GAUSS-MANIN CONNECTION IN DISGUISE: GENUS TWO CURVES

JIN CAO, HOSSEIN MOVASATI AND SHING-TUNG YAU

Abstract. We describe an algebra of meromorphic functions on the Siegel domain of genus two which contains Siegel modular forms for an arithmetic index six subgroup of the symplectic group, which is closed under three canonical derivations of the Siegel domain. The main ingredients of our study are the moduli of enhanced genus two curves, Gauss-Manin connection and the modular vector fields living on such moduli spaces.

1. Introduction

Igusa in [Igu67] computes 5 explicit generators $E_4, E_6, \chi_{10}, \chi_{12}, \chi_{35}$ of the ring of Siegel modular forms of genus 2 for $\mathrm{Sp}(4, \mathbb{Z})$. The first 4 are given by Eisenstein series and they generate the ring of Siegel modular forms of even weight. The first example of a differential equation for a Siegel modular form is due to Resnikoff in [Res70a, Res70b]. He computes in [Res70a, page 496] an order-eight differential equation for E_4 (in his notation ψ_4) and expresses the difficulty to find differential equations for E_6. Bertrand and Zudilin in [BZ03] show that the transcendental degree of the field generated by Siegel modular forms of genus g and their derivations is $2g^2 + g$. They further describe many differential equations involving theta constants, see also [Zud00, Theorem 1]. Several authors have also studied differential operators D on automorphic forms F such that DF restricted to some lower dimensional domain is also automorphic, see Ibukiyama's paper [Ibu99] and the references therein.

We are motivated by the general philosophy that for dealing with derivations of modular and automorphic forms, there is a moduli space equipped with canonical vector fields which is responsible for all the involved computations. In [Mov08] it is remarked that the Ramanujan's differential equation between Eisenstein series can be interpreted as a vector field on the moduli of elliptic curves enhanced with a suitable frame of the first cohomology bundle, see also [Mov12] for further details. Such a moduli space in the case of Calabi-Yau varieties is worked out in [Mov15, AMSY16, Mov12] and it is the building block of the new theory of Calabi-Yau modular forms. This is mainly inspired by many computations in string theory and in particular Yamaguchi and Yau's polynomials description in [YY04] of generating functions of genus g Gromov-Witten invariants, see also [Ali13] and the references therein. The case of principally polarized abelian varieties was initiated in [Mov13] and the construction of Ramanujan-type vector fields is done in [Mov20] for a complex moduli framework and in [Fon18] for a moduli stack framework. In this paper we work out such a moduli space for genus two curves with a marked

Weierstrass point and the outcome of our approach in terms of Siegel modular forms is the following.

Let Γ be the subgroup of $\mathrm{Sp}(4,\mathbb{Z})$ generated by four matrices:

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&-1&1\\0&1&1&-1\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&1&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&0&-1\\0&0&1&0\\0&0&0&1\end{pmatrix}, \tag{1.1}$$

which turns out to be of index 6, see §6.2, and $\mathbb{H}_2$ be the Siegel upper half plane of genus 2. We denote an element of $\mathbb{H}_2$ by $\tau=\begin{pmatrix}\tau_1&\tau_3\\\tau_3&\tau_2\end{pmatrix}$.

Theorem 1.1. *There are meromorphic functions $X_i,\ i=1,2,\dots,153$ on $\mathbb{H}_2$ with possible poles along the $\mathrm{Sp}(4,\mathbb{Z})$-transforms of the locus $\tau_3=0$ such that*

(1) 5 of X_i's are meromorphic Siegel modular forms of weights $1,2,3,4,5$ for Γ. These are denoted by $X_i=T_{4i},\ i=1,2,3,4,5,$ and we have a quadratic relation of the form $T_{16}^2=T_{20}T_{12}$. The Siegel modular form T_{20} is holomorphic and vanishes in $\tau_3=0$.

(2) The ideal I of all polynomial relations between X_i's is defined over $\mathbb{Q}$ (by the first item we have $X_4^2-X_5X_3\in I$).

(3) The affine variety $\mathrm{Spec}(\mathbb{Q}[X_1,\cdots,X_{153}]/I)$ is isomorphic to an open subset of the weighted projective space $\mathbb{P}^{6,8,10,3,3,3,3,1,1,1,1}$, which is 10-dimenisonal[1].Its complement is the zero set of a degree 4 homogeneous polynomial.

(4) The derivation $\frac{\partial X_i}{\partial \tau_k},\quad i=1,2,\dots,k=1,2,3$ multiplied with the Siegel modular form X_5 are polynomials in X with $\mathbb{Q}$ coefficients.

We remark here that the number of the meromorphic functions is obtained by the explicit computation. The proof of Theorem 1.1, and also the structure of the paper is as follows. In §2 we first use some classical statements on the moduli of genus two curves in order to construct the moduli of genus two curves endowed with a Weierstrass point. Then using a well-known basis of de Rham cohomologies, we compute the Gauss-Manin connection over such a moduli. In §3 we use the hypercohomology definition of de Rham cohomology and compute the cup product in de Rham cohomology. The content of these two sections are needed in order to construct the moduli space T of genus two curves enhanced with a basis of de Rham cohomologies with some compatibility conditions. This is done in §4. In §5 using the Gauss-Manin connection matrix on T, we construct three vector fields $\mathsf{R}_k,\ \ k=1,2,3$ in T which will be eventually interpreted as derivations $\frac{\partial}{\partial\tau_k},\ k=1,2,3$ in §7. The bridge between regular functions in T, and meromorphic functions X_i in the Siegel domain is the t-map which is explained in §7. Our functions X_i have functional equations with respect to the underlying monodromy group and in §6 we compute such a group explicitly. This turns out to be the group Γ. We would like to highlight that instead of genus two curves we could start with the Clingher-Doran family of

[1]The dimension of the affine variety is exactly the same as the expected transcendental degree of the field generated by Seigel modular forms of degree two and their derivations, which is showed in the work of Bertrand and Zudilin.

K3 surfaces in [CD12], as some of the ingredients of our work has been worked out in [DMWH16]. The computations in this case become heavier and this will be exploited in a future work.

The main body of the paper was written during the visit of the first two authors in CMSA, Harvard university. They would like to thank the institute for financial support and its lovely and stimulating ambient. The last section was worked out during the first two authors's stays at the Issac Newton Institute for Mathematical Sciences, Cambridge university. They are very grateful to the institution for the hospitality and financial support. The first author is partially supported by the National Science Foundation for Young Scientists of China (Grant No. 11901334).

2. A geometric set up

2.1. **Hyperelliptic curves.** Fix a positive integer d. Given $t=(t_1,\cdots,t_d)\in V:=\mathbb{C}^d$, we let L_t be the affine curve defined by $y^2=f(x)$, where

$$f(x)=x^d+\sum_{k=1}^{d}t_kx^{d-k}. \tag{2.1}$$

We set: $\deg x=2, \deg y=d$ and in this way P is a tame polynomial in the sense of [Mov19, Chapter 10]. Let $\alpha_i,\quad i=1\cdots d$ be the formal roots of $f(x)$, i.e., $f(x)=\prod_{i=1}^{d}(x-\alpha_i)$, and then we define the discriminant Δ of $f(x)$ as

$$\Delta=\prod_{1\leq i<j\leq d}(\alpha_i-\alpha_j)^2. \tag{2.2}$$

Because of the fundamental theorem of symmetric polynomials, it is known that $\Delta\in\mathbb{Q}[t]$. Hence, if $t\in V-\{\Delta=0\}$ then L_t is a smooth curve.

From now on assume that $d=2g+1$ is odd. For $t\in V$, we let Y_t be the completion of L_t which is obtained by adding a point at infinity $[0:1:0]\in\mathbb{P}^2$ to L_t, and then a desingularization of it by performing blow-ups at $[0:1:0]$. The projective curve Y_t is smooth of genus g which is a hyperelliptic curve and the complement of L_t in Y_t consists of single point that we denote it by ∞. It is known that $H^1_{dR}(L_t)\cong H^1_{dR}(Y_t)$ has a basis consisting of

$$\left[\frac{\mathrm{d}x}{y}\right],\left[\frac{x\mathrm{d}x}{y}\right],\cdots,\left[\frac{x^{2g-1}\mathrm{d}x}{y}\right], \tag{2.3}$$

from which the first g elements form a basis of $H^0(Y_t,\Omega^1_{Y_t})$, see [Mov19, Theorem 10.1].

Assume that d is even, say $2g+2$. The completion of (2.1) also gives us a hyperelliptic curve Y_t. However, L_t is corresponding to an open subvariety of Y_t by removing two points. In this case, $H^1_{dR}(L_t)$ also has a basis consisting of

$$\left[\frac{\mathrm{d}x}{y}\right],\left[\frac{x\mathrm{d}x}{y}\right],\cdots,\left[\frac{x^{2g}\mathrm{d}x}{y}\right]. \tag{2.4}$$

The first g elements are holomorphic, but the rest may have non-vanishing residues. In order to get a basis of $H^1_{dR}(Y_t)$, we need to find $2g$ elements in $H^1_{dR}(L_t)$ without residues. This basis can be found after choosing a coordinate function around the two points at infinity, §3.1. As an example for

$$y^2 = f(x) = x^6 + t_2x^4 + t_3x^3 + t_4x^2 + t_5x + t_6.$$

we have

$$\mathrm{Res}_\infty \frac{x^2\mathrm{d}x}{y} = -1, \quad \mathrm{Res}_\infty \frac{x^3\mathrm{d}x}{y} = 0, \quad \mathrm{Res}_\infty \frac{x^4\mathrm{d}x}{y} = \frac{t_2}{2},$$

where ∞ is one of the two points at infinity, and so, we may choose the following basis of $H^1_{dR}(Y_t)$:

$$\frac{\mathrm{d}x}{y}, \quad \frac{x\mathrm{d}x}{y}, \quad \frac{x^3\mathrm{d}x}{y}, \quad \frac{t_2}{2}\frac{x^2\mathrm{d}x}{y} + \frac{x^4\mathrm{d}x}{y}.$$

2.2. **Gauss-Manin connections.** We consider the family of hyperelliptic curves Y_t over V and let $H^1_{dR}(Y/V)$ be the first cohomology bundle, that is, its fiber over $t \in V$ is $H^1_{dR}(Y_t)$. By abuse of notation, we use ω_i to denote the global section of $H^1_{dR}(Y/V)$, whose value at t is consisiting of the basis of $H^1_{dR}(Y_t)$ as we describe in the previous subsection. After fixing these sections, the Gauss-Manin connection

$$\nabla : H^1_{dR}(Y/V) \to \Omega^1_V \otimes_{\mathcal{O}_V} H^1_{dR}(Y/V)$$

can be expressed as:

$$\nabla \begin{pmatrix} \omega_1 \\ \omega_2 \\ \cdots \\ \omega_{2g} \end{pmatrix} = \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,2g} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,2g} \\ \cdots & & & \\ b_{2g,1} & b_{2g,2} & \cdots & b_{2g,2g} \end{pmatrix} \otimes \begin{pmatrix} \omega_1 \\ \omega_2 \\ \cdots \\ \omega_{2g} \end{pmatrix}, \tag{2.5}$$

where $b_{i,j}$ are meromorphic differential 1-forms in $t_1, \cdots, t_d$ with pole order one along $\Delta = 0$, see for instance [Mov19, §12.5]. If we denote the matrix in the right hand side by B, then we may write B as

$$B = \widetilde{B}_1\mathrm{d}t_1 + \cdots + \widetilde{B}_d\mathrm{d}t_d, \tag{2.6}$$

where $\widetilde{B}_i$ are $2g \times 2g$ matrices with entries lying in $\frac{1}{\Delta}\mathbb{Q}[t_1, t_2, \cdots, t_d]$. For simplicity, we decompose the Gauss-Manin connection matrix B as

$$\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}. \tag{2.7}$$

where B_i's are $g \times g$ matrices. We also use the notation $B_i = \sum_{m=1}^d B_{m,i} \cdot \mathrm{d}t_m$, where $B_{m,i}$ are $g\times g$-matrices. In general the expressions of Gauss-Manin matrices are huge. However, it turns out that the inverses of Gauss-Manin matrices are much simpler. Below is the data of the Gauss-Manin connection for a family of hyperelliptic curves of genus 2 given by

$$Y_t : \quad y^2 = x^5 + t_2x^3 + t_3x^2 + t_4x + t_5. \tag{2.8}$$

These are computed by the procedure `gaussmaninmatrix` of the library `foliation.lib` [2] of SINGULAR, [GPS01], and the relevant algorithms are explained in [Mov11, Chapter 4], see also [Mov19, Chapter 12]. It mainly uses the notion of Brieskorn modules

[2] `http://w3.impa.br/~hossein/foliation-allversions/foliation.lib`

which captures the de Rham cohomology of families of affine varieties, and pole order reduction of differential forms, and hence, it is slightly different from Griffith-Dwork's method for compact hypersurfaces.

(2.9)

$$
\begin{aligned}
5^5\cdot\Delta \;=\;& 108t_2^5t_5^2-72t_2^4t_3t_4t_5+16t_2^4t_4^3+16t_2^3t_3^3t_5-4t_2^3t_3^2t_4^2-900t_2^3t_4t_5^2+825t_2^2t_3^2t_5^2\\
&+560t_2^2t_3t_4^2t_5-128t_2^2t_4^4-630t_2t_3^3t_4t_5+144t_2t_3^2t_4^3-3750t_2t_3t_5^3+2000t_2t_4^2t_5^2\\
&+108t_3^5t_5-27t_3^4t_4^2+2250t_3^2t_4t_5^2-1600t_3t_4^3t_5+256t_4^5+3125t_5^4,
\end{aligned}
$$

$$
\widetilde{B}_2^{-1}=\begin{pmatrix}
-\frac{72t_2^2t_5^2-56t_2t_3t_4t_5+12t_2t_4^3+16t_3^3t_5-4t_3^2t_4^2}{24t_2t_5^2-12t_3t_4t_5+3t_4^3} & \frac{6t_2^2t_4t_5-8t_2t_3^2t_5+2t_2t_3t_4^2+80t_3t_5^2-30t_4^2t_5}{24t_2t_5^2-12t_3t_4t_5+3t_4^3} & -\frac{40t_2t_5^2-28t_3t_4t_5+8t_4^3}{8t_2t_5^2-4t_3t_4t_5+t_4^3} & -\frac{6t_2t_4t_5-8t_3^2t_5+2t_3t_4^2}{24t_2t_5^2-12t_3t_4t_5+3t_4^3}\\
-\frac{24t_2t_3t_5^2-2t_2t_4^2t_5-8t_3^2t_4t_5+2t_3t_4^3}{24t_2t_5^2-12t_3t_4t_5+3t_4^3} & -\frac{36t_2^2t_5^2-16t_2t_3t_4t_5+4t_2t_4^3-20t_4t_5^2}{24t_2t_5^2-12t_3t_4t_5+3t_4^3} & \frac{2t_4^2t_5}{8t_2t_5^2-4t_3t_4t_5+t_4^3} & -\frac{60t_2t_5^2-32t_3t_4t_5+8t_4^3}{24t_2t_5^2-12t_3t_4t_5+3t_4^3}\\
\frac{152t_2t_4t_5^2+80t_3^2t_5^2-116t_3t_4^2t_5+24t_4^4}{24t_2t_5^2-12t_3t_4t_5+3t_4^3} & \frac{184t_2t_3t_5^2-10t_2t_4^2t_5-72t_3^2t_4t_5+18t_3t_4^3-400t_5^3}{24t_2t_5^2-12t_3t_4t_5+3t_4^3} & \frac{32t_2^2t_5^2-16t_2t_3t_4t_5+4t_2t_4^3-40t_4t_5^2}{8t_2t_5^2-4t_3t_4t_5+t_4^3} & -\frac{40t_3t_5^2-10t_4^2t_5}{24t_2t_5^2-12t_3t_4t_5+3t_4^3}\\
\frac{10t_5}{3} & \frac{8t_4}{3} & 2t_3 & \frac{4t_2}{3}
\end{pmatrix},
$$

$$
\widetilde{B}_3^{-1}=\begin{pmatrix}
\frac{6t_2t_4t_5-8t_3^2t_5+2t_3t_4^2}{4t_3t_5-t_4^2} & -\frac{12t_2t_3t_5-4t_2t_4^2}{4t_3t_5-t_4^2} & \frac{10t_4t_5}{4t_3t_5-t_4^2} & -\frac{20t_3t_5-8t_4^2}{4t_3t_5-t_4^2}\\
\frac{60t_2t_5^2-32t_3t_4t_5+8t_4^3}{4t_3t_5-t_4^2} & \frac{10t_2t_4t_5-24t_3^2t_5+6t_3t_4^2}{4t_3t_5-t_4^2} & -\frac{16t_2t_3t_5-4t_2t_4^2-100t_5^2}{4t_3t_5-t_4^2} & \frac{30t_4t_5}{4t_3t_5-t_4^2}\\
10t_5 & 8t_4 & 6t_3 & 4t_2\\
-\frac{32t_2^2t_5^2-16t_2t_3t_4t_5+4t_2t_4^3}{4t_3t_5-t_4^2} & -\frac{16t_2^2t_4t_5-32t_2t_3^2t_5+8t_2t_3t_4^2-40t_3t_5^2+10t_4^2t_5}{12t_3t_5-3t_4^2} & \frac{16t_2^2t_3t_5-4t_2^2t_4^2-160t_2t_5^2+32t_3t_4t_5-8t_4^3}{12t_3t_5-3t_4^2} & -\frac{16t_2t_4t_5-8t_3^2t_5+2t_3t_4^2}{4t_3t_5-t_4^2}
\end{pmatrix},
$$

$$
\widetilde{B}_4^{-1}=\begin{pmatrix}
\frac{20t_3t_5-8t_4^2}{3t_4} & \frac{10t_2t_5-2t_3t_4}{t_4} & -\frac{4t_2}{3} & \frac{50t_5}{3t_4}\\
-10t_5 & -8t_4 & -6t_3 & -4t_2\\
-\frac{16t_2t_3t_5-4t_2t_4^2}{3t_4} & -\frac{8t_2^2t_5-10t_4t_5}{t_4} & -\frac{4t_2^2-24t_4}{3} & -\frac{40t_2t_5-18t_3t_4}{3t_4}\\
\frac{16t_2t_4t_5-8t_3^2t_5+2t_3t_4^2}{3t_4} & -\frac{4t_2t_3t_5-4t_2t_4^2}{t_4} & \frac{6t_2t_3+10t_5}{3} & \frac{4t_2^2t_4-20t_3t_5+8t_4^2}{3t_4}
\end{pmatrix},
$$

$$
\widetilde{B}_5^{-1}=\begin{pmatrix}
-\frac{10t_5}{3} & -\frac{8t_4}{3} & -2t_3 & -\frac{4t_2}{3}\\
\frac{4t_2t_4}{5} & \frac{8t_2t_3-50t_5}{5} & \frac{12t_2^2-40t_4}{5} & -6t_3\\
\frac{40t_2t_5-18t_3t_4}{15} & \frac{20t_2t_4-36t_3^2}{15} & -\frac{18t_2t_3-50t_5}{5} & -\frac{4t_2^2-24t_4}{3}\\
-\frac{4t_2^2t_4-20t_3t_5+8t_4^2}{15} & -\frac{8t_2^2t_3-80t_2t_5+6t_3t_4}{15} & -\frac{4t_2^3-12t_2t_4}{5} & \frac{6t_2t_3+10t_5}{3}
\end{pmatrix}.
$$

We could also compute the Gauss-Manin connection matrices of the family $y^2 = x^6 + t_2x^4 + t_3x^3 + t_4x^2 + t_5x + t_6$ which is too big to fit into this paper. For the computer data of this see the second author's webpage.

