

GAUSS-MANIN CONNECTION IN DISGUISE: DWORK FAMILY

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Abstract

We study the moduli space T of the Calabi-Yau n -folds arising from the Dwork family and enhanced with bases of the n -th de Rham cohomology with constant cup product and compatible with Hodge filtration. We also describe a unique vector field R in T which contracted with the Gauss-Manin connection gives an upper triangular matrix with some non-constant entries which are natural generalizations of Yukawa couplings. For $n = 1, 2$ we compute explicit expressions of R and give a solution of R in terms of quasi-modular forms. The moduli space T is an affine variety and for $n = 4$ we give explicit coordinate system for T and compute the vector field R and the q -expansion of its solution.

1. Introduction

The project Gauss-Manin connection in disguise started in the articles [Mov15, AMSY16] and the book [Mov17] aims to unify modular and automorphic forms with topological string partition functions of string theory, see for instance [Ali13] and the references therein. Modular and automorphic forms have a vast amount of applications in number theory and so it is highly desirable to seek for such applications for q -expansions of Physics. The main ingredient of this unification is a natural generalization of Ramanujan relations between the Eisenstein series interpreted as a vector field in a moduli space of enhanced elliptic curves. This has been extensively used in transcendental number theory, see [NP01, Zud11] for an overview of some results. The starting point is either a Picard-Fuchs equation or a family of algebraic varieties. In [Mov17] the first author has described the construction of such vector fields attached to Calabi-Yau equations of the list in [AvEvSZ10], and in particular the well-known 14 family of Calabi-Yau threefolds

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in [DM06]. In this article we are going to consider the family of n -dimensional Calabi-Yau varieties $X = X_\psi$, $\psi \in \mathbb{P}^1 - \{0, 1, \infty\}$ obtained by a quotient and desingularization of the so-called Dwork family:

$$(1.1) \quad x_0^{n+2} + x_1^{n+2} + \cdots + x_{n+1}^{n+2} - (n+2)\psi x_0 x_1 \cdots x_{n+1} = 0,$$

where x_0, x_1, \dots, x_{n+1} are homogeneous coordinates of \mathbb{P}^{n+1} . From now on we call X_ψ a *mirror (Calabi-Yau) variety*, as for $n = 1, 2, 3, 4$ it is mirror to generic cubic, quartic, quintic and sextic hypersurfaces in \mathbb{P}^{n+1} and this is fairly explained in the literature, see [Dij95] for $n = 1$, [Dol96] for $n = 2$, [CdIOP91] for $n = 3$, [KP08] for $n = 4$ and [GMP95] for a discussion of an arbitrary n . In the present article we discuss the mentioned unification in the case of Dwork family, namely we explain a construction of a modular vector field $R_n = R$ attached to X_ψ such that for $n = 1, 2$ it has solutions in terms of (quasi)-modular forms, for $n = 3$ the topological partition functions are rational functions in the coordinates of a solution of R , and for $n \geq 4$ one gets q -expansions beyond the so-far well-known special functions. It is worth pointing out that we can consider the modular vector field R as an extension of the systems of differential equations introduced by G. Darboux [Dar78], G. H. Halphen [Hal81] and S. Ramanujan [Ram16], for more details see [Mov12], [Nik15, § 1].

For the purpose of Introduction, we need only to know that for a mirror variety X associated to the Dwork family, $\dim H_{\text{dR}}^n(X) = n + 1$, where $H_{\text{dR}}^n(X)$ is the n -th algebraic de Rham cohomology of X , and its Hodge numbers $h^{i,j}$, $i + j = n$, are all one. For $n = 3$ this is also called the family of mirror quintic. Let $\mathbb{T} = \mathbb{T}_n$ be the moduli of pairs $(X, [\alpha_1, \dots, \alpha_n, \alpha_{n+1}])$, where

$$\begin{aligned} \alpha_i &\in F^{n+1-i} \setminus F^{n+2-i}, \quad i = 1, \dots, n, n+1, \\ [\langle \alpha_i, \alpha_j \rangle] &= \Phi_n. \end{aligned}$$

Here F^i is the i -th piece of the Hodge filtration of $H_{\text{dR}}^n(X)$, $\langle \cdot, \cdot \rangle$ is the intersection form in $H_{\text{dR}}^n(X)$ and $\Phi = \Phi_n$ is the explicit constant matrix given by

$$(1.2) \quad \Phi_n := \begin{pmatrix} 0_{\frac{n+1}{2}} & J_{\frac{n+1}{2}} \\ -J_{\frac{n+1}{2}} & 0_{\frac{n+1}{2}} \end{pmatrix},$$

if n is an odd positive integer, and

$$(1.3) \quad \Phi_n := J_{n+1},$$

if n is an even positive integer, where by $0_k, k \in \mathbb{N}$, we mean a $k \times k$ block of zeros, and J_k is the following $k \times k$ block

$$(1.4) \quad J_k := \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix},$$

with $J_1 = 1$. We construct the universal family $\mathbb{X} \rightarrow \mathbb{T}$ together with global sections $\alpha_i, i = 1, \dots, n + 1$ of the relative algebraic de Rham cohomology $H_{\text{dR}}^n(\mathbb{X}/\mathbb{T})$. Let

$$\nabla : H_{\text{dR}}^n(\mathbb{X}/\mathbb{T}) \rightarrow \Omega_{\mathbb{T}}^1 \otimes_{\mathcal{O}_{\mathbb{T}}} H_{\text{dR}}^n(\mathbb{X}/\mathbb{T}),$$

be the algebraic Gauss-Manin connection on $H_{\text{dR}}^n(\mathbb{X}/\mathbb{T})$. Our main theorem is:

