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Quasi-modular forms attached to elliptic curves: Hecke operators

Hossein Movasati

Instituto de Matemática Pura e Aplicada, IMPA, Estrada Dona Castorina, 110, 22460-320, Rio de Janeiro, RJ, Brazil

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ABSTRACT

In this article we describe Hecke operators on the differential algebra of geometric quasi-modular forms. As an application for each natural number d we construct a vector field in six dimensions which determines uniquely the polynomial relations between the Eisenstein series of weight 2, 4 and 6 and their transformation under multiplication of the argument by d, and in particular, it determines uniquely the modular curve of degree d isogenies between elliptic curves.

1. Introduction

The theory of quasi-modular forms was first introduced by Kaneko and Zagier in [5] due to its applications in mathematical physics. One can describe quasi-modular forms in the framework of the algebraic geometry of elliptic curves, and in particular, the Ramanujan differential equation between Eisenstein series can be derived from the Gauss–Manin connection of families of elliptic curves, see for instance [7] and [9]. We call this the Gauss–Manin connection in disguise. The terminology arose from a private letter of

E-mail address: hossein@impa.br.
URL: http://www.impa.br/~hossein.

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Pierre Deligne to the author, see [4]. In the present article we describe Hecke operators for such quasi-modular forms, and certain differential ideals related to modular curves. In [3] the authors describe a differential equation in the j-invariant of two elliptic curves which is tangent to all modular curves of degree d isogenies of elliptic curves. This differential equation can be derived from the Schwarzian differential equation of the j-function and the latter can be calculated from the Ramanujan differential equation between Eisenstein series. This suggests that there must be a relation between Ramanujan differential equation and modular curves. In this article we also establish this relation. Another motivation behind this work is to prepare the ground for similar topics in the case of Calabi–Yau varieties, see [8].

Consider the Ramanujan ordinary differential equation

$$\mathsf{R}: \begin{cases} \dot{s}_1 = \frac{1}{12}(s_1^2 - s_2) \\ \dot{s}_2 = \frac{1}{3}(s_1 s_2 - s_3) \\ \dot{s}_3 = \frac{1}{2}(s_1 s_3 - s_2^2) \end{cases} \dot{s}_k = \frac{\partial s_k}{\partial \tau} \tag{1}$$

which is satisfied by the Eisenstein series:

$$s_i(\tau) = a_i E_{2i}(q) := a_i \left(1 + b_i \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{2i-1}\right) q^n\right),$$
$$i = 1, 2, 3, \ q = e^{2\pi i \tau}, \ \mathrm{Im}(\tau) > 0 \tag{2}$$

and

$$(b_1, b_2, b_3) = (-24, 240, -504), \quad (a_1, a_2, a_3) = (2\pi i, (2\pi i)^2, (2\pi i)^3).$$

The algebra of modular forms for $\mathrm{SL}(2,\mathbb{Z})$ is generated by the Eisenstein series E_4 and E_6 and all modular forms for congruence groups are algebraic over the field $\mathbb{C}(E_4, E_6)$, see for instance [14]. In a similar way the algebra of quasi-modular forms for $\mathrm{SL}(2,\mathbb{Z})$ is generated by E_2, E_4 and E_6, see for instance [6,9], and we have:

Theorem 1. *For $i = 1, 2, 3$ and $d \in \mathbb{N}$, there is a homogeneous polynomial $I_{d,i}$ of degree $i \cdot \psi(d)$, where $\psi(d) := d \prod_p (1 + \frac{1}{p})$ is the Dedekind ψ function and p runs through primes p dividing d, in the weighted ring*

$$\mathbb{Q}[t_i, s_1, s_2, s_3], \ \text{weight}(t_i) = i, \ \text{weight}(s_j) = j, \ j = 1, 2, 3 \tag{3}$$

and monic in the variable t_i such that $t_i(\tau) := d^{2i} \cdot s_i(d \cdot \tau), s_1(\tau), s_2(\tau), s_3(\tau)$ satisfy the algebraic relation:

$$I_{d,i}(t_i, s_1, s_2, s_3) = 0.$$

Moreover, for $i = 2, 3$ the polynomial $I_{d,i}$ does not depend on s_1.

The novelty of Theorem 1 is mainly due to the case $i = 1$. We consider s_i, t_i, $i = 1, 2, 3$ as indeterminate variables and for simplicity we do not introduce new notation in order to distinguish them from the Eisenstein series. We regard $(t, s) = (t_1, t_2, t_3, s_1, s_2, s_3)$ as coordinates of the affine variety $\mathbb{A}^6_{\mathsf{k}}$, where k is any field of characteristic zero and not necessarily algebraically closed. Ramanujan's ordinary differential equation (1) is considered as a vector field in $\mathbb{A}^3_{\mathsf{k}}$ with the coordinates (s_1, s_2, s_3). It can be shown that the curve given by $I_{d,2} = I_{d,3} = 0$ in the weighted projective space $\mathbb{P}^{(2,3,2,3)}_{\mathbb{C}}$ with the coordinates (t_2, t_3, s_2, s_3) is a singular model for the modular curve

$$X_0(d) := \Gamma_0(d)\backslash(\mathbb{H} \cup \mathbb{Q}), \quad \text{where} \quad \Gamma_0(d) := \{\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \mid a_3 \equiv 0 \pmod{d}\}.$$

Computing explicit equations for $X_0(d)$ in terms of the variables $j_1 = 1728\frac{t_2^3}{t_2^3 - t_3^2}$ and $j_2 = 1728\frac{s_2^3}{s_2^3 - s_3^2}$ has many applications in number theory and it has been done by many authors, see for instance [13] and the references therein.

Let R_t, respectively R_s, be the Ramanujan vector field in $\mathbb{A}^3_{\mathsf{k}}$ with coordinates (t_1, t_2, t_3), respectively (s_1, s_2, s_3). In $\mathbb{A}^6_{\mathsf{k}} = \mathbb{A}^3_{\mathsf{k}} \times \mathbb{A}^3_{\mathsf{k}}$ with the coordinates system (t, s) we consider the vector field:

$$\mathsf{R}_d := \mathsf{R}_t + d \cdot \mathsf{R}_s.$$

Let $\mathsf{T} := \mathbb{A}^3_{\mathsf{k}}\backslash\{t_2^3 - t_3^2 = 0\}$ and let V_d be the affine subvariety of $\mathsf{T} \times \mathsf{T}$ given by the ideal $\langle I_{d,1}, I_{d,2}, I_{d,3}\rangle \subset \mathsf{k}[s, t, \frac{1}{t_2^3 - t_3^2}, \frac{1}{s_2^3 - s_3^2}]$.