3. Cup product in de Rham cohomology

In $H^1_{\mathrm{dR}}(Y_t)$ we have a natural pairing which is $\langle \alpha, \beta \rangle := \frac{1}{2\pi i}\int_{Y_t} \alpha \cup \beta$ for $\alpha, \beta \in H^1_{\mathrm{dR}}(Y_t)$. In this section we want to compute this pairing in the basis ω_i:

$$\Omega := [\langle \omega_i, \omega_j \rangle] = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = \begin{bmatrix} 0 & \Omega_2 \\ -\Omega_2^{\mathrm{tr}} & \Omega_4 \end{bmatrix}, \tag{3.1}$$

where Ω_i's are $g \times g$ matrices. Note that the first g differential forms are holomorphic and then Riemann's bilinear relations lead to the vanishing of Ω_1. Applying the Gauss-Manin connection on Ω, we get

$$d\Omega = B\Omega + \Omega B^{\mathrm{tr}}, \tag{3.2}$$

where B is the Gauss-Manin connection matrix written in the basis ω_i. In order to get entries of Ω, one needs to solve the above equation. This seems to be a quite difficult task. In this section we give a direct way to compute Ω.

3.1. Coordinate functions at the infinity point. We write Y_t in homogeneous coordinates: $y^2z^{d-2} = F(x, z)$, where $F(x, 1) = f(x)$ and cover it with two open sets: $U_0 := \{z \neq 0\}$ and $U_1 := \{y \neq 0\}$. It is easy to see that $[0 : 1 : 0]$ is the only possible singular point of Y_t. It lies in U_1. We do blow-ups to determine the coordinate function at the desingularization of this point. We first perform $(x, z) = (x_1, t_1x_1)$. Then Y_t in U_1 is given by

$$(t_1x_1)^{d-2} = F(x_1, t_1x_1), \quad \text{that is ,} \quad t_1^{d-2} = x_1^2F(1, t_1).$$

For the next, we do the blow-up: $(x_1, t_1) = (x_2t_2, t_2)$. Then we get:

$$t_2^{d-2} = (x_2t_2)^2F(1, t_2), \quad \text{that is ,} \quad t_2^{d-4} = x_2^2F(1, t_2).$$

We do more blow-ups. If d is odd, we will stop with an equation $t_{\frac{d-1}{2}} = x^2_{\frac{d-1}{2}}F(1, t_{\frac{d-1}{2}})$, whose linear term is $t_{\frac{d-1}{2}}$. We may choose $x_{\frac{d-1}{2}}$ as the coordinate function. Pulling pack along all these blow-up maps and pulling back along the transformation map from the U_0-chart to the U_1-chart, we get the coordinate function $t = \frac{x^{\frac{d-1}{2}}}{y}$ written in the coordinate system of U_0. If d is even, we will stop with an equation $t^2_{\frac{d-2}{2}} = x^2_{\frac{d-2}{2}}F(1, t_{\frac{d-2}{2}})$ which shows that Y_t has two points at infinity. In this case, we may choose $x_{\frac{d-2}{2}}$ as a coordinate function. Pulling back will lead us to the coordinate function $t = \frac{x^{\frac{d-2}{2}}}{y}$.

3.2. De Rham cohomology as hypercohomology. In this subsection we focus on our genus two curve Y_t, given by (2.8). For a general genus two curve Y, the algebraic de Rham cohomology of Y can be described as:

$$H^1_{dR}(Y) \cong \frac{\{(\omega_0, \omega_1) \in \Omega^1_{U_0} \times \Omega^1_{U_1} | \omega_1 - \omega_0 \in \mathrm{d}(\Omega^0_{U_0 \cap U_1})\}}{\mathrm{d}\Omega^0_{U_0} \times \mathrm{d}\Omega^0_{U_1}}, \tag{3.3}$$

and

$$H^2_{dR}(Y) \cong \frac{\Omega^1_{U_0\cap U_1}}{\Omega^1_{U_0}+\Omega^1_{U_1}+\mathrm{d}\Omega^0_{U_0\cap U_1}}, \tag{3.4}$$

This follows from the de Rham-Cech double complex of classical hypercohomology defined by A. Grothendieck in [Gro66], see for instance [Mov12, §2.6]. Under the first isomorphisms (3.3), the basis $\omega_i = \frac{x^i\mathrm{d}x}{y}(i=0,\cdots,3)$ can be represented as

$$\omega_i \to \begin{cases} (\frac{x^i\mathrm{d}x}{y}|_{U_0}, \frac{x^i\mathrm{d}x}{y}|_{U_1}), & i=0,1,\\ (\frac{x^i\mathrm{d}x}{y}|_{U_0}, \frac{x^i\mathrm{d}x}{y}|_{U_1}+\mathrm{d}(P_i(\frac{x^2}{y}))), & i=2,3, \end{cases} \tag{3.5}$$

for some polynomials P_i such that the underlying sum becomes holomorphic at infinity. In the next subsection, we explain how to compute P_2, P_3.

3.3. **Computing the cup product.** In the $U_1 := \{y \neq 0\}$, the curve Y_t is given by:

$$z^3 = x^5 + t_2x^3z^2 + t_3x^2z^3 + t_4xz^4 + t_5z^5.$$

In this chart, $\frac{x^i\mathrm{d}x}{y}$ becomes $\frac{x^i\mathrm{d}x}{z^i} - \frac{x^{i+1}\mathrm{d}z}{z^{i+1}}$. We may choose the coordinate function $t = \frac{x^2}{y}$ on U_0 as explained at the beginning of this section. On the U_0-chart, along $t=0$, we may compute the local t-expansion of (x,y): assume that

$$x = t^{-2} + a_{-1}t^{-1} + a_0 + a_1t + a_2t^2 + \cdots, \tag{3.6}$$

then

$$y = t^{-5} + (\frac{5}{2})t^{-4} + (\frac{15}{8}a_{-1}^2 + \frac{5}{2}a_0)t^{-3} + (\frac{5}{16}a_{-1}^3 + \frac{15}{4}a_{-1}a_0 + \frac{5}{2}a_1) + \cdots. \tag{3.7}$$

Under the coordinate function $t = \frac{x^2}{y}$, we may solve these coefficients

$$a_{-1} = a_0 = a_1 = 0, a_2 = -t_2, a_3 = 0, \ldots.$$

Therefore, for $\frac{x^2\mathrm{d}x}{y}$, we may choose a representative $\frac{x^2\mathrm{d}x}{y} - 2\mathrm{d}((\frac{x^2}{y})^{-1})$, which is algebraic in U_1. Similarly for $\frac{x^3\mathrm{d}x}{y}$, we may choose a representative $\frac{x^3\mathrm{d}x}{y} - \frac{2}{3}\mathrm{d}((\frac{x^2}{y})^{-3})$. Hence, we have:

$$P_2(t) = -2t^{-1}, \qquad P_3(t) = -\frac{2}{3}t^{-3}.$$

According to the cup product formula given in [Mov12, §2.10], for any $0 \le i,j \le 3$, we have:

$$\frac{x^i\mathrm{d}x}{y} \cup \frac{x^j\mathrm{d}x}{y} = P_j(\frac{x^2}{y})\frac{x^i\mathrm{d}x}{y} - P_i(\frac{x^2}{y})\frac{x^j\mathrm{d}x}{y} + P_i\mathrm{d}P_j, \tag{3.8}$$

and

(3.9)
$$\left\langle \frac{x^i\mathrm{d}x}{y}, \frac{x^j\mathrm{d}x}{y} \right\rangle = \text{ the residue of } P_j(\frac{x^2}{y})\frac{x^i\mathrm{d}x}{y} - P_i(\frac{x^2}{y})\frac{x^j\mathrm{d}x}{y} + P_i\mathrm{d}P_j \text{ at } \infty.$$

We have $P_0 = P_1 = 0$ and P_2, P_3 are defined above and so we get:

$$\Omega = \begin{pmatrix} 0 & 0 & 0 & \frac{4}{3} \\ 0 & 0 & 4 & 0 \\ 0 & -4 & 0 & \frac{4}{3}t_2 \\ -\frac{4}{3} & 0 & -\frac{4}{3}t_2 & 0 \end{pmatrix}. \tag{3.10}$$

It can be easily verified that Ω satisfies the differential equation (3.2).

4. Moduli spaces

In this section we introduce three types of moduli spaces. The first one is the classical moduli of curves. For the second moduli, the curves are enhanced with differential 2-forms. This ultimately leads us to classical Siegel modular forms. In the third moduli, the curves are enhanced with a frame of differential 1-forms. This leads us to a natural geometric framework for derivations of Siegel modular forms.

4.1. **Moduli space I.** We recall that the moduli space $\mathcal{M}_2$ of curves of genus 2 can be described as the GIT quotient associated to the action of $GL_2(\mathbb{C})$ on the space of binary sextics with non-vanishing discriminants, see for instance [CFv17, §4] and the appendix of [EG98]. Precisely, such a binary sextics $\tilde{Q}(x_1, x_2)$ determines an affine equation $C : y^2 = Q(x)$, where $Q(x) = \tilde{Q}(x, 1)$, which is a curve of genus 2. Moreover, the curve C comes with a basis of differentials $\frac{\mathrm{d}x}{y}$ and $\frac{x\mathrm{d}x}{y}$. The group $GL_2(\mathbb{C})$ acts on these data by

$$x \to \frac{ax+b}{cx+d}, y \to \frac{y}{(cx+d)^3}, \begin{pmatrix} \frac{\mathrm{d}x}{y} \\ \frac{x\mathrm{d}x}{y} \end{pmatrix} \to \det(A) \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}x}{y} \\ \frac{x\mathrm{d}x}{y} \end{pmatrix}.$$

The six zeros of Q with $y = 0$ gives us 6 Weierstrass points of C. Now we assume that x divides $Q(x)$. After a change of variables:

$$x \to \frac{1}{x}, \quad y \to \frac{y}{x^3},$$

we get a curve $y^2 = f(x)$, where $f(x) = x^6 Q(\frac{1}{x})$ is a polynomial of degree 5. The subgroup B of $GL_2(\mathbb{C})$ fixing the root $x = 0$ of $f(x)$, consists of lower triangular matrices of the form $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$. It acts on the parameter space of $y^2 = f(x)$ (coefficients of the polynomial f) which is induced by

$$x \to \frac{c+dx}{a}, y \to \frac{y}{a^3}, \tag{4.1}$$

This is compatible with its action on the corresponding sextic. Notice that each orbit of $y^2 = f(x)$ under the action of B can be chosen as (2.8). Therefore, giving a quintic with non-vanishing discriminant will lead to a $GL_2(\mathbb{C})$-orbit of the equation $y^2 = xP(x)$ which is just a genus 2 curve with a Weierstrass point $(0, 0)$. For the family (2.8) the action of B reduces to the action of $\mathbb{G}_m = \mathbb{C}^*$ given by:

$$(x, y) \to (a^2 x, a^5 y), t \to t \bullet a : (t_2, t_3, t_4, t_5) \to (a^{-4}t_2, a^{-6}t_3, a^{-8}t_4, a^{-10}t_5) \tag{4.2}$$

for $a \in \mathbb{G}_m$. The curve (2.8) after the change of variables $y \to \frac{y}{x^3}$, $x \to \frac{1}{x}$ has the form $y^2 = t_5x^6 + t_4x^5 + t_3x^4 + t_2x^3 + x$, and so, for the moduli of genus two curve we have to remove $t_5 = 0$. We conclude that

Proposition 4.1. *The moduli space $\mathcal{M}_2(w)$ of hyperelliptic curves of genus two and with a marked Weierstrass point is $\mathbb{P}^{4,6,8,10}\backslash\{\Delta\cdot t_5 = 0\}$. The hyperelliptic curve together with its Weierstrass point over the point $t = [t_2 : t_3 : t_4 : t_5]$, is given by*

$$(y^2 = x^5 + t_2x^3 + t_3x^2 + t_4x + t_5, [0 : 1 : 0]).$$

It seems natural to choose the following modular coordinates on $\mathcal{M}_2(w)$:

$$j_2 := \frac{t_2^5}{t_5^2},\ j_3 := \frac{t_3^5}{t_5^3},\ j_4 := \frac{t_4^5}{t_5^3}. \tag{4.3}$$

4.2. **Moduli space II.** In this section we define:

$$T_4 := t_2,\quad T_8 := t_4,\quad T_{12} := t_3^2,\quad T_{16} := t_3t_5,\quad T_{20} := t_5^2. \tag{4.4}$$

We can write the discriminant Δ in (2.9) as a polynomial in T_i's and for simplicity we denote it again by Δ; being clear in the context whether it depends on t or T. For a curve Y of genus 2 the abstract wedge product $H^2_{\mathrm{dR}}(J(Y)) := \wedge H^1_{\mathrm{dR}}(Y)$ has a one dimensional subspace $F^2H^2_{dR}(J(Y))$ which is generated by the wedge product of holomorphic 1-forms in Y.

Proposition 4.2. *The moduli space $\mathbf{S}$ of triples (Y, P, ω), where (Y, P) is as before and $\omega \in F^2H^2_{dR}(J(Y))$, is*

$$\mathbf{S} = \mathrm{Spec}(\mathbb{C}[T_4, T_8, T_{12}, T_{16}, T_{20}, \frac{1}{T_{20}\Delta}]/\langle T_{16}^2 - T_{12}T_{20}\rangle).$$

The corresponding triple over a point is given by:

$$(y^2 = x^5 + t_2x^3 + t_3x^2 + t_4x + t_5, [0 : 1 : 0], \left[\frac{\mathrm{d}x}{y}\right] \wedge \left[\frac{x\mathrm{d}x}{y}\right]).$$

and so, we do not have a universal family over $\mathbf{S}$.

Proof. For a triple (Y, P, ω), by Proposition 4.1 we can assume that (Y, P) has the form

$$(y^2 = x^5 + t_2x^3 + t_3x^2 + t_4x + t_5, [0 : 1 : 0]).$$

Because of $\dim F^2H^2_{dR}(J(Y)) = 1$, $\omega = k[\frac{\mathrm{d}x}{y}] \wedge [\frac{x\mathrm{d}x}{y}]$ for some $k = \mathbb{C}^*$. Under the action of $a \in \mathbb{G}_m$, we have:

$$(t_2, t_3, t_4, t_5, k\left[\frac{\mathrm{d}x}{y}\right] \wedge \left[\frac{x\mathrm{d}x}{y}\right]) \to (a^{-4}t_2, a^{-6}t_3, a^{-8}t_4, a^{-10}t_5, a^{-4}k\left[\frac{\mathrm{d}x}{y}\right] \wedge \left[\frac{x\mathrm{d}x}{y}\right]).$$

Hence two triples (Y_1, P_1, ω_1) with t-coordinates (t_2, t_3, t_4, t_5) and (Y_2, P_2, ω_2) with t-coordinates $(\tilde{t}_2, \tilde{t}_3, \tilde{t}_4, \tilde{t}_5)$ are isomorphic if and only $t = \tilde{t}$ or $t = (\tilde{t}_2, -\tilde{t}_3, \tilde{t}_4, -\tilde{t}_5)$. Therefore, the moduli space of (Y, P, ω) is isomorphic to $\mathrm{Spec}(\mathbb{C}[t_2, t_3^2, t_4, t_5^2, t_3t_5, \frac{1}{t_5^2\Delta}])$. $\square$

4.3. **Moduli space III.** Now let us consider the moduli of

$$(Y, P, \alpha_1, \alpha_2, \alpha_3, \alpha_4),$$

where (Y, P) are as before, and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is a basis of $H^1_{dR}(Y)$ such that

- $\alpha_1, \alpha_2 \in F^1 H^1_{dR}(Y)$;
- the intersection form in the basis (α_i) is:

$$\Phi = [\langle \alpha_i, \alpha_j \rangle] = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{4.5}$$

.

In the following, we will describe such a moduli space, which is denoted by T. We recall that $\langle \alpha_i, \alpha_j \rangle := \mathbf{Tr}(\alpha_i \cup \alpha_j) = \frac{1}{2\pi i} \int_{Y(\mathbb{C})} \alpha_i \cup \alpha_j$. In order to construct the moduli space T, we take a 4×4 matrix $S = \begin{pmatrix} S_1 & 0 \\ S_3 & S_4 \end{pmatrix}$ with unknown entries and write $\alpha = S\omega$:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & 0 & 0 \\ s_{21} & s_{22} & 0 & 0 \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix}. \tag{4.6}$$

The constancy of the cup product in α_i's implies that $\Phi = S\Omega S^{\mathrm{tr}}$, where S^{tr} is the transpose matrix of S. Under the action $a \in \mathbb{G}_m$ on Y_t, for $i = 0, 1, 2, 3$, the pull back of ω_i is $a^{2i-3}\omega_i$ (on $Y_{t\bullet a}$). Hence, under this action we have the identification:

$$(Y_t, [0:1:0]\ ,\ \alpha) \cong (Y_{t\bullet a},\ [0:1:0],\ S \cdot \mathrm{diag}(a^{-3}, a^{-1}, a, a^3) \cdot \omega),$$

where $\mathrm{diag}(a^{-3}, a^{-1}, a, a^3)$ is a diagonal matrix. Therefore, we get that the moduli space T is isomorphic to

$$\left\{ (t_2, t_3, t_4, t_5, S) \in \mathbb{C}^{16} | \Phi = S\Omega S^{\mathrm{tr}}, t_5\Delta \neq 0, \det S \neq 0 \right\} / \mathbb{G}_m.$$

Proposition 4.3. *We have*

$$\mathsf{T} = \mathrm{Proj}\left(\frac{\mathbb{C}\left[t_2, t_3, t_4, t_5, s_{11}, s_{21}, s_{31}, s_{41}, s_{12}, s_{22}, s_{32}, s_{42},\ \frac{1}{t_5\Delta\cdot(s_{11}s_{22} - s_{12}s_{21})} \right]}{\langle s_{42}s_{21} - s_{41}s_{22} + s_{32}s_{11} - s_{31}s_{12} - \frac{t_2}{4} \rangle} \right) \subset \mathbb{P}^{4,6,8,10,3,3,3,3,1,1,1,1},$$

where we have considered the degrees $\deg(t_i) = 2i$, $\deg(s_{i1}) = 3$ *and* $\deg(s_{i2}) = 1$. *In particular,* T *is of dimension* 10.