Theorem 1.1. *There exist a unique vector field $R := R_n$ and regular functions $Y_i, i = 1, 2, \dots, n - 2$ in T such that the Gauss-Manin connection of the universal family of n -fold mirror variety \mathbb{X}/T composed with the vector field R , namely ∇_R , satisfies:*

$$(1.5) \quad \nabla_R \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \\ \alpha_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & Y_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & Y_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & Y_{n-2} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}}_Y \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \\ \alpha_{n+1} \end{pmatrix},$$

and $Y\Phi + \Phi Y^{\text{tr}} = 0$. In fact,

$$(1.6) \quad T := \text{Spec}(\mathbb{C}[t_1, t_2, \dots, t_d, \frac{1}{t_{n+2}(t_{n+2} - t_1^{n+2})\check{t}}]),$$

where

$$(1.7) \quad d = d_n = \begin{cases} \frac{(n+1)(n+3)}{4} + 1, & \text{if } n \text{ is odd} \\ \frac{n(n+2)}{4} + 1, & \text{if } n \text{ is even} \end{cases},$$

and \check{t} is a product of s variables among t_i 's, $i = 1, 2, \dots, d, i \neq 1, n + 2$ and $s = \frac{n-1}{2}$ if n is an odd integer and $s = \frac{n-2}{2}$ if n is an even integer.

In the proof of Theorem 1.1 we will show more than what we announce in its statement. Indeed, we will give the regular functions Y_i 's explicitly, and we will find an algorithm to express the modular vector field R . An explicit expression for R_3 has been given in [Mov15, Mov17] by the first author. In the next theorem we find R_1 and R_2 explicitly and

express their solutions in terms of quasi-modular forms. For a similar computation of R_1 and R_2 in the context of special geometry see [Ali17]. The modular structure for $n = 1, 2$, that is mirror cubic and quartic, has been discussed respectively in [CDF+97] and [LY96], and also in the GMCD context in [Ali17]. These references include discussions of the derivation of the modular group from the monodromies of the periods of the mirror variety. Modular properties of R_n for $n \geq 3$ is not known and after the present text was ready, the second author in [Nik20] was able to find $\mathfrak{sl}_2(\mathbb{C})$ Lie algebras involving the vector field R_n .

Theorem 1.2. *For $n = 1, 2$ the vector field R as an ordinary differential equation is respectively given by*

$$(1.8) \quad R_1 : \begin{cases} \dot{t}_1 = -t_1 t_2 - 9(t_1^3 - t_3) \\ \dot{t}_2 = 81t_1(t_1^3 - t_3) - t_2^2 \\ \dot{t}_3 = -3t_2 t_3, \end{cases} ,$$

where $\dot{*} = 3 \cdot q \cdot \frac{\partial *}{\partial q}$, and

$$(1.9) \quad R_2 : \begin{cases} \dot{t}_1 = t_3 - t_1 t_2 \\ \dot{t}_2 = 2t_1^2 - \frac{1}{2}t_2^2 \\ \dot{t}_3 = -2t_2 t_3 + 8t_1^3 \\ \dot{t}_4 = -4t_2 t_4, \end{cases} ,$$

where $\dot{*} = -\frac{1}{5} \cdot q \cdot \frac{\partial *}{\partial q}$, and the following polynomial equation holds among t_i 's

$$(1.10) \quad t_3^2 = 4(t_1^4 - t_4).$$

In the above theorem q is a free parameter. For a complex number τ with $\text{Im } \tau > 0$, if we set $q = e^{2\pi i \tau}$, then we find the following solutions of R_1 and R_2 respectively:

$$(1.11) \quad \begin{cases} t_1(q) = \frac{1}{3}(2\theta_3(q^2)\theta_3(q^6) - \theta_3(-q^2)\theta_3(-q^6)), \\ t_2(q) = \frac{1}{8}(E_2(q^2) - 9E_2(q^6)), \\ t_3(q) = \frac{\eta^9(q^3)}{\eta^3(q)}, \end{cases}$$

and

$$(1.12) \quad \begin{cases} \frac{10}{6}t_1\left(\frac{q}{10}\right) = \frac{1}{24}(\theta_3^4(q^2) + \theta_2^4(q^2)), \\ \frac{10}{4}t_2\left(\frac{q}{10}\right) = \frac{1}{24}(E_2(q^2) + 2E_2(q^4)), \\ 10^4 t_4\left(\frac{q}{10}\right) = \eta^8(q)\eta^8(q^2), \end{cases}$$

where E_2 , η and θ_i 's are the classical Eisenstein, eta and theta series, respectively, given as follows:

$$(1.13) \quad E_2(q) = 1 - 24 \sum_{k=1}^{\infty} \sigma(k)q^k \text{ with } \sigma(k) = \sum_{d|k} d,$$

$$(1.14) \quad \eta(q) = q^{\frac{1}{24}} \prod_{k=1}^{\infty} (1 - q^k),$$

$$(1.15) \quad \theta_2(q) = \sum_{k=-\infty}^{\infty} q^{\frac{1}{2}(\frac{k+1}{2})^2}, \quad \theta_3(q) = 1 + 2 \sum_{k=1}^{\infty} q^{\frac{1}{2}k^2}.$$

We have checked the statements given in (1.11) and (1.12) for the first 100 coefficients of q -expansions, and the proof can be done in a similar way as of the Ramanujan's or the Darboux's case. The first 16 coefficients of (1.11) and (1.12) are listed in Table 1. We study the coefficients of q -expansions of the solutions given in (1.11) and (1.12), and we find some interesting enumerative properties. For example, in (1.11) the coefficients of $t_1(q) = \sum_{k=0}^{\infty} t_{1,k}q^k$ have the following enumerative property. Let k be a non-negative integer. If $k = 4m$, $m \in \mathbb{N}$, then the equation $x^2 + 3y^2 = k$ has $3t_{1,k}$ integer solutions. Otherwise the equation has $t_{1,k}$ integer solutions. For more properties of this type see Section 8.

