
t_4 &= -(2a^2b\lambda+a^2\lambda^2+2ab^2\lambda+5ab\lambda^2+2a\lambda^3+b^2\lambda^2+2b\lambda^3),\\
&= -(2a(\lambda-a)^3+2b(\lambda-a)^3+7a^2(\lambda-a)^2+11ab(\lambda-a)^2+b^2(\lambda-a)^2\\
&+ 8a^3(\lambda-a)+18a^2b(\lambda-a)+4ab^2(\lambda-a)+3a^4+9a^3b+3a^2b^2),\\
t_5 &= a^2b\lambda^2+ab^2\lambda^2+2ab\lambda^3\\
&= 2ab(\lambda-a)^3+7a^2b(\lambda-a)^2+ab^2(\lambda-a)^2+8a^3b(\lambda-a)+2a^2b^2(\lambda-a)\\
&+ 3a^4b+a^3b^2.
\end{aligned}
$$

We further let λ goes to a and define

$$E_{a,b}: \ y^2=(x-a)(x-b)(x+3a+b),$$

Let $P,-P$ be the points in $E_{a,b}$ with the x-coordinate λ. When $\lambda\to a$, $E_{a,b,\lambda}$ becomes $E_{a,b}$ and the two points $P,-P$ converges to each other, see Figure 14.1.

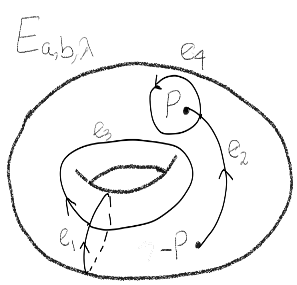

Fig. 14.1 An elliptic point with two marked points

The period matrix for the hyperelliptic curve $E_{a,b,\lambda}$ is

$$
\begin{bmatrix} x_{11} & x_{12}\\ x_{21} & x_{22}\\ x_{31} & x_{32}\\ x_{41} & x_{42}\end{bmatrix}
=
\begin{bmatrix}
\int_{e_1}\frac{\mathrm{d}x}{(x-\lambda)y} & \int_{e_1}\frac{x\mathrm{d}x}{(x-\lambda)y}\\
\int_{-P}^{P}\frac{\mathrm{d}x}{(x-\lambda)y} & \int_{-P}^{P}\frac{x\mathrm{d}x}{(x-\lambda)y}\\
\int_{e_3}\frac{\mathrm{d}x}{(x-\lambda)y} & \int_{e_3}\frac{x\mathrm{d}x}{(x-\lambda)y}\\
\int_{e_4}\frac{\mathrm{d}x}{(x-\lambda)y} & \int_{e_4}\frac{x\mathrm{d}x}{(x-\lambda)y}
\end{bmatrix},
$$

where $y^2=(x-a)(x-b)(x+a+b+2\lambda)$ by abuse of notation. Note that $\frac{x\mathrm{d}x}{(x-\lambda)y}=\frac{(x-\lambda)\mathrm{d}x}{(x-\lambda)y}+\lambda\frac{x\mathrm{d}x}{(x-\lambda)y}$. Hence if we replace the right column of the above matrix by $\int_{e_i}\frac{\mathrm{d}x}{y}$, there is no effect on the determinants of sub $2*2$ matrices. Therefore,

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \\ x_{41} & x_{42} \end{bmatrix} = \begin{bmatrix} \int_{e_1}\frac{\mathrm{d}x}{(x-\lambda)y} & \int_{e_1}\frac{\mathrm{d}x}{y} \\ \int_{-P}^{P}\frac{\mathrm{d}x}{(x-\lambda)y} & \int_{-P}^{P}\frac{\mathrm{d}x}{y} \\ \int_{e_3}\frac{\mathrm{d}x}{(x-\lambda)y} & \int_{e_3}\frac{\mathrm{d}x}{y} \\ \int_{e_4}\frac{\mathrm{d}x}{(x-\lambda)y} & \int_{e_4}\frac{\mathrm{d}x}{y} \end{bmatrix} = \begin{bmatrix} \int_{e_1}\frac{\mathrm{d}x}{(x-\lambda)y} & \int_{e_1}\frac{\mathrm{d}x}{y} \\ \int_{-P}^{P}\frac{\mathrm{d}x}{(x-\lambda)y} & \int_{-P}^{P}\frac{\mathrm{d}x}{y} \\ \int_{e_3}\frac{\mathrm{d}x}{(x-\lambda)y} & \int_{e_3}\frac{\mathrm{d}x}{y} \\ \frac{1}{\mu} & 0 \end{bmatrix}. \tag{14.2}$$

where $P=(\lambda,\mu)$ and $-P=(\lambda,-\mu)$. Here we use the facts that $\int_{e_4}\frac{\mathrm{d}x}{y}=0, \int_{e_4}\frac{\mathrm{d}x}{(x-\lambda)y}=\mathrm{Res}(\frac{\mathrm{d}x}{(x-\lambda)y},P)=\frac{1}{\mu}$. Note that if $\lambda\to a$, then $\int_{-P}^{P}\frac{\mathrm{d}x}{y}\to 0$. For simplicity, we denote $\det(i,j)$ the determinant of the above matrix consisting of the i and j rows. Then we have:

$$\tau=\begin{bmatrix} \frac{x_{11}x_{42}-x_{41}x_{12}}{x_{31}x_{42}-x_{41}x_{32}} & -\frac{x_{11}x_{32}-x_{31}x_{12}}{x_{31}x_{42}-x_{41}x_{32}} \\ \frac{x_{21}x_{42}-x_{41}x_{22}}{x_{31}x_{42}-x_{41}x_{32}} & -\frac{x_{21}x_{32}-x_{31}x_{22}}{x_{31}x_{42}-x_{41}x_{32}} \end{bmatrix} = \begin{bmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{bmatrix}.$$

When $\lambda=a$, we have $\tau_3=-\frac{\det(1,3)}{\det(3,4)}=\frac{\det(2,4)}{\det(3,4)}=\frac{x_{12}x_{42}-x_{22}x_{41}}{x_{31}x_{42}-x_{32}x_{41}}=\frac{x_{22}}{x_{32}}=0$. Note that

$$\det(3,4)=x_{31}x_{42}-x_{32}x_{41}=\frac{1}{\mu}\int_{e_3}\frac{dx}{y} \tag{14.3}$$

Even though the curve $E_{a,b\lambda}$ is nodal it seems that

Conjecture 14.1 *The period map $(a,b,\lambda)\to\tau$ is dominant, that is, its image is a dense open sebset of $\mathbb{H}_2$.*

Proof. This must be an easy exercise.

14.2 Computing Laurant series in τ_3

Recall from [CMY19] that

$$\begin{aligned} T_4 &= t_2\det(3,4), && (14.4)\\ T_8 &= t_4\det(3,4)^2, && (14.5)\\ T_{12} &= t_3^2\det(3,4)^3, && (14.6)\\ T_{16} &= t_3t_5\det(3,4)^4, && (14.7)\\ T_{20} &= t_5^2\det(3,4)^5. && (14.8) \end{aligned}$$

We want to compute the Laurant sereis of T_{4i}'s along $\lambda_a:=\lambda-a$ and find some homogeneous polynomials P in T_{4i}'s such that it is holomorphic in λ_a. First we note that

$$P(T_4,T_8,T_{12},T_{16},T_{20}) = P(t_2,t_4,t_3^2,t_3t_5,t_5^2)\det(3,4)^d, \quad d := deg(P).$$

From 14.3 and the fact that $\int_{e_3} \frac{\mathrm{d}x}{y}$ is holomorphic in λ_a it follows $P(T_4,T_8,T_{12},T_{16},T_{20})$ is holomorphic in λ_a if and only if $P(t_2,t_4,t_3^2,t_3t_5,t_5^2)$ is divisible by μ^d. We know that

$$\mu^2 = 3\lambda_a^3 + (7a-2b)\lambda_a^2 + (a-b)(4a+b)\lambda_a. \tag{14.9}$$

for some computations see Jin Cao's notes, (still under construction).

14.3 Relation with Jacobi forms

For Siegel modular forms of genus ≥ 2 there is a Fourier–Jacobi development which is attributed to Piatetski-Shapiro, for more details see van Geer's lectures [vdG08, §8] or Eichler-Zagier classical book. For $g = 2$ and changing the notation $\tau = \begin{pmatrix} \tau & z \\ z & \tilde{\tau} \end{pmatrix}$, a Siegel modular form f has an expansion of the form :

$$f(\tau) = \sum_{m=0}^{\infty} \phi_m(z,\tau) e^{2\pi i m \tilde{\tau}}$$

where ϕ_m's are Jacobi forms. It might be possible to see the limit $\mathrm{Im}(\tilde{\tau}) \to +\infty$ by a degeneration of a genus two curve to a nodal curve in §14.1. Note that after the degeneration we have an elliptic curve $E_{a,b,\lambda}$ with two marked non-ordered points $P = (\lambda,\mu)$ and $-P = (\lambda,-\mu)$ and with two differential forms $\frac{dx}{y}$, $\frac{dx}{(x-\lambda)y}$ such that the latter one has pole order one at the points P and $-P$ with the opposite residues at these points. If one constructs the moduli space of such triples, it must be a 2 : 1 covering of the moduli space which was intended to be discussed in Chapter 11.

Chapter 15
Differential equations with finite monodromy

During the corona visrus pandemy in 2020, I started to think about the general theory behind my work [MR06] with Stefan Reiter and my student Jorge Duque's thesis. I included parts of my thougths in §16.9 and §19.20 of my book [Mov19]. There, I find families of curves of degree $3,4$ and 6 and a family of $K3$ surfaces such that some differential forms in the middle cohomology integrated over any topolological cycles are algebraic, up to some π and Γ factors. It turned out that this might have something to do with generalizations of Chowla-Selberg theorem by P. Deligne and D. Gross, see [Gro78, Fre17, MR04, Col93], for which I did not have time and capacity to justify the statement I was observing (see [Mov19, Theorem 16.8]) by comparing to all these works.

For the proof of [Mov19, Theorem 16.8], I first find join Hodge cycles and use a consequence of Lefschetz $(1,1)$ theorem on the algbraicity of periods of Hodge cycles. I also compute Picard-Fuchs equations of some of these periods which has finite monodromy. I check this experimentally by computing the corresponding p-curvature for many primes and observe that it vanishes except for some small bad primes. I would like to thank T. Fonseca for providing many references used in this note, and S. Reiter for informing me that most of the Picard-Fuchs equations in this sections are hypergeometric.

15.1 Katz-Grothendieck conjecture

Let us consider a linear differential equation $L : Y' = AY, \quad ' = \frac{\partial}{\partial s}$ over $\mathbb{P}^1$ with the coordinate s, and with coefficients in $\mathbb{Q}$. The p-curvature of L is simply the matrix in A_n in $Y^{(n)} = A_n Y$ and it can be computed through the recursion $A_{n+1} = A_n' + A_n A, \quad A_1 = A$. It is an observation to Katz and Grothendieck that if L has finite monodromy then $A_p \equiv_p 0$ for all except a finite number of primes, that we call them bad primes, see for instance [Cha02, §3] The inverse of this statement is known as the Katz-Grothendieck p-curvature conjecture. For a linear differential equation $L : y^{(n)} - \sum_{i=0}^{n-1} a_i y^{(i)}$ we attach the canonical system $Y' = AY$ with $Y =$

$[y, y', \cdots, y^{(n-2)}]^{\mathsf{tr}}$ and

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_0 & a_1 & a_2 & \cdots & a_{n-1} \end{bmatrix}$$

In Table 15.1 we have gathered some differential equations with finite monodromy.

In 2003 when I started my collaboration with S. Reiter, I learned the following different version of Katz-Grothendieck conjecture:

Conjecture 15.1 If

$$(A_p)^n \equiv_p 0$$

for almost all prime p then $y' = Ay$ comes from geometry. Here, n is the dimension of A.

The most simple definition of 'coming from geometry' is that the solution space of $Y' = AY$ is spanend by $\int_\delta \omega$, where ω are sections of the cohomology bundle of a family of projective varieties over $s \in \mathbb{P}^1$ and δ's are homology classes, see also Totaro's expository article [Tot07] for other formulations of this under the name Simpson's conjecture and Bombieri-Dwork conjecture.

In `foliation.lib` I have written the procedure `BadPrD` which finds bad and good primes. For example, in Jorge Duque's thesis it appears that $F(4/3, -4/3, 8/3 : z)$ is not algebraic. We compute the bad and good primes in this case:

```
 ring r=(0,z),x,dp;
   matrix lde[1][3]; number a=4/3; number b=-4/3;  number c=8/3;
   lde=-a*b, c-(a+b+1)*z, z*(1-z); int ub=50; number parm=z;
   BadPrD(lde, parm, ub);



   matrix A_0[2][2]=c-1,-b,0,0; matrix A_1[2][2]=0,0,a,c-a-b-1;
   matrix lde=1/z*A_0+1/(z-1)*A_1;
   BadPrD(lde, parm, ub);


[1]:
   [1]:
      3
[2]:
   [1]:
      2
   [2]:
      5
   [3]:
      11
   [4]:
      17
   [5]:
      23
   [6]:
      29
   [7]:
      41
   [8]:
      47
[3]:
   [1]:
      7
   [2]:
      13
   [3]:
      19
   [4]:
      31
   [5]:
      37
   [6]:
```

```
    43




[1]:
   [1]:
      3
[2]:
   [1]:
      5
   [2]:
      11
   [3]:
      17
   [4]:
      23
   [5]:
      29
   [6]:
      41
   [7]:
      47
[3]:
   [1]:
      2
   [2]:
      7
   [3]:
      13
   [4]:
      19
   [5]:
      31
   [6]:
      37
   [7]:
      43
```

15.2 Some Picard-Fuchs equations with finite monodromy

In [Mov19, Theorem 16.8] we take one parameter families given by the tame polynomial $f = y^d - \tilde{f}(x) - s$, where $\tilde{f}$ is a polynomial in $x = (x_1, x_2, \ldots, x_{n+1})$ with non-zero discriminant. Moreover, the emphasis has been on families whose real locus consists of at least one oval. In this way, we can rewrite some of our integrals as usual claculus multiple integral in $\mathbb{R}^{n+2}$. For simplicity, we only discuss the two dimensional case. For instance, for

$$L_s : x^4 + y^4 + z^4 - x^2 - s = 0$$

for $s > -\frac{1}{4}$, $X(\mathbb{R})$ is empty, for $s = -\frac{1}{4}$ it is uinion of two points, for $-\frac{1}{4} < s < 0$ it is a union of two surfaces isomorphic to sphere. These surface for $s = 0$ toches each other forming a singularity, and for $s > 0$ it becomes a single surface isomorphic to the sphere, see Figure 15.1. Let δ_s be one of these ovals and Δ_s be its interior. The integrals

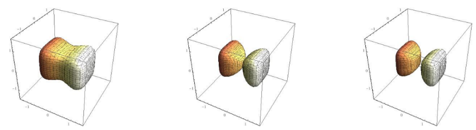

Fig. 15.1 Real level surfaces of $x^4+y^4+z^4-x^2=s,\ s=1/5,0,-1/50$

$$I(s) := \int_{\Delta_s} P(x,y)dx\wedge dy\wedge dz,\ \ P\in\mathbb{R}[x,y,z]$$

and its particular case $P=1$ which is the volume of Δ_s, are related to integrals in [Mov19, Theorem 16.8] in the following way:

$$\frac{\partial^m I}{\partial s^m} = \int_{\sigma(\delta_s)} \frac{P(x,y)dx\wedge dy\wedge dz}{(z^d-f(x,y))^m}$$

As another example take the family of curves $C_s : y^4-(x^2-1)(x^2-4)-s=0$, where s is a real parameter in the interval $[-4,\frac{9}{4}]$. The real locus $C_s(\mathbb{R})$ consists of three components, one oval and two curves coming from infinity and going to infinity. Let $I(s)$ be the integration of $y^2dx\wedge dx$ over the region surrounded by the oval. It follwos that $\Gamma(\frac{1}{4})^2\pi^{-3/2}\frac{\partial^2 I}{\partial^2 s}$ evaluated over any algebraic number is algebraic. We can verify that

$$\Gamma(\frac{1}{4})^2\pi^{-3/2}\frac{\partial^2 I}{\partial^2 s} = c\cdot(s+4)^{-\frac{4}{3}}(\frac{9}{4}-s)^{-\frac{4}{3}}$$

where c is algebraic number independent of s.

Periods which satisfy a first order differential equation can be solved easily. For those cases in Table (15.1), instead of the Picard-Fuchs equation we have have put the explicit formula for the period up to a constant c which depends on the domain of integration δ and not s.