Theorem 2. *The vector field R_d is tangent to the affine variety V_d.*

I do not know the complete classification of all R_d-invariant algebraic subvarieties of $\mathbb{A}^6_{\mathsf{k}}$. We consider R_d as a differential operator:

$$\mathsf{k}[t, s] \to \mathsf{k}[t, s], \ f \mapsto \mathsf{R}_d(f) := df(\mathsf{R}_d).$$

From Theorem 2 and the fact that V_d is irreducible (see Section 9), it follows that:

$$\mathsf{R}_d^j(I_{d,i}) \in \mathrm{Radical}\langle I_{d,1}, I_{d,2}, I_{d,3}\rangle, \quad i = 1, 2, 3, \quad j \in \mathbb{N} \cup \{0\}.$$

Note that the ideal $\langle I_{d,1}, I_{d,2}, I_{d,3}\rangle \subset \mathsf{k}[s, t, \frac{1}{t_2^3 - t_3^2}, \frac{1}{s_2^3 - s_3^2}]$ may not be radical. We can compute $I_{d,i}$'s using the q-expansion of Eisenstein series, see Section 11. This method works only for small degrees d. However, for an arbitrary d we can introduce some elements in the radical of the ideal generated by $I_{d,i}$, $i = 1, 2, 3$. Let

$$J_{d,i} = \det \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{m_{d,i}} \\ \mathsf{R}_d(\alpha_1) & \mathsf{R}_d(\alpha_2) & \cdots & \mathsf{R}_d(\alpha_{m_{d,i}}) \\ \vdots & \vdots & \cdots & \vdots \\ \mathsf{R}_d^{m_{d,i}-1}(\alpha_1) & \mathsf{R}_d^{m_{d,i}-1}(\alpha_2) & \cdots & \mathsf{R}_d^{m_{d,i}-1}(\alpha_{m_{d,i}}) \end{pmatrix},$$

where for $i = 1$, α_j's are the monomials:

$$t_i^{a_0} s_1^{a_1} s_2^{a_2} s_3^{a_3}, \ i \cdot \psi(d) = ia_0 + a_1 + 2a_2 + 3a_3, \ a_0, a_1, a_2, a_3 \in \mathbb{N}_0 \tag{4}$$

and for $i = 2, 3$, α_j's are the above monomials with $a_1 = 0$. Here, $m_{d,i}$ is the number of monomials α_j's. The polynomial $J_{d,i}$ is weighted homogeneous of degree

$$\begin{aligned} &i \cdot \psi(d) + i \cdot \psi(d) + 1 + i \cdot \psi(d) + 2 + \cdots + i \cdot \psi(d) + m_{d,i} - 1 \\ &= m_{d,i} \cdot i \cdot \psi(d) + \frac{m_{d,i} \cdot (m_{d,i} - 1)}{2}. \end{aligned}$$

Theorem 3. *We have*

$$J_{d,i} \in \mathrm{Radical}\langle I_{d,1}, I_{d,2}, I_{d,3} \rangle, \ i = 1, 2, 3,$$

and so

$$J_{d,i}(d^{2i} \cdot s_i(d \cdot \tau), s_1(\tau), s_2(\tau), s_3(\tau)) = 0, \ i = 1, 2, 3.$$

Our proofs of Theorems 1, 2, 3 use the notion of geometric quasi-modular forms, Hecke operators and the fact that the affine variety $\mathbb{A}^3_{\mathsf{k}} \backslash \{t_2^3 - t_3^2 = 0\}$ is a fine moduli of elliptic curves enhanced with elements in their de Rham cohomologies, see Theorem 4. It might be possible to modify our proofs in order to avoid any reference to Algebraic Geometry, however, one may lose the motivation for many arguments which mainly lie on the geometry of vector fields. Moreover, geometric approach to quasi-modular forms gives us a convenient context in order to work over rational numbers and this might lead us to connections of arithmetic of elliptic curves and quasi-modular forms.

Throughout the text we will state our results over a field k of characteristic zero and not necessarily algebraically closed. Such results are valid if and only if the same results are valid over the algebraic closure $\bar{\mathsf{k}}$ of k. By Lefschetz principle, see for instance [11] p. 164, it is enough to prove such results over the complex numbers. For a variety T defined over k, $\mathsf{T}(\mathsf{k})$ denotes the set of k-rational points of T.

The article is organized in the following way. In Section 2 and Section 3 we recall the definition of full quasi-modular forms in the framework of both algebraic geometry and complex analysis. In Section 4 we describe some facts relating isogenies and algebraic de Rham cohomology of elliptic curves. Using isogeny of elliptic curves we introduce geometric Hecke operators in Section 5 and in Section 6 we describe their translation into holomorphic Hecke operators. For our discussion of modular curves we need a refined version of Hecke operators that we discuss in Section 7. Theorem 1, Theorem 2 and Theorem 3 are respectively proved in Section 8, Section 9 and Section 10. Finally, in Section 11 we give some examples.

The main idea behind the proof of Theorem 3 is due to J.V. Pereira in [10]. Here, I would like to thank him for teaching me such an elegant and simple argument. Thanks go to J. Sijsling for his useful comments for the first draft of the present text. Finally, I would like to thank the referee whose critical comments improved the text.