The article is organized in the following way. In Section 2 we review and summarize some basic facts about the structure of the Dwork family from which the mirror variety X_ψ arises. In Section 3 we introduce the notion of moduli space of holomorphic n -form \mathbb{S} , and we see that \mathbb{S} is two dimensional and present a coordinate system for it. Section 4 deals with the calculation of intersection form matrix of a given basis of the de Rham cohomology of mirror variety. In Section 5 we present the moduli space \mathbb{T} and construct a complete coordinate system for \mathbb{T} . Section 6 is devoted to the computing of Gauss-Manin connection of the families X/\mathbb{S} and X/\mathbb{T} . In Section 7 Theorem 1.1 is proved and the modular vector field is explicitly computed for $n = 1, 2, 3, 4$. Finally, in Section 8 after finding the solutions of \mathbb{R}_1 and \mathbb{R}_2 in terms of quasi-modular forms, we proceed with the study of enumerative properties of the q -expansions of the solutions.

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2. Dwork family

Let f_ψ be the polynomial in the left hand side of (1.1). Let also W_ψ be the n -dimensional hypersurface in \mathbb{P}^{n+1} given by f_ψ . We know that the first Chern class of W_ψ is zero, from which follows that W_ψ is a Calabi-Yau manifold. Thus we have a family of Calabi-Yau manifolds given by $\pi : \mathcal{W} \rightarrow \mathbb{P}^1$, where $\mathcal{W} \subset \mathbb{P}^{n+1} \times \mathbb{C}$, $W_\psi = \pi^{-1}(\psi)$ and

$$W_\infty = \{(x_0, x_1, \dots, x_{n+1}) \mid x_0 x_1 \dots x_{n+1} = 0\}.$$

This family has been used by B. Dwork in order to develop the deformation theory of zeta functions of nonsingular hypersurfaces in a projective space, see [Dwo62, Dwo66]. One can easily see that the singular points of this family are $\psi^{n+2} = 1, \infty$. Let G be the following group

$$(2.1) \quad G := \{(\zeta_0, \zeta_1, \dots, \zeta_{n+1}) \mid \zeta_i^{n+2} = 1, \zeta_0 \zeta_1 \dots \zeta_{n+1} = 1\},$$

which acts on W_ψ as follows

$$(2.2) \quad (\zeta_0, \zeta_1, \dots, \zeta_{n+1}) \cdot (x_0, x_1, \dots, x_{n+1}) = (\zeta_0 x_0, \zeta_1 x_1, \dots, \zeta_{n+1} x_{n+1}).$$

We denote by $Y_\psi := W_\psi/G$ the quotient space of this action, which is quite singular. Indeed, Y_ψ is singular in any $x \in W_\psi$ that its stabilizer in G is nontrivial. For $\psi^{n+2} \neq 1, \infty$ there exists a resolution $X_\psi \rightarrow Y_\psi$ of singularities of Y_ψ , such that X_ψ is a Calabi-Yau n -fold with $h^{i,j}(X_\psi) = 1$, $i + j = n$. Therefore, we have a new family where the fibers are Calabi-Yau n -folds X_ψ which is the mirror family of W_ψ , see [GMP95]. The de Rham cohomology $H_{\text{dR}}^n(X_\psi)$ can be identified in a natural way with the equivariant cohomology $H_{\text{dR}}^n(W_\psi)_G$, and in practice one uses this, and the knowledge of resolution of singularities will not be used throughout the paper. The standard variable which is used in the literature is defined by

$$z := \psi^{-(n+2)}.$$

We have the families W_z given by $f_z = 0$, where

$$f_z(x_0, x_1, \dots, x_{n+1}) := z x_0^{n+2} + x_1^{n+2} + x_2^{n+2} + \dots + x_{n+1}^{n+2} \\ - (n+2)x_0 x_1 x_2 \dots x_{n+1},$$

and X_z is defined similar to X_ψ . The new set of singularities is given by $z = 0, 1$ and ∞ . From now on we call X_z (or X_ψ) the *mirror variety*. There is a global holomorphic $(n, 0)$ -form $\eta \in H_{\text{dR}}^n(X_z)$ which is given by

$$\eta := \frac{dx_1 \wedge dx_2 \wedge \dots \wedge dx_{n+1}}{df_z}$$

in the affine chart $\{x_0 = 1\}$. The periods $\int_{\delta} \eta$, $\delta \in H_n(X_z, \mathbb{Z})$ satisfy the well-known Picard-Fuchs equation

$$(2.3) \quad L \left(\int_{\delta} \eta \right) = 0, \quad \text{where}$$

$$(2.4) \quad L := \vartheta^{n+1} - z \left(\vartheta + \frac{1}{n+2} \right) \left(\vartheta + \frac{2}{n+2} \right) \dots \left(\vartheta + \frac{n+1}{n+2} \right), \quad \vartheta := z \frac{\partial}{\partial z}.$$

Note that if $n = 1, 2$ or 3 respectively, then X_z is a special family of elliptic curves, K3-surfaces and mirror quintic 3-folds, respectively. Note that X_z for $n = 1, 2$ is not the generic elliptic curve nor the generic K3 but rather the cubic curve and the quartic K3 surface.