Proof. From the equation $\Phi = S\Omega S^{\mathrm{tr}}$, we get:

$$S_1 \Omega_2 S_4^{\mathrm{tr}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.7}$$

and

$$S_4 \Omega_3 S_3^{\mathrm{tr}} + S_3 \Omega_2 S_4^{\mathrm{tr}} + S_4 \Omega_4 S_4^{\mathrm{tr}} = 0. \tag{4.8}$$

Using (4.7), we get identities: $S_4\Omega_3 = -(S_1^{\mathrm{tr}})^{-1}$ and $\Omega_2 S_4^{\mathrm{tr}} = S_1^{-1}$. Putting them in the equation (4.8), we have an equation:

$$-(S_1^{\mathrm{tr}})^{-1}S_3^{\mathrm{tr}} + S_3S_1^{-1} + S_4\Omega_4S_4^{\mathrm{tr}} = 0, \tag{4.9}$$

which is equivalent to the equation

$$s_{42}s_{21} - s_{41}s_{22} + s_{32}s_{11} - s_{31}s_{12} - \frac{t_2}{4} = 0. \tag{4.10}$$

□

In Proposition 4.3 we can discard the variable t_2 as this is equal to $4(s_{42}s_{21} - s_{41}s_{22} + s_{32}s_{11} - s_{31}s_{12})$. In this way we obtain the fact that T is an open subset of the weighted projective space $\mathbb{P}^{6,8,10,3,3,3,3,1,1,1,1}$ given by $t_5 \cdot \Delta \cdot (s_{11}s_{22} - s_{12}s_{21}) \neq 0$. This appears in Theorem 1.1, part 3.

4.4. **Ring of functions of T.** In this section we find the smallest N such that we can realize T as an affine subvariety of $\mathbb{C}^N$.

Proposition 4.4. *Let $\delta = \det S_1 = s_{11}s_{22} - s_{12}s_{21}$. There is an embedding from $\mathsf{T} \to \mathbb{C}^{153}$, which is given by sending $(t_2, t_3, t_4, t_5, s_{11}, s_{21}, s_{31}, s_{41}, s_{12}, s_{22}, s_{32}, s_{42})$ to*

$$\begin{aligned}
&T_4 = t_2/\delta, \quad T_8 = t_4/\delta^2, \quad T_{12} = t_3^2/\delta^3, \quad T_{16} = t_3t_5/\delta^4, \quad T_{20} = t_5^2/\delta^5,\\
&Q_{i_1i_2} = \frac{s_{i_11}s_{i_22}}{\delta}, 1 \le i_1, i_2 \le 4,\\
&Q_{i_1i_2i_3i_4} = \frac{(s_{12})^{i_1}(s_{22})^{i_2}(s_{32})^{i_3}(s_{42})^{i_4}}{\delta}, 0 \le i_1, i_2, i_3, i_4 \le 4, \sum_{j=1}^{4} i_j = 4,\\
&P_{i_1i_2i_3i_4} = \frac{(s_{11})^{i_1}(s_{21})^{i_2}(s_{31})^{i_3}(s_{41})^{i_4}}{\delta^3}, 0 \le i_1, i_2, i_3, i_4 \le 4, \sum_{j=1}^{4} i_j = 4,\\
&U_{3i_1i_2} = \frac{t_3s_{i_12}s_{i_22}}{\delta^2}, U_{5i_1i_2} = \frac{t_5s_{i_12}s_{i_22}}{\delta^3}, 1 \le i_1, i_2 \le 4,\\
&V_{3i_1i_2} = \frac{t_3s_{i_11}s_{i_21}}{\delta^3}, V_{5i_1i_2} = \frac{t_5s_{i_11}s_{i_21}}{\delta^4}, 1 \le i_1, i_2 \le 4.
\end{aligned} \tag{4.11}$$

Note that via the equation (4.10), we have:

$$T_4 = 4(Q_{24} - Q_{42} + Q_{13} - Q_{31})$$

and

$$Q_{21} - Q_{12} = 1,$$

and so we do not need two functions in (4.11) which consists of $153 + 2$ functions.

Proof. Let U be the open affine subvariety of $\mathbb{P}^{4,6,8,10,3,3,3,3,1,1,1,1}$, which is the complement of $\delta = 0$. By the definition of T, there is an embedding $\mathsf{T} \to U$. We just want to find enough invariants under the action of $\mathbb{G}_m$ generating the coordinate functions on U. According to the degrees of variables t_m, s_{ij}, if $\delta^{-n}\prod_{m=2}^{5} t_m^{n_m}\prod_{1\leq i\leq 4, 1\leq j\leq 2} s_{ij}^{n_{ij}}$ is a function on U, then we need to make sure that:

$$4n = 4n_2 + 6n_3 + 8n_4 + 10n_5 + 3\sum_{i=1}^{4} n_{i1} + \sum_{i=1}^{4} n_{i2}. \tag{4.12}$$

Moreover, for generators of the ring of regular functions on U, we can assume that $0 \leq n_m \leq 1, 0 \leq n_{ij} \leq 2$ for $2 \leq m \leq 5$ and $1 \leq i \leq 4, 1 \leq j \leq 2$. For the solutions of (4.12), we may divide into the following types:

- $4n = 4n_2 + 6n_3 + 8n_4 + 10n_5$;
- $4n = 3\sum_{i=1}^{4} n_{i1} + \sum_{i=1}^{4} n_{i2}$, where $n_{ij} \neq 0$;
- $4n = \sum_{i=1}^{4} n_{i2}$;
- $4n = 3\sum_{i=1}^{4} n_{i2}$;
- $4n = 6n_3 + 10n_5 + \sum_{i=1}^{4} n_{i2}$;
- $4n = 6n_3 + 10n_5 + 3\sum_{i=1}^{4} n_{i1}$.

In each case, we will get the solutions as stated in the proposition respectively. □

4.5. **Algebraic group G.** The algebraic group G

$$\mathsf{G} = \left\{ \begin{pmatrix} k & k' \\ 0 & k^{-\mathrm{tr}} \end{pmatrix} \in GL(4,\mathbb{C}) \middle| k(k')^{\mathrm{tr}} = (k(k')^{\mathrm{tr}})^{\mathrm{tr}} \right\} \subset \mathrm{Sp}(4,\mathbb{C})$$

acts from the left on T by the base change:

$$\mathsf{T} \times \mathsf{G} \to \mathsf{T}, t = ((Y, [0:1:0], \alpha), \mathbf{g}) \to t \bullet \mathbf{g} = (Y_t, [0:1:0], \alpha \cdot \mathbf{g}),$$

where we regard α as a 1×4 matrix and $\alpha \cdot \mathbf{g}$ is the usual multiplication of matrices. The algebraic group G is also called a Siegel parabolic subgroup of $\mathrm{Sp}(4,\mathbb{C})$. For $\mathbf{g} = \begin{bmatrix} k & k' \\ 0 & k^{-\mathrm{tr}} \end{bmatrix}$, we have $\delta(t \bullet \mathbf{g}) = \det(k) \cdot \delta(t)$. From this we get the functional equations for T_i:

$$T_i(t \bullet \mathbf{g}) = (\det k)^{-\frac{i}{4}} T_i(t), \quad t \in \mathsf{T}. \tag{4.13}$$

The functional equations of other regular functions on T are not as short as (4.13) and we do not reproduce them here.

5. Modular vector fields

In this section we describe three vector fields on T which are algebraic incarnation of the derivations/vector fields $\frac{\partial}{\partial \tau_i}, \quad i = 1, 2, 3$.

Theorem 5.1. *There are unique global vector fields $\mathsf{R}_k, \quad 1 \leq k \leq 3$ on T such that*

$$\nabla_{\mathsf{R}_k} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 & C_k \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}, \tag{5.1}$$

where $C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, C_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, C_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. *Here,* ∇ *is the Gauss-Manin connection of the family of genus two curves over* T.

Proof. Our proof actually computes R_k, $\quad k = 1, 2, 3$ and it is as follows. For the relevant computer code see Appendix A. Let us first consider a vector field

$$\mathsf{R}_k = \sum_{m=2}^{5} \mathsf{R}_{k,m} \frac{\partial}{\partial t_m} + \sum_{1 \le i \le 4, 1 \le j \le 2} \mathsf{R}_{k,ij} \frac{\partial}{\partial s_{ij}},$$

with unknown coefficients $\mathsf{R}_{k,m}$ and $\mathsf{R}_{k,ij}$. We have

$$\nabla_{\mathsf{R}_k} \alpha = \nabla_{\mathsf{R}_k}(S\omega) = (\nabla_{\mathsf{R}_k} S)\omega + SB(\mathsf{R}_k)\omega = (\mathrm{d}S(\mathsf{R}_k) + SB(\mathsf{R}_k))S^{-1}\alpha.$$

The equation (5.1) is equivalent to

$$\mathrm{d}S(\mathsf{R}_k) + SB(\mathsf{R}_k) = \check{C}_k S, \tag{5.2}$$

where $\check{C}_k = \begin{pmatrix} 0 & C_k \\ 0 & 0 \end{pmatrix}$. Recall that we use the notation $S = \begin{pmatrix} S_1 & 0 \\ S_3 & S_4 \end{pmatrix}$ and $B_m = \begin{pmatrix} B_{m,1} & B_{m,2} \\ B_{m,3} & B_{m,4} \end{pmatrix}$ for $2 \le m \le 5$. From the equation (5.2), we get an equation:

$$\begin{pmatrix} \mathsf{R}_k(S_1) & 0 \\ \mathsf{R}_k(S_3) & \mathsf{R}_k(S_4) \end{pmatrix} + \begin{pmatrix} S_1 & 0 \\ S_3 & S_4 \end{pmatrix} \sum_{m=2}^{5} \begin{pmatrix} B_{m,1} & B_{m,2} \\ B_{m,3} & B_{m,4} \end{pmatrix} \mathsf{R}_{k,m} = \begin{pmatrix} 0 & C_k \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ S_3 & S_4 \end{pmatrix}. \tag{5.3}$$

Considering the right-upper 2×2 matrices, we get its equivalent form:

$$\sum_{m=2}^{5} \mathsf{R}_{k,m} \cdot B_{m,2} = S_1^{-1} C_k S_4. \tag{5.4}$$

From the equation (4.7), the right-hand side of the above equation can be transformed into:

$$\begin{aligned} S_1^{-1} C_k S_4 &= S_1^{-1} C_k S_1^{-\mathrm{tr}} (\Omega_2)^{-\mathrm{tr}} \\ &= \frac{1}{(\det S_1)^2} \begin{pmatrix} s_{22}^2 & -s_{21}s_{22} \\ -s_{22}s_{21} & s_{21}^2 \end{pmatrix} (\Omega_2)^{-\mathrm{tr}} \\ &= \frac{1}{(\det(S_1))^2} \begin{pmatrix} -\frac{1}{4}s_{21}s_{22} & \frac{3}{4}s_{22}^2 \\ \frac{1}{4}s_{21}^2 & -\frac{3}{4}s_{21}s_{22} \end{pmatrix}, \end{aligned} \tag{5.5}$$

for $k = 1$. Similarly we do the computation for $k = 2, 3$ and define:

$$\begin{bmatrix} s_{1,-4} \\ s_{1,-6} \\ s_{1,-2} \end{bmatrix} = \frac{1}{(\det(S_1))^2} \begin{pmatrix} -\frac{1}{4}s_{21}s_{22} \\ \frac{3}{4}s_{22}^2 \\ \frac{1}{4}s_{21}^2 \end{pmatrix}, \quad \begin{bmatrix} s_{2,-4} \\ s_{2,-6} \\ s_{2,-2} \end{bmatrix} = \frac{1}{(\det(S_1))^2} \begin{pmatrix} -\frac{1}{4}s_{11}s_{12} \\ \frac{3}{4}s_{12}^2 \\ \frac{1}{4}s_{11}^2 \end{pmatrix}, \tag{5.6}$$

and

$$\begin{bmatrix} s_{3,-4} \\ s_{3,-6} \\ s_{3,-2} \end{bmatrix} = \frac{1}{(\det(S_1))^2} \begin{pmatrix} \frac{1}{4}(s_{12}s_{21} + s_{11}s_{22}) \\ -\frac{3}{2}s_{12}s_{22} \\ -\frac{1}{2}s_{11}s_{21} \end{pmatrix}. \tag{5.7}$$

The existence of R_k depends on the solution of

$$\begin{pmatrix} b_{13}^2 & b_{13}^3 & b_{13}^4 & b_{13}^5 \\ b_{14}^2 & b_{14}^3 & b_{14}^4 & b_{14}^5 \\ b_{23}^2 & b_{23}^3 & b_{23}^4 & b_{23}^5 \\ b_{24}^2 & b_{24}^3 & b_{24}^4 & b_{24}^5 \end{pmatrix} \begin{pmatrix} \mathsf{R}_{k,2} \\ \mathsf{R}_{k,3} \\ \mathsf{R}_{k,4} \\ \mathsf{R}_{k,5} \end{pmatrix} = \begin{pmatrix} s_{k,-4} \\ s_{k,-6} \\ s_{k,-2} \\ 3s_{k,-4} \end{pmatrix}. \tag{5.8}$$

The vector $[2t_2, 3t_3, 4t_4, 5t_5]^{\mathsf{tr}}$ is in the kernel of the above 4×4 matrix. This also follows from the fact that the Euler vector field $\sum_{m=2}^5 mt_m\frac{\partial}{\partial t_m}$ induces the zero vector field in the weighted projective space $\mathbb{P}^{4,6,8,10,3,3,3,3,1,1,1,1}$. The solution of (5.8) up to addition of the Euler vector field is unique and it is given by

$$\begin{aligned}
\mathsf{R}_{k,2} =&\frac{1}{75t_5}((24t_2^2t_4+450t_3t_5)s_{k,-2} \\
&+(-180t_2^2t_5+36t_2t_3t_4+600t_4t_5)s_{k,-4}+(-40t_2t_3t_5+16t_2t_4^2+250t_5^2)s_{k,-6}), \\
\mathsf{R}_{k,3} =&\frac{1}{75t_5}((-180t_2^2t_5+36t_2t_3t_4+600t_4t_5)s_{k,-2} \\
&+(-390t_2t_3t_5+54t_3^2t_4+750t_5^2)s_{k,-4}+(-20t_2t_4t_5-60t_3^2t_5+24t_3t_4^2)s_{k,-6}), \\
\mathsf{R}_{k,4} =&\frac{1}{75t_5}((-120t_2t_3t_5+48t_2t_4^2+750t_5^2)s_{k,-2} \\
&+(-60t_2t_4t_5-180t_3^2t_5+72t_3t_4^2)s_{k,-4}+(150t_2t_5^2-110t_3t_4t_5+32t_4^3)s_{k,-6}), \\
&\mathsf{R}_{k,5}=0.
\end{aligned}$$

Despite the fact that the Gauss-Manin connection matrix B is not simple, we can compute the matrices E_{-2}, E_{-4}, E_{-6}, which is defined through the equality

$$E(k) := \sum_{m=2}^{5} \mathsf{R}_{k,m}B_m = s_{k,-2}E_{-2}+s_{k,-4}E_{-4}+s_{k,-6}E_{-6}$$

for $k=1,2,3$, and they have rather simple expressions:

$$E_{-2} = \begin{pmatrix} -\frac{6t_2t_4}{25t_5} & -1 & 0 & 0 \\ \frac{4}{5}t_2 & -\frac{2t_2t_4}{25t_5} & 1 & 0 \\ 0 & \frac{8}{5}t_2 & \frac{2t_2t_4}{25t_5} & 3 \\ -t_4 & -2t_3 & -\frac{3}{5}t_2 & \frac{6t_2t_4}{25t_5} \end{pmatrix},$$

$$E_{-4} = \begin{pmatrix} (\frac{4}{5}t_2-\frac{9t_3t_4}{25t_5}) & 0 & 1 & 0 \\ \frac{6}{5}t_3 & (\frac{8}{5}t_2-\frac{3t_3t_4}{25t_5}) & 0 & 3 \\ -t_4 & \frac{2}{5}t_3 & -(\frac{3}{5}t_2-\frac{3t_3t_4}{25t_5}) & 0 \\ -2t_5 & -3t_4 & -\frac{2}{5}t_3 & -(\frac{9}{5}t_2-\frac{9t_3t_4}{25t_5}) \end{pmatrix},$$

$$E_{-6} = \begin{pmatrix} (\frac{2}{5}t_3-\frac{4t_4^2}{25t_5}) & \frac{1}{3}t_2 & 0 & 1 \\ \frac{1}{5}t_4 & (\frac{2}{15}t_3-\frac{4t_4^2}{75t_5}) & 0 & 0 \\ -\frac{2}{3}t_5 & \frac{1}{15}t_4 & -(\frac{2}{15}t_3-\frac{4t_4^2}{75t_5}) & 0 \\ 0 & -\frac{4}{3}t_5 & -\frac{1}{15}t_4 & -(\frac{2}{5}t_3-\frac{4t_4^2}{25t_5}) \end{pmatrix}.$$

Putting these solutions back to (5.3), we get $\mathsf{R}_{k,ij}$:

$$\begin{aligned}\begin{bmatrix} dS_1(\mathsf{R}_k) \\ dS_3(\mathsf{R}_k)\end{bmatrix} &= \begin{bmatrix} C_kS_3 \\ 0\end{bmatrix} - \begin{bmatrix} S_1(\sum_{m=2}^5 \mathsf{R}_{k,m}B_{m,1}) \\ S_3(\sum_{m=2}^5 \mathsf{R}_{k,m}B_{m,1}) + S_4(\sum_{m=2}^5 \mathsf{R}_{k,m}B_{m,3})\end{bmatrix} \\ &= \begin{bmatrix} C_kS_3 - S_1E(k)_1 \\ -S_3E(k)_1 - S_4E(k)_3\end{bmatrix},\end{aligned}$$

where we have used our convention $E(k) = \begin{bmatrix} E(k)_1 & E(k)_2 \\ E(k)_3 & E(k)_4\end{bmatrix}$. It remains to prove that R_k is tangent to the loci $F = 0$, where F is the polynomial (4.10). This follows from the equations:

$$\frac{dF(\mathsf{R}_1)}{F} = \frac{1}{25t_5\delta^2}(2t_2t_4s_{21}^2 + 15t_2t_5s_{21}s_{22} - 3t_3t_4s_{21}s_{22} - 10t_3t_5s_{22}^2 + 4t_4^2s_{22}^2),$$

$$\frac{dF(\mathsf{R}_2)}{F} = \frac{1}{25t_5\delta^2}(2t_2t_4s_{11}^2 + 15t_2t_5s_{11}s_{12} - 3t_3t_4s_{11}s_{12} - 10t_3t_5s_{12}^2 + 4t_4^2s_{12}^2),$$

$$\begin{aligned}\frac{dF(\mathsf{R}_3)}{F} =& \frac{1}{25t_5\delta^2}(-4t_2t_4s_{11}s_{21} - 15t_2t_5s_{11}s_{22} - 15t_2t_5s_{12}s_{21} + 3t_3t_4s_{11}s_{22}+ \\ & 3t_3t_4s_{12}s_{21} + 20t_3t_5s_{12}s_{22} - 8t_4^2s_{12}s_{22}).\end{aligned}$$

□

All R_k's are tangent to $\Delta = 0$ and this follows from the computation:

$$\frac{\mathrm{d}\Delta}{\Delta}(\mathsf{R}_k) = \frac{16t_2t_4}{5t_5}\cdot s_{k,-2} - \frac{60t_2t_5 - 24t_3t_4}{5t_5}\cdot s_{k,-4} - \frac{60t_3t_5 - 32t_4^2}{15t_5}\cdot s_{k,-6}. \tag{5.9}$$

Let us define

$$H_k := S_1^{-1}C_kS_3, \quad J_k := S_1^{-1}C_kS_4 = \begin{bmatrix} s_{k,-4} & s_{k,-6} \\ s_{k,-2} & 3s_{k,-4}\end{bmatrix}, \quad k = 1, 2, 3.$$

The following proposition might be useful for a better understanding of R_k's.