2. Geometric quasi-modular forms

In this section we recall some definitions and theorems in [6,7]. The reader is also referred to [9] for a complete account of quasi-modular forms in a geometric context. Note that in [9] the t parameter is in fact $(\frac{1}{12}t_1, 12\frac{1}{12^2}t_2, 8\frac{1}{12^3}t_3)$. Let k be any field of characteristic zero and let E be an elliptic curve over k. The first algebraic de Rham cohomology of E, namely $H^1_{\mathrm{dR}}(E)$, is a k-vector space of dimension two and it has a one dimensional space F^1 consisting of elements represented by regular differential 1-forms on E. Let us define

$$\mathsf{T} := \mathrm{Spec}(\mathsf{k}[t_1, t_2, t_3, \frac{1}{t_2^3 - t_3^2}])$$

Theorem 4. *(See [9], §5.5.) The affine variety* T *is the fine moduli of the pairs* (E, ω)*, where* E *is an elliptic curve and* $\omega \in H^1_{\mathrm{dR}}(E)\backslash F^1$*. For* $(t_1, t_2, t_3) \in \mathsf{T}(\mathsf{k})$*, the corresponding pair* (E, ω) *is given by*

$$E : 3y^2 = (x - t_1)^3 - 3t_2(x - t_1) - 2t_3, \quad \omega = \frac{1}{12}\frac{xdx}{y}. \tag{5}$$

From now on an element of $\mathsf{T}(\mathsf{k})$ is denoted either by (t_1, t_2, t_3) or (E, ω). We can regard t_i as a rule which for any pair (E, ω) as above it associates an element $t_i = t_i(E, \omega) \in \mathsf{k}$. We will also use t_i as an indeterminate variable or an element in k, being clear from the text which we mean. For m an even number, a full quasi-modular form f of weight m and differential order n is a homogeneous polynomial of degree $\frac{m}{2}$ in the k-algebra

$$M := \mathsf{k}[t_1, t_2, t_3], \ \mathrm{weight}(t_i) = i, \ i = 1, 2, 3,$$

with $\deg_{t_1} f \le n$. The set of such quasi-modular forms is denoted by M_m^n.

For a pair $(E, \omega) \in \mathsf{T}(\mathsf{k})$ we have also a unique element $\alpha \in F^1$ satisfying $\langle \alpha, \omega \rangle = 1$, where $\langle \cdot, \cdot \rangle$ is the intersection form in the de Rham cohomology, see for instance [9] §2.10. For this reason we sometimes use $(E, \{\alpha, \omega\})$ instead of (E, ω). The algebraic groups $\mathbb{G}_a := (\mathsf{k}, +)$ and $\mathbb{G}_m := (\mathsf{k} - \{0\}, \cdot)$ act from the right on $\mathsf{T}(\mathsf{k})$:

$$(E, \omega) \bullet k := (E, k\omega), \ k \in \mathbb{G}_m,$$
$$(E, \omega) \circ k := (E, \omega + k\alpha), \ k \in \mathbb{G}_a$$

and so they act from the left on M. It can be shown that M_m^n is invariant under these actions and the functions $t_i : \mathsf{T} \to \mathsf{k}$, $i = 1, 2, 3$ satisfy

$$k \circ t_1 = t_1 + k, \ k \circ t_i = t_i, \ i = 2, 3 \quad k \in \mathbb{G}_a, \tag{6}$$
$$k \bullet t_i = k^{-2i} t_i, \ i = 1, 2, 3 \quad k \in \mathbb{G}_m. \tag{7}$$

Let R be the Ramanujan vector field in T. It is the unique vector field in T which satisfies $\nabla_{\mathsf{R}}\alpha = -\omega$, $\nabla_{\mathsf{R}}\omega = 0$, where ∇ is the Gauss–Manin connection of the universal family of elliptic curves over T, see for instance [9] §2. The k-algebra of full quasi-modular forms has a differential structure which is given by:

$$M_m^n \to M_{m+2}^{n+1},\ t \mapsto \mathsf{R}(t) := \sum_{i=1}^{3} \frac{\partial t}{\partial t_i}\mathsf{R}_i,$$

where $\mathsf{R} = \sum_{i=1}^{3} \mathsf{R}_i \frac{\partial}{\partial t_i}$ is the Ramanujan vector field.

3. Holomorphic quasi-modular forms

Now, let us assume that $\mathsf{k} = \mathbb{C}$. The period domain is defined to be

$$\mathcal{P} := \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \mid x_i \in \mathbb{C},\ x_1x_4 - x_2x_3 = 1,\ \mathrm{Im}(x_1\overline{x_3}) > 0 \right\}. \tag{8}$$

We let the group $\mathrm{SL}(2,\mathbb{Z})$ act from the left on $\mathcal{P}$ by usual multiplication of matrices. In $\mathcal{P}$ we consider the vector field

$$X = -x_2\frac{\partial}{\partial x_1} - x_4\frac{\partial}{\partial x_3} \tag{9}$$

which is invariant under the action of $\mathrm{SL}(2,\mathbb{Z})$ and so it induces a vector field in the complex manifold $\mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{P}$. For simplicity we denote it again by X. The Poincaré upper half plane $\mathbb{H}$ is embedded in $\mathcal{P}$ in the following way:

$$\tau \to \begin{pmatrix} \tau & -1 \\ 1 & 0 \end{pmatrix}$$

and so we have a canonical map $\mathbb{H} \to \mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{P}$.

Theorem 5. *(See [9] §8.4 and §8.8.) The period map*

$$\mathsf{pm} : \mathsf{T}(\mathbb{C}) \to \mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{P}$$
$$t \mapsto \left[\frac{1}{\sqrt{-2\pi i}}\begin{pmatrix} \int_\delta \alpha & \int_\delta \omega \\ \int_\gamma \alpha & \int_\gamma \omega \end{pmatrix}\right]$$

is a biholomorphism, where $\{\delta,\gamma\}$ is a basis of $H_1(E,\mathbb{Z})$ with $\langle\delta,\gamma\rangle = -1$. Under this biholomorphism the Ramanujan vector field is mapped to X. The pull-back of t_i by the composition

$$\mathbb{H} \to \mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{P} \overset{\mathsf{pm}^{-1}}{\to} \mathsf{T}(\mathbb{C}) \hookrightarrow \mathbb{A}^3_{\mathbb{C}}, \tag{10}$$

is the Eisenstein series $a_iE_{2i}(e^{2\pi i\tau})$ in (2).