More examples can be found in these two places, here and here. We have used the following computer code.

```
LIB "foliation.lib"; LIB "latex.lib";
int n=2;//---The dimension
ring r=(0,s), (x(1..n),y),dp;
int d=4; //---The degree
poly f=y^d; for (int i=1;i<=n ;i=i+1){f=f+x(i)^d;}
f=f-x(1)^3*x(2)-x(1)*x(2)^2-par(1); vector ve=[1];
number disc=discriminant(f);
list ll=factorize(substpar(disc,par(1),x(1))); ll; poly hilf=1;
for (i=2;i<=size(ll[1]);i=i+1){hilf=hilf*subst(ll[1][i],x(1),par(1));}
disc=number(hilf);
matrix et1=etaof(f, disc, par(1))[2];
```

The Family $f=0$	ω	PF equation
$y^4-(x^2-1)(x^2-4)-s$	$\frac{y^2dx\wedge dy}{f^2}$	$c\cdot(s+4)^{-\frac{4}{3}}(\frac{9}{4}-s)^{-\frac{4}{3}}$
y^4+x^4-x-s	$\frac{y^2dx\wedge dy}{f^2}$	$1120sI+1344s^2I'+(256s^3+27)I''=0$
y^6+x^6-x-s	$\frac{y^2dx\wedge dy}{f^2}$	$3459456sI+8173440s^2I'+4633200s^3I''$ $+855360s^4I'''+(46656s^5+3125)I''''=0$
$y^4+x^4+z^4-xy^2-s$	$\frac{zdx\wedge dy\wedge dz}{f^2}$	$(7680s+21)I+(15360s^2+144s)I'$ $+(4096s^3+64s^2)I''=0$
$y^4+x^4+z^4-y^3-s$	$\frac{zdx\wedge dy\wedge dz}{f^2}$	$(10752s+231)I+(17408s^2+1080s)I'$ $+(4096s^3+432s^2)I''=0$
$y^4+x^4+z^4-xy-s$	$\frac{zdx\wedge dy\wedge dz}{f^2}$	$c(64s^3-s)^{-\frac{1}{2}}$

Table 15.1 Picard-Fuchs equations with finite monodromy.

```
list  aabb=infoof(f, disc, et1);

intvec mlist=d; int i;  for (i=1;i<=n; i=i+1){mlist=mlist,d;}
list li=MixedHodgeFermat(mlist)[3];
poly P; int k; int dr; int di; int check=0; int count=0;
for(int b=1;b<=size(li);b=b+1)
   {
   check=0;
   P=li[b]; k= ((deg(P)+n+1) div d)+1;//---The pole order
   matrix pf=PFequ(f, P,ve,aabb);
   dr=1;  while (pf[1,dr]==0){dr=dr+1;}
   matrix pf0=submat(pf,1,dr..ncols(pf));
   di=ncols(pf0)-1;  matrix A[di][di]; pf0=pf0/(-pf0[1,di+1]);
   for (i=1;i<=di-1;i=i+1){A[i,i+1]=1;}
   A[di,1..di]=pf0[1,1..di];
   for (i=1;i<=k-dr;i=i+1)
         {
         if (det(A)<>0){A=diffpar(A,s)*inverse(A)+A;}
         else{check=1;}//--Note that for some P det(A)==0 and we have to treat this P separately.
         }
    if (check==0 and  deg(P)+n+1-d*(k-1)<>0 ){count=count+1; count; P; pf; BadPrD(A, s, 50);}
   }
```

Remark 15.1 One explcitly write down a basis of the homology of L in [Mov19, Theorem 16.8] and compute the Mondromy group, see for instance [AGZV88] or [Mov19, Chapters 6,7]. In this way one can arrange a topological proof for the fact that the integrals there are algebraic functions in the parameters of f.

Remark 15.2 One might try to reproduce [Mov19, Theorem 16.8] in the context of tame polynomials. For a tame polynomial f with a basis $x^\beta,\ \beta\in I$ of its Milnor module, let

$$C_f:=\min\{A_\beta|\beta\in I\}=\sum_{i=1}^{n+1}\frac{\alpha_i}{d}=\sum_{i=1}^{n+1}\frac{1}{m_i}.$$

The second equality is for $f=x_1^{m_1}+\cdots+x_{n+1}^{n+1}+\cdots$. In our case $d:=m_1=m_2=\cdots=m_{n+1}$ and so $C_f=\frac{n+1}{d}$. If the following is not true

$$C_f>\frac{n+1}{2}-\begin{matrix}1, & m\text{ even},\\ \frac{1}{2}, & m\text{ odd}\end{matrix}$$

then the cycle δ_2 in the proof of [Mov19, Theorem 16.8] is a cycle at infinity and hence it gives a zero cycle in the compactification of $\{f-g=0\}$. For $n=0$ and $n=1$ this is always valid, however, for $n>1$ this is a non-trivial condition on degree and weights of variables x_i of f. For some examples in this context see Duque's thesis [Duq20].

Remark 15.3 S. Reiter analyzed many differential equations computed in this are hypergeometric. For example, the third example in 15.1 is a pull-back by s^5 of [BH89, Table 8.3 Nr 32]. Our discussion converged to the fact all integrals in [Mov19, Theorem 16.8] must be related in some way to hypergeometric functions.

15.3 A modification of Katz-Grothendieck conjecture [12/08/2020]

There are situations in which a linear differential equation has an algebraic solution, but not all its solutions are algebraic. We want to invent a new version of Katz-Grothendieck conjecture which detects this. A typical example is the differential equation of Gauss hypergeometric function $F(a,b,c|z),\ \ b,c\in\mathbb{Q}$ with $b=c$. In this case we know the algebraic solution $F(a,b,b|z)=(1-z)^{-a}$ and the other solution is not necessarily algebraic. In this example the Gauss hypergeometric equation is reducible over $\mathbb{Q}$:

$$z(1-z)\partial^2+(c-(a+b+1)z)\partial-ab=(z\partial+b)((1-z)\partial-a).$$

The linear combination of two hypergeometric functions in Jorge Duques thesis which turns out to be algebraic satisfy a fourth order differential equation and it seems that not all its solutions is algebraic. This is because both the bad and good primes form an infinite set (conjecturally). See the output of the cose below:

```
 LIB "foliation.lib";
 ring r=(0,z),x,dp;
number a=4/3; number b=-4/3; number c=8/3;
matrix E1[2][2]=0,1,a*b/(z*(1-z)),((a+b+1)*z-c)/(z*(1-z));
matrix E2[2][2]=0,1,a*(b+1)/(z*(1-z)),((a+b+2)*z-c)/(z*(1-z));
matrix B1[2][2]=6*(z^2-z+1), 0, 0, 1;
matrix C1=diffpar(B1,z);
matrix A1[2][2]=C1*inverse(B1)+B1*E1*inverse(B1);
matrix B2[2][2]=(2/3)*(z-2)*(5*z^2-2z+2), 0,0,1;
matrix C2=diffpar(B2, z);
matrix A2[2][2]=C2*inverse(B2)+B2*E2*inverse(B2);
// Matrix A satisfies W'=AW in the Hossein's 4-letter.
matrix A[6][6]=
0, 0, A1[1,1], A1[1,2], A2[1,1], A2[1,2],
0, 0, A1[2,1], A1[2,2], A2[2,1], A2[2,2],
0, 0, A1[1,1], A1[1,2], 0,       0,
0, 0, A1[2,1], A1[2,2], 0,       0,
0, 0, 0,       0,       A2[1,1], A2[1,2],
0, 0, 0,       0,       A2[2,1], A2[2,2];
matrix B[1][6]=1,0,0,0,0,0;
sysdif(A,B,z);

_[1,1]=0
_[1,2]=(700z5+2212z4-66876z3+187408z2-184496z+61152)
_[1,3]=(-1500z6-2118z5+97230z4-344454z3+491712z2-319488z+78624)
_[1,4]=(450z7+504z6-45432z5+208278z4-403416z3+397800z2-197496z+39312)
_[1,5]=(1350z8-2322z7-18360z6+77949z5-129168z4+109863z3-47736z2+8424z)
_[1,6]=0
_[1,7]=0

matrix lde=submat(sysdif(A,B,z),1, 1..5);
BadPrD(lde, z, 100);
```

```
[1]:
   [1]:
      3
[2]:
   [1]:
      2
   [2]:
      5
   [3]:
      11
   [4]:
      17
   [5]:
      23
   [6]:
      29
   [7]:
      41
   [8]:
      47
[3]:
   [1]:
      7
   [2]:
      13
   [3]:
      19
   [4]:
      31
   [5]:
      37
   [6]:
      43
```

Trying to find out that this differential equation is reducible over $\mathbb{Q}$ (or worse over $\bar{\mathbb{Q}}$) might be difficult. After an hour thinking I came to the conclusion that the following must be the correct generalization of Katz-Grothendieck conjecture for the situation describe above.

Conjecture 15.2 *A linear differential system $\frac{\partial Y}{\partial z} = A(z)Y$ defined over $\mathbb{Q}$ has an algebraic solution if and only if for almost all primes p, the detrminant of the p-curvature is zero, that is $\det(A_p) = 0$. More generally, the dimension of the $\mathbb{C}$-vector space of algebraic solutions is the same as the rank of A_p over $\mathbb{F}_p(z)$.*

The direction $\rightarrow$ must be easy to see. We can check this for the Gauss hypergeometric equation and Jorge Duque's example.

```
   LIB "foliation.lib";
   ring r=(0,z),x,dp;
   number a=4/3; number b=-4/3;  number c=8/3; int pr=47;
   matrix lde[1][3]=-a*b, c-(a+b+1)*z, z*(1-z); BadPrD(lde, z, 100);
   matrix lde[1][5]=0, (700z5+2212z4-66876z3+187408z2-184496z+61152),
   (-1500z6-2118z5+97230z4-344454z3+491712z2-319488z+78624),
   (450z7+504z6-45432z5+208278z4-403416z3+397800z2-197496z+39312),
   (1350z8-2322z7-18360z6+77949z5-129168z4+109863z3-47736z2+8424z);



   int i; int j; int di=ncols(lde)-1; matrix A[di][di];
   if (ncols(lde)<>nrows(lde))
      {
       lde=lde/(-lde[1,di+1]);
       for (i=1;i<=di-1;i=i+1){A[i,i+1]=1;}
       A[di,1..di]=lde[1,1..di];
      }
   else
      {A=lde;}

  number disc=denominator(cleardenommat(A)[1]);
  matrix B=A;
      for (i=1;i<=pr-1;i=i+1)
         {
         B=diffpar(B,z)+B*A;
         }
      B=B*disc^(pr);
ring rr=pr,z,dp;
```

```
matrix B=imap(r,B);
int nc=deg(B);  matrix Bf=subst(B,z,0); matrix Bd=B;
 for (i=1;i<=nc;i=i+1)
     {
     Bd=diff(Bd,z);
     Bf=concat(Bf, subst(Bd,z,0));
     }
 B;
 det(B);
```

Chapter 16
Quadric hypersurfaces

Once these notes were written, I understood that I have missed Deligne's article [Del73]. Before writing any thing more in this chapter this article must be studied.

A hypersurface of degree one in $\mathbb{P}^{n+1}$ is isomorphic to $\mathbb{P}^n$, and so nothing to do. A smooth hypersurface X of degree 2 is given by a homogeneous polynomial which can be written in the form

$$X: \ f = 0, \qquad f := x^{\mathsf{tr}} \cdot A \cdot x. \tag{16.1}$$

where $x^{\mathsf{tr}} = [x_0, x_1, \cdots, x_{n+1}]$ and A is a $(n+2) \times (n+2)$ matrix with coefficients in $\mathbb{C}$ and $A^{\mathsf{tr}} = A$.

Proposition 16.1 *The hypersurface X is smooh if and only if* $\det(A) \neq 0$.

Quadric hypersurfaces from the point of view of complex geometry do not seem to be interesting. However, their arithmetic seems to be highly non-trivial and for this we refer to B. Totaro's articles [Tot08, Tot09] and the references therein. Over of a field k of characteristic$\neq 0$, we can always find a linear transformation $By := x$ such that $B^{\mathsf{tr}}AB$ is the diagonal matrix $\mathrm{diag}[a_0, \cdots, a_{n+1}]$ and hence

$$f = a_0 y_0^2 + a_1 y_1^2 + \cdots + a_{n+1} y_{n+1}^2$$

where $a_i \in \mathsf{k}^*$.

16.1 Topology of quadratic forms

Let Y be a smooth hyperplane section of X which itself is a quadric and $U = X \backslash Y$. For instance, we take Y to be given by $x_0 = 0$, and so U is the subset of $\mathbb{C}^{n+1}$ given by $f(1, x_1, \cdots, x_{n+1}) = 0$. We know that $H_i(U, \mathbb{Z}) = 0$ for all i except for $i = 0, n$. In fact $H_n(U) = \mathbb{Z}\delta$ where δ is a vanishing cycle. From this we can derive the following.

Theorem 16.1 *For $m \neq n$ we have $H_m(X, \mathbb{Z}) = 0$ for m odd and $H_m(X, \mathbb{Z}) \cong \mathbb{Z}$ for m even. Further, if n is odd then $H_n(X, \mathbb{Z}) = 0$ and if n is even then $H_n(X, \mathbb{Z}) \cong \mathbb{Z}^2$.*

For further details see [Mov19] Chapter 5 and Chapter 6.

For n even and for a smooth quartic we have in total $1\cdot 3\cdot 5\cdots(n+1)\cdot 2^{\frac{n}{2}}$ linear cycles $\mathbb{P}^{\frac{n}{2}}$ inside X.

Proposition 16.2 ([Mov19], §17.6) *Let us consider two linear algebraic cycle $\mathbb{P}^{\frac{n}{2}}$ and $\check{\mathbb{P}}^{\frac{n}{2}}$ with $\mathbb{P}^{\frac{n}{2}}\cap\check{\mathbb{P}}^{\frac{n}{2}}=\mathbb{P}^m$ inside the quartic hypersurface of dimension n. Then*

$$\mathbb{P}^{\frac{n}{2}}\cdot\check{\mathbb{P}}^{\frac{n}{2}}=\begin{cases}0 & \text{if } m \text{ is odd}\\ 1 & \text{if } m \text{ is even}\end{cases}$$

It follows that

Proposition 16.3 *For $n=4k$ and m odd (resp. $n=4k+2$ and m even) the homology $H_n(X,\mathbb{Z})$ is freely generated by the homology classes of $\mathbb{P}^{\frac{n}{2}}$ and $\check{\mathbb{P}}^{\frac{n}{2}}$. The intersection matrices are given respectively by*

$$\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

16.2 Periods of quadric hypersurfaces

The de Rham cohomology $H^n_{\mathrm{dR}}(U)$ is a one dimensional space generated by the differential form

$$\eta:=\sum_{i=1}^{n+1}(-1)^i x_i\widehat{dx_i} \tag{16.2}$$

which is written in the affine chart $x_0=1$. For n an even number, it is also generated by the following form which is written in the projective coordinates:

$$\text{Residue}\left(\frac{\sum_{i=0}^{n+1}(-1)^i x_i\widehat{dx_i}}{f^{\frac{n}{2}+1}}\right) \tag{16.3}$$

The Gauss-Manin connection for the full family of quadrics is

$$\nabla\eta=a\frac{d\Delta}{\Delta}\otimes\eta$$

where a is rational number. Using Picard-Lefschetz formula and the self intersection formula of a vanishing cycle, see [Mov19] §6.6, we know that for n even it is in $\mathbb{Z}+\frac{1}{2}$ and for n odd it is in $\mathbb{Z}$. we can also write the above equality as:

$$\int_\delta\eta=b\Delta^a$$

where $b \in \mathbb{C}$ is a constant (independent of X). The Taylor series of the above period is basically derived from the Taylor series of $\frac{1}{1-x}$ or $\sqrt{1+x}$. Such a series in a more general context is computed in [Mov19] §13.9.

16.3 Moduli of union of a quartic and linear hypersurfaces

Let us consider the space T of singular hypersurfaces in $\mathbb{P}^{n+1}$ given by $f_1 f_2 = 0$, where f_i is a homogeneous polynomial of degree i. We further assume that the quartic hypersurface in $\mathbb{P}^{n+1}$ given by $f_2 = 0$ is smooth and it intersects $f_1 = 0$ transversely. Up to the action of $\mathrm{GL}(n+2,\mathbb{C})$ any such hypersurface is given by

$$x_0\left(x_0 L + x_0^2 + x_1^2 + \cdots + x_{n+1}^2\right) = 0$$

where $L = \sum_{i=1}^{n+1} a_i x_i$ is a homogeneous polynomia of degree 1 and independent of x_0. Therefore, L depends on $n+1$ parameters.

Proposition 16.4 *The moduli of a union $X_1 \cup X_2$ of two smooth hypersurfaces in $\mathbb{P}^{n+1}$ of degrees one and two which intersect each other transversely is*

$$\mathrm{Spec}\left(\mathbb{C}\left[a_1, a_2, \ldots, a_{n+1}, \frac{1}{a_1^2 + a_2^2 + \cdots + a_{n+1}^2 - 4}\right]\right).$$

Proof. It is left to the reader as an exercise!

Accordong to computations in [Mov19] Chapter 18, if we consider two linear cycles $\mathbb{P}^{\frac{n}{2}} \subset X_1$ and $\check{\mathbb{P}}^{\frac{n}{2}} \subset X_2$ with $\mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \mathbb{P}^m$ for $-1 << m \leq \frac{n}{2} - 2$, the Hodge locu of $\mathbb{P}^{\frac{n}{2}} - \check{\mathbb{P}}^{\frac{n}{2}}$ is bigger then the expected one.