Proposition 5.2. *We have the following equations:*

$$\begin{aligned}\mathsf{R}_{\check{k}}(J_k) &= H_{\check{k}}J_k - E(\check{k})_1J_k - H_kE(\check{k})_2 - J_kE(\check{k})_4, \\ \mathsf{R}_{\check{k}}(H_k) &= H_{\check{k}}H_k - E(\check{k})_1H_k - H_kE(\check{k})_1 - J_kE(\check{k})_3,\end{aligned}$$

for $\check{k} = 1, 2, 3$.

Proof. We only check the first equation. Similarly, one can verify the other.

$$\begin{aligned}
\mathsf{R}_{\check{k}}(J_k) &= \mathsf{R}_{\check{k}}(S_1^{-1}C_kS_4) = \mathsf{R}_{\check{k}}(S_1^{-1})C_kS_4 + S_1^{-1}C_k\mathsf{R}_{\check{k}}(S_4) \\
&= S_1^{-1}\mathsf{R}_{\check{k}}(S_1)S_1^{-1}C_kS_4 + S_1^{-1}C_k\mathsf{R}_{\check{k}}(S_4) \\
&= S_1^{-1}(C_{\check{k}}S_3 - \sum_m \mathsf{R}_{\check{k},m}S_1B_{m,1})S_1^{-1}C_kS_4 + \\
&\quad S_1^{-1}C_k(-\sum_m \mathsf{R}_{\check{k},m}(S_3B_{m,2} + S_4B_{m,4})) \\
&\overset{(5.3)}{=} S_1^{-1}C_{\check{k}}S_3J_k - \sum_m \mathsf{R}_{\check{k},m}(B_{m,1}J_k + (S_1^{-1}C_kS_3)B_{m,2} + J_kB_{m,4}) \\
&= H_{\check{k}}J_k - \sum_m \mathsf{R}_{\check{k},m}(B_{m,1}J_k + H_kB_{m,2} + J_kB_{m,4}) \\
&= H_{\check{k}}J_k - E(\check{k})_1J_k - H_kE(\check{k})_2 - J_kE(\check{k})_4.
\end{aligned}$$

□

5.1. **The description of R_k's in an affine chart.** Let f be a homogeneous polynomial in $\mathbb{C}[x_0, \cdots, x_n]$ with $\deg x_i = \alpha_i \in \mathbb{N}$ and $\mathbb{P}^\alpha := \mathrm{Proj}(\mathbb{C}[x_0, \cdots, x_n])$ be the weighted projective space. The coordinate ring of $U := \mathbb{P}^\alpha - \{f = 0\}$ is given by the polynomial ring generated by $\frac{x_0^{a_0}\cdots x_n^{a_n}}{f^k}$ such that $\sum_{i=0}^n a_i\alpha_i = k\deg(f)$. We find generators $X = (X_i)_{i=1,\cdots,m}$ of the coordinate ring of U and we have $\mathcal{O}(U) = \mathrm{Spec}(\mathbb{C}[X_1, \cdots, X_m]/I)$, where I is the ideal generated by the polynomials P with variables in X such that $P(X) = 0$. A vector field on $\mathbb{P}^\alpha$ is given by $v = \sum_{i=0}^n v_i(x)\frac{\partial}{\partial x_i}$, where $v_i(x) = \frac{f_i}{g_i}$ can be written as the ratio of two homogeneous polynomials f_i and g_i such that $\deg f_i - \deg g_i = \alpha_i$. Then we can write $\mathrm{d}X_i(v)$ in terms of X_i, denoted by $Q_i(X)$, and then we have:

$$v|_U = \sum_{i=1}^m Q_i \frac{\partial}{\partial X_i}. \tag{5.10}$$

This can be used to describe the vector fields R_k's in the complement U of $\delta = 0$ in $\mathbb{P}^{4,6,8,10,3,3,3,3,1,1,1,1}$. Note that we have already chosen coordinate functions for U in §4.4.

6. The monodromy group of genus 2 hyperelliptic curves

In this section, we want to compute the monodromy group of the moduli space of genus 2 hyperelliptic curves with a marked Weierstrass point.

6.1. **Picard-Lefschetz theory.** First we recall some general statements. We denote the moduli space of hyperelliptic curves of genus g by $\mathcal{C}_g$. For each hyperelliptic curve g, we can associate it with a polynomial of degree d with a non-vanishing discriminant such that the affine part of Y is given by $y^2 = P(x)$. Then the genus of Y is $g = [\frac{d-1}{2}]$. We fix a base point $b = 0 \in \mathcal{C}_g$, which is corresponding to the hyperelliptic curve whose affine part is given by

$$Y_b : y^2 = x^d + 1. \tag{6.1}$$

We denote the monodromy map:

$$\pi_1(\mathcal{C}_g, b) \to Aut(H_1(Y_b,\mathbb{Z}),\cdot) \cong \mathrm{Sp}(2g,\mathbb{Z})$$

by h, where $\cdot$ denotes the intersection pairing.

We define $f:\mathbb{C}^2\to\mathbb{C}$ by $f(x,y)=-y^2+P(x)$. Then we let $\{p_1,p_2,\cdots,p_{d-1}\}$ be the critical points of f, i.e., $p_i=(a_i,0)$ such that $P'(a_i)=0$. Moreover, we denote the critical values of f by $\{c_1,c_2,\cdots,c_{d-1}\}$. One may choose a set of vanishing cycles $\delta_1,\delta_2,\cdots,\delta_{d-1}$ such that their intersection numbers are given by the Dynkin diagram of the polynomial $P(x)$, i.e.

$$\begin{aligned}&\delta_1\cdot\delta_2=-\delta_2\cdot\delta_1=\delta_2\cdot\delta_3=-\delta_3\cdot\delta_2=\cdots\delta_{d-2}\cdot\delta_{d-1}=-\delta_{d-1}\cdot\delta_{d-2}=1\\&\text{otherwise}\quad \delta_i\cdot\delta_j=0,\end{aligned}\tag{6.2}$$

see for instance [Mov19, Chapter 7]. Moreover if d is odd, we can use linear combinations of δ_i to form a symplectic basis e_i of $H_1(Y_b,\mathbb{Z})$:

$$\begin{aligned}&e_1=\delta_1, e_2=\delta_3,\cdots,e_{\frac{d-1}{2}}=\delta_{d-2},\\&e_{\frac{d+1}{2}}=\delta_2+\delta_4, e_{\frac{d+3}{2}}=\delta_4+\delta_6,\cdots,e_{d-2}=\delta_{d-3}+\delta_{d-1}, e_{d-1}=\delta_{d-1}.\end{aligned}\tag{6.3}$$

If d is even, the similar e_i consists of a symplectic basis of $H_1(Y_b,\mathbb{Z})$ and $\delta_1+\delta_3+\cdots+\delta_{d-1}=0$ in $H_1(Y_b,\mathbb{Z})$.

We recall the Picard-Lefschetz formula (see for instance [AGZV88] or [Mov19, §6.6]). Suppose that f is a holomorphic map from a projective complex manifold Y of dimension n to the projective line $\mathbb{P}^1$. Also suppose that all critical points are non-degenerate (not necessarily in different fibers). We let b be a regular value of f. Then the monondromy h around the critical value c_i is given by the Picard-Lefschetz formula

$$h(\delta)=\delta-\sum_j\langle\delta,\delta_j\rangle\delta_j,\quad \delta\in H_1(Y_b),\tag{6.4}$$

where j runs through all the Lefschetz vanishing cycles in the singularities of Y_{c_i} and $\langle\cdot,\cdot\rangle$ denotes the intersection number of two cycles in Y_b.

6.2. **Monodromy of genus two curves.** Now we assume $d=5$. Then there are 4 critical points c_i. For simplicity, we choose a symplectic basis of $H_1(Y_b,\mathbb{Z})$:

$$e_1=\delta_1, e_2=\delta_3, e_3=\delta_2+\delta_4, e_4=\delta_4.\tag{6.5}$$

Using the Picard-Lefschetz formula and the symplectic basis $\{e_i\}\in H_1(Y_b,\mathbb{Z})$, we get:

$$h_{c_1}(e_i)=e_i-\langle e_i,\delta_1\rangle\delta_1.$$

Under the basis (e_i), h_{c_1} corresponds to the matrix

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&0&1&0\\0&0&0&1\end{pmatrix}.$$

Similarly we will get other 3 matrices and hence the monodromy group of genus 2 hyperelliptic curves with a Weierstrass point is generated by four generators of Γ

in (1.1). Let us call these matrices A, B, C and D, respectively. Note that if we consider the moduli space of genus 2 hyperelliptic curves, then the similar discussion implies that $\mathrm{Sp}(4,\mathbb{Z})$ is generated by the above 4 matrices together with

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}. \tag{6.6}$$

We let Γ_2^0 be the mapping class group associated with the orbifold fundamental group of the moduli space of nondegenerated Riemann surfaces of genus 2. Then it is well know that Γ_2^0 has the following presentation:

$$\begin{aligned} \text{generators} :&\zeta_1, \cdots, \zeta_5, \\ \text{relations} :&[\zeta_i, \zeta_j] = 1, \mid i-j \mid > 1 \\ &\zeta_i\zeta_{i+1}\zeta_i = \zeta_{i+1}\zeta_i\zeta_{i+1}, \\ &(\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5)^6 = 1, \zeta^2 = 1, [\zeta, \zeta_i] = 1, \end{aligned} \tag{6.7}$$

where $\zeta = \zeta_1\zeta_2\zeta_3\zeta_4\zeta_5^2\zeta_4\zeta_3\zeta_2\zeta_1$, see for instance [BL94, Page 577]. Hence we may view Γ_2^0 as a natural quotient of the braid group B_6, whose generators are also denoted by ζ_i. It also known that the map $\Gamma_2^0 \to \mathrm{Sp}(4,\mathbb{Z})$ is surjective and the image of ζ_1 (resp $\zeta_2, \zeta_3, \zeta_4, \zeta_5$) is A (resp B, C, D, E). Therefore, A, B, C, D, E generate the whole group $\mathrm{Sp}(4,\mathbb{Z})$. We let Γ be the subgroup of $\mathrm{Sp}(4,\mathbb{Z})$ generated by A, B, C, D. Note that $E^2 = (ABC)^4 \in \Gamma$. We denote the principal congruence subgroup of $\mathrm{Sp}(4,\mathbb{Z})$ of level two by $\Gamma(2)$, i.e., $\Gamma(2) = \mathrm{Ker}(\mathrm{Sp}(4,\mathbb{Z}) \to \mathrm{Sp}(4,\mathbb{Z}/2\mathbb{Z}))$.

Proposition 6.1. *Γ contains the congruence group $\Gamma(2)$ as a subgroup.*

Proof. Note that by our definition of A, B, C, D, E, there is a natural map from the 6-th braid group B_6 to $\mathrm{Sp}(4,\mathbb{Z})$:

$$\phi : B_6 \to \mathrm{Sp}(4,\mathbb{Z}),$$

given by $\phi(\zeta_1) = A, \phi(\zeta_2) = B, \phi(\zeta_3) = C, \phi(\zeta_4) = D$ and $\phi(\zeta_5) = E$, see [KT08, §1.1]. Moreover we have the following commutative diagram:

$$\begin{tikzcd} P_6 \arrow[r] \arrow[d] & B_6 \arrow[r, "\pi"] \arrow[d, "\phi"] & S_6 \arrow[d, "\cong"] \\ \Gamma(2) \arrow[r] & \mathrm{Sp}(4,\mathbb{Z}) \arrow[r] & \mathrm{Sp}(4,\mathbb{Z}/2\mathbb{Z}) \end{tikzcd}$$

where π is the map sending ζ_i to $s_i = (i, i+1)$ and $P_6 = \mathrm{Ker}(\pi)$. We want to show that $\Gamma(2) \subset \Gamma = \langle A, B, C, D\rangle$. Note that a set of generators of P_6 also provides a set of generators of $\Gamma(2)$. Now we can consider a set of generators of P_6 given by Corollary 1.19 in [KT08, Page 21] and then we want to show that the corresponding generators of $\Gamma(2)$ can be expressed by A, B, C, D. Besides the ones lying in Γ, we need to show that:

$$EDCBA^2B^{-1}C^{-1}D^{-1}E^{-1}, EDCB^2C^{-1}D^{-1}E^{-1}, EDC^2D^{-1}E^{-1}, ED^2E^{-1} \in \Gamma.$$

By the explicit computation, we find that:

$$ED^2E^{-1} = (ABCD)^5(AB)^3D^{-2}(ABC)^{-4} \in \Gamma,$$

and

$$EDC^2D^{-1}E^{-1} = (CD)^3(ABCD)^{-5}(AB)^{-3}(DA)^2 \in \Gamma.$$

Moreover, we also have an identity:

$$(EDCBA^2B^{-1}C^{-1}D^{-1}E^{-1})^{-1} = EDCB^2C^{-1}D^{-1}E^{-1}[A^2(CD)^3(ABCD)^5].$$

This implies that we only need to show that $EDCBA^2B^{-1}C^{-1}D^{-1}E^{-1} \in \Gamma$. After a lot of computations, we get that:

$$EDCBA^2B^{-1}C^{-1}D^{-1}E^{-1} = (DCB)^4 \in \Gamma.$$

Therefore, we have $\Gamma(2) \subset \Gamma$. $\square$

Corollary 6.2. *The index of* Γ *in* $\mathrm{Sp}(4, \mathbb{Z})$ *is* 6. *Moreover we have a short exact sequence*

$$1 \to \Gamma(2) \to \Gamma \to S_5 \to 1. \tag{6.8}$$

Proof. We have the following diagram:

$$\begin{tikzcd}
\mathrm{Ker}(\pi_\Gamma) \arrow[r] \arrow[d] & \Gamma \arrow[r, "\pi_\Gamma"] \arrow[d] & S_5 \arrow[d] \\
\Gamma(2) \arrow[r] & \mathrm{Sp}(4,\mathbb{Z}) \arrow[r] & \mathrm{Sp}(4,\mathbb{Z}/2\mathbb{Z}) \cong \Sigma_6
\end{tikzcd}$$

Note that we use the fact that the map $\Gamma \to \mathrm{Sp}(4, \mathbb{Z}) \to \mathrm{Sp}(4, \mathbb{Z}/2\mathbb{Z}) \cong \Sigma_6$ sends A, B, C, D to $(12), (23), (34), (45)$, which implies that the image is S_5. On the other hand, we have shown that $\Gamma(2) \to \Gamma$ is injective and this map factors through $\mathrm{Ker}(\pi_\Gamma)$. This means that $\Gamma(2) \to \mathrm{Ker}(\pi_\Gamma)$ is injective and hence is an isomorphism. Then we get a short exact sequence

$$1 \to \Gamma(2) \to \Gamma \to S_5 \to 1.$$

We may take $ABCDE, BCDE, CDE, DE, E, I$ as representatives of cosets of Γ in $\mathrm{Sp}(4, \mathbb{Z})$. $\square$

Remark 6.3. Note that there is an equality of matrices: $(ABCD)^5 = -I \in \Gamma(2)$. In geometric terms, $ABCD$ is the monodromy matrix around the singularity $y^2 - x^5 = 0$.

Remark 6.4. We recall that the theta group $\Gamma_{2,\theta}$ of genus 2 is defined as

$$\Gamma_{2,\theta} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(4, \mathbb{Z}) \middle| CD^{\mathrm{tr}} \text{ and } AB^{\mathrm{tr}} \text{ have even diagonal} \right\},$$

which is a congruence subgroup of the full modular group $Sp(4, \mathbb{Z})$ with index 10, see [Fre11, Remark 8.1, page 459]. It is also called Igusa's congruence subgroup and denoted by $\Gamma[1, 2]$, see [Fre11, page 462]. Therefore $\Gamma_{2,\theta}$ is neither the same as Γ, nor a subgroup of Γ.