3. Moduli space of holomorphic n -forms

We denote by \mathbb{S} the moduli of pairs (X, α) , where X is an n -dimensional mirror variety and α is a holomorphic n -form on X . We know that the family X_z is a one parameter family and the n -form α is unique, up to multiplication by a constant, therefore $\dim \mathbb{S} = 2$. The multiplicative group $\mathbb{G}_m := (\mathbb{C}^*, \cdot)$ acts on \mathbb{S} by:

$$(X, \alpha) \bullet k = (X, k^{-1} \alpha), \quad k \in \mathbb{G}_m, \quad (X, \alpha) \in \mathbb{S}.$$

We present a chart (t_1, t_{n+2}) for \mathbb{S} . To do this, for any $(t_1, t_{n+2}) \in \mathbb{C}^2$ we define the following polynomial

$$f_{t_1, t_{n+2}}(x_0, x_1, \dots, x_{n+1}) := t_{n+2} x_0^{n+2} + x_1^{n+2} + x_2^{n+2} + \dots + x_{n+1}^{n+2} - (n+2)t_1 x_0 x_1 x_2 \dots x_{n+1}.$$

Note that t_{n+2} is only multiplied with the monomial x_0^{n+2} and not all the monomials in the expression except the last one. This will be essential in the proof of Proposition 3.1. The choice of the monomial x_0^{n+2} is not relevant. In fact, if we choose to define $f_{t_1, t_{n+2}}$ with $t_{n+2} x_1^{n+2}$ then the two families of varieties are isomorphic under the linear transformation $[x_0 : x_1 : x_2 : \dots : x_{n+1}] \mapsto [t_{n+2}^{\frac{1}{n+2}} x_0 : t_{n+2}^{-\frac{1}{n+2}} x_1 : x_2 : \dots : x_{n+1}]$. The discriminant of $f_{t_1, t_{n+2}}$ is given by $\Delta_{t_1, t_{n+2}} = (t_{n+2} - t_1^{n+2}) t_{n+2}$. Let $W_{t_1, t_{n+2}}$ be the following two parameter family of Calabi-Yau manifolds

$$W_{t_1, t_{n+2}} := \{(x_0, x_1, \dots, x_{n+1}) \mid f_{t_1, t_{n+2}}(x_0, x_1, \dots, x_{n+1}) = 0\} \subset \mathbb{P}^{n+1}.$$

$W_{t_1, t_{n+2}}$ is singular if and only if $\Delta_{t_1, t_{n+2}} = 0$. For any

$$(t_1, t_{n+2}) \in \mathbb{C}^2 \setminus \{(t_1, t_{n+2}) \mid \Delta_{t_1, t_{n+2}} = 0\}$$

we let $X_{t_1, t_{n+2}}$ to be the resolution of the singularities of $W_{t_1, t_{n+2}}/G$ where the group G and the group action are given by (2.1) and (2.2). Next we fix the n -form ω_1 on the family $X_{t_1, t_{n+2}}$, where ω_1 in the affine space $\{x_0 = 1\}$ is given by

$$\omega_1 := \frac{dx_1 \wedge dx_2 \wedge \dots \wedge dx_{n+1}}{df_{t_1, t_{n+2}}}.$$

Note that for $(t_1, t_{n+2}) = (1, z)$ we have $(X_{t_1, t_{n+2}}, \omega_1) = (X_z, \eta)$.

Proposition 3.1. *We have*

$$S = \text{Spec} \left(\mathbb{C} \left[t_1, t_{n+2}, \frac{1}{(t_1^{n+2} - t_{n+2})t_{n+2}} \right] \right)$$

and the morphism $X \rightarrow S$ is the universal family of (X, α) , where X is an n -dimensional mirror variety and α is a holomorphic n -form on X . Moreover, the \mathbb{G}_m -action on S is given by

$$(3.1) \quad (t_1, t_{n+2}) \bullet k = (kt_1, k^{n+2}t_{n+2}), \quad (t_1, t_{n+2}) \in S, \quad k \in \mathbb{G}_m.$$

Proof. We have a map which sends a point $(t_1, t_{n+2}) \in S$ to the pair $(X_{t_1, t_{n+2}}, \omega_1)$ in the moduli space S as a set. Its inverse is given by

$$(X_z, a\eta) \mapsto (a^{-1}, za^{-(n+2)}), \quad a \in \mathbb{G}_m.$$

Note that $(X_{t_1, t_{n+2}}, \omega_1)$ and $(X_z, t_1^{-1}\eta)$, where $z = \frac{t_{n+2}}{t_1^{n+2}}$, in the moduli space S represent the same element. The affirmation concerning the \mathbb{G}_m -action follows from the isomorphism:

$$(3.2) \quad (X_{kt_1, k^{n+2}t_{n+2}}, k\omega_1) \cong (X_{t_1, t_{n+2}}, \omega_1), \\ (x_1, x_2, \dots, x_{n+1}) \mapsto (k^{-1}x_1, k^{-1}x_2, \dots, k^{-1}x_{n+1}),$$

given in the affine coordinates $x_0 = 1$.

q.e.d.