The algebra of full holomorphic quasi-modular forms is the pull-back of $\mathsf{k}[t_1, t_2, t_3]$ under the composition (10). We can also introduce it in a classical way using functional equations plus growth conditions: a holomorphic function f on $\mathbb{H}$ is called a (holomorphic) quasi-modular form of weight m and differential order n if the following two conditions are satisfied:

1. There are holomorphic functions $f_i(z)$, $i = 0, 1, \dots, n$ on $\mathbb{H}$ such that

$$(cz+d)^{-m} f(Az) = \sum_{i=0}^{n} \binom{n}{i} c^i (cz+d)^{-i} f_i(z), \ \forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}). \tag{11}$$

2. $f_i(z)$, $i = 0, 1, 2, \dots, n$ have finite growths when $\mathrm{Im}(z)$ tends to $+\infty$, i.e.

$$\lim_{\mathrm{Im}(z) \to +\infty} f_i(z) = a_{i,\infty} < \infty, \ a_{i,\infty} \in \mathbb{C}.$$

For the proof of the equivalence of both notions of quasi-modular forms see [9] §8.11. We have $f_0 = f$ and the associated functions f_i are unique. In fact, f_i is a quasi-modular form of weight $m - 2i$ and differential order $n - i$ and with associated functions $f_{ij} := f_{i+j}$. It is useful to define

$$f||_m A := (\det A)^{m-1} \sum_{i=0}^{n} \binom{n}{i} (\frac{-c}{\det(A)})^i (cz+d)^{i-m} f_i(Az),$$
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R}), \ f \in M_m^n. \tag{12}$$

In this way, the equality (11) is written in the form

$$f = f||_m A, \forall A \in \mathrm{SL}(2, \mathbb{Z}) \tag{13}$$

and we have

$$f||_m A = f||_m (BA), \ \forall A \in \mathrm{GL}(2, \mathbb{R}), \ B \in \mathrm{SL}(2, \mathbb{Z}), \ f \in M_m^n, \tag{14}$$
$$f^k||_{km} A = (\det A)^{k-1} (f||_m A)^k, \quad \forall A \in \mathrm{GL}(2, \mathbb{R}), \quad k \in \mathbb{N}. \tag{15}$$

It can be proved that the algebra of full quasi-modular forms is generated by the Eisenstein series E_{2i}, $i = 1, 2, 3$. For further details on holomorphic quasi-modular forms see [6,7,9].

4. Isogeny of elliptic curves

Let $(E_1, 0_1)$ and $(E_2, 0_2)$ be two elliptic curves over the field k. Here, $0_i \in E_i(\mathsf{k})$, $i = 1, 2$ is the neutral element of the group $E_i(\mathsf{k})$. We say that E_1 is isogenous to E_2

over k if there is a non-constant morphism of algebraic curves over k $f: E_1 \to E_2$ which sends 0_1 to 0_2. It can be shown that f induces a morphism of groups $E_1(\mathsf{k}) \to E_2(\mathsf{k})$. We also say that f is an isogeny between E_1 and E_2 over k. For all points $p \in E(\bar{\mathsf{k}})$ except a finite number, $\#f^{-1}(p)$ is a fixed number which we denote it by $\deg(f)$. Here, we have considered f as a map from $E_1(\bar{\mathsf{k}})$ to $E_2(\bar{\mathsf{k}})$. Since f is a morphism of groups, for a point $q \in f^{-1}(p)$ the map $x \mapsto x + q$ induces a bijection $f^{-1}(0_2) \cong f^{-1}(p)$. We conclude that for all points $p \in E_2(\bar{\mathsf{k}})$, the set $f^{-1}(p)$ has $\deg(f)$ points (and hence f has no ramification points).

Proposition 1. *Let $f: E_1 \to E_2$ be an isogeny of degree d. Then for all $\omega, \alpha \in H^1_{\mathrm{dR}}(E_2)$ we have*

$$\langle f^*\omega, f^*\alpha\rangle = d \cdot \langle \omega, \alpha\rangle.$$

Here, $\langle\cdot,\cdot\rangle : H^1_{\mathrm{dR}}(E) \times H^1_{\mathrm{dR}}(E) \to \mathsf{k}$ is the intersection form in the de Rham cohomology, see [9] §2.10.

Proof. It is enough to prove the proposition over an algebraically closed field. Since the above formula is k-linear in both ω and α, it is enough to prove it in the case $\omega = \frac{dx}{y}$, $\alpha = \frac{xdx}{y}$, where x, y are the Weierstrass coordinates of E_2. Since $\langle \frac{dx}{y}, \frac{xdx}{y}\rangle = 1$, we have to prove that $\langle f^*(\frac{dx}{y}), f^*(\frac{xdx}{y})\rangle = d$. Let $f^{-1}(0_2) = \{p_1, p_2, \dots, p_d\}$. The differential form $f^*(\frac{dx}{y})$ is again a regular differential form and $f^*(\frac{xdx}{y})$ has poles of order two at each p_i. Consider the covering $U = \{U_0, U_1\}$ of E_1, where $U_0 = E_1 \backslash f^{-1}(0_2)$ and U_1 is any other open set which contains $f^{-1}(0_2)$. The differential forms $f^*(\frac{x^i dx}{y})$, $i = 0, 1$ as elements in $H^1_{\mathrm{dR}}(E_1)$ are represented by the pairs

$$(\frac{d\tilde{x}}{\tilde{y}}, \frac{d\tilde{x}}{\tilde{y}}), \quad (\frac{\tilde{x}d\tilde{x}}{\tilde{y}}, \frac{\tilde{x}d\tilde{x}}{\tilde{y}} - \frac{1}{2}d(\frac{\tilde{y}}{\tilde{x}})),$$

where $\tilde{x} = f^*x$, $\tilde{y} = f^*y$. We have $\frac{d\tilde{x}}{\tilde{y}} \cup \frac{\tilde{x}d\tilde{x}}{\tilde{y}} = \{\omega_{01}\}$, where $\omega_{01} = \frac{-1}{2}\frac{d\tilde{x}}{\tilde{x}}$ and so

$$\langle f^*(\frac{dx}{y}), f^*(\frac{xdx}{y})\rangle = \langle \frac{d\tilde{x}}{\tilde{y}}, \frac{\tilde{x}d\tilde{x}}{\tilde{y}}\rangle = \sum_{i=1}^{d} \mathrm{Residue}(\frac{-1}{2}\frac{d\tilde{x}}{\tilde{x}}, p_i) = \sum_{i=1}^{d} 1 = d. \qquad \square$$

Proposition 2. *We have:*

1. *Let $f: E_1 \to E_2$ be an isogeny defined over k. The induced map $f^*: H^1_{\mathrm{dR}}(E_2) \to H^1_{\mathrm{dR}}(E_1)$ is an isomorphism.*
2. *Let $[d]_E : E \to E$ be the multiplication by $d \in \mathbb{N}$ map. We have $[d]_E^* : H^1_{\mathrm{dR}}(E) \to H^1_{\mathrm{dR}}(E)$, $\omega \mapsto d \cdot \omega$.*

Proof. In the complex context, $E = \mathbb{C}/\langle\tau, 1\rangle$ and $[d]_E$ is induced by $\mathbb{C} \to \mathbb{C}$, $z \mapsto d \cdot z$. Moreover, a basis of the C^∞ de Rham cohomology is given by dz, $d\bar{z}$. This proves the second part of the proposition. For the first part we take the dual isogeny and use the first part. □

For E an elliptic curve over an algebraically closed field k of characteristic zero, the number of isogenies $f : E_1 \to E$ of degree d and up to canonical isomorphisms is equal to $\sigma_1(d) := \sum_{c|d} c$. To prove this we may work in the complex context and assume that $E = \mathbb{C}/\langle\tau, 1\rangle$. The number of such isogenies is the number of subgroups of order d of $(\mathbb{Z}/d\mathbb{Z})^2$, which is known to be $\sigma_1(d)$.