7. Differential Siegel modular forms

7.1. **The t map.** We denote the moduli space of curves of genus 2 (resp. principally polarized abelian surfaces) by $\mathcal{M}_2$ (resp. $\mathcal{A}_2$). Recall that we can view $\mathcal{M}_2$ as an open subspace of $\mathcal{A}_2$ and its complement is the moduli of product of two elliptic curves. The period map gives us an isomorphism $\mathcal{A}_2 \cong \mathrm{Sp}(4,\mathbb{Z})\backslash\mathbb{H}_2$ and under this isomorphism, the the complement of $\mathcal{M}_2$ in $\mathcal{A}_2$ is given by

$$\widehat{\{\tau_3 = 0\}} := \pi^{-1} \circ \pi\{\tau_3 = 0\}, \quad \text{where } \pi : \mathbb{H}_2 \to \mathrm{Sp}(4,\mathbb{Z})\backslash\mathbb{H}_2. \tag{7.1}$$

We have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{M}_2(w) & \longrightarrow & \Gamma\backslash\mathbb{H}_2 \\ \downarrow & & \downarrow \\ \mathcal{M}_2 & \longrightarrow & \mathrm{Sp}(4,\mathbb{Z})\backslash\mathbb{H}_2 \cong \mathcal{A}_2 \end{array}$$

where the horizontal maps are injective and the vertical maps are both 6 to 1 covering maps. By our construction $\mathcal{M}_2(w) \cong \mathbb{P}^{4,6,8,10}\backslash\{t_5\Delta = 0\}$. Let us define

$$\overline{\mathcal{M}_2(w)} \ := \ \left(\mathbb{P}^{4,6,8,10}\backslash\{t_5 = 0\}\right) \cup \mathrm{smooth}(\Delta = 0).$$

We observe that the map $\overline{\mathcal{M}_2(w)} \to \mathcal{A}_2$ is ramified over smooth$(\Delta = 0)$, and in this loci it is a 5 to 1 map.

Proposition 7.1. *The moduli of triples (Y, P, e), where Y is a genus two curve, P is a Weierstrass point of Y and $e := (e_1, e_2, e_3, e_4)$ is a symplectic basis of $H_1(Y,\mathbb{Z})$, is isomorphic to a disjoint union of six copies of $\mathbb{H}_2 - \widehat{\{\tau_3 = 0\}}$.*

Proof. Let $Y_0 : y^2 = x^5+1$ be the hyperelliptic curve in (6.1) and e be the symplectic basis of $H_1(Y_0,\mathbb{Z})$ defined in (6.5). This data together with the Weierstrass point $P := [0 : 1 : 0]$ at infinity, gives us a point in the moduli space of the proposition. Let $\tilde{\mathbb{H}}_2$ be the connected component of this moduli containing this point. We have the following commutative diagram

$$\begin{array}{ccc} \mathcal{M}_2(w) & \longrightarrow & \mathcal{M}_2 \\ \uparrow & & \uparrow \\ \tilde{\mathbb{H}}_2 & \longrightarrow & \mathbb{H}_2 - \widehat{\{\tau_3 = 0\}} \end{array}$$

Since $\tilde{\mathbb{H}}_2/\Gamma \cong \mathcal{M}_2(w)$ and $(\mathbb{H}_2/\mathrm{Sp}(4,\mathbb{Z})) - \widehat{\{\tau_3 = 0\}} \cong \mathcal{M}_2$ and $\mathcal{M}_2(w) \to \mathcal{M}_2$ is 6 to 1 map and Γ has index 6 in $\mathrm{Sp}(4,\mathbb{Z})$ we get the fact the canonical covering $\tilde{\mathbb{H}}_2 \to \mathbb{H}_2 - \widehat{\{\tau_3 = 0\}}$ is one to one, and hence, an isomorphism. This is enough to finish the proof. $\square$

From now on we regard $\mathbb{H}_2 - \widehat{\{\tau_3 = 0\}}$ as the connected component of the moduli of triples (Y, P, e) containing the marked point as in the beginning of the proof of

Proposition 7.1. We have a natural map

$$\mathsf{t}:\mathbb{H}_2-\widehat{\{\tau_3=0\}}\to\mathsf{T},$$

which is defined in the following way. For $\tau\in\mathbb{H}_2-\widehat{\{\tau_3=0\}}$ corresponding to (Y,p,e), there is a unique basis α of $H^1_{\rm dR}(Y)$ such that (Y,α) is enhanced and we have the following format of the period matrix

$$\left[\int_{e_i}\alpha_j\right]_{i,j=1,2,3,4}=\begin{bmatrix}\tau & -I_{2\times 2}\\ I_{2\times 2} & 0_{2\times 2}\end{bmatrix}. \tag{7.2}$$

For this we first take an arbitrary enhancement (Y,α) and use the fact that there is a unique $\mathbf{g}\in\mathsf{G}$ such that the period matrix of $\alpha\mathbf{g}$ is of the desired form, see for instance [Mov13, §4.1]. We define $\mathsf{t}(\tau)=(Y,\alpha)$. We can now interpret all the functions in Proposition 4.4, and in particular T_i's, as holomorphic functions in $\mathbb{H}_2-\widehat{\{\tau_3=0\}}$. This is obtained by taking the pull-back $T_i\circ\mathsf{t}$. The $\mathbb{G}_m$-action on $\mathbf{S}$, which is basically given by (4.13), is translated into the automorphic property of T_i's:

$$\det(c\tau+d)^{-\frac{i}{4}}T_i((a\tau+b)\cdot(c\tau+d)^{-1})=T_i(\tau),\quad \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma. \tag{7.3}$$

The functional equation of other functions on T are more complicated and can be obtained from the action of G. For computing these functional equations in the case of elliptic curves (resp. mirror quintic) see [Mov12, §8.5] ([Mov17, Theorem 7]).

In principle, the pull-back of regular functions in T by the t-map might be meromorphic in $\tau_3=0$. Therefore, we are looking for generators of the algebra of differential Siegel modular forms for Γ:

$$\mathrm{DM}(\Gamma):=\left\{f\in\mathcal{O}_{\mathsf{T}}\;\middle|\; f\circ\mathsf{t}\text{ is holomorphic along }\tau_3=0\right\}, \tag{7.4}$$

where for an affine variety V we have denoted by $\mathcal{O}_V$ the ring of global regular functions in V. The algebra of Siegel modular forms for Γ is defined similarly:

$$\mathrm{M}(\Gamma):=\left\{f\in\mathcal{O}_{\mathbf{S}}\;\middle|\; f\circ\mathsf{t}\text{ is holomorphic along }\tau_3=0\right\}. \tag{7.5}$$

Note that using the canonical projection $\mathsf{T}\to\mathbf{S}$ we have considered $\mathcal{O}_{\mathbf{S}}$ as a subalgebra of $\mathcal{O}_{\mathsf{T}}$. We note that $T_{20},\Delta\in\mathrm{DM}(\Gamma)$. More precisely, these are holomorphic Siegel modular forms for Γ of weights 5 and 10, respectively. Both of them vanish along $\tau_3=0$. If not, then $\frac{1}{T_{20}}$ and $\frac{1}{\Delta}$ would be holomorphic Siegel modular form for Γ of negative weight which is a contradiction.

Theoretical arguments as in [Mov20, Chapter 11] and [Fon18] show that modular vector fields in the case of abelian varieties are holomorphic in the whole moduli. However our vector fields in the present text have pole order one along $t_5=0$. This translates into meromorphicity along $\tau_3=0$ and it implies that the pull-back of regular functions in T to the Siegel domain are not all holomorphic along $\tau_3=0$.

7.2. **Comparison with Igusa's invariants.** In order to compare our results in this paper with Igusa's results in [Igu67], we also need to define the moduli space $\check{\mathsf{S}}$ and $\check{\mathsf{T}}$ which have the same definition as S and T but without the presence of Weierstrass points. There is a natural forgetful map:

$$\mathsf{S} \to \check{\mathsf{S}}, \mathsf{T} \to \check{\mathsf{T}}, \quad (Y, P, \alpha) \to (Y, \alpha). \tag{7.6}$$

We also need the ring of invariants of binary sextics, which is given by

$$I = \mathbb{C}[A, B, C, D, E]/\langle E^2 - P(A, B, C, D)\rangle), \tag{7.7}$$

where P is an explicit polynomial and we have natural weights $\deg(A) = 2,\ \deg B = 4, \deg(C) = 6, \deg(D) = 10,\ \deg(E) = 15$, see [Bol87]. We also need the moduli T_{av} and $\mathbf{S}_{\mathrm{av}}$ which are the moduli of enhanced principally polarized abelian surfaces, see for instance [Mov13, §4.1] or [Mov20, Chapter 11]. Then we have the following diagram:

$$\begin{tikzcd}
\mathrm{M}(\Gamma) \arrow[r, hook] & \mathcal{O}_{\mathbf{S}} \arrow[r, hook] & \mathcal{O}_{\mathsf{T}} & \mathrm{DM}(\Gamma) \arrow[l, hook'] \\
\mathrm{M}(\mathrm{Sp}(4,\mathbb{Z})) \cong \mathcal{O}_{\mathbf{S}_{\mathrm{av}}} \arrow[u, hook] \arrow[r, hook] & \mathcal{O}_{\check{\mathbf{S}}} \arrow[u, hook] \arrow[r, hook] & \mathcal{O}_{\check{\mathsf{T}}} \arrow[u, hook] & \mathcal{O}_{\mathsf{T}_{\mathrm{av}}} \cong \mathrm{DM}(\mathrm{Sp}(4,\mathbb{Z})) \arrow[u, hook] \arrow[l, hook'] \\
 & I \arrow[u, hook] & &
\end{tikzcd}$$

In [Igu67, Page 848], Igusa shows that the intersection of of $\mathrm{M}(\mathrm{Sp}(4,\mathbb{Z}))$ with I, is generated by

$$E_4 := B,\ E_6 := 4AB - 3C,\ \chi_{10} := D,\ \chi_{12} := AD,\ \chi_{35} := D^2E$$

up to constants. Note that in the proof of Theorem 4.2, the $\mathbb{G}_m$-action multiplies $[\frac{\mathrm{d}x}{y}]\wedge[\frac{x\mathrm{d}x}{y}]$ with a^{-4} for $a \in \mathbb{G}_m$. This implies that the degree in our case is 4 times the canonical degree of Siegel modular forms. Let F be the composition of two inclusions $I \hookrightarrow \mathcal{O}_{\check{\mathbf{S}}} \hookrightarrow \mathcal{O}_{\mathbf{S}}$. Looking I inside $\mathcal{O}_{\mathbf{S}}$, we know that E_4 (resp. $E_6, \chi_{10}, \chi_{12}, \chi_{35}$) are degree 16 (resp. $24, 40, 48, 140$) polynomials in variables T_4, T_8, T_{12}, T_{16} and T_{20}, which are defined in Remark 4.4. We can use invariant theory to determine these expressions explicitly. This is as follows. The curve given by $y^2 = x^5 + t_2x^3 + t_3x^2 + t_4x + t_5$ corresponds to the binary sextic $f = t_5x^6 + t_4x^5y + t_3x^4y^2 + t_2x^3y^3 + xy^5$.

The result up to multiplication with constants is

$$
\begin{aligned}
A(f) &= -3t_2^2 - 20t_4,\\
B(f) &= -3t_2t_3^2 + 9t_2^2t_4 - 20t_4^2 + 75t_3t_5,\\
C(f) &= 12t_2^3t_3^2 + 18t_3^4 - 36t_2^4t_4 - 13t_2t_3^2t_4 - 88t_2^2t_4^2 +\\
&\quad 160t_4^3 - 165t_2^2t_3t_5 - 800t_3t_4t_5 - 1125t_2t_5^2,\\
D(f) &= \Delta \quad \text{is defined in (2.9)},\\
E(f) &= 729t_2^{10}t_5^2 - 486t_2^9t_3t_4t_5 + 108t_2^8t_3^3t_5 + 81t_2^8t_3^2t_4^2 - 18225t_2^8t_4t_5^2 - 36t_2^7t_3^4t_4 +\\
&\quad 12150t_2^7t_3^2t_5^2 + 10800t_2^7t_3t_4^2t_5 + 4t_2^6t_3^6 - 9720t_2^6t_3^3t_4t_5 - 1800t_2^6t_3^2t_4^3 +\\
&\quad 135000t_2^6t_4^2t_5^2 + 1584t_2^5t_3^5t_5 + 2015t_2^5t_3^4t_4^2 - 175500t_2^5t_3^2t_4t_5^2 - 60000t_2^5t_3t_4^3t_5 +\\
&\quad 928125t_2^5t_5^4 - 623t_2^4t_3^6t_4 + 60000t_2^4t_3^4t_5^2 + 92500t_2^4t_3^3t_4^2t_5 + 10000t_2^4t_3^2t_4^4 -\\
&\quad 1012500t_2^4t_3t_4t_5^3 - 250000t_2^4t_4^3t_5^2 + 59t_2^3t_3^8 - 45050t_2^3t_3^5t_4t_5 - 17500t_2^3t_3^4t_4^3 +\\
&\quad 225000t_2^3t_3^3t_5^3 + 850000t_2^3t_3^2t_4^2t_5^2 - 2812500t_2^3t_4t_5^4 + 5700t_2^2t_3^7t_5 + 10825t_2^2t_3^6t_4^2 -\\
&\quad 478125t_2^2t_3^4t_4t_5^2 - 50000t_2^2t_3^3t_4^3t_5 + 1875000t_2^2t_3^2t_5^4 + 1250000t_2^2t_3t_4^2t_5^3 -\\
&\quad 2610t_2t_3^8t_4 + 93750t_2t_3^6t_5^2 + 32500t_2t_3^5t_4^2t_5 - 1187500t_2t_3^3t_4t_5^3 + 216t_3^{10} -\\
&\quad 9000t_3^7t_4t_5 + 625t_3^6t_4^3 + 175000t_3^5t_5^3 + 15625t_3^4t_4^2t_5^2 - 390625t_3^2t_4t_5^4 - 9765625t_5^6,
\end{aligned}
$$

which can be easily rewritten in terms of T_i's. The equalities for A, B, C, D are taken from [Tay14, §3.3 page 47]. Igusa's generator E is the first invariant with the odd degree and we have computed it from the classical theory of transvectants. Up to constant, it is j_{15} defined in [Dra14, page 53].

For our main purposes, it would be essential to generalize Igusa's results and find generators for the $\mathrm{DM}(\Gamma)$. For this one has to understand in a geometric way why the pull-back of $A = \frac{AD}{D}$ and $E = \frac{D^2E}{D^2}$ by the t-map has a pole order 2 and 4 along $\tau_3 = 0$, respectively. Note that D has a zero of order two along $\tau_3 = 0$, see for instance [Igu67, Page 849]. One possible way to investigate this is as follows. We take a three parameters family of genus two curves with a marked Weierstrass point (for instance the family Y_t in (2.8) with some t_i equal to zero). We also take the canonical basis ω of $H^1_{\mathrm{dR}}(Y_t)$ in (2.3). We can express the generators of $\mathcal{O}_{\mathsf{T}}$ in terms of the periods of ω and then we investigate when such period expressions are holomorphic around nodal curves. This is going to be explained in §7.3. Similar computations has been explained in the case of elliptic curve in [Mov12, §8.7] and mirror quintic in [Mov17, §4.9].

7.3. **Expressions of the image of t-map via abelian integrals.** We choose the following loci L of T consisting of points (t, S_0) of the form:

$$
(t_2, t_3, t_4, t_5),\ S_0 = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & \frac{t_2}{4} & 0 & \frac{3}{4}\\ 0 & 0 & \frac{1}{4} & 0 \end{pmatrix}. \tag{7.8}
$$

We could restrict ourselves to a three dimensional subspace in $t = (t_2, t_3, t_4, t_5)$, however, for lack of a canonical choice we do not do it. Let

$$x_{ij} := \int_{e_i} \omega_j,$$

where e_i (resp. ω_j) are the symplectic basis (resp. the canonical de Rham cohomology basis) of genus two curves determined by (2.8). We have

$$\left[\int_{e_i} \alpha_j\right] = [x_{ij}] S_0^{\mathsf{tr}} = \begin{bmatrix} \tau & -I \\ I & 0 \end{bmatrix} \cdot \mathbf{g},$$

where

(7.9)

$$\tau = \begin{bmatrix} \frac{x_{11}x_{42}-x_{41}x_{12}}{x_{31}x_{42}-x_{41}x_{32}} & -\frac{x_{11}x_{32}-x_{31}x_{12}}{x_{31}x_{42}-x_{41}x_{32}} \\ \frac{x_{21}x_{42}-x_{41}x_{22}}{x_{31}x_{42}-x_{41}x_{32}} & -\frac{x_{21}x_{32}-x_{31}x_{22}}{x_{31}x_{42}-x_{41}x_{32}} \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} x_{31} & x_{32} & \frac{1}{4}t_2x_{32} + \frac{3}{4}x_{34} & \frac{1}{4}x_{33} \\ x_{41} & x_{42} & \frac{1}{4}t_2x_{42} + \frac{3}{4}x_{44} & \frac{1}{4}x_{43} \\ 0 & 0 & \frac{x_{42}}{x_{31}x_{42}-x_{32}x_{41}} & \frac{-x_{41}}{x_{31}x_{42}-x_{32}x_{41}} \\ 0 & 0 & \frac{-x_{32}}{x_{31}x_{42}-x_{32}x_{41}} & \frac{x_{31}}{x_{31}x_{42}-x_{32}x_{41}} \end{bmatrix} \in \mathsf{G}.$$

This implies that $\mathsf{t}(\tau) = (t, S_0) \bullet \mathbf{g}^{-1}$, where (t, S_0) is the point in T given by (7.8). This means that the image of τ as above (viewed as an element in the Siegel domain $\mathbb{H}_2$) under the t-map is given by the entries of the action of $\mathbf{g}^{-1}$ on the point (t, S_0) in (7.8), where $t = (t_2, t_3, t_4, t_5)$. This is (t, S), where

$$S := \mathbf{g}^{-\mathrm{tr}} S_0 =$$

(7.10)

$$\begin{bmatrix} \frac{x_{42}}{x_{31}x_{42}-x_{32}x_{41}} & -\frac{x_{41}}{x_{31}x_{42}-x_{32}x_{41}} & 0 & 0 \\ -\frac{x_{32}}{x_{31}x_{42}-x_{32}x_{41}} & \frac{x_{31}}{x_{31}x_{42}-x_{32}x_{41}} & 0 & 0 \\ \frac{3x_{31}x_{32}x_{44}-3x_{31}x_{34}x_{42}+x_{32}^2x_{43}-x_{32}x_{33}x_{42}}{4x_{31}x_{42}-4x_{32}x_{41}} & -\frac{3x_{31}^2x_{44}+x_{31}x_{32}x_{43}-3x_{31}x_{34}x_{41}-x_{32}x_{33}x_{41}}{4x_{31}x_{42}-4x_{32}x_{41}} & \frac{x_{32}}{4} & \frac{3x_{31}}{4} \\ \frac{3x_{32}x_{41}x_{44}+x_{32}x_{42}x_{43}-x_{33}x_{42}^2-3x_{34}x_{41}x_{42}}{4x_{31}x_{42}-4x_{32}x_{41}} & -\frac{3x_{31}x_{41}x_{44}+x_{31}x_{42}x_{43}-x_{33}x_{41}x_{42}-3x_{34}x_{41}^2}{4x_{31}x_{42}-4x_{32}x_{41}} & \frac{x_{42}}{4} & \frac{3x_{41}}{4} \end{bmatrix}$$