4. Intersection form and Gauss-Manin connection

Let X be an n -dimensional mirror variety and $\xi_1, \xi_2 \in H_{\text{dR}}^n(X)$. Then in the context of de Rham cohomology, the *intersection form* of ξ_1 and ξ_2 , denoted by $\langle \xi_1, \xi_2 \rangle$, is given by

$$\langle \xi_1, \xi_2 \rangle = \frac{1}{(2\pi i)^n} \int_X \xi_1 \wedge \xi_2.$$

We recall that $\langle \cdot, \cdot \rangle$ is a non-degenerate $(-1)^n$ -symmetric form, and

$$(4.1) \quad \langle F^i, F^j \rangle = 0, \quad i + j \geq n + 1,$$

where

$$F^\bullet : \{0\} = F^{n+1} \subset F^n \subset \dots \subset F^1 \subset F^0 = H_{\text{dR}}^n(X), \quad \dim F^i = n+1-i,$$

is the Hodge filtration of $H_{\text{dR}}^n(X)$.

Let

$$(4.2) \quad \nabla : H_{\text{dR}}^n(X/S) \rightarrow \Omega_S^1 \otimes_{\mathcal{O}_S} H_{\text{dR}}^n(X/S)$$

be the Gauss-Manin connection of the two parameter family of varieties X/S , and $\frac{\partial}{\partial t_1}$ be a vector field on the moduli space S . For simplicity, we use the same notation $\frac{\partial}{\partial t_1}$, to show $\nabla \frac{\partial}{\partial t_1}$ which is the composition of

the Gauss-Manin connection ∇ with the vector field $\frac{\partial}{\partial t_1}$. Now we define new n -forms ω_i , $i = 1, 2, \dots, n + 1$, as follows

$$(4.3) \quad \omega_i := \frac{\partial^{i-1}}{\partial t_1^{i-1}}(\omega_1).$$

Later, in Proposition 4.1 we will see that $\omega_1, \omega_2, \dots, \omega_{n+1}$ form a basis of $H_{\text{dR}}^n(X)$ compatible with its Hodge filtration, i.e.

$$(4.4) \quad \omega_i \in F^{n+1-i} \setminus F^{n+2-i}, i = 1, 2, \dots, n + 1.$$

We write the Gauss-Manin connection of X/S in the basis ω as follows

$$(4.5) \quad \nabla \omega = B\omega,$$

and we denote by

$$(4.6) \quad \Omega = \Omega_n := (\langle \omega_i, \omega_j \rangle)_{1 \leq i, j \leq n+1},$$

the intersection form matrix in the basis ω . We have

$$(4.7) \quad d\Omega = B\Omega + \Omega B^{\text{tr}}.$$

The entries of B and Ω are respectively regular differential 1-forms and functions in S . For arbitrary n , we do not have a general formula for Ω and B . We have only an algorithm which computes the entries of Ω and B recursively. For $n = 1, 2, 3, 4$ the Picard-Fuchs equation associated with the n -form ω_1 is given by

$$(4.8) \quad \begin{aligned} \frac{\partial^{n+1}}{\partial t_1^{n+1}} &= -S_2(n+2, n+1) \frac{t_1^{n+1}}{t_1^{n+2} - t_{n+2}} \frac{\partial^n}{\partial t_1^n} \\ &\quad - S_2(n+2, n) \frac{t_1^n}{t_1^{n+2} - t_{n+2}} \frac{\partial^{n-1}}{\partial t_1^{n-1}} \\ &\quad - \dots - S_2(n+2, 2) \frac{t_1^2}{t_1^{n+2} - t_{n+2}} \frac{\partial}{\partial t_1} - S_2(n+2, 1) \frac{t_1}{t_1^{n+2} - t_{n+2}}, \end{aligned}$$

where $S_2(r, s)$, $r, s \in \mathbb{N}$, refers to Stirling number of the second kind which is given by

$$(4.9) \quad S_2(r, s) = \frac{1}{s!} \sum_{i=0}^s (-1)^i \binom{s}{i} (s-i)^r.$$

For details of the computation in the mirror quintic case ($n = 3$) see [Mov17, §3.8]. The equation (4.8) must be true for arbitrary n , however, we are only interested to compute this for explicit n 's and so we do not provide a proof for arbitrary n .

Proposition 4.1. *We have*

- (i): $\langle \omega_i, \omega_j \rangle = 0$, if $i + j \leq n + 1$.
- (ii): $\langle \omega_1, \omega_{n+1} \rangle = (-n + 2)^n \frac{c_n}{t_1^{n+2} - t_{n+2}}$, where c_n is a constant.

- (iii): $\langle \omega_j, \omega_{n+2-j} \rangle = (-1)^{j-1} \langle \omega_1, \omega_{n+1} \rangle$, for $j = 1, 2, \dots, n+1$.
 (iv): We can determine all the rest of $\langle \omega_i, \omega_j \rangle$'s in a unique way.

Proof. Note that the intersection form is well-defined for all points in S , and so, $\langle \omega_i, \omega_j \rangle$'s are regular functions in S . This implies that they have poles only along $t_{n+2} = 0$ and $t_{n+2} - t_1^{n+2} = 0$.

- (i): The Griffiths transversality implies that

$$\omega_i \in F^{n+1-i}, i = 1, 2, \dots, n+1.$$

This property and the property given in (4.1) complete the proof of (i).