5. Geometric Hecke operators

In this section all the algebraic objects are defined over k unless it is mentioned explicitly. Let d be a positive integer. The Hecke operator T_d acts on the space of full quasi-modular forms as follows:

$$T_d : M_m^n \to M_m^n,$$
$$T_d(t)(E,\omega) = \frac{1}{d} \sum_{f:E_1\to E,\ \deg(f)=d} t(E_1, f^*\omega), \quad t \in M_m^n$$

where the sum runs through all isogenies $f : E_1 \to E$ of degree d defined over $\bar{\mathsf{k}}$. Since (E,ω) and t are defined over k, $T_d(t)(E,\omega)$ is invariant under $\mathrm{Gal}(\bar{\mathsf{k}}/\mathsf{k})$ and so it is in the field k. This implies that $T_d(t)$ is defined over k. The statement $T_d \in \mathsf{k}[t_1,t_2,t_3]$ is not at all clear. In order to prove this, we assume that $\mathsf{k} = \mathbb{C}$ and we prove the same statement for holomorphic quasi-modular forms, see Section 6. The functional equation of $T_d t$ with respect to the action of the algebraic groups $\mathbb{G}_m$ and $\mathbb{G}_a$ can be proved in the algebraic context as follows:

Proposition 3. *The action of $\mathbb{G}_m$ commutes with Hecke operators, that is,*

$$k \bullet T_d(t) = T_d(k \bullet t),\ t \in M,\ k \in \mathbb{G}_m \tag{16}$$

and the action of $\mathbb{G}_a$ satisfies:

$$k \circ T_d(t) = T_d((d \cdot k) \circ t),\ t \in M,\ k \in \mathbb{G}_a.$$

Proof. The first equality is trivial:

$$(k \bullet T_d(t))(E,\omega) = T_d(t)(E,k\omega) = \frac{1}{d}\sum t(E_1, f^*(k\omega))$$
$$= \frac{1}{d}\sum (k \bullet t)(E_1, f^*(\omega)) = T_d(k \bullet t)(E,\omega)$$

For the second equality we use Proposition 1:

$$
\begin{aligned}
(k\circ T_d(t))(E,\omega) &= T_d(t)(E,\omega+k\alpha) = \frac{1}{d}\sum t(E_1, f^*(\omega+k\alpha)) \\
&= \frac{1}{d}\sum t(E_1, f^*(\omega)+d\cdot k f^*(\frac{1}{d}\alpha)) = \frac{1}{d}\sum((d\cdot k)\circ t)(E_1, f^*(\omega)) \\
&= T_d(d\cdot k\circ t)(E,\omega). \qquad \square
\end{aligned}
$$

We can also define the Hecke operators in the following way:

$$
T_d(t)(E,\omega) = d^{m-1}\sum_{g:E\to E_1,\ \deg(g)=d} t(E_1, g_*\omega),\ \ t\in M_m^n
$$

where the sum runs through all isogenies $g : E \to E_1$ of degree d defined over $\bar{\mathsf{k}}$. Both definitions of $T_d(t)$ are equivalent: for an isogeny $f : E_1 \to E$ of degree d defined over $\bar{\mathsf{k}}$ we have a unique dual isogeny $g : E \to E_1$ such that

$$
f\circ g = [d]_E, \quad g\circ f = [d]_{E_1}.
$$

Therefore by Proposition 2 we have $d\cdot g_*\omega = [d]_{E_1}^*(g_*\omega) = f^*\omega$ and so

$$
t(E_1, f^*\omega) = t(E_1, d\cdot g_*\omega) = d^m t(E_1, g_*\omega).
$$

It can be shown that the geometric Eisenstein modular form G_k (see [9] §6.5) is an eigenform with eigenvalue

$$
\sigma_{k-1}(d) := \sum_{c|d} c^{k-1}
$$

for the Hecke operator T_d, that is

$$
T_d G_k = \sigma_{k-1}(d) G_k, \quad d\in\mathbb{N}, \quad k\in 2\mathbb{N}
$$

see for instance [6].

The differential operator $\mathsf{R} : M \to M$ and the Hecke operator T_d commute, that is

$$
\mathsf{R}\circ T_d = T_d\circ\mathsf{R}, \ \forall d\in\mathbb{N}.
$$

For the proof we may assume that $\mathsf{k} = \mathbb{C}$. In this way using Theorem 5 it is enough to prove the same statement for holomorphic quasi-modular forms, see for instance [7] Proposition 4.

6. Holomorphic Hecke operators

In this section we want to use the biholomorphism in Theorem 5 and describe the Hecke operators on holomorphic quasi-modular forms. Let us take $\mathsf{k} = \mathbb{C}$ and let $\mathrm{Mat}_d(2,\mathbb{Z})$ be the set of 2×2 matrices with coefficients in $\mathbb{Z}$ and with determinant d.

Proposition 4. *The d-th Hecke operator on the vector space of quasi-modular forms of weight m and differential order n is given by*

$$T_d : M_m^n \to M_m^n, \quad T_d f = \sum_A f||_m A,$$

where A runs through the set $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{Mat}_d(2,\mathbb{Z})$ and $||$ is the double slash operator (12) for quasi-modular forms.