16.4 On a Taylor series

In this section we further analyze a Taylor series in [Mov19]§13.9 for the following family of curves

$$x^2 + y^2 - t_0 - t_1 x - \cdots - t_n x^n = 0. \tag{16.4}$$

where $t := (t_0, t_1, \ldots, t_n) \in (\mathbb{R}^{n+1}, 0)$. In the projective space $\mathbb{P}^2$ we ca interpret this as a deformation of

$$z^{n-2}(x^2 + y^2 - 1) = 0$$

which is a union of a line of multiplicity $n-2$ and a quartic. For $t_0 > 0$ it is a closed curve in $\mathbb{R}^2$ which we orient it anti-clockwise and name it δ_t. Up to multiplication by a constant we have

$$\int_{\delta_t} \frac{dx}{y} = \sum_{a_0,a_1,\ldots,a_n \in \mathbb{N}\cup\{0\},\, a^* := \frac{a_1+2a_2+\cdots+na_n}{2} = a_0+a_1+\cdots+a_n} \left(\frac{1}{2}\right)_{a^*} \frac{t_0^{a_0} t_1^{a_1} \cdots t_n^{a_n}}{a_0! a_1! \cdots a_n!} \quad (16.5)$$

where $(x)_y := x(x+1)(x+2)\cdots(x+y-1)$ is the Pochhammer symbol. This might have some applications in a problem investigated in [Chicone-Jacobs 1989 Tans. AMS].

Chapter 17
Zero dimensional integrals

17.1 Introduction

To understand better the problems related to the zeros of Abelian, and in general multiple, integrals which arise in Algebraic Geometry and Differential Equations, one may try to solve similar problems in the case of zero dimensional integrals. These integrals are in fact algebraic functions and the word integral is used just because of their similarities with higher dimensional integrals. Surprisingly, all the topics which we are going to discuss in an arbitrary dimension fit well into the dimension zero. Since in this case we do not need the topology of varieties, this chapter can be understood without any advanced information in (co)homology theory of varieties. Our objective in this chapter is to analyze some problems for zero dimensional integrals whose counterparts in higher dimensional cases are difficult to treat. Our observation is that zero dimensional integrals can be studied in a more arithmetic context and this helps us to understand their behavior better. The idea of this chapter comes from [GM07]. While the mentioned paper mainly discusses the infinitesimal Hilbert problem in zero dimension, this chapter emphasizes the arithmetic properties of such integrals. I wrote this chapter in the same time as the joint paper **??**. A basic knowledge of the classical Galois theory will be useful for understanding the contents of this chapter. In this direction, we have used the book [Mil]. The higher dimensional version of the present chapter is discusses in [Mov19, Chapter 10] under the name 'tame polynomial'.

17.2 Zero dimensional Abelian integrals

For a finite discrete set M we denote by $\mathbb{Z}[M]$ the free $\mathbb{Z}$-module generated by the elements of M. The degree of $\delta = \sum_i r_i x_i \in \mathbb{Z}[M], r_i \in \mathbb{Z},\ x_i \in M$ is

$$\deg(\delta) := \sum_i r_i.$$

We use the reduced 0-th homology and cohomology for the set M:

$$H_0(M,\mathbb{Z}) = \{\delta \in \mathbb{Z}[M] \mid \deg(\delta) = 0\},$$

$$H^0(M,\mathbb{Z}) := \check{H}_0(M,\mathbb{Z}),$$

where $\check{}$ means dual. In $H_0(M,\mathbb{Z})$ we have the intersection form induced by

$$\langle x,y\rangle = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \quad x,y \in M.$$

By definition $\langle\cdot,\cdot\rangle$ is a symmetric form in $H_0(M,\mathbb{Z})$, i.e. for all $\delta_1,\delta_2 \in H_0(M,\mathbb{Z})$ we have $\langle\delta_1,\delta_2\rangle = \langle\delta_2,\delta_1\rangle$. Let

$$f = t_d x^d + t_{d-1}x^{d-1} + \cdots + t_1 x + t_0 \tag{17.1}$$

be a polynomial of degree d in variable x and with coefficient $t := (t_0,\ldots,t_{d-1},t_d)$, $t_d \neq 0$ in R and

$$\{f = 0\} = L_f = L_t := \{x \in \bar{\mathsf{k}} \mid f(x) = 0\} = \{x_1,x_2,\ldots,x_d\}.$$

We assume that t_d is invertible in R. An element of $H_0(L_t,\mathbb{Z})$ is called a cycle. A canonical basis of $H_0(L_t,\mathbb{Z})$ is given by

$$\delta_i = x_{i+1} - x_i,\ i = 1,2,\ldots,d-1.$$

The intersection matrix of f with respect to the above basis is:

$$\Psi_0 := \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \tag{17.2}$$

For a cycle δ and $\omega \in \mathsf{R}[x]$ we define

$$\int_\delta \omega := \sum_i r_i \omega(x_i),$$

where

$$\delta = \sum_i r_i x_i,\ r_i \in \mathbb{Z},\ x_i \in \{f = 0\}$$

and call them (zero dimensional Abelian) integrals or periods. Following the terminology in higher dimension, we call ω a 0-form and denote by

$$\Omega^0_{\mathbb{U}_1/\mathbb{U}_0} := \mathsf{R}[x]$$

the R-module of 0-forms. Particularly we are interested on simple cycles $\delta = x_1 - x_2$, where x_1 and x_2 are two simple roots of $f(x) = 0$. We look at

$$\mathsf{P}(\{f=0\},\omega) := \{\int_\delta \omega \mid \delta \in H_0(\{f=0\},\mathbb{Z})\}$$

as a $\mathbb{Z}$-module and call it the period module.

Remark 17.1 When there is a danger of confusion between the complex number $x_1 - x_2$ and the cycle $\delta = x_1 - x_2$, we will write $\delta = [x_1] - [x_2]$. The first one can be obtained by integration of the 0-form x on δ.

Remark 17.2 We frequently use

$$\mathbb{Z}_a[t, \frac{1}{t_d}] = \mathbb{Z}_a[t_0, t_1, \ldots, t_{d-1}, t_d, \frac{1}{t_d}],\ a \in \mathbb{N}$$

instead of the general ring R and we consider t_i's in (17.1) as parameters. Using topological arguments, we may prove an statement for $\mathbb{Z}_a[t, \frac{1}{t_d}]$ and then we replace t by elements in an arbitrary ring R and obtain the same statement for arbitrary R (a must be invertible in R). It is useful in this case to consider $\mathbb{Z}_a[t, \frac{1}{t_d}]$ as a weighted ring with $\deg(t_i) = d - i,\ i = 0, 1\ldots, d-1$. We need the localization $\mathbb{Z}_a$ of $\mathbb{Z}$ over a because for some arguments we need to divide on a. In many cases we put $t_d = 1$.

Remark 17.3 The notions irreducibility, irreducible decomposition, division and so on will be used in the ring $\mathsf{k}[x]$. For instance when we write $g \mid f,\ f, g \in \mathsf{R}[x]$, we mean that there exists $q \in \mathsf{k}[x]$ such that $f = gq$.

17.3 Discriminant of a polynomial

For a monic polynomial $f = x^d + t_{d-1}x^{d-1} + \cdots + t_1 x + t_0 \in \mathsf{R}[x]$ we define the discriminant of f

$$\Delta = \Delta_f := \prod_{1 \le i \ne j \le d} (x_i - x_j) = \prod_{i=1}^{d} f'(x_i) \in \mathsf{R},$$

where $f' = \frac{\partial f}{\partial x}$ is the derivative of f. The discriminat of $af, a \in \mathsf{R},\ a \neq 0$ is defined to be the discriminant of f. Recall Remark (17.2). For $\mathsf{R} = \mathbb{Z}[t]$ the discriminant Δ_f is a homogeneous polynomial of degree $d(d-1)$ with $\mathbb{Z}$ coefficients in the graded ring $\mathbb{Z}[t],\ \deg(t_i) = d - i$. For instance, for $f = x^d - t$ we have

$$\Delta_f = t^{d-1} \prod_{0 \le i \ne j \le d-1} (\zeta_d^i - \zeta_d^j) = d^d(-t)^{d-1}.$$

Proposition 17.1 *If* $\mathsf{R} = \mathbb{Z}[t]$ *and* $f = x^d + t_{d-1}x^{d-1} + \cdots + t_1 x + t_0$ *then* Δ_f *is an irredecible polynomial in* $\mathbb{C}[t]$.

Proof. Consider the map

$$\alpha : \mathbb{C}^d \to \mathbb{C}^d$$

$$\alpha(x_1, x_2, \ldots, x_d) =$$

$$(t_0, \ldots, t_{d-2}, t_{d-1}) = ((-1)^d x_1 x_2 \cdots x_d, \cdots, \sum_{i \neq j} x_i x_j, -\sum x_i).$$

α maps $\{x_1 = x_2\}$ onto $\{\Delta = 0\}$. Since the first variety is irreducible, the second one is also irreducible.

The Milnor module associated to f is the quotient

$$\mathsf{V}_f := \frac{\mathsf{R}[x]}{f' \cdot \mathsf{R}[x]}.$$

It is also useful to define the quotient

$$\mathsf{W}_f := \frac{\mathsf{R}[x]}{f' \cdot \mathsf{R}[x] + f \cdot \mathsf{R}[x]}.$$

Proposition 17.2 *Let* $f \in \mathsf{R}[x]$ *be a monic polynomial and assume that* $d = \deg(f)$ *is invertible in* R. *Then*

1. V_f *is a free* R*-module with the basis*

$$I := \{1, x, x^2, \ldots, x^{d-2}\}.$$

2. *Let* A *be the multiplication by* f R*-linear map in* V_f. *We have the following identity*

$$\Delta_f = d^d \cdot \det(A).$$

3. Δ_f *is a zero divisor of* W_f, *i.e.*

$$\Delta_f \cdot \mathsf{W}_f = 0.$$

Proof. The first part of the proposition is easy and is left to the reader. Recall Remark (17.2). For the second part, it is enough to prove it for the case

$$\mathsf{R} = \mathbb{Z}_d[t],\ f = x^d + t_{d-1}x^{d-1} + t_{d-2}x^{d-2} + \cdots + t_0.$$

We first prove that Δ_f and $\det(A)$ have the same zero set in $\bar{\mathbb{Q}}^d$. The polynomial f_t, $t \in \bar{\mathbb{Q}}^d$ has multiple roots in $\bar{\mathbb{Q}}$ if and only if there are polynomials $p(x), q(x) \in \bar{\mathbb{Q}}[x]$, $\deg(p) \leq d-2$ such that $f \cdot p = f' \cdot q$. This is equivalent to the fact that there is a $\bar{\mathbb{Q}}$-linear relation between $f, xf, \ldots, x^{d-2}f$ in V_f and so $\det(A) = 0$.

By Proposition 17.1, we have $\det(A) = a \cdot \Delta_f^n$ for some $a \in \bar{\mathbb{Q}}$ and $n \in \mathbb{N}$. It remains to prove that $a = d^{-d}$ and $n = 1$. Since $\Delta_f := \prod_{i=1}^d f'(x_i)$, Δ_f as apolynomial in t_0 is of degree $d-1$ and it is with the leading coefficient d^d. From another side, we look the matrix of A in the basis I and see that the term t_0 appears only in the diagonal entries of of A and it has the leading coefficient 1. Therefore $\det(A)$, as a polynomial in t_0, is of degree $d-1$ and it is with the leading coefficient 1.

The third part follows from

$$\det(A - f \cdot I_{\mu\times\mu}) \cdot \mathsf{V}_f = 0.$$

Note that if $f = t_d x^d + t_{d-1}x^{d-1} + \cdots + t_0$ is not monic then the the corresponding multiplication by f linear map has determinant $(dt_d)^d \Delta_{\frac{f}{t_d}}$. Bellow there is a table of discriminants for $d \leq 4$ and $\mathsf{R} = \mathbb{Z}[t]$.

$f = x^d + t_{d-1}x^{d-1} + t_{d-2}x^{d-2} + \cdots + t_0$

d	Δ
2	$4t_0 - t_1^2$
3	$27t_0^2 - 18t_0t_1t_2 + 4t_0t_2^3 + 4t_1^3 - t_1^2t_2^2$
4	$256t_0^3 - 192t_0^2t_1t_3 - 128t_0^2t_2^2 + 144t_0^2t_2t_3^2 - 27t_0^2t_3^4 +$ $144t_0t_1^2t_2 - 6t_0t_1^2t_3^2 - 80t_0t_1t_2^2t_3 + 18t_0t_1t_2t_3^3 + 16t_0t_2^4 -$ $4t_0t_2^3t_3^2 - 27t_1^4 + 18t_1^3t_2t_3 - 4t_1^3t_3^3 - 4t_1^2t_2^3 + t_1^2t_2^2t_3^2$

The above table is obtained by the command `discriminant` from `foliation.lib`. Note that this proceedure calculates $\det(A)$ and so in order to obtain the above table, we have to multiply its output with d^d. Throughout this chapter we assume that d is invertible in R. Therefore, we will freely use Proposition 17.2.

17.4 Gelfand-Leray form

We denote by $\Omega^1_{\mathbb{U}_1/\mathbb{U}_0}$ the space of 1-forms $\omega := p dx,\ p \in \mathsf{R}[x]$. According to Proposition 17.2 there are $q_1, q_2 \in \mathsf{R}[x]$ such that

$$\Delta \cdot dx = df \cdot q_1 + f \cdot q_2 dx.$$

The Gelfand-Leray form of $\omega = p dx$ is a 0-form given by

$$\frac{\omega}{df} := \frac{pq_1}{\Delta} \in \mathsf{R}[x]_\Delta.$$

By integration of a 1-form ω on a cycle $\delta \in H_0(\{f=0\}, \mathbb{Z})$ we mean the integration of the Gelfand-Leray form $\frac{\omega}{df}$ on δ, i.e.

$$\int_\delta \omega := \int_\delta \frac{\omega}{df}.$$

17.5 De Rham cohomology or Brieskorn modules

The global Brieskorn modules associated to the polynomial $f \in \mathsf{R}[x]$ are the quotients

$$\mathsf{H}' = \mathsf{H}'_f := \frac{\mathsf{R}[x]}{f\cdot \mathsf{R}[x] + \mathsf{R}}$$

and

$$\mathsf{H}'' = \frac{\Omega^1_{\mathbb{U}_1/\mathbb{U}_0}}{f\cdot \Omega^1_{\mathbb{U}_1/\mathbb{U}_0} + \mathsf{R}\cdot df}.$$

They play the role of the de Rham cohomology of the zero dimensional variety $\{f=0\}$ [i]. More precisely, the map $\mathsf{H}' \to \mathsf{H}''$, $\omega \mapsto \omega df$ is an inclusion which gives us the isomorphism $\mathsf{H}' \otimes_{\mathsf{R}} \mathsf{k} \to \mathsf{H}'' \otimes_{\mathsf{R}} \mathsf{k}$ of k-vector spaces. Its inverse is given by the Gelfand-Leray map $\omega \mapsto \frac{\omega}{df}$. The mentioned k-vector space is the de Rham cohomology of $\{f=0\}$.

The sets H' and H'' are R-modules in a canonical way. By H we mean one of H' or H''. It turns out that the integrals $\int_\delta \omega,\ \delta \in H_0(L_f,\mathbb{Z}),\ \omega \in \mathsf{H}$ are well-defined.

Proposition 17.3 *The* R*-module* H' *(resp.* H''*) is a free* R*-module of rank* $d-1$ *generated by* $x, x^2, \ldots, x^{d-1}$ *(resp.* $dx, xdx, \cdots, x^{d-2}dx$*).*

Proof. The proof is easy and is left to the reader.

The basis of H given in Proposition 17.3 is called the canonical basis of H.

17.6 An operation

For polynomials $f, \omega \in \mathsf{R}[x]$ we define the following polynomial

$$\omega * f(x) := (x-\omega(x_1))(x-\omega(x_2))\cdots(x-\omega(x_d)) \in \mathsf{R}[x],$$

where $f(x) = (x-x_1)(x-x_2)\cdots(x-x_d)$. We have the following trivial identities:

$$\begin{aligned}
\omega_1 * (\omega_2 * f) &= (\omega_1 \circ \omega_2) * f,\\
\omega * (f_1 \cdot f_2) &= (\omega * f_1)\cdot(\omega * f_2),\\
(\omega_1 \cdot f + \omega_2) * f &= \omega_2 * f,\ c * f = (x-c)^{\deg(f)},\\
x^p * (x^q - 1) &= (x^{\frac{q}{(p,q)}} - 1)^{(p,q)}
\end{aligned}$$

for

$$\omega, \omega_1, \omega_2, f, f_1, f_2 \in \mathsf{R}[x],\ c \in \mathsf{R},\ p,q \in \mathbb{N}.$$

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[i] The classical defintion of Brieskorn modules, see [Bri70], is $H' := \frac{\mathsf{R}[x]}{\mathsf{R}[f]}$, $H'' = \frac{\Omega^1_{\mathbb{U}_1/\mathbb{U}_0}}{\mathsf{R}[f]\cdot df}$ for the case $\mathsf{R} = \mathbb{C}$. These are $\mathsf{R}[f]$-modules and are introduced for the study of the monodromy of the fibration $\{f-s=0\}$.