- (ii): If we present the Picard-Fuchs equation associated with the holomorphic n -form η as follows:

$$(4.10) \quad \vartheta^{n+1} = a_0(z) + a_1(z)\vartheta + \dots + a_n(z)\vartheta^n,$$

then because of (2.4) we find

$$a_n(z) = \frac{n+1}{2} \frac{z}{1-z}.$$

One can verify the differential equation given below

$$\vartheta \langle \eta, \vartheta^n \eta \rangle + \frac{2}{n+1} a_n(z) \langle \eta, \vartheta^n \eta \rangle = 0,$$

from which we get $\langle \eta, \vartheta^n \eta \rangle = c_n \exp\left(-\frac{2}{n+1} \int_0^z a_n(v) \frac{dv}{v}\right)$, where c_n is a constant. This yields

$$(4.11) \quad \langle \eta, \vartheta^n \eta \rangle = \frac{c_n}{1-z}.$$

On the other hand in Section 3 we saw $z = \frac{t_{n+2}}{t_1^{n+2}}$, which implies $\vartheta = z \frac{\partial}{\partial z} = -\frac{1}{n+2} t_1 \frac{\partial}{\partial t_1}$. One can easily see that $\eta = t_1 \omega_1$, hence

$$\begin{aligned} \vartheta^n \eta &= \left(-\frac{1}{n+2} t_1 \frac{\partial}{\partial t_1}\right)^n (t_1 \omega_1) \\ &= b_1 \omega_1 + \dots + b_n \omega_n + \left(-\frac{1}{n+2}\right)^n t_1^{n+1} \omega_{n+1}, \end{aligned}$$

where b_j 's are rational functions in t_1, t_{n+1} . Therefore, (i) implies

$$\langle \eta, \vartheta^n \eta \rangle = \langle t_1 \omega_1, \left(-\frac{1}{n+2}\right)^n t_1^{n+1} \omega_{n+1} \rangle,$$

which completes the proof of (ii).

- (iii): By (i) we have $\langle \omega_j, \omega_{n+1-j} \rangle = 0$, $j = 1, 2, \dots, n$. Thus we get

$$\begin{aligned} \frac{\partial}{\partial t_1} \langle \omega_j, \omega_{n+1-j} \rangle &= \left\langle \frac{\partial}{\partial t_1} \omega_j, \omega_{n+1-j} \right\rangle + \left\langle \omega_j, \frac{\partial}{\partial t_1} \omega_{n+1-j} \right\rangle \\ &= \langle \omega_{j+1}, \omega_{n+1-j} \rangle + \langle \omega_j, \omega_{n+2-j} \rangle = 0, \end{aligned}$$

hence we obtain $\langle \omega_{j+1}, \omega_{n+1-j} \rangle = -\langle \omega_j, \omega_{n+2-j} \rangle$, $j = 1, 2, \dots, n$, from which follows (iii).

(iv): We present the desired algorithm. So far, we have computed the first row of the matrix Ω . Suppose that we have the i -th row of Ω , $1 \leq i \leq n$, and then determine $(i+1)$ -th row. To compute $\langle \omega_{i+1}, \omega_j \rangle$, $n+2-i \leq j \leq n+1$, we apply $\frac{\partial}{\partial t_1} \langle \omega_i, \omega_j \rangle$, which implies

$$\langle \omega_{i+1}, \omega_j \rangle = \frac{\partial}{\partial t_1} \langle \omega_i, \omega_j \rangle - \langle \omega_i, \omega_{j+1} \rangle.$$

Note that if $j = n+1$, then $\omega_{n+2} = \frac{\partial^{n+1}}{\partial t_1^{n+1}}(\omega_1)$ and we compute it by using the Picard-Fuchs equation given in (4.8). q.e.d.

The intersection form matrix for $n = 1, 2, 3, 4$ are respectively given as follows:

$$\Omega_1 = \begin{pmatrix} 0 & -\frac{3c_1}{t_1^3-t_3} \\ \frac{3c_1}{t_1^3-t_3} & 0 \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} 0 & 0 & \frac{16c_2}{t_1^4-t_4} \\ 0 & -\frac{16c_2}{t_1^4-t_4} & \frac{32c_2t_1^3}{(t_1^4-t_4)^2} \\ \frac{16c_2}{t_1^4-t_4} & \frac{32c_2t_1^3}{(t_1^4-t_4)^2} & \frac{-16c_2t_1^2(5t_1^4-t_4)}{(t_1^4-t_4)^3} \end{pmatrix},$$

$$\Omega_3 = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{625(t_1^5-t_5)} \\ 0 & 0 & \frac{1}{625(t_1^5-t_5)} & -\frac{t_1^4}{625(t_1^5-t_5)^2} \\ 0 & -\frac{1}{625(t_1^5-t_5)} & 0 & \frac{t_1^3}{625(t_1^5-t_5)^2} \\ \frac{1}{625(t_1^5-t_5)} & \frac{t_1^4}{625(t_1^5-t_5)^2} & -\frac{t_1^3}{625(t_1^5-t_5)^2} & 0 \end{pmatrix},$$