Now, a differential Siegel modular form $f \in \mathcal{O}_{\mathsf{T}}$ evaluated at (t, S) above has an expression in terms of abelian integrals x_{ij}. If we regard $f(\tau)$ as a function in τ, this means that we replace τ in (7.9) in $f(\tau)$ and we get such period expressions. For instance, we have the equality:

$$\text{(7.11)} \quad T_{4i}\left(\begin{bmatrix} \frac{x_{11}x_{42}-x_{41}x_{12}}{x_{31}x_{42}-x_{41}x_{32}} & -\frac{x_{11}x_{32}-x_{31}x_{12}}{x_{31}x_{42}-x_{41}x_{32}} \\ \frac{x_{21}x_{42}-x_{41}x_{22}}{x_{31}x_{42}-x_{41}x_{32}} & -\frac{x_{21}x_{32}-x_{31}x_{22}}{x_{31}x_{42}-x_{41}x_{32}} \end{bmatrix}\right) = a_i \cdot (x_{31}x_{42} - x_{41}x_{32})^i, \quad i = 1, 2, 3, 4, 5,$$

where $(a_1, a_2, a_3, a_4, a_5) = (t_2, t_4, t_3^2, t_3t_5, t_5^2)$. This follows form the definition of T_i's in §4.4 and we have regarded them as functions on $\mathbb{H}_2$. The periods x_{ij} are functions in t, and if we write them in terms of the modular parameters (4.3) and then these parameters as functions of τ then we may arrive in formulas which are natural generalization of some classical identities such as Fricke and Klein's formula in the case of elliptic curves:

$$\sqrt[4]{E_4(\tau)} = F\left(\frac{1}{12}, \frac{5}{12}, 1; \frac{1728}{j(\tau)}\right),$$

see for instance [KZ01, §2.3]. See also [Mov12, page 364] for more examples of this type. Finally note that we have six relations between x_{ij}:

$$
\begin{aligned}
x_{12}x_{31} - x_{11}x_{32} + x_{22}x_{41} - x_{21}x_{42} &= 0, \\
x_{13}x_{31} - x_{11}x_{33} + x_{23}x_{41} - x_{21}x_{43} &= 0, \\
x_{14}x_{31} - x_{11}x_{34} + x_{24}x_{41} - x_{21}x_{44} &= \frac{4}{3}, \\
x_{13}x_{32} - x_{12}x_{33} + x_{23}x_{42} - x_{22}x_{43} &= 4, \\
x_{14}x_{32} - x_{12}x_{34} + x_{24}x_{42} - x_{22}x_{44} &= 0, \\
x_{14}x_{33} - x_{13}x_{34} + x_{24}x_{43} - x_{23}x_{44} &= \frac{4}{3}t_2.
\end{aligned}
\tag{7.12}
$$

These equalities correspond to the entries $(1,2),(1,3),(1,4),(2,3),(2,4)$ and $(3,4)$ of the equality $[x_{ij}]^{\mathsf{tr}}\Phi^{-\mathsf{tr}}[x_{ij}] = \Omega$, where Φ and Ω are given in (4.5) and (3.10). This comes from the duality of intersection form in homology and cup product in cohomology, see for instance [Mov17, §4.1].

7.4. **Proof of Theorem 1.1.** The main ingredient of the proof is the t-map which translates most of the algebraic machinery introduced in this paper into complex analysis of holomorphic functions on the Siegel domain. The 153 quantities mentioned in the theorem are just the pull-back of functions constructed in §4.4. Part 1 is established in §7.1. Note that the $\mathbb{G}_m$-action on $\mathbf{S}$ and the functional equation (4.13) via the t-map is translated into the functional equation (7.3). Part 2 follows from the fact that T is an affine variety defined over $\mathbb{Q}$. The ideal of polynomial relations among X_i can be computed easily from their algebraic definition in §4.4. For part 3 note that the affine variety $\mathbb{Q}[X]/I$ is just the ring of functions that we have described in §4.4. For part 4 note that $\frac{\partial X_i}{\partial \tau_i}$ is just $dX_i(\mathsf{R}_k)$ and the computation of this is explained in §5.1. In §5.1 we have explained how to write R_k's in the affine chart given by X_i's. Since R_k is meromorphic in t_5, the vector field in X_i variables has a pole order one in $X_5 = T_{20} = t_5^2$.

8. Compatibility with Resnikoff's computation

In this section, we compare our construction of differential equations of Siegel modular forms with few differential equations obtained by Resnikoff in [Res70a, Res70b]. Let

$$\partial := \det \begin{bmatrix} \mathsf{R}_1 & \frac{1}{2}\mathsf{R}_3 \\ \frac{1}{2}\mathsf{R}_3 & \mathsf{R}_2 \end{bmatrix} = \mathsf{R}_1 \circ \mathsf{R}_2 - \frac{1}{4}\mathsf{R}_3 \circ \mathsf{R}_3,$$

where we consider $\mathsf{R}_i : \mathcal{O}_{\mathsf{T}} \to \mathcal{O}_{\mathsf{T}}$ as derivations. For $f \in \mathcal{O}_{\mathbf{S}}$ of degree $4w$ (of weight w if f is interpreted as a Siegel modular form) let also

$$D^{\mathrm{Res}} f = \frac{4w-1}{4w^2} f\partial f + \frac{1-2w}{8w^2}\partial f^2. \tag{8.1}$$

It turns out that D^{Res} maps $\mathcal{O}_{\mathbf{S}}$ to the set of cuspidal forms in $\mathcal{O}_{\mathbf{S}}$. It seems to be interesting to prove this statement in our geometric framework. For

$$\widehat{E}_2 = A/\delta^2, \widehat{E}_4 = B/\delta^4, \widehat{E}_6 = (4AB-3C)/\delta^6, \widehat{\chi}_{10} = D/\delta^{10}, \widehat{\chi}_{12} = AD/\delta^{12}.$$

we have verified the following equalities which confirms the above statement:

$$D^{\mathrm{Res}}\widehat{E}_4 = \frac{984375}{1024}\widehat{\chi}_{10}; \tag{8.2}$$

$$D^{\mathrm{Res}}\widehat{E}_6 = \frac{2165625}{64}\widehat{E}_4\widehat{\chi}_{10}; \tag{8.3}$$

$$D^{\mathrm{Res}}\widehat{\chi}_{10} = -\frac{3}{6400}\widehat{\chi}_{10}\widehat{\chi}_{12}; \tag{8.4}$$

$$D^{\mathrm{Res}}\widehat{\chi}_{12} = -\frac{49}{2304}\widehat{E}_6\widehat{\chi}_{10}^2 + \frac{37}{2304}\widehat{E}_4\widehat{\chi}_{10}\widehat{\chi}_{12}. \tag{8.5}$$

$$D^{\mathrm{Res}}\widehat{E}_2 = \frac{3}{256}\widehat{E}_2^3 - \frac{1}{16}\widehat{E}_2\widehat{E}_4 - \frac{1}{8}\widehat{E}_6. \tag{8.6}$$

Except for the last one, these are also obtained in [Res70a, Theorem 1]). Note that up to multiplication with a constant $\widehat{E}_4$ (resp. $\widehat{E}_6, \widehat{\chi}_{10}, \widehat{\chi}_{12}$) is equal to E_4 (resp. $E_6, \chi_{10}, \chi_{12}$) and this is the reason why the constants in the above equalities are different from Resnikoff's constants.

9. Final comments

The vector fields R_k's together with the vector fields coming from the action of G on T provide natural foliations for Humbert surfaces in the moduli of principally polarized abelian varieties. The general theory is being formulated in [Mov20] and explicit computations of R_k's in the present text can be used for computing explicit equations for such surfaces, see [DMWH16]. In [Igu67], Igusa defined the map from the ring of Siegel modular forms to the ring of invariants via the theta functions. It seems also reasonable to use theta functions to understand the differential Siegel modular forms in our sense. For this we must rewrite the content of this paper for the family $y^2 = \prod_{i=1}^{6}(x - t_i)$. In our geometric setting, we have not written (differential) Siegel modular forms as Poincaré series and one might ask for their Fourier expansions. This can be done using the explicit expression of R_k's, which turns out to be recursion in the coefficients of such Fourier expansions. For such a computation in the case of Calabi-Yau modular forms, one may find in [Mov17, Chapter 5].

Appendix A. Our computer code

For the convenience of the reader we reproduce here our computer code in order to compute the Gauss-Manin connection matrices B_i, $i = 2, 3, 4, 5$ and the vector fields R_k, $k = 1, 2, 3$. The library `foliation.lib` of Singular can be downloaded from the second author's webpage.

```
LIB "foliation.lib"; LIB "latex.lib";
ring r=(0,t(2..5)),(x,y),wp(2,5);
poly f=y^2-x^5-t(2)*x^3-t(3)*x^2-t(4)*x-t(5);
list GM1=gaussmaninmatrix(f,list(t(2..5)), 2);
matrix S[4][4]=0,0,0,1,0,0,1,0,0,1,0,0,1,0,0,0; print(S);
     for (int i=2; i<=5;i=i+1){ GM1[i]=S*GM1[i]*inverse(S);}
list B=0; for (int i=1; i<=4;i=i+1){ B=insert(B, GM1[1]*GM1[i+1], size(B));}
poly Delta=1/GM1[1];
//-----B is the list of Gauss-Manin connection matrices-----------
matrix Om[4][4]= 0,0,0,4/3,0,0,4,0,0,-4,0,(4/3)*t(2),-4/3,0,
```

```
-(4/3)*t(2), 0;   print(Om);
//-----Om is the cup product pairing in de Rham cohomology
//-----Checking the differential equation of Om--------------
for (int i=2; i<=5;i=i+1){ print(B[i]*Om+Om*transpose(B[i]));}
//-------A check for the structure of M matrix----------
matrix M[4][4]=B[2][1,3],B[3][1,3], B[4][1,3], B[5][1,3],
B[2][1,4],B[3][1,4], B[4][1,4], B[5][1,4], B[2][2,3],
B[3][2,3], B[4][2,3], B[5][2,3], B[2][2,4], B[3][2,4], B[4][2,4], B[5][2,4];
matrix kn[1][4]=-3,0,0,1; print(kn*M);
//-------Computing R_k's in t_i's---------------------------
ring r2=(0,t(2..5), s(-2), s(-4),s(-6),s(1..4)(1..2)),(x(2..5)),dp;
matrix M=imap(r, M);
matrix X[4][1]=x(2..5);  matrix S[4][1]=s(-4),s(-6),s(-2), 3*s(-4);
ideal I=M*X-S; I=std(I); int i; list Ra=0;
for (i=2; i<=5;i=i+1){ Ra=insert(Ra, reduce(x(i), I),size(Ra));}
for (i=2; i<=5;i=i+1){Ra[i]=subst(Ra[i],x(5),0); }
list B=imap(r, B);
matrix E=Ra[2]*B[2]+Ra[3]*B[3]+Ra[4]*B[4]+Ra[5]*B[5]; //-Computing E-2, E-4, E-6--
matrix S1[2][2]=s(1)(1),s(1)(2),s(2)(1),s(2)(2);
matrix S3[2][2]=s(3)(1),s(3)(2),s(4)(1),s(4)(2);
poly delta= det(S1);
matrix Om2[2][2]= 0,4/3,4,0;
matrix S4 =inverse(transpose(S1))*inverse(transpose(Om2));
//---------Computing R_1, R_2, R_3--------------------------
list Cl; matrix C[2][2]=1,0,0,0; Cl=C;
C=0,0,0,1; Cl=insert(Cl, C, size(Cl));
C=0,1,1,0; Cl=insert(Cl, C, size(Cl));
list s4l=-1/4*s(2)(1)*s(2)(2)/(delta^2), -1/4*s(1)(1)*s(1)(2)/(delta^2),
                                 1/4*(s(1)(2)*s(2)(1)+s(1)(1)*s(2)(2))/(delta^2);
list s6l=3/4*s(2)(2)^2/(delta^2),       3/4*s(1)(2)^2/(delta^2),        -3/2*s(1)(2)*s(2)(2)/(delta^2);
list s2l=1/4*s(2)(1)^2/(delta^2),       1/4*s(1)(1)^2/(delta^2),        -1/2*s(1)(1)*s(2)(1)/(delta^2);
matrix Eh=E; int k;  matrix E1[2][2]; matrix E3[2][2]; matrix R[4][2]; list Ral; list Rh; poly komak;
for (k=1; k<=3;k=k+1)
     {
     Rh=Ra[2],Ra[3],Ra[4],Ra[5];
     for (i=1; i<=size(Rh);i=i+1)
         {
         komak=Rh[i];
         komak=substpar(komak, s(-2), s2l[k]); komak=substpar(komak, s(-4), s4l[k]);
         komak=substpar(komak, s(-6), s6l[k]);
         Rh[i]=komak;
         }
     Eh=substpar(E, s(-2), s2l[k]); Eh=substpar(Eh, s(-4), s4l[k]); Eh=substpar(Eh, s(-6), s6l[k]);
     E1=submat(Eh, 1..2, 1..2); E3=submat(Eh, 3..4, 1..2);
     R=Cl[k]*S3-S1*E1, -S3*E1-S4*E3;
     Rh=Rh+list(R[1,1], R[1,2], R[2,1], R[2,2], R[3,1], R[3,2], R[4,1], R[4,2]);
     Ral=insert(Ral, Rh, size(Ral));
     }
//--------------Tangency to Delta=0 for this we can use s(-4),s(-6), s(-2)------------
poly Delta=imap(r, Delta);
(diffpar(Delta,t(2))*Ra[2]+diffpar(Delta,t(3))*Ra[3]+diffpar(Delta,t(4))*Ra[4]+diffpar(Delta,t(5))*Ra[5])/Delta;
//-------------------Three proceedure----------------------------
proc dvpar(poly P, list parl, list vfl)
{
poly Q; for (int i=1; i<=size(parl);i=i+1){Q=Q+diffpar(P, parl[i])*vfl[i];} return(Q);
}
proc Resnikoffd(poly P)
{
poly Q= dvpar( dvpar(P, lp, Ral[1]), lp, Ral[2])-(1/4)*dvpar( dvpar(P, lp, Ral[3]), lp, Ral[3]); return(Q);
}
proc ResnikoffD(poly P, int w)
{
```

```
poly Q= (8*w-2)*P*Resnikoffd(P)-(2*w-1)*Resnikoffd(P^2); Q=Q/(8*w^2); return(Q);
}
//------verfying that F=0 is invariant-----------------
poly F=s(4)(2)*s(2)(1)-s(4)(1)*s(2)(2)+s(3)(2)*s(1)(1)-s(3)(1)*s(1)(2)-t(2)/4;
list lp=t(2..5), s(1)(1), s(1)(2), s(2)(1), s(2)(2), s(3)(1), s(3)(2), s(4)(1), s(4)(2);
for (k=1; k<=3;k=k+1)
        {
        delta^2*25*t(5)*dvpar(F, lp, Ral[k])/F;
        }
//----------------Verifying Resnikoff's differential equation-----------
poly Ai=(-3*t(2)^2-20*t(4))/delta^2;
poly Bi= (-3*t(2)*t(3)^2+9*t(2)^2*t(4)-20*t(4)^2+75*t(3)*t(5))/(delta^4);
poly Ci=(12*t(2)^3*t(3)^2+18*t(3)^4-36*t(2)^4*t(4)-13*t(2)*t(3)^2*t(4)-88*t(2)^2*t(4)^2+160*t(4)^3-
165*t(2)^2*t(3)*t(5)-800*t(3)*t(4)*t(5)-1125*t(2)*t(5)^2)/(delta^6);
poly Di=Delta/delta^10;
poly Ei=
(729*t(2)^10*t(5)^2-486*t(2)^9*t(3)*t(4)*t(5)+108*t(2)^8*t(3)^3*t(5)+81*t(2)^8*t(3)^2*t(4)^2
-18225*t(2)^8*t(4)*t(5)^2-36*t(2)^7*t(3)^4*t(4)+12150*t(2)^7*t(3)^2*t(5)^2+10800*t(2)^7*t(3)*t(4)^2*t(5)
+4*t(2)^6*t(3)^6-9720*t(2)^6*t(3)^3*t(4)*t(5)-1800*t(2)^6*t(3)^2*t(4)^3+135000*t(2)^6*t(4)^2*t(5)^2
+1584*t(2)^5*t(3)^5*t(5)+2015*t(2)^5*t(3)^4*t(4)^2-175500*t(2)^5*t(3)^2*t(4)*t(5)^2-60000*t(2)^5*t(3)*t(4)^3*t(5)
+928125*t(2)^5*t(5)^4-623*t(2)^4*t(3)^6*t(4)+60000*t(2)^4*t(3)^4*t(5)^2+92500*t(2)^4*t(3)^3*t(4)^2*t(5)
+10000*t(2)^4*t(3)^2*t(4)^4-1012500*t(2)^4*t(3)*t(4)*t(5)^3-250000*t(2)^4*t(4)^3*t(5)^2+59*t(2)^3*t(3)^8
-45050*t(2)^3*t(3)^5*t(4)*t(5)-17500*t(2)^3*t(3)^4*t(4)^3+225000*t(2)^3*t(3)^3*t(5)^3
+850000*t(2)^3*t(3)^2*t(4)^2*t(5)^2-2812500*t(2)^3*t(4)*t(5)^4+5700*t(2)^2*t(3)^7*t(5)+10825*t(2)^2*t(3)^6*t(4)^2
-478125*t(2)^2*t(3)^4*t(4)*t(5)^2-50000*t(2)^2*t(3)^3*t(4)^3*t(5)+1875000*t(2)^2*t(3)^2*t(5)^4
+1250000*t(2)^2*t(3)*t(4)^2*t(5)^3-2610*t(2)*t(3)^8*t(4)+93750*t(2)*t(3)^6*t(5)^2+32500*t(2)*t(3)^5*t(4)^2*t(5)
-1187500*t(2)*t(3)^3*t(4)*t(5)^3+216*t(3)^10-9000*t(3)^7*t(4)*t(5)+625*t(3)^6*t(4)^3+175000*t(3)^5*t(5)^3
+15625*t(3)^4*t(4)^2*t(5)^2-390625*t(3)^2*t(4)*t(5)^4-9765625*t(5)^6)/delta^15;

poly DBi=ResnikoffD(Bi,4);
substpar(DBi/Di,t(2), 4*F+t(2));
poly DAD=ResnikoffD(Di,10);
substpar(DAD/(Di*Ai*Di), t(2), 4*F+t(2));
poly P=ResnikoffD(Ai*Bi+(-3/4)*Ci,6);
poly X1=substpar(P/(Bi*Di),t(2), 4*F+t(2));

poly DK12=ResnikoffD(Ai*Di,12);
poly X1=substpar(DK12/(Di)^2,t(2), 4*F+t(2));
X1=X1*delta^6;
ring r3=(0,t(2..5)), (s(1..4)(1..2)), dp;
ideal I=(-t(2)+4*s(1)(1)*s(3)(2)-4*s(1)(2)*s(3)(1)+4*s(2)(1)*s(4)(2)-4*s(2)(2)*s(4)(1))/4;
I=std(I);  poly X1=imap(r2, X1); reduce(X1, I);
```