Proposition 17.4 *Suppose that $\Delta_f \neq 0$. Then there are $D_\omega \in \mathsf{R}$ and $E_\omega \in \mathsf{R}[x]$ such that*

$$(\omega * f) \circ \omega = E_\omega \cdot f \tag{17.3}$$

and

$$\Delta_f \cdot D_\omega^2 = \Delta_{\omega * f}.$$

Proof. For the first part note that $(\omega * f) \circ \omega(x_i) = 0,\ i = 1, 2, \cdots, d$ and the multiplicity of $(\omega * f) \circ \omega$ at x_i is at least the multiplicity of f at x_i. In the second part D_ω is explicitly given by

$$D_\omega := \prod_{1 \le i < j \le d} \frac{\int_{\delta_{ij}} \omega}{\int_{\delta_{ij}} x} \in \mathsf{R}, \tag{17.4}$$

where $\delta_{ij} = x_i - x_j \in H_0(L_t, \mathbb{Z})$.

The following Proposition is taken from [GM07, Proposition 7]

Proposition 17.5 *Let $f \in \mathsf{R}[x]$ be an irreducible polynomial, d be invertible in R and $\omega \in \mathsf{R}[x]$. Then $\omega * f = g^k$ for some $k \in \mathbb{N}$ and irreducible polynomial $g \in \mathsf{R}[x]$. Moreover, if for some simple cycle $\delta \in H_0(\{f = 0\}, \mathbb{Z})$ we have $\int_\delta \omega = 0$ then $k \ge 2$.*

Proof. We define the equivalence relation $\sim$ on L_f:

$$x_i \sim x_j \Leftrightarrow \omega(x_i) = \omega(x_j).$$

Let G_f be the Galois group of the splitting field of f. For $\sigma \in G_f$ we have

$$x_i \sim x_j \Rightarrow \sigma(x_i) \sim \sigma(x_j) \tag{17.5}$$

Since f is irreducible over k, the action of G_f on I is transitive (see for instance [Mil] Prop. 4.4). This and (17.5) imply that G_f acts on $I/\sim$ and each equivalence class of $I/\sim$ has the same number of elements as others. Let $I/\sim = \{v_1, v_2, \ldots, v_e\},\ e \mid d$ and $c_i := \omega(v_i)$. Define

$$g(x) := (x - c_1)(x - c_2) \cdots (x - c_e).$$

We have

$$g^k = f * \omega \in \mathsf{R}[x],$$

where $k = \frac{d}{e}$. Let $g = x^e + a_1 x^{e-1} + \cdots + a_e$. We have $ka_1 \in \mathsf{R}$ and we calculate the coefficients of g in terms of the coefficients of the right hand side of the above equality. A simple induction implies that all the coefficients of g lies in R. Note that here we use the fact that d, and hence k, is invertible in R. Since G_f acts transitively on the roots of g, we conclude that g is irreducible over k. $\square$

Following the notations of Proposition 17.5, we have the morphism

$$\{f=0\} \stackrel{\alpha_\omega}{\to} \{g=0\},\ \alpha_\omega(x)=\omega(x)$$

defined over R. Since $f \mid g\circ\omega$ over R, it defines a well-defined map

$$\alpha_\omega^* : \mathsf{H}'_g \to \mathsf{H}'_f,\ \alpha_\omega(\omega')=\omega'\circ\omega.$$

Suppose that f is irreducible and there is no simple cycle $\delta\in H_0(\{f=0\},\mathbb{Z})$ such that $\int_\delta \omega = 0$. According to Proposition 17.5 the polynomial $g:=\omega * f$ is also irreducible and the morphism α_ω is topologically an isomorphism.

Proposition 17.6 *Assume that D_ω in (17.4) is invertible in R. Under the above hypothesis, there is an $\eta\in\mathsf{R}[x]$, such that*

$$\eta\circ\omega = f\cdot q + x, \textit{ for some } q\in\mathsf{R}[x]$$

and hence the inverse of α_ω is given by α_η.

Proof. Let $x_i,\ i=1,2,\dots,d$ be the roots of f and $\omega(x_i)=c_i$. We are looking for a polynomial $\eta=r_0+r_1x+\cdots+r_{d-1}x^{d-1},\ r_i\in\mathsf{R}$ such that $\eta(c_i)=x_i$. This gives the equation $A(r_0,r_1,\cdots,r_{d-1})^{\mathsf{tr}}=(x_1,x_2,\cdots,x_d)^{\mathsf{tr}}$, where A is the Vandermonde matrix formed by c_i's. Since $\det(A)^2=\Delta_f D_\omega^2$ and $A^{-1}(x_1,x_2,\cdots,x_d)^{\mathsf{tr}}$ is symmetric in $x_1,x_2,\cdots,x_d$, we conclude that $r_i\in\mathsf{R}$. Now, the facts that $\eta\circ\omega(x_i)=x_i,\ i=1,2,\dots,d$ and f is irreducible finishes the proof.

Note that the topologically identity map $\{f^n=0\}\to\{f=0\},\ n\geq 2,\ x\mapsto x$ does not induce an isomorphim between the corresponding Brieskorn modules.

17.7 Zero dimensional Fermat variety

Let

$$f := x^d - 1 \in \mathbb{Z}[x].$$

We call $\{f=0\}$ the Fermat variety of dimension zero. Let also

$$f := x^d - 1 = \prod_{i\mid d} p_i(x), \tag{17.6}$$

be the decomposition of x^d-1 into irreducible component over $\mathbb{Q}$. We have

$$p_i(x) = \prod_{\gcd(a,i)=1,\ 1\leq a\leq i} (x-\zeta_d^a)\in\mathbb{Z}[x],$$

$p_1(x)=x-1$. The polynomial p_i is called the i-th cyclotomic polynomial. Using Proposition 17.5 one concludes that for all $i\mid d$ the morphisms

$$\{p_d(x)=0\} \to \{p_{\frac{d}{i}}(x)=0\},\ x\mapsto x^i$$

are well-defined over $\mathbb{Z}$.

Let $\phi(d):=p_d(1)$, the sum of the coefficients of p_d. Derivating (17.6) and putting $x=1$, we concludes that

$$\prod_{i|d,i\neq 1}\phi(i)=d.$$

This implies that

$$\phi(d)=\begin{cases} p, & \text{if for some prime } p, d=p^\alpha \\ 1, & \text{otherwise.}\end{cases}.$$

The following function

$$\sigma_d:\mathbb{Z}[\zeta_d]\to\mathbb{Z}/\phi(d)\mathbb{Z},$$
$$\sum_{i=0}^{d-1}a_i\zeta_d^i\mapsto\sum_{i=0}^{d-1}a_i$$

is well defined, where $\mathbb{Z}[\zeta_d]$ is the ring of integers of $\mathbb{Q}(\zeta_d)$ (the sum of coefficients of any $p_d(x)q(x),\ q\in\mathbb{Z}[x]$ is congruent to 0 modulo $\phi(d)$). We conclude that:

$$\mathsf{P}(\{x^d-1=0\},x)=\ker(\sigma_d).$$

Note that if $a=\sum_i a_i\zeta_d^i\in\mathbb{Z}[\zeta_d]$ with $\sum_i a_i=\phi(d)k,\ k\in\mathbb{N}_0$ then $a=\sum_i a_i\zeta_d^i-p_d(\zeta_d)k=\sum_i b_i\zeta_d^i$ with $\sum_i b_i=0$.

For a $n\in\mathbb{N}$ we want to determine $\mathsf{P}(\{f=0\},\omega)$, where $\omega=x^n$ or $=x^{n-1}dx$. We note that if $d':=\frac{d}{\gcd(n,d)}$ and $n'=\frac{n}{\gcd(n,d)}$ then the morphisim

$$\alpha:\{x^d-1=0\}\to\{x^{d'}-1=0\},\ \alpha(x)=x^{(n,d)}$$

is defined over $\mathbb{Z}$ and has the property $\alpha^*(x^{n'})=x^n$. Therefore,

$$\mathsf{P}(\{x^d-1=0\},x^n)=\mathsf{P}(\{x^{d'}-1=0\},x^{n'}).$$

For $\gcd(d',n')=1$, the automorphism

$$\beta:\{x^{d'}-1=0\}\to\{x^{d'}-1=0\},\ \beta(x)=x^{n'}$$

is an isomorphism. We conclude that:

Proposition 17.7 *We have*

$$\mathsf{P}(\{x^d-1=0\},x^n)=\ker(\sigma_{d'}),$$

where $d'=\frac{d}{\gcd(n,d)}$. *In particular*

$$\mathsf{P}(\{x^d-1=0\},x^n)\otimes_{\mathbb{Z}}\mathbb{Q}=\mathbb{Q}(\zeta_{d'})$$

and if two distinct prime numbers divide d' then

$$\mathsf{P}(\{x^d-1=0\},x^n)=\mathbb{Z}[\zeta_{d'}].$$

For $f=x^d-t\in\mathbb{Z}[t][x]$ we have

$$\mathsf{P}(x^d-t=0,\ x^n)=t^{\frac{n}{d}}\mathsf{P}(x^d-1=0,\ x^n).$$

We have $f'\cdot x+f\cdot(-d)=d\cdot t$ and so we have

$$\int\frac{x^{n-1}dx}{df}=\frac{1}{d\cdot t}\int x^n. \tag{17.7}$$

Therefore,

$$\mathsf{P}(x^d-t=0,\ x^{n-1}dx)=\frac{1}{d}t^{\frac{n}{d}-1}\mathsf{P}(x^d-1=0,\ x^n).$$

17.8 Zeros of Abelian integrals and contraction of varieties

Let $f,g,\omega\in\mathsf{R}[x]$ be such that

$$g\circ\omega=q\cdot f,\ \ \text{for some } q\in\mathsf{k}[x]. \tag{17.8}$$

We have the morphism

$$\{f=0\}\stackrel{\alpha_\omega}{\to}\{g=0\},\ \alpha_\omega(x)=\omega(x)$$

defined over R. Let $\delta\in H_0(\{f=0\},\mathbb{Z})$ such that $(\alpha_\omega)_*(\delta)=0$, where $(\alpha_\omega)_*$ is the induced map in homology. For instance, if $\deg(g)<\deg(f)$ then because of (17.8), there exist two zeros x_1,x_2 of f such that $\int_\delta\omega=\omega(x_1)-\omega(x_2)=0$ and so the topological cycle $\delta:=x_1-x_2$ has the desired property. Note that the 0-form ω on $\{f=0\}$ is the pull-back of the 0-form x by α_ω. The following theorem discusses the inverse of the above situation:

Theorem 17.1 *Let $f,\omega\in\mathsf{R}[x]$. Assume that f is monic, the degree of each irreducible component of f is invertible in* R *and*

$$\int_\delta\omega=0 \tag{17.9}$$

for some simple cycle $\delta\in H_0(\{f=0\},\mathbb{Z})$. Then there exists a polynomial $g\in\mathsf{R}[x]$ such that

1. $\deg(g)<\deg(f)$*;*
2. *the degree of each irreducible components of g divides the degree of some irreducible component of f;*

3. $g \circ \omega = fq$ *for some* $q \in \mathsf{k}[x]$, *the morphism* $\alpha_\omega : \{f = 0\} \to \{g = 0\}$ *is surjective and* $(\alpha_\omega)_*(\delta) = 0$.

Proof. Let $f = f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_r^{\alpha_r}$ (resp. $\omega * f = g_1^{\beta_1} g_2^{\beta_2} \cdots g_s^{\beta_s}$) be the decomposition of f (resp. $\omega * f$) into irreducible components (in $\mathsf{k}[x]$). By Proposition 17.5 and the second equality in (17.3), we have $s \le r$ and we can assume that $\omega * f_i = g_i^{k_i}$ for $i = 1, 2, \ldots, s$ and some $k_i \in \mathbb{N}$. For some $a \in \mathsf{R}$ the polynomial $g := a g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_s^{\alpha_s}$ is in $\mathsf{R}[x]$ and we claim that it is the desired one. Except the first item and $(\alpha_\omega)_*(\delta) = 0$, all other parts of the theorem are satisfied by definition.

Let $\delta = x_1 - x_2$. We consider two cases: First let us assume that x_1 and x_2 are two distinct roots of an irreducible component of f, say f_1. By Proposition 17.5 we have $\omega * f_1 = g_1^{k_1},\ k_1 > 1$ and so $\deg(g) < \deg(f)$. Now assume that x_1 is a zero of f_1 and x_2 is a zero of f_2. Let $\omega * f_1 = g_1^{k_1},\ \omega * f_2 = g_2^{k_2},\ k_1, k_2 \in \mathbb{N}$. The number $\omega(x_1) = \omega(x_2)$ is a root of both $g_i,\ i = 1,2$ and G_f acts transitively on the roots of both $g_i,\ i = 1,2$. This implies that $g_1 = b g_2$ for some $b \in \mathsf{k}$ and so $\deg(g) < \deg(f)$.

Remark 17.4 Let $f \in \mathsf{R}[x]$ as before and $\delta \in H_0(\{f = 0\}, \mathbb{Z})$. We define

$$\Omega_\delta := \{\omega \in \mathsf{H} \mid \int_\delta \omega = 0\}.$$

It is a left $\mathsf{R}[x]$-module by the usual composition of polynomials:

$$\omega \in \Omega_\delta,\ p \in \mathsf{R}[x] \Rightarrow p \circ \omega \in \Omega_\delta.$$

For d a prime number, f irreducible and δ simple, Theorem 17.1 implies that the integral $\int_\delta \omega,\ 0 \neq \omega \in \mathsf{H}$ never vanishes and so $\Omega_\delta = 0$.

We may want to formulate theorems like Theorem 17.1 for the collection of 0-forms Ω_δ. Since H is a freely generated R-module, its subset Ω_δ is finitly generated. Let $\omega_i,\ i = 1, 2, \ldots, s$ generate the R-module Ω_δ. Applying Theorem 17.1 to each ω_i we find varieties $\{g_i = 0\},\ i = 1, 2, \ldots, s$. Now the morphism

$$\alpha : \{f = 0\} \to Y := \{g_1 = 0\} \times \{g_2 = 0\} \times \cdots \times \{g_s = 0\},$$

$$\alpha = (\alpha_{\omega_1}, \alpha_{\omega_2}, \ldots, \alpha_{\omega_s})$$

has the property that $\alpha_*(\delta) = 0$ and Ω_δ is the pull-back of a set of 0-forms on Y.

Remark 17.5 Starting from a field k, polynomials f, ω over k and $\delta \in H_0(\{f = 0\}, \mathbb{Z})$, we may integrate and obtain an element $\int_\delta \omega$ in $\bar{\mathsf{k}}$. A natural question is that whether $\int_\delta \omega$ can be a non-zero element of k. The answer is no for $\mathrm{char}(\mathsf{k}) = 0$, an irreducible polynomial f and a simple cycle δ (this is a part of a general philosophy that by integrating over topological cycles either we get zero or some element beyond the base filed). The reason is as follows: If $0 \neq \omega(x_1) - \omega(x_2) = r \in \mathsf{k}$ then we replace f with g, where $g^k = \omega * f$ is as in Proposition 17.5, and assume that $\omega = x$ and so $x_1 - x_2 = r$. Since the action of the Galois group G_f of f on the roots of f is transitive, there is a sequence $x_1, x_2, x_3, \ldots$ of roots of f

such that $x_i - x_{i+1} = r$ and some $\sigma_i \in G_f$ sends x_i to x_{i+1} and x_{i+1} to x_{i+2} for all $i = 1,2,3,\ldots$. All x_i's are not distinct and at the end one get $nr = 0$ for some $n \in \mathbb{N}$. Since $\mathrm{char}(\mathsf{k}) = 0$ we obtain $r = 0$.

Remark 17.6 Let f be a polynomial over k without multiple roots. If $\int_\delta \omega = 0$ for some $0 \neq \omega \in \mathsf{H}$ and a cycle $0 \neq \delta \in H_0(L_f, \mathbb{Z})$ (not necessarily simple) then the Galois group G_f of f is not the full permutation group of the roots of f. The reason is as follows: For δ a simple cycle an argument similar to the one in the proof of Proposition 17.5 implies that there is a partition $\{x_1, x_2, \cdots, x_d\} = A_1 \cup A_2 \cup \cdots \cup A_s, s > 1$ of the roots of f such that the action of G_f on x_i's induces an action on each $A_j,\ j = 1,2,\ldots,s$. Therefore, G_f does not contain all possible permutations of x_i's. For an arbitrary cycle let us assume by contradiction that G_f contains all basic permutations $\sigma_{i,j}$: σ_{ij} permutes x_i and x_j and fixes other roots. We write $\delta = \sum_{i=1}^d a_i x_i,\ a_i \in \mathbb{Z}$ and let σ_{ij} to act on $\sum_{i=1}^d a_i \omega(x_i) = 0$. We conclude that either $a_i = a_j$ or $\int_{[x_i]-[x_j]} \omega = 0$. The second case is already treated and so we get $\delta = a_1 \sum_{i=1}^d x_i$ which is in contradiction with the definition of the 0-th reduced homology of f.