Proof. Let us consider two points $(E_i, \{\alpha_i, \omega_i\})$, $i = 1, 2$ in the moduli space T. Let us also consider a d-isogeny $f : E_1 \to E_2$ with

$$f^*\omega_2 = \omega_1, \ f^*\alpha_2 = d \cdot \alpha_1.$$

We can take a symplectic basis δ_1, γ_1 of $H_1(E_1,\mathbb{Z})$ and δ_2, γ_2 of $H_1(E_2,\mathbb{Z})$ such that

$$f_*[\delta_1, \gamma_1]^{\mathsf{tr}} = A[\delta_2, \gamma_2]^{\mathsf{tr}},$$

where $A \in \mathrm{Mat}_d(2,\mathbb{Z})$ and tr means transpose of a matrix. From another side we have

$$[\alpha_1, \omega_1]B = f^*[\alpha_2, \omega_2], \quad \text{where } B = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore, if the period matrix associated to $(E_i, \{\alpha_i, \omega_i\}, \{\delta_i\, \gamma_i\})$, $i = 1, 2$ is denoted respectively by x' and x then

$$x'B = Ax.$$

Using Theorem 5, the d-th Hecke operator acts on the space of $\mathrm{SL}(2,\mathbb{Z})$-invariant holomorphic functions on $\mathcal{P}$ by:

$$T_d F(x) = \frac{1}{d} \sum_A F(AxB^{-1}), \ x \in \mathcal{P}$$

where $A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ runs through $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{Mat}_d(2,\mathbb{Z})$. Let $f \in M_m^n$ be a holomorphic quasi-modular form defined on the upper half plane. By definition there is a geometric modular form $\tilde{f} \in \mathbb{C}[t_1, t_2, t_3]$ such that f is the pull-back of $\tilde{f}$ by the composition (10).

Let F be the holomorphic function on $\mathcal{P}$ obtained by the push-forward of $\tilde{f}$ by the period map and then its pull-back by $\mathcal{P} \to \mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{P}$. We have

$$\begin{aligned}
T_d f(\tau) &= T_d F\begin{pmatrix}\tau & -1\\ 1 & 0\end{pmatrix}\\
&= \frac{1}{d}\sum_A F(A\begin{pmatrix}\tau & -1\\ 1 & 0\end{pmatrix}B^{-1})\\
&= \frac{1}{d}\sum_A F(\begin{pmatrix}A\tau & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}d^{-1}(c_1\tau+d_1) & -c_1\\ 0 & (c_1\tau+d_1)^{-1}d\end{pmatrix})\\
&= \frac{1}{d}\sum_A d^m(c_1\tau+d_1)^{-m}\sum_{i=0}^{n}\binom{n}{i}(-c_1)^i(c_1\tau+d_1)^i d^{-i} f_i(A(\tau))\\
&= \sum_A f||_m A. \qquad \square
\end{aligned}$$

In the passage from the second to third equality we have used

$$A\begin{pmatrix}\tau & -1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}A\tau & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}c_1\tau+d_1 & -c_1\\ 0 & (\det A)(c_1\tau+d_1)^{-1}\end{pmatrix}.$$

For the passage from the third to fourth equality we have used the functional equation of $\tilde{f}$ (and hence the $\mathrm{SL}(2,\mathbb{Z})$-invariant function F) with respect to the actions in (6) and (7), see [7] Proposition 6. One can take the representatives

$$\{A_i\} := \left\{\begin{pmatrix}\frac{d}{c} & b\\ 0 & c\end{pmatrix}\middle|\, c \mid d,\ 0 \le b < c\right\}$$

for the quotient $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{Mat}_d(2,\mathbb{Z})$ and so

$$T_d f(\tau) = \frac{1}{d}\sum_{cc'=d,\quad 0\le b<c} c'^m f\Big(\frac{c'\tau+b}{c}\Big). \tag{17}$$

In a similar way to the case of modular forms (see [1] §6) one can check that

$$T_p \circ T_q = \sum_{d|(p,q)} d^{m-1} T_{\frac{pq}{d^2}}.$$

If we write $f = \sum_{n=0}^{\infty} f_n q^n$ then we have:

$$(T_d f)_n = \sum_{c|(d,n)} c^{m-1} f_{\frac{nd}{c^2}}.$$

In particular if we set $n = 0$ then the constant term of $T_d(f)$ is $f_0\sigma_{m-1}(d)$. If f is an eigenvector of T_d and the constant term of f is non-zero then the corresponding eigenvalue is $\sigma_{m-1}(d)$.

7. Refined Hecke operators

Let W_d be the set of subgroups of $\frac{\mathbb{Z}}{d\mathbb{Z}} \times \frac{\mathbb{Z}}{d\mathbb{Z}}$ of order d and S_d be the set (up to isomorphism) of abelian finite groups of order d and generated by at most two elements. We have a canonical surjective map $W_d \to S_d$.

Proposition 5. *We have bijections*

$$\mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{Mat}_d(2,\mathbb{Z}) \cong W_d, \tag{18}$$

$$\mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{Mat}_d(2,\mathbb{Z})/\mathrm{SL}(2,\mathbb{Z}) \cong S_d, \tag{19}$$

both given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{\mathbb{Z}^2}{\mathbb{Z}(a,b)+\mathbb{Z}(c,d)}.$$

If d is square-free then both sides of the second bijection are single points.

Proof. First note that the induced maps are both well-defined. The first bijection is already proved and used in Section 6. The action of $\mathrm{SL}(2,\mathbb{Z})$ from the left on $\mathrm{Mat}_d(2,\mathbb{Z})$ corresponds to the base change in the lattice $\mathbb{Z}(a,b)+\mathbb{Z}(c,d)$ in the right hand side of the bijection. The action of $\mathrm{SL}(2,\mathbb{Z})$ from the right on $\mathrm{Mat}_d(2,\mathbb{Z})$ corresponds to the isomorphism of finite groups in the right hand side of the bijection. □

Any element in S_d is isomorphic to the group $\frac{\mathbb{Z}}{d_2\mathbb{Z}} \times \frac{\mathbb{Z}}{d_1 d_2\mathbb{Z}}$ for some $d_1, d_2 \in \mathbb{N}$ with $d = d_2^2 d_1$. In the right hand side of (19) the corresponding element is represented by the matrix $\begin{pmatrix} d_1 d_2 & 0 \\ 0 & d_2 \end{pmatrix}$. In the geometric context, this means that any isogeny of elliptic curves $E_1 \to E_2$ over an algebraically closed field can be decomposed into $E_1 \xrightarrow{\alpha} E_1 \xrightarrow{\beta} E_2$, where α is the multiplication by d_2 and β is a degree d_1 isogeny with cyclic center. Note that

$$\sigma_1(d) = \sum_{d=d_2^2 d_1} \psi(d_1).$$

We conclude that we have a natural decomposition of both geometric and holomorphic Hecke operators:

$$T_d t = \sum_{d=d_2^2 d_1} d_2^{-m-2} \cdot T^0_{d_1} t, \quad t \in M^n_m \tag{20}$$

where in the geometric context

$$T^0_d : M^n_m \to M^n_m,\ T^0_d(t)(E,\omega) = \frac{1}{d} \sum_{f:E_1\to E,\ \deg(f)=d, \ker(f) \text{ is cyclic}} t(E_1, f^*\omega),$$

and in the holomorphic context

$$T_d^0 : M_m^n \to M_m^n, \quad T_d^0 f = \sum_{A \in (\mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{Mat}_d(2,\mathbb{Z}))^0} f||_m A.$$