Appendix B. Three modular vector fields

In this appendix, we reproduce our computation of modular vector fields $\mathsf{R}_i,\ i = 1,2,3$ corresponding to classical derivations $\frac{\partial}{\partial \tau_i}$ in the Siegel domain for $g=2$. They are written in the weighted homogeneous coordinate system $t_2, t_3, t_4, t_5,\ s_{11}, s_{12}, s_{21},$

$s_{22}, s_{31}, s_{32}, s_{41}, s_{42}$ with $\delta := s_{11}s_{22} - s_{12}s_{21}$.

$$
\begin{aligned}
&\mathsf{R}_1 = \\
&[(4t_2^2t_4s_{21}^2 + 30t_2^2t_5s_{21}s_{22} - 6t_2t_3t_4s_{21}s_{22} - 20t_2t_3t_5s_{22}^2 + 8t_2t_4^2s_{22}^2 + 75t_3t_5s_{21}^2 \\
&-100t_4t_5s_{21}s_{22} + 125t_5^2s_{22}^2)/(50t_5\delta^2)]\partial/\partial t_2 \\
&+[(-30t_2^2t_5s_{21}^2 + 6t_2t_3t_4s_{21}^2 + 65t_2t_3t_5s_{21}s_{22} - 10t_2t_4t_5s_{22}^2 - 9t_3^2t_4s_{21}s_{22} \\
&-30t_3^2t_5s_{22}^2 + 12t_3t_4^2s_{22}^2 + 100t_4t_5s_{21}^2 - 125t_5^2s_{21}s_{22})/(50t_5\delta^2)]\partial/\partial t_3 \\
&+[(-20t_2t_3t_5s_{21}^2 + 8t_2t_4^2s_{21}^2 + 10t_2t_4t_5s_{21}s_{22} + 75t_2t_5^2s_{22}^2 + 30t_3^2t_5s_{21}s_{22} \\
&-12t_3t_4^2s_{21}s_{22} - 55t_3t_4t_5s_{22}^2 + 16t_4^3s_{22}^2 + 125t_5^2s_{21}^2)/(50t_5\delta^2)]\partial/\partial t_4 \\
&+[0]\partial/\partial t_5 \\
&+[(6t_2t_4s_{11}s_{21}^2 + 20t_2t_5s_{11}s_{21}s_{22} - 20t_2t_5s_{12}s_{21}^2 - 9t_3t_4s_{11}s_{21}s_{22} - 30t_3t_5s_{11}s_{22}^2 \\
&+30t_3t_5s_{12}s_{21}s_{22} + 12t_4^2s_{11}s_{22}^2 - 15t_4t_5s_{12}s_{22}^2 + 100t_5s_{11}^2s_{22}^2s_{31} \\
&-200t_5s_{11}s_{12}s_{21}s_{22}s_{31} + 100t_5s_{12}^2s_{21}^2s_{31})/(100t_5\delta^2)]\partial/\partial s_{11} \\
&+[(2t_2t_4s_{12}s_{21}^2 - 25t_2t_5s_{11}s_{22}^2 + 40t_2t_5s_{12}s_{21}s_{22} - 3t_3t_4s_{12}s_{21}s_{22} - 10t_3t_5s_{12}s_{22}^2 \\
&+4t_4^2s_{12}s_{22}^2 + 100t_5s_{11}^2s_{22}^2s_{32} - 200t_5s_{11}s_{12}s_{21}s_{22}s_{32} + 25t_5s_{11}s_{21}^2 \\
&+100t_5s_{12}^2s_{21}^2s_{32})/(100t_5\delta^2)]\partial/\partial s_{12} \\
&+[(6t_2t_4s_{21}^3 - 9t_3t_4s_{21}^2s_{22} + 12t_4^2s_{21}s_{22}^2 - 15t_4t_5s_{22}^3)/(100t_5\delta^2)]\partial/\partial s_{21} \\
&+[(2t_2t_4s_{21}^2s_{22} + 15t_2t_5s_{21}s_{22}^2 - 3t_3t_4s_{21}s_{22}^2 - 10t_3t_5s_{22}^3 + 4t_4^2s_{22}^3 \\
&+25t_5s_{21}^3)/(100t_5\delta^2)]\partial/\partial s_{22} \\
&+[(6t_2t_4s_{11}s_{21}^2s_{22}s_{31} - 6t_2t_4s_{12}s_{21}^3s_{31} - 20t_2t_5s_{11}s_{21}^2s_{22}s_{32} + 20t_2t_5s_{11}s_{21}s_{22}^2s_{31} \\
&+20t_2t_5s_{12}s_{21}^3s_{32} - 20t_2t_5s_{12}s_{21}^2s_{22}s_{31} - 9t_3t_4s_{11}s_{21}s_{22}^2s_{31} + 9t_3t_4s_{12}s_{21}^2s_{22}s_{31} \\
&+30t_3t_5s_{11}s_{21}s_{22}^2s_{32} - 30t_3t_5s_{11}s_{22}^3s_{31} - 30t_3t_5s_{12}s_{21}^2s_{22}s_{32} + 30t_3t_5s_{12}s_{21}s_{22}^2s_{31} \\
&+12t_4^2s_{11}s_{22}^3s_{31} - 12t_4^2s_{12}s_{21}s_{22}^2s_{31} - 15t_4t_5s_{11}s_{22}^3s_{32} + 15t_4t_5s_{12}s_{21}s_{22}^2s_{32} \\
&+25t_4t_5s_{21}^2s_{22} - 50t_5^2s_{21}s_{22}^2)/(100t_5\delta^3)]\partial/\partial s_{31} \\
&+[(2t_2t_4s_{11}s_{21}^2s_{22}s_{32} - 2t_2t_4s_{12}s_{21}^3s_{32} + 40t_2t_5s_{11}s_{21}s_{22}^2s_{32} - 25t_2t_5s_{11}s_{22}^3s_{31} \\
&-40t_2t_5s_{12}s_{21}^2s_{22}s_{32} + 25t_2t_5s_{12}s_{21}s_{22}^2s_{31} + 10t_2t_5s_{21}^3 - 3t_3t_4s_{11}s_{21}s_{22}^2s_{32} \\
&+3t_3t_4s_{12}s_{21}^2s_{22}s_{32} - 10t_3t_5s_{11}s_{22}^3s_{32} + 10t_3t_5s_{12}s_{21}s_{22}^2s_{32} + 35t_3t_5s_{21}^2s_{22} \\
&+4t_4^2s_{11}s_{22}^3s_{32} - 4t_4^2s_{12}s_{21}s_{22}^2s_{32} - 55t_4t_5s_{21}s_{22}^2 + 75t_5^2s_{22}^3 + 25t_5s_{11}s_{21}^2s_{22}s_{31} \\
&-25t_5s_{12}s_{21}^3s_{31})/(100t_5\delta^3)]\partial/\partial s_{32} \\
&+[(24t_2t_4s_{11}s_{21}^2s_{22}s_{41} - 24t_2t_4s_{12}s_{21}^3s_{41} - 80t_2t_5s_{11}s_{21}^2s_{22}s_{42} + 80t_2t_5s_{11}s_{21}s_{22}^2s_{41} \\
&+80t_2t_5s_{12}s_{21}^3s_{42} - 80t_2t_5s_{12}s_{21}^2s_{22}s_{41} - 36t_3t_4s_{11}s_{21}s_{22}^2s_{41} + 36t_3t_4s_{12}s_{21}^2s_{22}s_{41} \\
&+120t_3t_5s_{11}s_{21}s_{22}^2s_{42} - 120t_3t_5s_{11}s_{22}^3s_{41} - 120t_3t_5s_{12}s_{21}^2s_{22}s_{42} + 120t_3t_5s_{12}s_{21}s_{22}^2s_{41} \\
&+48t_4^2s_{11}s_{22}^3s_{41} - 48t_4^2s_{12}s_{21}s_{22}^2s_{41} - 25t_4t_5s_{11}s_{21}s_{22} - 60t_4t_5s_{11}s_{22}^3s_{42} - 75t_4t_5s_{12}s_{21}^2 \\
&+60t_4t_5s_{12}s_{21}s_{22}^2s_{42} + 50t_5^2s_{11}s_{22}^2 + 150t_5^2s_{12}s_{21}s_{22})/(400t_5\delta^3)]\partial/\partial s_{41} \\
&+[(8t_2t_4s_{11}s_{21}^2s_{22}s_{42} - 8t_2t_4s_{12}s_{21}^3s_{42} - 40t_2t_5s_{11}s_{21}^2 + 160t_2t_5s_{11}s_{21}s_{22}^2s_{42} \\
&-100t_2t_5s_{11}s_{22}^3s_{41} - 160t_2t_5s_{12}s_{21}^2s_{22}s_{42} + 100t_2t_5s_{12}s_{21}s_{22}^2s_{41} - 12t_3t_4s_{11}s_{21}s_{22}^2s_{42} \\
&+12t_3t_4s_{12}s_{21}^2s_{22}s_{42} + 10t_3t_5s_{11}s_{21}s_{22} - 40t_3t_5s_{11}s_{22}^3s_{42} - 150t_3t_5s_{12}s_{21}^2 \\
&+40t_3t_5s_{12}s_{21}s_{22}^2s_{42} + 16t_4^2s_{11}s_{22}^3s_{42} - 16t_4^2s_{12}s_{21}s_{22}^2s_{42} - 5t_4t_5s_{11}s_{22}^2 + 225t_4t_5s_{12}s_{21}s_{22} \\
&-300t_5^2s_{12}s_{22}^2 + 100t_5s_{11}s_{21}^2s_{22}s_{41} - 100t_5s_{12}s_{21}^3s_{41})/(400t_5\delta^3)]\partial/\partial s_{42}
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{R}_2 = \\
&[(4t_2^2t_4s_{11}^2 + 30t_2^2t_5s_{11}s_{12} - 6t_2t_3t_4s_{11}s_{12} - 20t_2t_3t_5s_{12}^2 + 8t_2t_4^2s_{12}^2 + 75t_3t_5s_{11}^2 \\
&-100t_4t_5s_{11}s_{12} + 125t_5^2s_{12}^2)/(50t_5\delta^2)]\partial/\partial t_2 \\
&+[(-30t_2^2t_5s_{11}^2 + 6t_2t_3t_4s_{11}^2 + 65t_2t_3t_5s_{11}s_{12} - 10t_2t_4t_5s_{12}^2 - 9t_3^2t_4s_{11}s_{12} - 30t_3^2t_5s_{12}^2 \\
&+12t_3t_4^2s_{12}^2 + 100t_4t_5s_{11}^2 - 125t_5^2s_{11}s_{12})/(50t_5\delta^2)]\partial/\partial t_3 \\
&+[(-20t_2t_3t_5s_{11}^2 + 8t_2t_4^2s_{11}^2 + 10t_2t_4t_5s_{11}s_{12} + 75t_2t_5^2s_{12}^2 + 30t_3^2t_5s_{11}s_{12} - 12t_3t_4^2s_{11}s_{12} \\
&-55t_3t_4t_5s_{12}^2 + 16t_4^3s_{12}^2 + 125t_5^2s_{11}^2)/(50t_5\delta^2)]\partial/\partial t_4 \\
&+[0]\partial/\partial t_5 \\
&+[(6t_2t_4s_{11}^3 - 9t_3t_4s_{11}^2s_{12} + 12t_4^2s_{11}s_{12}^2 - 15t_4t_5s_{12}^3)/(100t_5\delta^2)]\partial/\partial s_{11} \\
&+[(2t_2t_4s_{11}^2s_{12} + 15t_2t_5s_{11}s_{12}^2 - 3t_3t_4s_{11}s_{12}^2 - 10t_3t_5s_{12}^3 + 4t_4^2s_{12}^3 \\
&+25t_5s_{11}^3)/(100t_5\delta^2)]\partial/\partial s_{12} \\
&+[(6t_2t_4s_{11}^2s_{21} - 20t_2t_5s_{11}^2s_{22} + 20t_2t_5s_{11}s_{12}s_{21} - 9t_3t_4s_{11}s_{12}s_{21} + 30t_3t_5s_{11}s_{12}s_{22} \\
&-30t_3t_5s_{12}^2s_{21} + 12t_4^2s_{12}^2s_{21} - 15t_4t_5s_{12}^2s_{22} + 100t_5s_{11}^2s_{22}^2s_{41} - 200t_5s_{11}s_{12}s_{21}s_{22}s_{41} \\
&+100t_5s_{12}^2s_{21}^2s_{41})/(100t_5\delta^2)]\partial/\partial s_{21} \\
&+[(2t_2t_4s_{11}^2s_{22} + 40t_2t_5s_{11}s_{12}s_{22} - 25t_2t_5s_{12}^2s_{21} - 3t_3t_4s_{11}s_{12}s_{22} - 10t_3t_5s_{12}^2s_{22} \\
&+4t_4^2s_{12}^2s_{22} + 25t_5s_{11}^2s_{21} + 100t_5s_{11}^2s_{22}^2s_{42} - 200t_5s_{11}s_{12}s_{21}s_{22}s_{42} + 100t_5s_{12}^2s_{21}^2s_{42})/ \\
&(100t_5\delta^2)]\partial/\partial s_{22} \\
&+[(24t_2t_4s_{11}^3s_{22}s_{31} - 24t_2t_4s_{11}^2s_{12}s_{21}s_{31} - 80t_2t_5s_{11}^3s_{22}s_{32} + 80t_2t_5s_{11}^2s_{12}s_{21}s_{32} \\
&+80t_2t_5s_{11}^2s_{12}s_{22}s_{31} - 80t_2t_5s_{11}s_{12}^2s_{21}s_{31} - 36t_3t_4s_{11}^2s_{12}s_{22}s_{31} + 36t_3t_4s_{11}s_{12}^2s_{21}s_{31} \\
&+120t_3t_5s_{11}^2s_{12}s_{22}s_{32} - 120t_3t_5s_{11}s_{12}^2s_{21}s_{32} - 120t_3t_5s_{11}s_{12}^2s_{22}s_{31} + 120t_3t_5s_{12}^3s_{21}s_{31} \\
&+48t_4^2s_{11}s_{12}^2s_{22}s_{31} - 48t_4^2s_{12}^3s_{21}s_{31} + 75t_4t_5s_{11}^2s_{22} - 60t_4t_5s_{11}s_{12}^2s_{22}s_{32} + 25t_4t_5s_{11}s_{12}s_{21} \\
&+60t_4t_5s_{12}^3s_{21}s_{32} - 150t_5^2s_{11}s_{12}s_{22} - 50t_5^2s_{12}^2s_{21})/(400t_5\delta^3)]\partial/\partial s_{31} \\
&+[(8t_2t_4s_{11}^3s_{22}s_{32} - 8t_2t_4s_{11}^2s_{12}s_{21}s_{32} + 160t_2t_5s_{11}^2s_{12}s_{22}s_{32} + 40t_2t_5s_{11}^2s_{21} \\
&-160t_2t_5s_{11}s_{12}^2s_{21}s_{32} - 100t_2t_5s_{11}s_{12}^2s_{22}s_{31} + 100t_2t_5s_{12}^3s_{21}s_{31} - 12t_3t_4s_{11}^2s_{12}s_{22}s_{32} \\
&+12t_3t_4s_{11}s_{12}^2s_{21}s_{32} + 150t_3t_5s_{11}^2s_{22} - 40t_3t_5s_{11}s_{12}^2s_{22}s_{32} - 10t_3t_5s_{11}s_{12}s_{21} + 40t_3t_5s_{12}^3s_{21}s_{32} \\
&+16t_4^2s_{11}s_{12}^2s_{22}s_{32} - 16t_4^2s_{12}^3s_{21}s_{32} - 225t_4t_5s_{11}s_{12}s_{22} + 5t_4t_5s_{12}^2s_{21} + 300t_5^2s_{12}^2s_{22} \\
&+100t_5s_{11}^3s_{22}s_{31} - 100t_5s_{11}^2s_{12}s_{21}s_{31})/(400t_5\delta^3]\partial/\partial s_{32} \\
&+[(6t_2t_4s_{11}^3s_{22}s_{41} - 6t_2t_4s_{11}^2s_{12}s_{21}s_{41} - 20t_2t_5s_{11}^3s_{22}s_{42} + 20t_2t_5s_{11}^2s_{12}s_{21}s_{42} \\
&+20t_2t_5s_{11}^2s_{12}s_{22}s_{41} - 20t_2t_5s_{11}s_{12}^2s_{21}s_{41} - 9t_3t_4s_{11}^2s_{12}s_{22}s_{41} + 9t_3t_4s_{11}s_{12}^2s_{21}s_{41} \\
&+30t_3t_5s_{11}^2s_{12}s_{22}s_{42} - 30t_3t_5s_{11}s_{12}^2s_{21}s_{42} - 30t_3t_5s_{11}s_{12}^2s_{22}s_{41} + 30t_3t_5s_{12}^3s_{21}s_{41} \\
&+12t_4^2s_{11}s_{12}^2s_{22}s_{41} - 12t_4^2s_{12}^3s_{21}s_{41} - 25t_4t_5s_{11}^2s_{12} - 15t_4t_5s_{11}s_{12}^2s_{22}s_{42} + 15t_4t_5s_{12}^3s_{21}s_{42} \\
&+50t_5^2s_{11}s_{12}^2)/(100t_5\delta^3)]\partial/\partial s_{41} \\
&+[(2t_2t_4s_{11}^3s_{22}s_{42} - 2t_2t_4s_{11}^2s_{12}s_{21}s_{42} - 10t_2t_5s_{11}^3 + 40t_2t_5s_{11}^2s_{12}s_{22}s_{42} - 40t_2t_5s_{11}s_{12}^2s_{21}s_{42} \\
&-25t_2t_5s_{11}s_{12}^2s_{22}s_{41} + 25t_2t_5s_{12}^3s_{21}s_{41} - 3t_3t_4s_{11}^2s_{12}s_{22}s_{42} + 3t_3t_4s_{11}s_{12}^2s_{21}s_{42} - 35t_3t_5s_{11}^2s_{12} \\
&-10t_3t_5s_{11}s_{12}^2s_{22}s_{42} + 10t_3t_5s_{12}^3s_{21}s_{42} + 4t_4^2s_{11}s_{12}^2s_{22}s_{42} - 4t_4^2s_{12}^3s_{21}s_{42} + 55t_4t_5s_{11}s_{12}^2 \\
&-75t_5^2s_{12}^3 + 25t_5s_{11}^3s_{22}s_{41} - 25t_5s_{11}^2s_{12}s_{21}s_{41})/(100t_5\delta^3)]\partial/\partial s_{42}
\end{aligned}
$$