17.9 Zero locus of integrals

In this section we work with $\mathsf{R} = \mathbb{Z}_d[t]$ and $f = x^d + t_{d-1}x^{d-1} + \cdots + t_1 x + t_0$. For $\omega \in \mathsf{R}[x]$ we have defined the element

$$D_\omega := \sqrt{\frac{\Delta_{\omega * f}}{\Delta_f}} \in \mathsf{R}$$

in §17.6. The set $\{D_\omega = 0\}$ is the locus of parameters $t \in \mathbb{C}^d$ such that $\int_\delta \omega = 0$ for some simple cycle $\delta \in H_0(\{f = 0\}, \mathbb{Z})$.

Let $s = (s_1, s_2, \ldots, s_k)$ be a parameter, Ω be a R-submodule of H generated by $\omega_i \in \mathsf{R}[x],\ i = 1,2,\ldots,k$ and $\omega := s_1\omega_1 + s_2\omega_2 + \cdots + s_k\omega_k$. We write the polynomial expansion of D_ω in the variable s

$$D_\omega = \sum_\alpha D_\alpha \cdot s^\alpha,\ D_\alpha \in \mathsf{R},$$

where α runs through

$$S := \{\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}_0^k,\ \sum_{i=1}^k \alpha_i = \frac{d(d-1)}{2}\}.$$

Let us define the ideal

$$I_\Omega := \langle D_\alpha \mid \alpha \in S \rangle \subset \mathsf{R}.$$

We conclude that

Proposition 17.8 *The zero set of the ideal I_Ω is the locus of parameters $t \in \mathbb{C}^d$ such that there exists a simple cycle $\delta \in H_0(\{f=0\},\mathbb{Z})$ such that $\int_\delta \Omega = 0$.*

The algebraic group $\mathbb{C}^*$ acts on $\mathbb{C}^d$ in the following way

$$k \bullet t = (k^d t_0, k^{d-1} t_1, \ldots, k t_{d-1})$$

and $x, x^2, \ldots, x^{d-1}$ are eigen 0-forms under this action. This implies that if Ω is generated by a subset of $\{x, x^2, \ldots, x^{d-1}\}$ then $Z(I_\omega)$ is invariant under the action of $\mathbb{C}^*$. Note also that $0 \in Z(I_\Omega)$ and each irredcible component of $Z(I_\Omega)$ passes through 0.

Example 17.2 For the case $f = x^3 + t_2 x^3 + t_1 x + t_0$ the integral $\int_\delta x^2$ is zero when

$$D_{x^2} = t_0 - t_1 t_2 = 0.$$

In such parameters, we have the contraction

$$\alpha : \{f=0\} \to \{x + t_1 = 0\},\ \alpha(x) = x^2.$$

Example 17.3 For $d = 4$ we have the following table:

$f = x^4 + t_3 x^3 + t_2 x^2 + t_1 x + t_0$

ω	D_ω
x^2	$-t_1^2 + t_1 t_2 t_3 - t_0 t_3^2$
x^3	$t_0^3 - 2t_0^2 t_2^2 - t_1^2 t_2^3 + t_0 t_2^4 - 3t_0^2 t_1 t_3 + t_0 t_1 t_2^2 t_3 +$ $3t_0 t_1^2 t_3^2 + t_1^2 t_2^2 t_3^2 - t_0 t_2^3 t_3^2 - t_1^3 t_3^3$

The ideal $I_{\langle x^3 - x, x^2 \rangle}$ is generated by 7 polynomials $p_i,\ i = 0, 2, \ldots, 6$ which can be calculated by a computer. Note that for $f = x^4 - 1$ we have $\int_{[1]-[-1]} \{x^2, x^3 - x\} = 0$ and so $(0,0,0,-1) \in Z(I_{\langle x^3 - x, x \rangle}) = Z(p_0, p_1 \ldots, p_6)$.

17.10 The connection of H

In this section we work with $\mathsf{R} = \mathbb{Z}_d[t]$ and $f = x^d + t_{d-1} x^{d-1} + \cdots + t_1 x + t_0$. We construct a connection on the R-module H'. A similar construction for H'' can be done easily and is left to the reader. A zero $x(t)$ of f can be seen as a holomorphic multi valued function on the affine space $\mathbb{C}^d \backslash \{\Delta = 0\}$, where $\Delta = \Delta_f$ is the discriminant of f. In particular, it is common to say that $\delta = \delta_t = x_1(t) - x_2(t) \in H_0(\{f=0\},\mathbb{Z})$ is a continuous family of simple cycles.

Consider the differential map

$$d : \mathsf{R} \to \Omega^1_{\mathbb{U}_0},$$

where

$$\Omega^1_{\mathbb{U}_0} = \Omega^1_{\mathsf{R}} := \{\sum_{i=0}^{d-1} p_i dt_i \mid p_i \in \mathsf{R}\}$$

is the set of differential 1-forms of R (for simplicity we have written Ω^1_{R} instead of $\Omega^1_{\mathsf{R}/\mathbb{Z}_d}$, see [Har77], p.17). The set of vector fields is given by

$$\mathscr{D} = \mathscr{D}_{\mathbb{U}_0} := \{\sum_{i=0}^{d-1} p_i \frac{\partial}{\partial t_i} \mid p_i \in \mathsf{R}\}$$

and we have the canonical R-bilinear map

$$\mathscr{D}_{\mathbb{U}_0} \times \Omega^1_{\mathbb{U}_0} \to \mathsf{R},\ (\partial, \eta) \mapsto \eta(\partial)$$

defined by the rule $dt_i(\frac{\partial}{\partial t_j}) = 1$ if $i = j$ and $= 0$ otherwise. One can look R as a (left) $\mathscr{D}$-module (differential module) in the following way:

$$\partial p := dp(\partial),\ \partial \in \mathscr{D},\ p \in \mathsf{R}.$$

The differential $d : \mathsf{R} \to \Omega^1_{\mathbb{U}_0}$ extends to $d : \mathsf{k} \to \Omega_{\mathsf{k}}$. We can consider k as a (left) $\mathscr{D}_{\mathbb{U}_0}$-module in a canonical way. Let $x(t)$ be a root of the polynomial f, $f(x(t)) = 0$. Then

$$d(x(t)) \cdot f'(x(t)) + (dt_{d-1}) \cdot x^{d-1} + (dt_{d-2}) \cdot x^{d-2} + \cdots + dt_0 = 0. \tag{17.10}$$

According to the third part of Proposition 17.2, there exists polynomial $p \in \mathsf{R}[x]$ such that

$$\Delta = p \cdot f' \text{ in } \mathsf{H}'. \tag{17.11}$$

This combined with (17.10) suggests to define the connection

$$\nabla : \mathsf{H}' \to \Omega_T \otimes_{\mathsf{R}} \mathsf{H}'$$

$$\omega \mapsto \frac{-1}{\Delta} \cdot (dt_{d-1} \otimes x^{d-1} + dt_{d-2} \otimes x^{d-2} + \cdots + dt_0 \otimes 1)\omega' \cdot p,$$

where

$$T := \mathbb{U}_0 \backslash \{\Delta = 0\}.$$

Our motivation of the definition of ∇ is the following identity:

$$d(\int_\delta \omega) = \int_\delta \nabla \omega,\ \delta \in H_0(\{f = 0\}, \mathbb{Z}), \tag{17.12}$$

which follows from (17.10) and (17.11). The operator ∇ satisfies the Leibniz rule, i.e.

$$\nabla(p \cdot \omega) = p \cdot \nabla(\omega) + \omega \otimes dp,\ p \in \mathsf{R},\ \omega \in \mathsf{H}'$$

and so it is a connection on the module H'. It defines the operators

$$\nabla_i = \nabla : \Omega_T^i \otimes_{\mathsf{R}} \mathsf{H}' \to \Omega_T^{i+1} \otimes_{\mathsf{R}} \mathsf{H}'.$$

If there is no danger of confusion we will use the symbol ∇ for these operators too. The connection ∇ is an integrable connection, i.e.

$$\nabla \circ \nabla = 0.$$

Using the connection ∇, the Brieskorn module H'_Δ turns into a $\mathscr{D}$-module. The notion of integration extends to the elements of $\Omega_T^i \otimes_T \mathsf{H}'$ in a trivial way:

$$\int_\delta \eta \otimes \omega = \eta \cdot (\int_\delta \omega) \in \Omega_{\mathsf{k}}^i,$$

$$\omega \in \mathsf{H}',\ \eta \in \Omega_T^i,\ \delta \in H_0(\{f=0\},\mathbb{Z}).$$

The construction of the connection ∇ for the polynomial $f = t_d x^d + \cdots + t_0 \in \mathbb{Z}_d[t, \frac{1}{t_d}]$ is similar. Every element of H defines a section of the cohomology bundle of f. By (17.12) every continuous family of cycles is a locally constant section of the homology bundle, which means that ∇ coincides with the Gauss-Manin connection. Therefore, we will call ∇ the Gauss-Manin connection.

Remark 17.7 The construction of ∇ works essentially for the general ring R. If $\mathsf{R} = \mathbb{Z}[t]$ then apart from derivations with respect to the parameters in t we have the map $\delta : \mathbb{Z} \to \mathbb{Z},\ a \mapsto \frac{a-a^p}{p}$ p a prime number, which is called the Fermat quotient operator and can be considered as the derivation of integers because it satisfies $\delta(ab) = a\delta(b) + b\delta(a) \mod p,\ a,b \in \mathbb{Z}$. For more information the reader is referred to [Bui05].

Note also that if $\mathsf{R} = \mathbb{Q}[e] \subset \mathbb{C}$ is a transcendent extension of $\mathbb{Q}$, where e is a collection of algebraically independent transcendent numbers, we have the derivation with respect to each transcendent number and so we can define again the connection ∇.

Example 17.4 Let[ii]

$$f = 4x^3 - g_2 x - g_3.$$

A straightforward and elementary computation implies: In the Brieskorn module H' the following identity holds

$$\nabla \begin{pmatrix} x \\ x^2 \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} \frac{d\Delta}{6} & -3\delta \\ -\frac{g_2\delta}{2} & \frac{d\Delta}{3} \end{pmatrix} \begin{pmatrix} x \\ x^2 \end{pmatrix},$$

where

$$\delta = 3g_3 dg_2 - 2g_2 dg_3,\ \Delta = g_2^3 - 27g_3^2.$$

Example 17.5 Let

$$f = t_3 x^3 + t_2 x^2 + t_1 x + t_0.$$

[ii] The notation g_i instead of t_i has historical reasons. They appear as Eisenstein series in the Weierstrass uniformization theorem for the family of elliptic curve $y^2 - f(x) = 0$.

The Gauss-Manin connection in the basis $\omega=[dx,xdx]^{\mathsf{tr}}$ of H'' is given by

$$\nabla\omega=\frac{1}{\tilde{\Delta}}(\sum_{i=0}^{3}A_i dt_i)\otimes\omega,$$

where

$$\tilde{\Delta}=27t_0^2t_3^2-18t_0t_1t_2t_3+4t_0t_2^3+4t_1^3t_3-t_1^2t_2^2$$

and

$$A_0=\begin{pmatrix}-18t_0t_3^2+8t_1t_2t_3-2t_2^3 & 6t_1t_3^2-2t_2^2t_3\\ 3t_0t_2t_3-4t_1^2t_3+t_1t_2^2 & -9t_0t_3^2+t_1t_2t_3\end{pmatrix},$$

$$A_1=\begin{pmatrix}3t_0t_2t_3-4t_1^2t_3+t_1t_2^2 & -9t_0t_3^2+t_1t_2t_3\\ 6t_0t_1t_3-2t_0t_2^2 & 6t_0t_2t_3-2t_1^2t_3\end{pmatrix},$$

$$A_2=\begin{pmatrix}6t_0t_1t_3-2t_0t_2^2 & 6t_0t_2t_3-2t_1^2t_3\\ -9t_0^2t_3+t_0t_1t_2 & 3t_0t_1t_3-4t_0t_2^2+t_1^2t_2\end{pmatrix},$$

$$A_3=\begin{pmatrix}-9t_0^2t_3+t_0t_1t_2 & 3t_0t_1t_3-4t_0t_2^2+t_1^2t_2\\ 6t_0^2t_2-2t_0t_1^2 & -18t_0^2t_3+8t_0t_1t_2-2t_1^3\end{pmatrix}.$$

17.11 Period map

In this section we work with $\mathsf{R}=\mathbb{Z}[t]$ and $f=x^d+t_{d-1}x^{d-1}+\cdots+t_1x+t_0$. Let $\omega=(\omega_1,\omega_2,\ldots,\omega_{d-1})^{\mathsf{tr}}$ be the canonical basis of H. In this basis we can write the matrix of the connection ∇[iii]:

$$\nabla\omega=A\otimes\omega,\ A\in\mathrm{Mat}^{\mu\times\mu}(\Omega_T^1). \tag{17.13}$$

We will call A the Gauss-Manin connection matrix of f in the basis ω. A fundamental matrix of solutions for the linear differential equation

$$dY=A\cdot Y$$

(with Y unknown) is given by $Y=\mathbf{P}^{\mathsf{tr}}$, where

$$\mathbf{P}=[\int_\delta\omega^{\mathsf{tr}}]=\begin{pmatrix}\int_{\delta_1}\omega_1 & \int_{\delta_1}\omega_2 & \cdots & \int_{\delta_1}\omega_{d-1}\\ \int_{\delta_2}\omega_1 & \int_{\delta_2}\omega_2 & \cdots & \int_{\delta_2}\omega_{d-1}\\ \vdots & \vdots & \vdots & \vdots\\ \int_{\delta_{d-1}}\omega_1 & \int_{\delta_{d-1}}\omega_2 & \cdots & \int_{\delta_{d-1}}\omega_{d-1}\end{pmatrix}$$

[iii] Note that in the zero dimensional case the entries of the Gauss-Manin connection matrix are in the localization of $\mathbb{Z}[t]$ over Δ. In higher dimensions to obtain such a matrix we will need that every integer to be invertible in R.

is the period matrix and $\delta=(\delta_1,\delta_2,\dots,\delta_{d-1})^{\mathrm{tr}}$ is a basis of $H_0(\{f=0\},\mathbb{Z})$. This follows form the equalities (17.12) and (17.13). We look at $\delta_i=\delta_{i,t}$ as continuous family of cycles. In this way $\mathbf{P}=\mathbf{P}(t)$ is a multi $\mathrm{Mat}^{d\times d}(\mathbb{C})$ valued holomorphic function defined in $\mathbb{C}^d\setminus\{\Delta=0\}$ and we call it also a period map.

Example 17.6 For a natural number d let

$$\mathbf{P}_d:=\frac{1}{d}\left(\zeta_d^{(\beta_1+1)(\beta_1'+1)}-\zeta_d^{\beta_1(\beta_1'+1)}\right)_{0\le\beta_1,\beta_1'\le d-2}=$$

$$\frac{1}{d}\begin{pmatrix}\zeta_d-1 & \zeta_d^2-1 & \cdots & \zeta_d^{d-1}-1\\ \zeta_d^2-\zeta_d & \zeta_d^4-\zeta_d^2 & \cdots & \zeta_d^{2(d-1)}-\zeta_d^{d-1}\\ \zeta_d^3-\zeta_d^2 & \zeta_d^6-\zeta_d^4 & \cdots & \zeta_d^{3(d-1)}-\zeta_d^{2(d-1)}\\ \vdots & \vdots & \vdots & \vdots\\ \zeta_d^{d-1}-\zeta_d^{d-2} & \zeta_d^{(d-1)2}-\zeta_d^{(d-2)2} & \cdots & \zeta_d^{(d-1)(d-1)}-\zeta_d^{(d-2)(d-1)}\end{pmatrix}.$$

According to the equality (17.7), the period matrix of $f=x^d-t$ associated to $\omega=(dx,xdx,\dots,x^{d-1}dx)$ and $\delta=(x_2-x_1,x_3-x_2,\dots,x_d-x_{d-1})$ is given by $\frac{1}{t}\mathbf{P}_d$.