$$\Omega_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{6^4c_4}{t_1^6-t_6} \\ 0 & 0 & \frac{6^4c_4}{t_1^6-t_6} & -\frac{3 \times 6^4c_4t_1^5}{(t_1^6-t_6)^2} \\ 0 & -\frac{6^4c_4}{t_1^6-t_6} & -\frac{3 \times 6^4c_4t_1^5}{(t_1^6-t_6)^2} & \frac{6^4c_4t_1^4(14t_1^6-5t_6)}{(t_1^6-t_6)^3} \\ \frac{6^4c_4}{t_1^6-t_6} & \frac{9 \times 6^4c_4t_1^5}{(t_1^6-t_6)^2} & \frac{6^4c_4t_1^4(7t_1^6+20t_6)}{(t_1^6-t_6)^3} & \frac{-6^4c_4t_1^3(56t_1^{12}+35t_1^6t_6-10t_6^2)}{(t_1^6-t_6)^4} \\ & & & \frac{6^4c_4}{t_1^6-t_6} \\ & & & \frac{9 \times 6^4c_4t_1^5}{(t_1^6-t_6)^2} \\ & & & \frac{6^4c_4t_1^4(7t_1^6+20t_6)}{(t_1^6-t_6)^3} \\ & & & \frac{-6^4c_4t_1^3(56t_1^{12}+35t_1^6t_6-10t_6^2)}{(t_1^6-t_6)^4} \\ & & & \frac{6^4c_4t_1^2(273t_1^{18}+238t_1^{12}t_6+217t_1^6t_6^2+t_6^3)}{(t_1^6-t_6)^5} \end{pmatrix}.$$

5. Moduli space of enhanced Calabi-Yau varieties

Let $\mathbb{T} = \mathbb{T}_n$ be the moduli of pairs $(X, [\alpha_1, \alpha_2, \dots, \alpha_{n+1}])$, where X is an n -fold mirror variety and $\{\alpha_1, \alpha_2, \dots, \alpha_{n+1}\}$ is a basis of $H_{\text{dR}}^n(X)$ compatible with its Hodge filtration, and such that the intersection form matrix of this basis is constant, that is,

$$(5.1) \quad (\langle \alpha_i, \alpha_j \rangle)_{1 \leq i, j \leq n+1} = \Phi.$$

If we denote by $d_n := \dim \mathbb{T}_n$, then from [Nik15, Theorem 1] we get (1.7). The objective of this section is to construct a coordinate system for \mathbb{T} .

In Section 4 we have fixed the basis $\{\omega_1, \omega_2, \dots, \omega_{n+1}\}$ of $H_{\text{dR}}^n(X)$ that is compatible with its Hodge filtration. Let $S = (s_{ij})_{1 \leq i, j \leq n+1}$ be a lower triangular matrix, whose entries are indeterminates s_{ij} , $i \geq j$ and $s_{11} = 1$. We define

$$\alpha := S\omega,$$

where

$$\omega := (\omega_1 \ \omega_2 \ \dots \ \omega_{n+1})^{\text{tr}}.$$

We assume that $(\langle \alpha_i, \alpha_j \rangle)_{1 \leq i, j \leq n+1} = \Phi$, and so, we get the following equation

$$(5.2) \quad S\Omega S^{\text{tr}} = \Phi.$$

If we set $\Psi = (\Psi_{ij})_{1 \leq i, j \leq n+1} := S\Omega S^{\text{tr}}$, then Ψ is a $(-1)^n$ -symmetric matrix and $\Psi_{ij} = 0$ for $i = 1, 2, \dots, n$ and $j \leq n+1-i$. Moreover, in the case that n is an odd integer we get $\Psi_{ii} = 0$, $i = 1, 2, \dots, n+1$. Therefore, the equation (5.2) gives us $d_0 := \frac{(n+2)(n+1)}{2} - d - 2$ equations, where d is given by (1.7). These equations are independent from each other and so we can express d_0 numbers of parameters s_{ij} 's in terms of other $d - 2$ parameters that we fix them as *independent parameters*. For simplicity we write the first class of parameters as $\check{t}_1, \check{t}_2, \dots, \check{t}_{d_0}$ and the second class as $t_2, t_3, \dots, t_{n+1}, t_{n+3}, \dots, t_d$. We put the independent parameters t_i inside S according to the following rule which is not canonical: t_i 's are written in S from left to right and top to bottom in the entries (i, j) for $i+j < n+2$ if n is even and $i+j \leq n+2$ if n is odd. The position of \check{t}_i 's inside S can be chosen arbitrarily. For instance, for $n = 1, 2, 3, 4$ we have:

$$\begin{pmatrix} 1 & 0 \\ t_2 & \check{t}_1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ t_2 & \check{t}_2 & 0 \\ \check{t}_4 & \check{t}_3 & \check{t}_1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_2 & t_3 & 0 & 0 \\ t_4 & t_6 & \check{t}_2 & 0 \\ t_7 & \check{t}_4 & \check{t}_3 & \check{t}_1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ t_2 & t_3 & 0 & 0 & 0 \\ t_4 & t_5 & \check{t}_3 & 0 & 0 \\ t_7 & \check{t}_7 & \check{t}_5 & \check{t}_2 & 0 \\ \check{t}_9 & \check{t}_8 & \check{t}_6 & \check{t}_4 & \check{t}_1 \end{pmatrix}.$$

Note that we have already used t_1, t_{n+2} as coordinate system of \mathbb{S} in Section 3.

Proposition 5.1. *The equation $S\Omega S^{\text{tr}} = \Phi$ yields*

$$(5.3) \quad s_{(n+2-i)(n+2-i)} = \frac{(-1)^{n+i+1} t_1^{n+2} - t_{n+2}}{c_n(n+2)^n s_{ii}},$$

where $i = 1, 2, \dots, \frac{n+1}{2}$ if n is an odd integer, and $i = 1, 2, \dots, \frac{n+2}{2}$ if n is an even integer. Moreover, one can compute \check{t}_i 's in terms of t_i 's.