Here, $(\mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{Mat}_d(2,\mathbb{Z}))^0$ is the fiber of the map

$$\mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{Mat}_d(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{Mat}_d(2,\mathbb{Z})/\mathrm{SL}(2,\mathbb{Z})$$

over the matrix $\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$. The factor d_2^{-m} in (20) comes from the functional equation of t with respect to the action of $\mathbb{G}_m$ and the second part of Proposition 2. Note that in the geometric context using the Galois action $\mathrm{Gal}(\bar{\mathsf{k}}/\mathsf{k})$, we can see that T_d^0 is defined over the field k and not its algebraic closure. We call T_d^0 the refined Hecke operator. The refined Hecke operator T_d^0 will be used in the next sections. Note that if d is square free then $T_d^0 = T_d$.

8. Proof of Theorem 1

For a holomorphic quasi-modular form of weight m we associate the polynomial

$$P_f^0(x) := \prod_{A \in (\mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{Mat}_d(2,\mathbb{Z}))^0} (x - d \cdot f||_m A) = \sum_{j=0}^{\psi(d)} P_{f,j}^0 x^j.$$

Proposition 6. *$P_{f,j}^0$ is a full quasi-modular form of weight $(\psi(d) - j) \cdot m$.*

Proof. The coefficient $P_{f,j}^0$ of x^j is a homogeneous polynomial with rational coefficients and of degree $\psi(d) - j$ in

$$T_d^0(f^k), \ k = 1, 2, \ldots, \psi(d) - j, \ \mathrm{weight}(T_d^0(f^k)) = k,$$

where $T_d^0 : M \to M$ is the refined d-th Hecke operator defined in Section 7. Here, we have used (15). For instance, the coefficient of $x^{\psi(d)-1}$ is $-d \cdot T_d^0 f$ and the coefficient of $x^{\psi(d)-2}$ is $\frac{d^2}{2}(T_d^0 f)^2 - \frac{d}{2} T_d^0 f^2$. Now the assertion follows from the fact that the Hecke operator T_d^0 sends a quasi-modular form of weight m to a quasi-modular form of weight m. $\square$

Using the fact that the algebra of quasi-modular forms over $\mathbb{Q}$ is isomorphic to $\mathbb{Q}[E_2, E_4, E_6]$ we conclude that $P_f^0(x)$ is a homogeneous polynomial of degree $\psi(d) \cdot m$ in the ring

$$\mathbb{Q}[x, E_2, E_4, E_6], \ \mathrm{weight}(E_{2k}) = k, \quad k = 1, 2, 3, \quad \mathrm{weight}(x) = m.$$

The geometric definition of the polynomial $P_f^0(x)$ is:

$$P_f^0(x)(E,\omega) = \prod (x - f(E_1, g^*\omega)), \tag{21}$$

where the product is taken over all degree d isogenies $g : E_1 \to E$ with cyclic kernel.

Proof of Theorem 1. Let us regard s_i's as holomorphic functions on the upper half plane and t_i's as variables. We define

$$I_{d,i} := P_{s_i}^0(t_i), \quad i = 1, 2, 3.$$

For $i = 1, 2, 3$, $T_d^0 s_i^k$ is a homogeneous polynomial of degree ki in $\mathbb{Q}[s_1, s_2, s_3]$, $\mathrm{weight}(s_i) = i$, and hence, from Proposition 6 it follows that $I_{d,i}(t_i, s_1, s_2, s_3)$ is a homogeneous polynomial of degree $i \cdot \psi(d)$ in the weighted ring (3). We have

$$s_i||_{2i} \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} = d^{2i-1} s_i(d\tau)$$

and so $I_{d,i}(d^{2i} \cdot s_i(d \cdot \tau), s_1(\tau), s_2(\tau), s_3(\tau)) = 0$. By definition $I_{d,i}$ is monic in t_i for $i = 1, 2, 3$. Finally, note that for $i = 2, 3$, $T_d^0 s_i^k$ is a homogeneous polynomial of degree ki in $\mathbb{Q}[s_2, s_3]$, $\mathrm{weight}(s_i) = i$, and so, $I_{d,i}(t_i, s_1, s_2, s_3)$ does not depend on s_1. $\square$

9. Proof of Theorem 2

Since the period map pm is a biholomorphism, it is enough to prove the same statement for the push-forward of R and V_d under the product of two period maps:

$$\mathsf{pm} \times \mathsf{pm} : \mathsf{T} \times \mathsf{T} \to \mathrm{SL}(2, \mathbb{Z}) \backslash \mathcal{P} \times \mathrm{SL}(2, \mathbb{Z}) \backslash \mathcal{P}.$$

First we describe the push-forward of V_d. Using Theorem 4 and the comparison of Hecke operators in both algebraic and complex context, we have:

$$V_d = \{((E_1, \omega_1), (E_2, \omega_2)) \in \mathsf{T} \times \mathsf{T} \mid \exists f : E_1 \to E_2,\ f^*\omega_2 = \omega_1, \\ \ker(f) \text{ is cyclic of order } d\}.$$

Let us now consider the elliptic curves E_i, $i = 1, 2$ as complex curves. For a d-isogeny $f : E_1 \to E_2$ such that $\ker(f)$ is cyclic, we can take a symplectic basis δ_1, γ_1 of $H_1(E_1, \mathbb{Z})$ and δ_2, γ_2 of $H_1(E_2, \mathbb{Z})$ such that

$$f_*\delta_1 = d \cdot \delta_2, \quad f_*\gamma_1 = \gamma_2.$$

Therefore, if the period matrix associated to $(E_i, \{\alpha_i, \omega_i\}, \{\delta_i\, \gamma_i\})$, $i = 1, 2$ is denoted respectively by x and y then

$$x = \pi_d(y) := \begin{pmatrix} y_1 & dy_2 \\ d^{-1}y_3 & y_4 \end{pmatrix}.$$

Therefore, the push-forward of V_d under $\mathsf{pm} \times \mathsf{pm}$ and then its pull-back to $\mathcal{P} \times \mathcal{P}$ is given by:

$$V_d^* = \{(\pi_d(y), y) \mid y \in \mathcal{P}\}.$$

The push-forward of the vector field R_d by $\mathsf{pm} \times \mathsf{pm}$ and then its pull-back in $\mathcal{P} \times \mathcal{P}$ is given by the vector field

$$\mathsf{R}^* = d(y_2\frac{\partial}{\partial y_1} + y_4\frac{\partial}{\partial y_3}) - (x_2\frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_3})$$

where we have used the coordinates (x, y) for $\mathcal{P} \times \mathcal{P}$. Now, it can be easily shown that the above vector field R^* is tangent to V_d^*.