Example 17.7 Let us take $f=x^2+t_1x+t_0$. Then $\Delta=4t_0-t_1^2$. We take the cycle $\delta=[-\frac{1}{2}t_1+\sqrt{\frac{1}{4}t_1^2-t_0}]-[-\frac{1}{2}t_1-\sqrt{\frac{1}{4}t_1^2-t_0}]$ and we have

$$\int_\delta x=\sqrt{-\Delta},$$

$$\nabla([x])=\frac{1}{\Delta}(t_1dt_1-\frac{1}{2}dt_0)\otimes[x]=\frac{1}{2}\frac{d\Delta}{\Delta}\otimes[x].$$

17.12 Monodromy group

We continue the notations of the previous section. For a fixed $p\in T:=\mathbb{U}_0\backslash\{\Delta=0\}$, we have a canonical action

$$\pi_1(T,p)\times H_0(L_p,\mathbb{Z})\to H_0(L_p,\mathbb{Z})$$

of the homotopy group $\pi_1(T,p)$ on the $\mathbb{Z}$-module $H_0(L_p,\mathbb{Z})$, defined by the continuation of the roots of f along a path in $\pi_1(T,p)$. The image $\Gamma_{\mathbb{Z}}$ of $\pi_1(T,p)$ in $\mathrm{Aut}_{\mathbb{Z}}(H_0(L_p,\mathbb{Z}))$ is usually called the monodromy group. To calculate it we proceed as follows:

The polynomial $f=(x-1)(x-2)\cdots(x-d)$ has $\mu:=d-1$ distinct critical values, namely $c_1,c_2,\dots,c_\mu$. We consider f as a function from $\mathbb{C}$ to itself and take a distinguished set of paths $\lambda_i,\ i=1,2,\dots,\mu$ in $\mathbb{C}$ which connects $b:=\sqrt{-1}$ to the critical values of f (see Figure 17.1). This mean that the paths λ_i do not intersect each other except at b and the order $\lambda_1,\lambda_2,\dots,\lambda_\mu$ around b is anti-clockwise. The

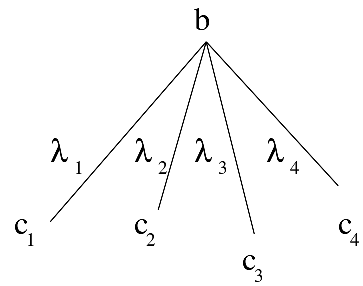

Fig. 17.1 A distinguished set of paths

cycle $\delta_i = x_{i+1} - x_i, i = 1, 2, \ldots, \mu$ vanishes along the path λ_i and $\delta = (\delta_1, \delta_2, \ldots, \delta_\mu)$ is called a distinguished set of vanishing cycles in $H_0(L_f, \mathbb{Z})$. Now, the monodromy around the critical value c_i is given by

$$\delta_j \mapsto \begin{cases} \delta_j & j \neq i-1, i, i+1 \\ -\delta_j & j = i \\ \delta_j + \delta_i & j = i-1, i+1 \end{cases}.$$

For example, the monodromy group in the canonical basis $\delta_i, i = 1, 2, \ldots, d-1,\ d = 5$ is generated by the matrices:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Let Ψ_0 be the intersection matrix in the basis δ. It is given in (17.2). The monodromy group keeps the intersection form in $H_0(L_p, \mathbb{Z})$. In other words:

$$\Gamma_{\mathbb{Z}} \subset \{A \in \mathrm{GL}(\mu, \mathbb{Z}) \mid A\Psi_0 A^{\mathsf{tr}} = \Psi_0\}. \tag{17.14}$$

Example 17.8 Consider the case $d = 3$. We choose the basis $\delta_1 = x_2 - x_1,\ \delta_2 = x_3 - x_2$ for $H_0(L_f, \mathbb{Z})$. In this basis the intersection matrix is given by

$$\Psi_0 := \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

There are two critical points for f for which the monodromy is given by:

$$\delta_1 \mapsto -\delta_1,\ \delta_2 \mapsto \delta_2 + \delta_1,$$

$$\delta_2 \mapsto -\delta_2,\ \delta_1 \mapsto \delta_2 + \delta_1.$$

Let $g_1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$, $g_2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$. The monodromy group satisfies the equalities:

$$\begin{aligned} \Gamma_{\mathbb{Z}} &= \langle g_1, g_2 \mid g_1^2 = g_2^3 = I, g_1g_2g_1 = g_2g_1g_2 \rangle \\ &= \{I, g_1, g_2, g_1g_2g_1, g_2g_1, g_1g_2\} \\ &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \right. \\ &\qquad \left. \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \right\}. \end{aligned}$$

For this example (17.14) turns out to be an equality (one obtains equations like $(a-b)^2 + a^2 + b^2 = 2$ for the entries of the matrix A and the calculation is explicit).

Remark 17.8 For a polynomial f over a field k, the Galois group $\mathrm{Gal}(\bar{\mathsf{k}}/\mathsf{k})$ acts on $H_0(L_f, \mathbb{Z})$ in a canonical way:

$$\sigma \cdot \delta = \sigma(x_i) - \sigma(x_j),$$

$$\sigma \in \mathrm{Gal}(\bar{\mathsf{k}}/\mathsf{k}),\ \delta = x_i - x_j \in H_0(L_f, \mathbb{Z}).$$

We denote the image of $\mathrm{Gal}(\bar{\mathsf{k}}/\mathsf{k})$ in $\mathrm{Aut}_{\mathbb{Z}}(H_0(L_f, \mathbb{Z}))$ by Γ_f. By definition we have:

$$\sigma(\int_\delta \omega) = \int_{\sigma\cdot\delta} \omega,\ \omega \in \mathsf{H},\ \delta \in H_0(L_f, \mathbb{Z}).$$

Therefore, if for some cycle δ we have $\int_\delta \omega = 0$ then $\int_{\sigma\cdot\delta} = 0$.

For $\mathsf{k} = \mathbb{Q}(t)$ we have the inclusion $\Gamma_{\mathbb{Z}} \subset \Gamma_f$ obtained in the following way: The action of a homotopy class $\gamma \in \pi_1(T, b)$ on the roots of f is obtained by analytic continuation of the roots of f along γ and so this action extends as an automorphism of the splitting field of f. Any such automorphism extends to an element of $\mathrm{Gal}(\bar{\mathsf{k}}/\mathsf{k})$.

17.13 Modular foliations

In this section we take $f = x^d + t_{d-1}x^{d-1} + \cdots + t_1x + t_0$ and $\mathsf{R} = \mathbb{Q}[t]$.

Definition 17.9 A modular foliation $\mathscr{F}_\eta$ associated to $\eta \in \mathsf{H}$ is a foliation in T given locally by the constant locus of the integrals $\int_{\delta_t} \eta, \delta_t \in H_1(L_t, \mathbb{Z})$, i.e. along the leaves of $\mathscr{F}_\eta$ the integral $\int_{\delta_t} \eta$ as a holomorphic function in t is constant.

By definition the period sets $\mathsf{P}(L_t, \eta)$ associated to to the points of a leaf of a modular foliation are the same. The algebraic description of a modular foliation $\mathscr{F}_\eta$ is as follows: We write $\nabla\eta = [\eta_1, \eta_2, \cdots, \eta_\mu]\omega,\ \eta_i \in \Omega_T^1,\ i = 1, 2, \ldots, \mu$, where

$\omega = (\omega_1, \omega_2, \ldots, \omega_\mu)^{\mathsf{tr}}$ is a basis of H. It is left to the reader to verify that:

$$\mathscr{F}_\eta : \eta_1 = 0, \eta_2 = 0, \cdots, \eta_\mu = 0.$$

Therefore, a modular foliation extends to an algebraic singular foliation in $\mathbb{U}_0$. The singular set of $\mathscr{F}_\eta$ is defined as follows:

$$\mathrm{Sing}(\mathscr{F}_\eta) := \{a \in \mathbb{U}_0 \mid \eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_\mu \mid_{\{a\}} = 0\}.$$

In practice one does as follows: Let us write $\eta = p\omega,\ p = (p_1, p_2, \ldots, p_\mu) \in \mathsf{R}^\mu$. If $\nabla\omega = A\omega$ is the Gauss-Manin connection of the polynomial f with respect to the basis ω then

$$\nabla(\eta) = \nabla(p\omega) = (dp + pA)\omega$$

and so

$$\mathscr{F}_\eta : dp_j + \sum_{i=1}^{\mu} p_i \omega_{ij} = 0,\ j = 1, 2, \ldots, \mu. \tag{17.15}$$

where $A = [\omega_{ij}]_{1\le i,j\le\mu}$. In particular, the foliation $\mathscr{F}_{\omega_i}$ is given by the differential forms of the i-th row of A.

Since $\mu = d-1$ differential forms define a modular foliation $\mathscr{F}_\eta$ in $\mathbb{U}_0$ with $\dim(\mathbb{U}_0) = d$, there is a vector-field $X_\eta = \sum_{i=0}^{d-1} p_i \frac{\partial}{\partial t_i},\ p_i \in \mathsf{R}$, where p_i's have no common factors, which is tangent to the leaves of $\mathscr{F}_\eta$. For many examples, it is possible to prove that $\mathscr{F}_\eta$ is a foliation by curves and so $\mathscr{F}_\eta$ is given by the solutions of the vector field X_η.

Example 17.10 For the polynomial $f = x^d + t_{d-1}x^{d-1} + \cdots + t_1 x + t_0$, a leaf of the foliation $\mathscr{F}_x, x \in \mathsf{H}'$ is given by the coefficients of x^i's in $(x+s)^d + a_{d-2}(x+s)^{d-2} + \cdots + a_1(x+s) + a_0$, where a_i's are some constant complex numbers and s is a parameter. In fact, $\mathscr{F}_x$ is given by the solutions of the vector field:

$$t_1 \frac{\partial}{\partial t_0} + 2t_2 \frac{\partial}{\partial t_1} + 3t_3 \frac{\partial}{\partial t_2} + \cdots + (d-1)t_{d-1} \frac{\partial}{\partial t_{d-2}} + d \frac{\partial}{\partial t_{d-1}}.$$

Example 17.11 For $x^3 + t_2x^2 + t_1x + t_0$ we have

$$\mathscr{F}_{x^2} : (-t_1)\frac{\partial}{\partial t_2} + (-t_2t_1 + 3t_0)\frac{\partial}{\partial t_1} + (2t_2t_0 - t_1^2)\frac{\partial}{\partial t_0}, \tag{17.16}$$

$\mathrm{Sing}(\mathscr{F}_{x^2}) = \{t \in \mathbb{C}^3 \mid t_1 = t_0 = 0\}$ and and the zero locus $\int x^2 = 0$ is given by $t_0 - t_2t_1 = 0$ which is $\mathscr{F}_{x^2}$-invariant.

Example 17.12 For $x^4 + t_3x^3 + t_2x^2 + t_1x + t_0$ we have

$$\mathscr{F}_{x^2} : (-2t_0t_2 + t_1^2)\frac{\partial}{\partial t_0} + (-3t_0t_3 + t_1t_2)\frac{\partial}{\partial t_1} + (-4t_0 + t_1t_3)\frac{\partial}{\partial t_2} + t_1\frac{\partial}{\partial t_3}$$

Example 17.13 We can take an arbitrary polynomial in some function field and define a modular foliation. For example, let $\mathsf{R} = \mathbb{Q}[s_1, s_2, \cdots, s_{d-1}, t]$, $f = g - t, g \in \mathbb{Q}[x]$, $\deg(g) = d$ and $\omega = s_1 x + s_2 x_2 + \cdots + s_{d-1} x^{d-1}$. The foliation $\mathscr{F}_\omega$ in $\mathbb{U}_0$ is given by the vector field:

$$p_1 \frac{\partial}{\partial s_1} + p_2 \frac{\partial}{\partial s_2} + \cdots + p_{d-1} \frac{\partial}{\partial s_{d-1}} - \frac{\partial}{\partial t},$$

where $\nabla_{\frac{\partial}{\partial t}} \omega = p_1 x + p_2 x^2 + \cdots + p_{d-1} x^{d-1}$.

Remark 17.9 For $\mathsf{R} = \mathbb{Q}[t]$, $f = x^d + t_{d-1} x^{d-1} + \cdots + t_1 x + t_0$ and $\eta \in \mathsf{H}$, $\mathscr{F}_\eta$ is defined over $\mathbb{Q}$ and so the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the leaf space of $\mathscr{F}_\eta$. This implies that for a $\mathbb{Q}$-rational point $p \in \mathbb{U}_0 \backslash \mathrm{Sing}(\mathscr{F}_\eta)$ the closure $\bar{L}_p$ of the leaf of $\mathscr{F}_\eta$ through p, which is an affine subvariety of $\mathbb{U}_0$, is defined over $\mathbb{Q}$. The reason to this fact is as follows: Since p is $\mathbb{Q}$-rational, $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ sends p to p and so it sends $\bar{L}_p$ to another leaf of $\mathscr{F}_\eta$ which crosses p. By our choice, p is not a singular point of $\mathscr{F}_\eta$ and so there is a unique leaf of $\mathscr{F}_\eta$ through p. This implies that $\bar{L}_p$ is mapped to itself under the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and so $\bar{L}_p$ is defined over $\mathbb{Q}$.

17.14 Period domain and the inverse of the period map

Let $f = x^d + t_{d-1} x^{d-1} + \cdots + t_1 x + t_0$, $\mathsf{R} = \mathbb{Q}[t]$ and $\Pi := \mathrm{GL}(\mu, \mathbb{C})$. The period map defines

$$\mathbf{P} : T \to \mathscr{L} := \Gamma_{\mathbb{Z}} \backslash \Pi$$

which we call it again the period map and use the same notation $\mathbf{P}$ as before. Here $\Gamma_{\mathbb{Z}}$ is the monodromy group. Without the danger of confusion, both complex manifolds $\mathscr{L}$ and Π are also called the period domain. The complex manifold $\mathscr{L}$ is of dimension $(d-1)^2$ and the affine variety $\mathbb{U}_0$ is of dimension d. Therefore, the image of the period map is of codimension $\geq (d-1)^2 - d$. One of the problems which we will face in higher dimensions is to determine the ideal of the of image of $\mathbf{P}$. Every holomorphic global function in $\mathscr{L}$ which vanishes on the image of $\mathbf{P}$ gives us a relation between the periods of f. To begin with, we have to construct holomorphic functions on $\mathscr{L}$.

Let $\mathscr{O}_\Pi$ (resp. $\mathscr{O}_{\mathscr{L}}$) be the space of global holomorphic functions in Π (resp. $\mathscr{L}$). We want to construct some elements of $\mathscr{O}_{\mathscr{L}}$. The relation (17.14) implies that $\det(A) = \pm 1$, $A \in \Gamma_{\mathbb{Z}}$ and so the function $det(x)^2$ is a one valued function on $\mathscr{L}$. There are two methods of constructing elements in $\mathscr{O}_{\mathscr{L}}$. The first method is as follows:

Using the relation (17.14), one can see easily that for $[x] \in \Pi$ the entries of $g = x^{\mathsf{tr}} \Psi_0^{-\mathsf{tr}} x$ does not depend depend on the choice of x in the class $[x]$, where Ψ_0 is the intersection matrix in (17.2). Therefore, the entries of g are global holomorphic functions in $\mathscr{L}$. In the same way, the entries of the matrix $\tilde{g} = x^{\mathsf{tr}} \Psi_0^{-\mathsf{tr}} \bar{x}$ are real analytic functions on $\mathscr{L}$.

The second method for producing holomorphic functions on $\mathscr{L}$ is as follows: Define

$$\tilde{\cdot} : \mathscr{O}_\Pi \to \mathscr{O}_\Pi,\ \tilde{p}(x) := \sum_{A \in \Gamma_\mathbb{Z}} p(Ax).$$

The new function $\tilde{p}$ is $\Gamma_\mathbb{Z}$ invariant and hence induce an element in $\mathscr{O}_\mathscr{L}$, which we denote it again with $\tilde{p}$. Note that the sum is finite and so we do not have the convergence problem. For higher dimensional abelian integrals the monodromy group will be an infinite group and so one has to verify the convergency.

Example 17.14 For Example 17.8, the functions on $\mathscr{L}$ obtained by the first method are given bellow:

$$x^{\mathsf{tr}} \Psi_0^{-\mathsf{tr}} x =$$

$$\frac{1}{3}\begin{pmatrix} 2x_1{}^2 + 2x_1x_3 + 2x_3{}^2 & 2x_1x_2 + x_1x_4 + x_2x_3 + 2x_3x_4 \\ 2x_1x_2 + x_1x_4 + x_2x_3 + 2x_3x_4 & 2x_2{}^2 + 2x_2x_4 + 2x_4{}^2 \end{pmatrix},$$

$$x^{\mathsf{tr}} \Psi_0^{-\mathsf{tr}} \bar{x} =$$

$$\frac{1}{3}\begin{pmatrix} 2x_1\bar{x}_1 + x_1\bar{x}_3 + x_3\bar{x}_1 + 2x_3\bar{x}_3 & 2x_1\bar{x}_2 + x_1\bar{x}_4 + x_3\bar{x}_2 + 2x_3\bar{x}_4 \\ 2x_2\bar{x}_1 + x_2\bar{x}_3 + x_4\bar{x}_1 + 2x_4\bar{x}_3 & 2x_2\bar{x}_2 + x_2\bar{x}_4 + x_4\bar{x}_2 + 2x_4\bar{x}_4 \end{pmatrix}.$$

Using the second method we have produced the following table:

p	$\tilde{p}$
x_1	0
x_1x_2	$4x_1x_2 + 2x_1x_4 + 2x_2x_3 + 4x_3x_4$
x_1x_3	$-2x_1^2 - 2x_1x_3 - 2x_3^2$
x_1x_4	$-2x_1x_2 - x_1x_4 - x_2x_3 - 2x_3x_4$

Since $d\mathbf{P} = \mathbf{P}A^{\mathsf{tr}}$, where A is the Gauss-Manin connection of f in the basis ω, the fact that for all $t \in T$ and v in the tangent space of T at t, $\det(A(t)(v)) \neq 0$ (equivalently for all $t \in T$ the matrices $A_i(t)$, where $A = \frac{1}{\Delta}\sum_{i=0}^{d-1} A_i dt_i$, are $\mathbb{C}$-linear independent), implies that $\mathbf{P}$ is a local biholomorphism. This can be regarded as the infinitesimal Torelli problem in dimension zero. The global Torelli problem is whether $\mathbf{P}$ is a biholomorphism between T and its image

Example 17.15 Let us consider the situation in Example (17.5). For a linear differential equation $Y' = AY$, we have $\det(Y)' = det(A)Y$. We use this fact and conclude that

$$\det(\mathbf{P}) = c \cdot \tilde{\Delta}^{\frac{-1}{2}}$$

for some constant c. The period map $\mathbf{P}$ in the basis $(dx, xdx)^{\mathsf{tr}}$ is a local biholomorphism. Let us denote the image of $\mathbf{P}$ by $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$. We look at the period map as a function sending $t = (t_0, t_1, t_2, t_3)$ to (x_1, x_2, x_3, x_4) and calculate the derivative of its (local) inverse F:

$$DF(x) = \frac{1}{\det(x)} \cdot$$

$$\begin{pmatrix} -2t_0x_4 & -t_0x_3-t_1x_4 & 2t_0x_2 & t_0x_1+t_1x_2 \\ 3t_0x_3-t_1x_4 & -2t_2x_4 & -3t_0x_1+t_1x_2 & 2t_2x_2 \\ 2t_1x_3 & t_2x_3-3t_3x_4 & -2t_1x_1 & -t_2x_1+3t_3x_2 \\ t_2x_3+t_3x_4 & 2t_3x_3 & -t_2x_1-t_3x_2 & -2t_3x_1 \end{pmatrix}.$$

It is obtained from the equality $d\mathbf{P} = \mathbf{P}A^{\mathrm{tr}}$. If $\mathbf{P} : T \to \mathscr{L}$ is a global biholomorphism then we have four holomorphic functions $F_i, i = 0,1,2,3$ on $\mathscr{L}$, where $F = \mathbf{P}^{-1} = (F_0, F_1, F_2, F_3)$. They satisfy the differential equation obtained by the above matrix (replace t_i with F_i).