$\mathsf{R}_3 =$

$[(-4t_2^2t_4s_{11}s_{21} - 15t_2^2t_5s_{11}s_{22} - 15t_2^2t_5s_{12}s_{21} + 3t_2t_3t_4s_{11}s_{22} + 3t_2t_3t_4s_{12}s_{21}$
$+20t_2t_3t_5s_{12}s_{22} - 8t_2t_4^2s_{12}s_{22} - 75t_3t_5s_{11}s_{21} + 50t_4t_5s_{11}s_{22} + 50t_4t_5s_{12}s_{21}$
$-125t_5^2s_{12}s_{22})/(25t_5\delta^2)]\partial/\partial t_2$
$+[(60t_2^2t_5s_{11}s_{21} - 12t_2t_3t_4s_{11}s_{21} - 65t_2t_3t_5s_{11}s_{22} - 65t_2t_3t_5s_{12}s_{21} + 20t_2t_4t_5s_{12}s_{22}$
$+9t_3^2t_4s_{11}s_{22} + 9t_3^2t_4s_{12}s_{21} + 60t_3^2t_5s_{12}s_{22} - 24t_3t_4^2s_{12}s_{22} - 200t_4t_5s_{11}s_{21} + 125t_5^2s_{11}s_{22}$
$+125t_5^2s_{12}s_{21})/(50t_5\delta^2)]\partial/\partial t_3$
$+[(20t_2t_3t_5s_{11}s_{21} - 8t_2t_4^2s_{11}s_{21} - 5t_2t_4t_5s_{11}s_{22} - 5t_2t_4t_5s_{12}s_{21} - 75t_2t_5^2s_{12}s_{22}$
$-15t_3^2t_5s_{11}s_{22} - 15t_3^2t_5s_{12}s_{21} + 6t_3t_4^2s_{11}s_{22} + 6t_3t_4^2s_{12}s_{21} + 55t_3t_4t_5s_{12}s_{22} - 16t_4^3s_{12}s_{22}$
$-125t_5^2s_{11}s_{21})/(25t_5\delta^2)]\partial/\partial t_4$
$+[0]\partial/\partial t_5$
$+[(-12t_2t_4s_{11}^2s_{21} - 20t_2t_5s_{11}^2s_{22} + 20t_2t_5s_{11}s_{12}s_{21} + 9t_3t_4s_{11}^2s_{22} + 9t_3t_4s_{11}s_{12}s_{21}$
$+30t_3t_5s_{11}s_{12}s_{22} - 30t_3t_5s_{12}^2s_{21} - 24t_4^2s_{11}s_{12}s_{22} + 30t_4t_5s_{12}^2s_{22} + 100t_5s_{11}^2s_{22}^2s_{41}$
$-200t_5s_{11}s_{12}s_{21}s_{22}s_{41} + 100t_5s_{12}^2s_{21}^2s_{41})/(100t_5\delta^2)]\partial/\partial s_{11}$
$+[(-4t_2t_4s_{11}s_{12}s_{21} + 10t_2t_5s_{11}s_{12}s_{22} - 40t_2t_5s_{12}^2s_{21} + 3t_3t_4s_{11}s_{12}s_{22} + 3t_3t_4s_{12}^2s_{21}$
$+20t_3t_5s_{12}^2s_{22} - 8t_4^2s_{12}^2s_{22} - 50t_5s_{11}^2s_{21} + 100t_5s_{11}^2s_{22}^2s_{42} - 200t_5s_{11}s_{12}s_{21}s_{22}s_{42}$
$+100t_5s_{12}^2s_{21}^2s_{42})/(100t_5\delta^2)]\partial/\partial s_{12}$
$+[(-12t_2t_4s_{11}s_{21}^2 + 20t_2t_5s_{11}s_{21}s_{22} - 20t_2t_5s_{12}s_{21}^2 + 9t_3t_4s_{11}s_{21}s_{22} + 9t_3t_4s_{12}s_{21}^2$
$-30t_3t_5s_{11}s_{22}^2 + 30t_3t_5s_{12}s_{21}s_{22} - 24t_4^2s_{12}s_{21}s_{22} + 30t_4t_5s_{12}s_{22}^2 + 100t_5s_{11}^2s_{22}^2s_{31}$
$-200t_5s_{11}s_{12}s_{21}s_{22}s_{31} + 100t_5s_{12}^2s_{21}^2s_{31})/(100t_5\delta^2)]\partial/\partial s_{21}$
$+[(-4t_2t_4s_{11}s_{21}s_{22} - 40t_2t_5s_{11}s_{22}^2 + 10t_2t_5s_{12}s_{21}s_{22} + 3t_3t_4s_{11}s_{22}^2 + 3t_3t_4s_{12}s_{21}s_{22}$
$+20t_3t_5s_{12}s_{22}^2 - 8t_4^2s_{12}s_{22}^2 + 100t_5s_{11}^2s_{22}^2s_{32} - 200t_5s_{11}s_{12}s_{21}s_{22}s_{32} - 50t_5s_{11}s_{21}^2$
$+100t_5s_{12}^2s_{21}^2s_{32})/(100t_5\delta^2)]\partial/\partial s_{22}$
$+[(-48t_2t_4s_{11}^2s_{21}s_{22}s_{31} + 48t_2t_4s_{11}s_{12}s_{21}^2s_{31} + 160t_2t_5s_{11}^2s_{21}s_{22}s_{32} - 80t_2t_5s_{11}^2s_{22}^2s_{31}$
$-160t_2t_5s_{11}s_{12}s_{21}^2s_{32} + 80t_2t_5s_{12}^2s_{21}^2s_{31} + 36t_3t_4s_{11}^2s_{22}^2s_{31} - 36t_3t_4s_{12}^2s_{21}^2s_{31}$
$-120t_3t_5s_{11}^2s_{22}^2s_{32} + 240t_3t_5s_{11}s_{12}s_{22}^2s_{31} + 120t_3t_5s_{12}^2s_{21}^2s_{32} - 240t_3t_5s_{12}^2s_{21}s_{22}s_{31}$
$-96t_4^2s_{11}s_{12}s_{22}^2s_{31} + 96t_4^2s_{12}^2s_{21}s_{22}s_{31} + 120t_4t_5s_{11}s_{12}s_{22}^2s_{32} - 175t_4t_5s_{11}s_{21}s_{22}$
$-120t_4t_5s_{12}^2s_{21}s_{22}s_{32} - 25t_4t_5s_{12}s_{21}^2 + 150t_5^2s_{11}s_{22}^2 + 250t_5^2s_{12}s_{21}s_{22})/(400t_5\delta^3)]\partial/\partial s_{31}$
$+[(-16t_2t_4s_{11}^2s_{21}s_{22}s_{32} + 16t_2t_4s_{11}s_{12}s_{21}^2s_{32} - 160t_2t_5s_{11}^2s_{22}^2s_{32} + 200t_2t_5s_{11}s_{12}s_{22}^2s_{31}$
$-80t_2t_5s_{11}s_{21}^2 + 160t_2t_5s_{12}^2s_{21}^2s_{32} - 200t_2t_5s_{12}^2s_{21}s_{22}s_{31} + 12t_3t_4s_{11}^2s_{22}^2s_{32} - 12t_3t_4s_{12}^2s_{21}^2s_{32}$
$+80t_3t_5s_{11}s_{12}s_{22}^2s_{32} - 290t_3t_5s_{11}s_{21}s_{22} - 80t_3t_5s_{12}^2s_{21}s_{22}s_{32} + 10t_3t_5s_{12}s_{21}^2$
$-32t_4^2s_{11}s_{12}s_{22}^2s_{32} + 32t_4^2s_{12}^2s_{21}s_{22}s_{32} + 225t_4t_5s_{11}s_{22}^2 + 215t_4t_5s_{12}s_{21}s_{22} - 600t_5^2s_{12}s_{22}^2$
$-200t_5s_{11}^2s_{21}s_{22}s_{31} + 200t_5s_{11}s_{12}s_{21}^2s_{31})/(400t_5\delta^3)]\partial/\partial s_{32}$
$+[(-48t_2t_4s_{11}^2s_{21}s_{22}s_{41} + 48t_2t_4s_{11}s_{12}s_{21}^2s_{41} + 160t_2t_5s_{11}^2s_{21}s_{22}s_{42} - 80t_2t_5s_{11}^2s_{22}^2s_{41}$
$-160t_2t_5s_{11}s_{12}s_{21}^2s_{42} + 80t_2t_5s_{12}^2s_{21}^2s_{41} + 36t_3t_4s_{11}^2s_{22}^2s_{41} - 36t_3t_4s_{12}^2s_{21}^2s_{41}$
$-120t_3t_5s_{11}^2s_{22}^2s_{42} + 240t_3t_5s_{11}s_{12}s_{22}^2s_{41} + 120t_3t_5s_{12}^2s_{21}^2s_{42} - 240t_3t_5s_{12}^2s_{21}s_{22}s_{41}$
$-96t_4^2s_{11}s_{12}s_{22}^2s_{41} + 96t_4^2s_{12}^2s_{21}s_{22}s_{41} + 25t_4t_5s_{11}^2s_{22} + 175t_4t_5s_{11}s_{12}s_{21}$
$+120t_4t_5s_{11}s_{12}s_{22}^2s_{42} - 120t_4t_5s_{12}^2s_{21}s_{22}s_{42} - 250t_5^2s_{11}s_{12}s_{22} - 150t_5^2s_{12}^2s_{21})/$
$(400t_5\delta^3)]\partial/\partial s_{41}$

$$+[(-16t_2t_4s_{11}^2s_{21}s_{22}s_{42}+16t_2t_4s_{11}s_{12}s_{21}^2s_{42}+80t_2t_5s_{11}^2s_{21}-160t_2t_5s_{11}^2s_{22}^2s_{42}$$
$$+200t_2t_5s_{11}s_{12}s_{22}^2s_{41}+160t_2t_5s_{12}^2s_{21}^2s_{42}-200t_2t_5s_{12}^2s_{21}s_{22}s_{41}+12t_3t_4s_{11}^2s_{22}^2s_{42}$$
$$-12t_3t_4s_{12}^2s_{21}^2s_{42}-10t_3t_5s_{11}^2s_{22}+290t_3t_5s_{11}s_{12}s_{21}+80t_3t_5s_{11}s_{12}s_{22}^2s_{42}$$
$$-80t_3t_5s_{12}^2s_{21}s_{22}s_{42}-32t_4^2s_{11}s_{12}s_{22}^2s_{42}+32t_4^2s_{12}^2s_{21}s_{22}s_{42}-215t_4t_5s_{11}s_{12}s_{22}$$
$$-225t_4t_5s_{12}^2s_{21}+600t_5^2s_{12}^2s_{22}-200t_5s_{11}^2s_{21}s_{22}s_{41}+200t_5s_{11}s_{12}s_{21}^2s_{41})/(400t_5\delta^3)]\partial/\partial s_{42}$$

References

[AGZV88] V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko. *Singularities of differentiable maps. Monodromy and asymptotics of integrals Vol. II*, volume 83 of *Monographs in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1988.

[Ali13] Murad Alim. Lectures on mirror symmetry and topological string theory. 6:1–44, 2013.

[AMSY16] M. Alim, H. Movasati, E. Scheidegger, and S.-T. Yau. Gauss-Manin connection in disguise: Calabi-Yau threefolds. *Comm. Math. Phys.*, 334(3):889–914, 2016.

[BL94] Alan Brownstein and Ronnie Lee. Cohomology of the symplectic group $\mathrm{Sp}(4,\mathbb{Z})$. II: Computations at the prime 2. *Mich. Math. J.*, 41(1):181–208, 1994.

[Bol87] O. Bolza. Darstellung der rationalen ganzen invarianten der binarform sechsten grades durch die nullwerte der zugehörigen θ-functionen. *Math. Ann*, 30:478–495, 1887.

[BZ03] D. Bertrand and W. Zudilin. On the transcendence degree of the differential field generated by Siegel modular forms. *J. Reine Angew. Math.*, 554:47–68, 2003.

[CD12] Adrian Clingher and Charles F. Doran. Lattice polarized K3 surfaces and Siegel modular forms. *Adv. Math.*, 231(1):172–212, 2012.

[CFv17] Fabien Cléry, Carel Faber, and Gerard van der Geer. Covariants of binary sextics and vector-valued Siegel modular forms of genus two. *Math. Ann.*, 369(3-4):1649–1669, 2017.

[DMWH16] Ch. Doran, H. Movasati, U. Whitcher, and A. Harder. Humbert surfaces and the moduli of lattice polarized k3 surfaces. *Proceedings of Symposia in Pure Mathematics, String-Math conference proceeding 2014*, 93, 2016.

[Dra14] M. Draissma. Invariants of binary forms. *Ph.D Thesis*, 2014.

[EG98] Dan Edidin and William Graham. Equivariant intersection theory (With an appendix by Angelo Vistoli: The Chow ring of $\mathcal{M}_2$). *Invent. Math.*, 131(3):595–644, 1998.

[Fon18] Tiago J. Fonseca. Higher Ramanujan equations and periods of abelian varieties. *arXiv e-prints*, page arXiv:1807.11044, Jul 2018.

[Fre11] Eberhard Freitag. *Complex analysis. 2*. Universitext. Springer, Heidelberg, 2011. Riemann surfaces, several complex variables, abelian functions, higher modular functions.

[GPS01] G.-M. Greuel, G. Pfister, and H. Schönemann. Singular 2.0. A Computer Algebra System for Polynomial Computations, Centre for Computer Algebra, University of Kaiserslautern, 2001. `http://www.singular.uni-kl.de`.

[Gro66] Alexander Grothendieck. On the de Rham cohomology of algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, (29):95–103, 1966.

[Ibu99] Tomoyoshi Ibukiyama. On differential operators on automorphic forms and invariant pluri-harmonic polynomials. *Comment. Math. Univ. St. Paul.*, 48(1):103–118, 1999.

[Igu67] Jun-ichi Igusa. Modular forms and projective invariants. *Amer. J. Math.*, 89:817–855, 1967.

[KT08] Christian Kassel and Vladimir Turaev. *Braid groups. With the graphical assistance of Olivier Dodane.*, volume 247. New York, NY: Springer, 2008.

[KZ01] Maxim Kontsevich and Don Zagier. Periods. In *Mathematics unlimited—2001 and beyond*, pages 771–808. Springer, Berlin, 2001.

[Mov08] H. Movasati. On differential modular forms and some analytic relations between Eisenstein series. *Ramanujan J.*, 17(1):53–76, 2008.

[Mov11] H. Movasati. *Multiple Integrals and Modular Differential Equations*. 28th Brazilian Mathematics Colloquium. Instituto de Matemática Pura e Aplicada, IMPA, 2011.

[Mov12] H. Movasati. Quasi-modular forms attached to elliptic curves, I. *Ann. Math. Blaise Pascal*, 19(2):307–377, 2012.

[Mov13] H. Movasati. Quasi-modular forms attached to Hodge structures. *Fields Communication Series, 67*, 2013.

[Mov15] H. Movasati. Modular-type functions attached to mirror quintic Calabi-Yau varieties. *Math. Zeit.*, 281, Issue 3, pp. 907-929(3):907–929, 2015.

[Mov17] H. Movasati. Gauss-Manin connection in disguise: Calabi-Yau modular forms. *Surveys of Modern Mathematics, Int. Press, Boston.*, 2017.

[Mov19] H. Movasati. *A Course in Hodge Theory: with Emphasis on Multiple Integrals*. Available at author's webpage. 2019.

[Mov20] H. Movasati. *Modular and Automorphic Forms & Beyond*. Book under preparation, see author's webpage. 2020.

[Res70a] H.L. Resnikoff. A differential equation for an Eisenstein series of genus two. *Proc. Natl. Acad. Sci. USA*, 65:495–496, 1970.

[Res70b] H.L. Resnikoff. Generating some rings of modular forms in several complex variables by one element. *Rice Univ. Stud.*, 56(2):115–126, 1970.

[Tay14] D. Taylor. Moduli of hyperelliptic curves and invariants of binary forms. *Ph.D Thesis*, 2014.

[YY04] Satoshi Yamaguchi and Shing-Tung Yau. Topological string partition functions as polynomials. *J. High Energy Phys.*, (7):047, 20 pp. (electronic), 2004.

[Zud00] V. V. Zudilin. Theta constants and differential equations. *Mat. Sb.*, 191(12):77–122, 2000.

Jin Cao, Yau Mathematical Sciences Center, Tsinghua University, Beijing, China
Email address: `caojin@mail.tsinghua.edu.cn`

Hossein Movasati, Instituto de Matemática Pura e Aplicada, IMPA, Estrada Dona Castorina, 110, 22460-320, Rio de Janeiro, RJ, Brazil.
Email address: `hossein@impa.br`

Shing-Tung Yau, Mathematics Department, Harvard University, 1 Oxford Street, Cambridge MA, 02138 USA.
Email address: `yau@math.harvard.edu`