Remark 1. The locus V_d^* contains the one dimensional locus:

$$\tilde{\mathbb{H}} := \{\left(\begin{pmatrix} \tau & -d \\ d^{-1} & 0 \end{pmatrix}, \begin{pmatrix} \tau & -1 \\ 1 & 0 \end{pmatrix}\right) \mid \tau \in \mathbb{H}\}. \tag{22}$$

Note also that the push-forward of the Ramanujan vector field R is tangent to the image of $\mathbb{H} \to \mathcal{P}$ and restricted to this locus it is $\frac{\partial}{\partial \tau}$. Therefore, R^* is tangent to the locus $\tilde{\mathbb{H}}$ and restricted to there is again $\frac{\partial}{\partial \tau}$.

10. Proof of Theorem 3

From Theorem 1 it follows that $I_{d,1}$ is a linear combination of the monomials (4) and $I_{d,i}$, $i = 2, 3$ is a linear combination of the same monomials with $a_1 = 0$. The proof is a slight modification of an argument in holomorphic foliations, see [10]. We prove that $J_{d,i}$'s restricted to V_d are identically zero. We know that $I_{d,i}$ is a linear combination of α_j's with $\mathbb{C}$ (in fact $\mathbb{Q}$) coefficients:

$$I_{d,i} = \sum c_j \alpha_j.$$

Since R_d is tangent to the variety V_d, we conclude that $\mathsf{R}_d^r(\sum c_j \alpha_j) = \sum c_j \mathsf{R}_d^r \alpha_j$ restricted to V_d is zero. This in turn implies that the matrix used in the definition of $J_{d,i}$ restricted to V_d has non-zero kernel and so its determinant restricted to V_d is zero.

11. Examples and final remarks

In order to calculate $I_{d,i}$'s using Theorem 3 we can proceed as follows: We use the Gröbner basis algorithm and find the irreducible components of the affine variety given by the ideal $\langle J_{d,1}, J_{d,2}, J_{d,3}\rangle$ and among them identify the variety V_d. In practice this algorithm fails even for the simplest case $d = 2$. In this case we have $\deg(J_{2,1}) = 42$, $\deg(J_{2,2}) = 40$, $\deg(J_{2,3}) = 69$ and calculating the Gröbner basis of the ideal $\langle J_{d,1}, J_{d,2}, J_{d,3}\rangle$ is a huge amount of computations. We use the q-expansion of t_i's and we calculate $I_{d,i}$, $i = 1, 2, 3$, $d = 2, 3$. We have written powers of t_i in the first row and the corresponding coefficients in the second row. For more examples see the author's web-page.[1]

$$I_{2,1} : \begin{pmatrix} t_1^3 & t_1^2 & t_1 & 1 \\ 1 & -6s_1 & 12s_1^2 - 3s_2 & -8s_1^3 + 6s_1s_2 - 2s_3 \end{pmatrix}$$

$$I_{2,2} : \begin{pmatrix} t_2^3 & t_2^2 & t_2 & 1 \\ 1 & -18s_2 & 33s_2^2 & 484s_2^3 - 500s_3^2 \end{pmatrix}$$

$$I_{2,3} : \begin{pmatrix} t_3^3 & t_3^2 & t_3 & 1 \\ 1 & -66s_3 & -1323s_2^3 + 1452s_3^2 & 10\,584s_2^3s_3 - 10\,648s_3^3 \end{pmatrix}$$

$$I_{3,1} : \begin{pmatrix} t_1^4 & t_1^3 & t_1^2 & t_1 & 1 \\ 1 & -12s_1 & 54s_1^2 - 24s_2 & -108s_1^3 + 144s_1s_2 - 64s_3 & 81s_1^4 - 216s_1^2s_2 - 48s_2^2 + 192s_1s_3 \end{pmatrix}$$

$$I_{3,2} : \begin{pmatrix} t_2^4 & t_2^3 & t_2^2 & t_2 & 1 \\ 1 & -84s_2 & 246s_2^2 & 63\,756s_2^3 - 64\,000s_3^2 & 576\,081s_2^4 - 576\,000s_2s_3^2 \end{pmatrix}$$

$$I_{3,3} : \begin{pmatrix} t_3^4 & t_3^3 & t_3^2 & t_3 & 1 \\ 1 & -732s_3 & -169\,344s_2^3 + 171\,534s_3^2 & 11\,007\,360s_2^3s_3 - 11\,009\,548s_3^3 & -502\,020\,288s_2^6 + 939\,266\,496s_2^3s_3^2 - 437\,245\,479s_3^4 \end{pmatrix}$$

The polynomials $I_{d,i}$, $i = 2, 3$ have integral coefficients. This follows from the formula (21) and the fact that geometric modular forms can be defined over a field of characteristic p. Our computations show that $I_{d,1}$ has also integral coefficients. However, the notion of a geometric quasi-modular form is only elaborated over a field of characteristic zero (see [9]). The proof of such a statement for $I_{d,1}$ would need a reformulation of the definition of geometric quasi-modular forms.

One of the fascinating applications of modular curves $X_0(d)$ is formulated in the Shimura–Taniyama conjecture, now the modularity theorem. It states that any elliptic curve over $\mathbb{Q}$ must appear in the decomposition of the Jacobian of $X_0(d)$, where d is the conductor of the elliptic curve (see [12] for the case of semi-stable elliptic curves and [2] for the case of all elliptic curves). It would be interesting to know whether the differential equation approach of the present article to modular curves has some implications in this direction.

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