The period map sends a modular foliation

$$\mathscr{F}_\eta,\ \eta = \sum_{i=0}^{\mu} s_i\omega_i,\ s_i \in \mathbb{Q}$$

to trivial foliations in $\mathscr{L}$ in the following sense: We define the following sub-algebra of $\mathscr{O}_\Pi$:

$$\mathscr{O}_\eta := \mathbb{C}[s_1x_{i,1} + s_2x_{i,2} + \cdots + s_\mu x_{i,\mu} \mid i = 1,2,\ldots,\mu] \subset \mathscr{O}_\Pi.$$

The locus of points $x \in \mathscr{L}$ in which $\mathscr{O}_\eta$ is constant is a foliation in $\mathscr{L}$. Pulling back this foliation by the period map, we obtain the foliation $\mathscr{F}_\eta$ in T.

Chapter 18
On a problem posed by Andreas Braun

This chapter started from email communications with Andreas Braun who was looking for certain Hodge cycles in the Fermat sextic fourfold. The first email goes back 21/06/2020, but I started to think seriously on this problem starting from 05/01/2021.

18.1 Looking for a Hodge cycle

18.1 (A. Braun, 21 July 2020) In physics, one incarnation of Hodge cycles concerns the question of the dimensionality of the Hodge locus. In particular, one is interested in Hodge cycles for which the Hodge locus has the maximal codimension, which I believe you called 'general Hodge cycles'. If I remember correctly, you make the remark that you knew of no general method to find such Hodge cycles beyond trying 'random' combinations of algebraic cycles. I did this exercise for the Fermat sextic fourfold, and can indeed find many instances of linear combinations of 23 (or more) of the 'linear algebraic cycles' that do the job. Thats the absolute lower bound I think, as each one of them fixes 19 complex structure moduli, so that we need at least 23 to get up to 426 from the sub-additivity property.

In physics, one is not just interested in a general Hodge cycle, but also its square (under the canoncial inner form given by the cup product). This number cannot exceed a certain bound in physics, which is about 200 for the Fermat sextic fourfold. Working through the choices of general Hodge cycles I found, I find that their square is consistently about double of what is permitted by physics. This is somewhat in line with an intuition I had from other examples, which tells me that this 'physics constraint on general Hodge cycles' is very tight and will rule out a generic choice of a Hodge cycle such as the ones I found by experimenting. As this point appears to be somewhat underappreciated by physicists (I think), it would be amazing if anything more precise was known. Do you know of any systematic results, in particular for the Fermat sextic fourfold? To be more precise, the question that I am most interested in is: given a general Hodge cycle delta on the Fermat sextic fourfold, what is the lower bound on $\int_X \delta\wedge\delta$?

18.2 (A. Braun, 23 July 2020) Take X the Fermat sextic fourfold and consider the lattice $H^4(X,\mathbb{Z})\cap H^{2,2}(X)$ (or $\mathrm{Hodge}_4(X,\mathbb{Z})_{\mathrm{alg}}$ as a variation). For every lattice element δ there is an associated matrix $[\mathsf{p}_{i+j}]$ which has a rank $\xi(\delta)$.[i] At the same time, we can compute δ^2, e.g. for the linear cycles we have $\delta^2=21$, $\xi(\delta)=19$. If you add two linear cycles δ_1 and δ_2, you notice that $\xi(\delta_1+\delta_2)$ only depends on $\delta_1\cdot\delta_2$, i.e. on $(\delta_1+\delta_2)^2$. For two linear algebraic cycles, the table is
$\delta_1\cdot\delta_2$, $\xi(\delta_1+\delta_2)$
21, 19
-4, 32
1, 38
0, 38

I checked all the above using my own code in sage/python, but maybe I should get around to using the things you wrote in singular, as they are certainly more efficient. I thought about iterating this, but became discouraged by having to go up to a linear combination of at least 23 cycles in order for ξ to be maximal.

We can then ask more generally how $\xi(\delta)$ behaves with δ^2. In particular, the lattice of primitive Hodge cycles is positive definite, so that there must be a smallest n such that there exists a δ with

$$\xi(\delta)=maximal=426,\quad \delta^2=n.$$

Let me now explain how this question arises in physics, which unfortunately gives a few bonus complications. In M/F-Theory, you can specify a (consistent) background by specifying a Calabi-Yau 4-fold X, together with a 4-form γ, subject to the following requirements:

1. $\gamma+c_2(X)/2\in H^4(X,\mathbb{Z})\cap H^{2,2}(X)$, where $c_2(X)$ is the 2nd Chern class of X.
2. γ is primitive.
3. $\int_X\gamma^2\le\chi(X)/12$.

Depending on the choice of X,γ the resulting physics can be quite different. In particular, there are massless degrees of freedom associated with the Hodge locus of γ. If it is empty, i.e. if γ is a general Hodge cycle, there are none. This is the desirable situation of which essentially no non-trivial examples (with more than a handful of complex structure moduli) exists.

The question I was asking above arose by ignoring 1) and 2) but sticking to 3). As $\chi(X)=2610$, condition 3) says that $\gamma^2\le 217.5$. For the sextic 4-fold, $4c_2(X)=15H^2$ with H the hyperplane class, so condition 1) is non-trivial unfortunately.

I wrote some code that randomly chooses linear combinations Δ of k linear algebraic cycles (with coefficients ±1). I then work out $\xi(\Delta)$ and observed that I my record for $\xi=426$ was $k=23$. I can then make sure that 1) holds and check 2) which kills some of my examples but not all (the conditions is $\Delta\cdot H^2=3mod6$). Finally, I can check 3) which always fails roughly by a factor of 2. Here is an example:

[i] For these notations see my Hodge Theory book I, Chapter 16

```
[[1, [0, 4, 3], [0, 1, 2, 3, 4, 5]], [1, [3, 1, 4], [0, 1, 2, 4, 3, 5]],
[1, [3, 2, 0], [0, 1, 2, 5, 3, 4]], [1, [2, 1, 2], [0, 2, 1, 3, 4, 5]],
[-1, [3, 5, 5], [0, 2, 1, 4, 3, 5]], [-1, [4, 3, 4], [0, 2, 1, 5, 3,
4]], [1, [3, 4, 4], [0, 3, 1, 2, 4, 5]], [-1, [4, 5, 0], [0, 3, 1, 4, 2,
5]], [-1, [4, 3, 1], [0, 3, 1, 5, 2, 4]], [-1, [3, 3, 2], [0, 4, 1, 2,
3, 5]], [1, [1, 0, 3], [0, 4, 1, 3, 2, 5]], [1, [0, 4, 5], [0, 4, 1, 5,
2, 3]], [1, [1, 5, 5], [0, 5, 1, 2, 3, 4]], [1, [3, 0, 5], [0, 5, 1, 3,
2, 4]], [-1, [5, 5, 1], [0, 5, 1, 4, 2, 3]], [1, [5, 3, 5], [0, 1, 2, 3,
4, 5]], [-1, [1, 5, 5], [0, 1, 2, 4, 3, 5]], [-1, [5, 1, 0], [0, 1, 2,
5, 3, 4]], [-1, [4, 0, 3], [0, 2, 1, 3, 4, 5]], [1, [5, 2, 5], [0, 2, 1,
4, 3, 5]], [-1, [2, 3, 0], [0, 2, 1, 5, 3, 4]], [1, [4, 3, 3], [0, 3, 1,
2, 4, 5]], [1, [1, 1, 3], [0, 3, 1, 4, 2, 5]]]
```

$\Delta = \sum_i c_i \delta_i$ where each linear algebraic cycle δ_i is given by a permutation π_i and three sixth roots of unity. The format is

```
[[c_1,[roots of unity_1],[pi_1]],[c_2,[roots of unity_2],[pi_2]],...]
```

and the numbers give the roots of unity as powers of $\exp(\pi i/3)$. In the example above $\Delta \cdot H^2 = 3$, so we can satisfy 1) and 2) by $\gamma = \Delta + 1/2H^2$ and work out $\Delta^2 = 465$ and $\gamma^2 = 939/2$. One can make similar examples by checking similar random choices and unsurprisingly γ^2 ends up roughly of the same size. Either there are no significantly shorter cases or some conspiracy must happen.

Note that there is no big difference between Δ^2 and γ^2, which is why I dropped 1) and 2) from the discussion first, making the question cleaner.

In 26 March 2021, Andrea Braun gave a talk in GADEPs regarding this problem that you can find it

https://www.youtube.com/watch?v=fZl56S89SDE.

18.2 Making linear algebra

The Hodge theory of the Fermat variety is just a heavy linear algebra, see [Mov19, Chapter 15]. In the following I will just present the linear algebra part without mentining the underlying advanced topics, only some names are borrowed from the origin. The procedures mentioned in this section are in `foliation.lib` of SINGULAR. For further details how to use them see [Mov19]

Let $d, n \in \mathbb{N}$ and

$$
\begin{aligned}
I &:= \{\beta \in \mathbb{Z}^{n+1} \mid 0 \le \beta_i \le d-2\},\ \ \mu := \# I = (d-1)^{n+1} \\
A_\beta &:= \sum_{i=1}^{n+1} \frac{(\beta_i+1)}{d}, \quad \beta \in I \\
(x)_y &:= x(x-1)\cdots(x-y+1), \quad \text{Pochhamer symbol} \\
\mathbf{P}_{\beta,\beta'} &:= \prod_{i=1}^{n+1} \left(\zeta_d^{(\beta_i+1)(\beta_i'+1)} - \zeta_d^{\beta_i(\beta_i'+1)} \right) \\
\omega_{\beta'} &:= \text{ a vector in } \mathbb{C}^I \text{ whose } \beta\text{-entry is } \mathbf{P}_{\beta,\beta'} \\
\mathbf{P} &:= \left[\mathbf{P}_{\beta,\beta'}\right]_{\beta,\beta' \in I} \text{ period matrix.} \\
\frac{\beta_0+1}{d} &:= \frac{n}{2} + 1 - A_\beta \\
I_1 &:= \Bigg\{ \beta \in I \Bigg| \frac{\beta_i+1}{d} + \frac{\beta_{\sigma(i)}+1}{d} = 1, \ \ i = 0,1,2,\ldots,n+1, \\
&\qquad \text{for some permutation } \sigma \text{ of } 0,1,\ldots,n+1 \text{ without fixed point and with } \sigma^2 \text{ being identity } \Bigg\}, \\
B_\beta &:= (2\pi i)^{-\frac{n}{2}} \frac{\Gamma(\frac{\beta_1+1}{d})\Gamma(\frac{\beta_2+1}{d})\cdots,\Gamma(\frac{\beta_{n+1}+1}{d})}{\Gamma(\frac{\beta_1+1}{d}+\frac{\beta_2+1}{d}+\cdots+\frac{\beta_{n+1}+1}{d})} \\
&= (2\pi i)^{-\frac{n}{2}-1} \frac{e^{\pi i (A_\beta - \frac{n}{2})} - e^{-\pi i (A_\beta - \frac{n}{2})}}{(A_\beta - \frac{n}{2})_{\frac{n}{2}}} \prod_{i=0}^{n+1} \Gamma\left(\frac{\beta_i+1}{m_i}\right). \\
B_\beta &:= \left((A_\beta - \frac{n}{2})_{\frac{n}{2}} \prod_{j \in A} \left(\zeta_{2d}^{\beta_j+1} + \zeta_{2d}^{\beta_{\sigma(j)}+1} \right) \right)^{-1}, \ \ \beta \in I_1 \\
&\qquad \text{where } A \text{ is a set of cardinality } \tfrac{n}{2} \text{ such that } A \cup \sigma(A) \cup \{0,\sigma(0)\} = \{0,1,\ldots,n+1\}. \\
V &:= \mathbb{Z}^I \text{ affine cycles} \\
V_0 &:= \left\{ \delta \in \mathbb{Z}^I \ \Big| \ \delta \cdot \omega_\beta = 0, \ \forall \beta, \ A_\beta \notin \mathbb{N} \right\}, \ \text{cycles at infinity} \\
W &:= V/V_0 \text{ cycles} \\
V_2 &:= \left\{ \delta \in \mathbb{Z}^I \ \Big| \ \delta \cdot \omega_\beta = 0, \ \forall \beta, \ (A_\beta \notin \mathbb{N}) \text{ and } (A_\beta < \frac{n}{2}) \right\}, \ \text{affine primitive Hodge cycles} \\
V_1 &:= \left\{ \delta \in \mathbb{Z}^I \ \Big| \ \delta \cdot \omega_\beta = 0, \ \forall \beta, (A_\beta \notin \mathbb{N}) \ \text{ and } \left((A_\beta < \frac{n}{2}) \text{ or } \left(\frac{n}{2} < A_\beta < \frac{n}{2}+1 \ \text{ and } \ \beta \notin I_1 \right) \right) \right\}, \\
&\qquad \text{affine linear Hodge cycles} \\
&\qquad V_0 \subset V_1 \subset V_2 \\
W_2 &:= V_2/V_0, \quad \text{primitive linear Hodge cycles} \\
W_1 &:= V_1/V_0, \quad \text{primitive Hodge cycles} \\
\Psi &:= [\beta \bullet \beta']_{\beta,\beta' \in I}, \quad \text{intersection matrix, where } \beta \cdot \beta' \text{ is defined using the following rules} \\
&\qquad \beta \bullet \beta' = (-1)^n \beta' \cdot \beta, \quad \forall \beta,\beta' \in I, \\
&\qquad \beta \bullet \beta = (-1)^{\frac{n(n-1)}{2}} (1+(-1)^n), \quad \forall \beta \in I \\
&\qquad \beta \bullet \beta' = (-1)^{\frac{n(n+1)}{2}} (-1)^{\Sigma_{k=1}^{n+1} \beta_k' - \beta_k}
\end{aligned}
$$

for those $\beta,\beta' \in I$ such that for all $k = 1,2,\ldots,n+1$ we have $\beta_k \le \beta_k' \le \beta_k + 1$ and $\beta \ne \beta'$.

In the remaining cases, except those arising from the previous ones by a permutation, we have $\beta \bullet \beta' = 0$.

In the space of cycles V we consider the bilinear form giveb by Ψ:

$$\langle \delta, \delta' \rangle := \delta \Psi \delta'^{\mathsf{tr}}, \ \ \delta, \delta' \in V.$$

This bilinear map is not non-degenerate. Actually, we have

Proposition 18.1 *The space of cycles $\delta \in V$ orthogonal to all elements of V is the space of cycles at infinity, that is,*

$$V^{\perp} := \{\delta \in V | \langle \delta, V \rangle = 0\} = V_0.$$

For now, I do not know an elementary proof for this. The proof goes thorough cohomological interpretation of all what is here in [Mov19, Chapter 15]. It follows that the bilinear map $\langle \cdot, \cdot \rangle$ can be transported to the space $W_2 \subset W_1 \subset W$. We will need the map

$$f : W \to \mathbb{Z}, \ \ f(\delta) := \langle \delta, \delta \rangle.$$

We call $f(\delta)$ the self intersection of δ.

Proposition 18.2 *For a primitive Hodge cycle $\delta \in W$ and ω_β with $\frac{n}{2} < A_\beta < \frac{n}{2} + 1$ we have either $\delta \cdot \omega_\beta = 0$ or $B_\beta \in \bar{\mathbb{Q}}$.*

We introduce periods of (primitive) Hodge cycles:

$$\mathsf{p}_\beta := (B_\beta \delta \cdot \omega_\beta, \ \ \frac{n}{2} < A_\beta < \frac{n}{2} + 1)$$

which has entries in $\bar{\mathbb{Q}}$. For any other β which is not in the range $\frac{n}{2} < A_\beta < \frac{n}{2} + 1)$ we set $\mathsf{p}_\beta = 0$. Let $[\mathsf{p}_{i+j}]$ be the matrix whose rows and columns are indexed by $i \in I$ with $\frac{n}{2} - 1 < A_i < \frac{n}{2}$ and $j \in I$ with $\frac{n+2}{d} < \ A_j \ < 1 + \frac{n+2}{d}$, respectively, and in its (i, j) entry we have p_{i+j}. Our next invariant of Hodge cycles is

$$g : W_1 \to \mathbb{Z}, \ \ g(\delta) = \mathrm{rank}([\mathsf{p}_{i+j}])$$

A Hodge cycle δ such that $g(\delta)$reachs its canonical bound (the small number among the number of rows or columns) is called a general Hodge cylce.

18.1. Find a primitive general Hodge cycle such that the norm of its self intersection is minimal.

```
LIB "foliation.lib";
  ring r=0,x,dp; //Any ring
  intvec mlist=4,4,4,4,4; int n=size(mlist)-1; int d=lcm(mlist);
  list wlist;    //weight of the variables
  for (int i=1; i<=size(mlist); i=i+1)
    { wlist=insert(wlist, (d div mlist[i]), size(wlist));}
  ring r=(0,z), (x(1..n+1)),wp(wlist[1..n+1]);
  poly cp=cyclotomic(2*d); int degext=deg(cp) div deg(var(1));
  cp=subst(cp, x(1),z);
  minpoly =number(cp); //z is the 2d-th primitive root of unity


  list ll=MixedHodgeFermat(mlist);  list J=ll[1][1];
  int hFn=size(ll[1][1]); int nh=n div 2;
  for (i=2; i<=nh; i=i+1){hFn=hFn+size(ll[1][i]); J=J+ll[1][i];}
  list Hn2n2=ll[1][nh+1]; list Pn2=LinearCoho(mlist,0);
  J=J+RemoveList(ll[1][nh+1], Pn2[1]);
  list Jli; for (i=1; i<=size(J); i=i+1)
              {Jli=insert(Jli, leadexp(J[i]),size(Jli));}
```

```
list  bhc=BasisHodgeCycles(mlist, z^2,Jli);
matrix BHC=bhc[2];
matrix X=bhc[1]; //--The rows of this matrix are Hodge cycles
//-----we clear de denominator of the rows of X
  for (i=1; i<=nrows(X); i=i+1)
      {X[i,1..ncols(X)]=cleardenommat(submat(X, i,1..ncols(X)))[2];}

matrix Psi=IntersectionMatrix(ll[3]);
matrix im=X*Psi*transpose(X);

//We insert the algebraic numbers due to B_beta's.
int mhn=ncols(BHC);    int s; int N; int j; int ab;


for (i=1; i<=size(Hn2n2); i=i+1)
    {  for (j=1; j<=size(Pn2[1]); j=j+1)
            {  if (Hn2n2[i]==Pn2[1][j])
                  {
                     for (s=2; s<=nh+1; s=s+1)
                         { N=int( d*Pn2[2][j][s]);
                                 for (ab=1; ab<=nrows(BHC); ab=ab+1)
                                {BHC[ab,i]=BHC[ab,i]/(z^N+z^(d-N));}
                         }
                 }
          }
     }
matrix P[1][mhn];
for (i=1; i<=nrows(BHC); i=i+1)
  {
  P=BHC[i,1..mhn];
  rank( Matrixpij(mlist,P)), im[i,i];
  }
  nrows(Matrixpij(mlist,P)); ncols(Matrixpij(mlist,P));
```

```
  21 6061574227824
21 23043244987648
21 2272108928
21 730380081658
21 1199458042236
21 3955089122232
21 27141700396728
21 66707997837304
21 17128950211304
21 4243521548668
21 253456675908
21 9672088514960
21 2774718701560
21 5024014983160
21 5142297195656
21 4358168996488
21 3237670215872
21 4033972870512
21 5844489528104
21 10347565916800
21 4266309439544
21 1099213076292
21 4510646302528
21 430146850984
21 2116275781226
21 2201596224454
21 2817820485336
21 8283312651256
21 5096806153536
21 6228022939920
21 5280508206368
21 3960825275472
21 2160216680
21 247516662632
21 47464090088872
21 434696613198
21 3699868606816
21 11888077167304
21 2966168602560
21 3681989963264
21 30789520208104
21 11194488269696
21 6953920231720
21 24142293900872
21 365330673540
21 8436731063104
21 22260205818648
21 614363569160
21 519934605896
21 17062563768104
21 6168608160872
21 1341978237704
```

```
21 21423744042728
21 4657144582864
21 5692403153360
21 17325305743800
21 308561843406
21 8063169628024
21 21310355829720
21 1875622607060
21 5083860127408
21 4977352456
21 4225032767648
21 17845426560872
21 13271611874856
21 1061482216
21 4656021495552
21 26678017678104
21 5403013682760
21 29475213705800
21 5992201128464
21 26096049064328
21 1199163056
21 50256528620456
21 6590017858320
21 1956799845728
21 19912082036808
21 15663957428520
21 18102440324440
21 4158641524408
21 14821474823976
21 6065891351640
21 29504582535672
21 6248731257752
21 1439437191632
21 23111887523368
21 1195009178900
21 37507220100264
21 3330838314512
21 28919658259768
21 676793395582
21 64522111198872
21 1322195832618
21 434442679564
21 15622715166072
21 7088846381184
21 370930372626
21 12886353576
21 6460150791496
21 161345961278
21 28064765002792
21 4478862028656
21 5747483472704
21 24483887745288
21 1151799296740
21 9033208023464
21 9832461976968
21 267381433830
21 3810482586464
21 23548642385368
21 1030265428846
21 9064123233776
21 15447608104280
21 547150603104
21 1035926477262
21 42256501668600
21 2246891669360
21 4597984525232
21 28441730935144
21 31240303857624
21 242452456318
21 9105525448432
21 795046546596
21 2427281030900
21 8600971284912
21 6833520971000
21 676780252220
21 2107912288534
21 432351542232
21 7772431593744
21 332528238060
21 2392986167892
21 274049886914
21 249814832886
21 244090104904
21 6582065253352
21 8090687429616
21 405191639814
21 332873431544
21 3537552811712
21 4470443698952
```

```
21 4342438556400
21 1160728831794
21 83061744239016
21 680893566512
21 341008462712
21 5120979490640
21 1051347416242
21 942180219936
21 1259794757006
21 27835617179352
21 58471411096392
21 25327724557848
21 666272458440
21 5650491305272
21 5170597044344
21 4652201282824
21 1571465962308
21 504335920700
21 35089100032168
21 13525077282600
21 22986167954168
21 1261155008750
21 11968271205680
21 3551357897560
21 23723092896200
21 4621302728376
21 312685258802
21 18051412575192
21 69627578134792
21 23663381748136
21 1935162066622
21 166896689102
21 909577122318
21 7362233258016
21 1463036264334
21 5908745921240
21 5262945491496
21 7616236327216
21 2689351231142
21 273918854110
21 915664545494
21 1879395038596
21 4546374085784
21 2931852814880
21 499045286040
21 2810784888656
21 1984838761280
21 7283739381088
21 1697038852792
21 859972372522
21 6879909456640
21 5816286440680
21 1015604644418
21 3557565448920
21 1441429081552
21 1167131314650
21 4044441552424
21 233320979716
21 2840066197724
21 2988523069360
>    nrows(Matrixpij(mlist,P)); ncols(Matrixpij(mlist,P));
21
90
```

Chapter 19
L-function of CY modular forms

In this chapter we would like to discuss possible generelizations of Hecke's L-functions to CY modular forms introduced in [Mov17a, AMSY16]. The classical L-functions must be reformulated in terms of elliptic integrals and mirror map. This has been done in [Mov21, Chapter: Riemann's Zeta Function].

19.1 Upper half plane

The following ia taken from [Mov17a, Section 4.6]. The discussion is valid for an arbitrary family of Calabi-Yau varieties which contains all deformations of its fibers (the discussion must be fruitful only with this hypothesis). However, we restrict ourselves to the mirror quintic family $X_z,\ z \in \mathbb{P}^1$. We only need to know that its singular fibers are over $z = 0, 1$, and for $z \neq 0, 1$, $H^3(X_z, \mathbb{Z})$ is of rank 4 and carries Hodge numbers all equal to one. The fiber at infinity $z = \infty$ is not-singular. It has an automorphism of order 5.

Let $\tilde{\mathbb{H}}$ be the moduli of the pairs (X, δ), where X is a mirror quintic Calabi-Yau threefold and $\delta = \{\delta_1, \delta_2, \delta_3, \delta_4\}$ is a basis of $H_3(X, \mathbb{Z})$ such that the intersection matrix in this basis is

$$\Psi := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

that is, $[\langle \delta_i, \delta_j \rangle] = \Psi$. The set $\tilde{\mathbb{H}}$ has a canonical structure of a Riemann surface, not necessarily connected. We denote by $\mathbb{H}$ the connected component of $\tilde{\mathbb{H}}$ which contains the particular pair (X_z, δ) such that the monodromies around $z = 0$ and $z = 1$ are respectively given by the matrices M_0 and M_1:

$$M_0 := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 5 & 5 & 1 & 0 \\ 0 & -5 & -1 & 1 \end{pmatrix}, \quad M_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{19.1}$$

We call δ the standard basis. It is already well-known that in the monodromy group $\Gamma := \langle M_0, M_1 \rangle$ the only relation between M_0 and M_1 is $(M_0M_1)^5 = I$, see [BT14] (the monodromy around infinity is given by $M_\infty := M_0M_1$). This is equivalent to say that $\mathbb{H}$ is biholomorphic to the upper half plane. In the following we do not need this.

Remark 19.1 Despite the fact that $\mathbb{H} \cong \{\tau \in \mathbb{C} | \mathrm{Im}(\tau) > 0\}$, this biholomorphism is not given by any function related to X_z. In the case of elliptic curves such a biholomorphism is given by elliptic integrals, see [Mov17a, Section 10.1].

By definition, the monodromy group Γ acts on $\mathbb{H}$ by base change in δ. From now on by w we denote a point (X, δ) of $\mathbb{H}$. We use the following meromorphic functions on $\mathbb{H}$:

$$\tau_i : \mathbb{H} \to \mathbb{C},\ i = 0, 1, 2,$$

$$\tau_0(w) = \frac{\int_{\delta_1} \alpha_1}{\int_{\delta_2} \alpha_1},\ \tau_1(w) = \frac{\int_{\delta_3} \alpha_1}{\int_{\delta_2} \alpha_1},\ \tau_2(w) = \frac{\int_{\delta_4} \alpha_1}{\int_{\delta_2} \alpha_1},$$

where α_1 is a holomorphic differential form on X. They do not depend on the choice of α_1. There is a useful meromorphic function z on $\mathbb{H}$ which is obtained by identifying X with some X_z. It has a pole of order 5 at elliptic points which are the pairs (X, δ) with $X = X_\infty$. In this way, we have a well-defined holomorphic function

$$\psi = z^{-\frac{1}{5}} : \mathbb{H} \to \mathbb{C}.$$

19.2 Mixed Hodge structures

Similar to the case of elliptic curves, it is desirable to enlarge $\mathbb{H}^* = \mathbb{H} \cup C$ with a set C together with the action of Γ such that $\Gamma \backslash \mathbb{H}^* \cong \mathbb{P}^1$ given by the function z. This can be done by considering the mixed Hodge structure of singular Calabi-Yau threefolds X_0 and X_1. This might get too comlicated. In this section we describe a more intuitional approach.

A neighborhhood of $i\infty$ in $\mathbb{H}$ contains all (X_z, δ) with z near to 0 and δ is M_0^n(standard basis) for some n. The function τ_0 maps such a neighborhood to a neighborhood of $i\infty$ of the upper half plane. According to [Mov21, Chapter: Riemann's zeta function] the path of integration of a possible L-function must be a path from ∞ to a point $p = (X_\infty, *) \in \mathbb{H}$ with $M_\infty(p) = p$ and then going to $M_\infty(i\infty)$. Since $M_0(i\infty) = i\infty$, the conclusion is that in the $z \in \mathbb{P}^1$ coordinate this path is just the path of Monodromy around $z = 1$.

19.3 Classical point of view

In the classical point of view, one attaches to a modular form $\sum_{n=0}^{\infty} f_n q^n$ the L-function $\sum_{n=1}^{\infty} \frac{f_n}{n^s}$. As for modular forms f_n grows like n^k with some $k \in \mathbb{N}$ related to the weight of f, such L-functons are convergent in some right planes in $s \in \mathbb{C}$. For CY modular forms this does not work. In [**?**, Section 7.1] it is argued that the coefficients of the topological string partition function F_g of genus g (which is a modular form) grwos like

$$f_n \sim n^{2g-3}(\log n)^{2g-2} e^{2\pi n \tau_0(1)}$$

Here, $\tau_0(1)$ (in their notation $\alpha = t(1)$) is the constant which is obtained by the limit of the mirror map near $z = 1$. We must have $\tau_0(1) > 0$ otherwise f_n converges to zero. In explicit examples f_n's are integers converging to infinity. We conclude that $\sum_{n=1}^{\infty} \frac{f_n}{n^s}$ is always divergent.

19.4 A generalization of Hecke's L-function

The R-function attached to the Calabi-Yau modular forms t_0 and $\frac{\partial t_0}{\partial \tau_0}$ is

$$R(s, t_0) := \int_{\gamma} \left(\frac{\psi_1}{\psi_0} \right)^{s-1} (\psi_0 - 1) d \left(\frac{\psi_1}{\psi_0} \right) \tag{19.2}$$

$$R(s, \frac{\partial t_0}{\partial \tau_0}) := \int_{\gamma} \left(\frac{\psi_1}{\psi_0} \right)^{s-1} d\left(\psi_0\right) \tag{19.3}$$

where γ is a path in $\mathbb{P}^1 \backslash \{0, 1, \infty\}$ starting from 0 turning around $z = 1$ anticlockwise and ending at $z = 0$ and

$$\psi_0(\tilde{z}) = \sum_{m=0}^{\infty} \frac{(5m)!}{(m!)^5} \tilde{z}^m$$

$$\psi_1(\tilde{z}) = \ln(\tilde{z})\psi_0(\tilde{z}) + 5\tilde{\psi}_1(\tilde{z}), \quad \tilde{\psi}_1(\tilde{z}) = \sum_{m=1}^{\infty} \frac{(5m)!}{(m!)^5} (\sum_{k=m+1}^{5m} \frac{1}{k}) \tilde{z}^m$$

$\tilde{z} = \frac{z}{5^5}$. The R-function attahed to t_4 is better (as it is like a cusp form. No constant term)

$$R(s, t_4) := \int_{\gamma} \left(\frac{\psi_1}{\psi_0} \right)^{s-1} z \psi_0^5 d \left(\frac{\psi_1}{\psi_0} \right) \tag{19.4}$$

Proposition 19.1 *The three R-functions as above are holomorphic in*

$$
\begin{aligned}
t_0 &: \mathrm{Re}(s) > \frac{9}{2}\\
\frac{\partial t_0}{\partial \tau_0} &: \mathrm{Re}(s) > \frac{5}{2}\\
t_4 &: \ \textit{the whole } s \in \mathbb{C}.
\end{aligned}
$$

Proof. First we verify the convergence at the start point of γ. The asymptotic of integrand in each case is

$$
\begin{aligned}
t_0 &\sim (\ln(z))^{s-1} z \frac{1}{z}\\
\frac{\partial t_0}{\partial \tau_0} &\sim (\ln(z))^{s-1}\\
t_4 &\sim (\ln(z))^{s-1} z \frac{1}{z}
\end{aligned}
$$

which means that at the start point all three integrals are convergent. We use the convergence criterion in [Mov21, section: Big O notation] For the convergence at the end point of γ, we note that after the monodromy M_1 around $z = 1$ we have the following transformations

$$
\psi_1 \to \psi_1, \quad \psi_0 \to \ln(z)^3(\cdot) + \ln(z)^2(\cdots) + \ln(z)(\cdots) + \cdots
$$

where $\cdots$ means holomorphic. Therefore, the asymptotic of integrand at the end point is

$$
\begin{aligned}
t_0 &\sim (\ln(z))^{-2(s-1)} (\ln(z)^3) \frac{\ln(z)^3}{z}\\
\frac{\partial t_0}{\partial \tau_0} &\sim (\ln(z))^{-2(s-1)} \frac{\ln(z)^2}{z}\\
t_4 &\sim (\ln(z))^{-2(s-1)} (z \ln(z)^{15}) \frac{\ln(z)^3}{z}
\end{aligned}
$$

□

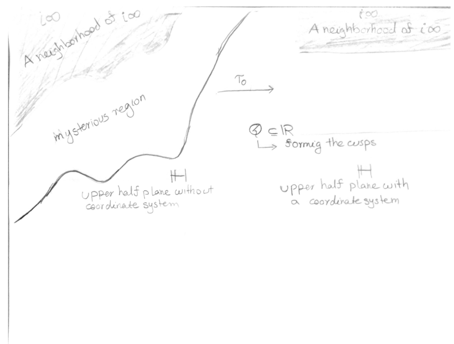

Fig. 19.1 A change of coordinate.

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