Proof. Let us first count the number of equalities that we get from $S\Omega S^{\text{tr}} = \Phi$. This is $\frac{(n+1)(n+2)}{2} + 1 - d$. Note that the left upper triangle of this equality consists of trivial equalities $0 = 0$. The equality (5.3) follows from the $(i, n+2-i)$ -th entry of $S\Omega S^{\text{tr}} = \Phi$. We have plugged the parameters $\check{t}_k = s_{ij}$ inside S such that the equality corresponding to the $(n+2-j, i)$ -th entry of $S\Omega S^{\text{tr}} = \Phi$ gives us an equation which computes \check{t}_k in terms of \check{t}_r , $r < k$ and t_s 's. Note that only divisions by s_{ii} 's, $t_{n+2} - t_1^{n+2}$ and t_{n+2} occurs. Another way to see this is to redefine $S := S^{-1}$ and so we will have the equality $S\Phi S^{\text{tr}} = \Omega$. q.e.d.

For $n = 1, 2, 3, 4$, we express \check{t}_j 's in terms of t_i 's as follows:

$n = 1$:

$$\check{t}_1 = -\frac{1}{3c_1}(t_1^3 - t_3).$$

$n = 2$:

$$(5.4) \quad \check{t}_1 = \frac{1}{16c_2}(t_1^4 - t_4), \quad \check{t}_2^2 = -\frac{1}{16c_2}(t_1^4 - t_4),$$

$$\check{t}_3 = \frac{1}{16c_2}(-16c_2t_2\check{t}_2 + 2t_1^3), \quad \check{t}_4 = \frac{1}{32c_2}(-16c_2t_2^2 + t_1^2).$$

$n = 3$:

$$\check{t}_1 = -625t_1^5 + 625t_5, \quad \check{t}_2 = \frac{625t_1^5 - 625t_5}{t_3},$$

$$\check{t}_3 = \frac{-625t_1^5t_2 + 625t_2t_5 - 3125t_1^4t_3}{t_3}, \quad \check{t}_4 = -t_2t_6 + t_3t_4 - 3125t_1^3.$$

$n = 4$:

$$(5.5) \quad \check{t}_1 = \frac{t_1^6 - t_6}{1296c_4}, \quad \check{t}_2 = -\frac{t_1^6 - t_6}{1296c_4t_3}, \quad \check{t}_3^2 = \frac{t_1^6 - t_6}{1296c_4},$$

$$\check{t}_4 = \frac{t_1^6t_2 + 9t_1^5t_3 - t_2t_6}{1296c_4t_3}, \quad \check{t}_5 = \frac{-432c_4t_5\check{t}_3 - t_1^5}{432c_4t_3},$$

$$\check{t}_6 = \frac{1296c_4t_2t_5\check{t}_3 - 1296c_4t_3t_4\check{t}_3 + 3t_1^5t_2 + 20t_1^4t_3}{1296c_4t_3},$$

$$\check{t}_7 = \frac{-1296c_4t_5^2 - 5t_1^4}{2592c_4t_3},$$

$$\check{t}_8 = \frac{1296c_4t_2t_5^2 - 2592c_4t_3^2t_7 - 2592c_4t_3t_4t_5 + 5t_1^4t_2 + 20t_1^3t_3}{2592c_4t_3},$$

$$\check{t}_9 = \frac{-2592c_4t_2t_7 - 1296c_4t_4^2 + t_1^2}{2592c_4}.$$

6. Gauss-Manin connection

We return to the Gauss-Manin connection ∇ , that was introduced in (4.2), and we proceed with the computation of the Gauss-Manin connection matrix \mathbf{B} , which is given in (4.5).

If we denote by $A(z)$ the Gauss-Manin connection matrix of the family $\mathbf{X}_{1,z}$ in the basis $\{\eta, \frac{\partial\eta}{\partial z}, \dots, \frac{\partial^n\eta}{\partial z^n}\}$, that is

$$\nabla \left(\eta \quad \frac{\partial\eta}{\partial z} \quad \dots \quad \frac{\partial^n\eta}{\partial z^n} \right)^{\text{tr}} = A(z)dz \otimes \left(\eta \quad \frac{\partial\eta}{\partial z} \quad \dots \quad \frac{\partial^n\eta}{\partial z^n} \right)^{\text{tr}},$$

then we get

$$A(z) = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ b_1(z) & b_2(z) & b_3(z) & b_4(z) & \dots & b_{n+1}(z) \end{pmatrix},$$

where the functions $b_i(z)$'s are the coefficients of the Picard-Fuchs equation (2.4) associated with the n -form η that we write in the following format

$$\frac{\partial^{n+1}\eta}{\partial z^{n+1}} = b_1(z)\eta + b_2(z)\frac{\partial\eta}{\partial z} + \dots + b_{n+1}(z)\frac{\partial^n\eta}{\partial z^n}, \quad \text{modulo exact forms.}$$

We calculate ∇ with respect to the basis (4.3) of $H_{\text{dR}}^n(\mathbf{X}/S)$. For this purpose we return back to the one parameter case. For $z := \frac{t_{n+2}}{t_1^{n+2}}$, consider the map

$$g : \mathbf{X}_{t_1, t_{n+2}} \rightarrow \mathbf{X}_{1,z},$$

given by (3.2) with $k = t_1^{-1}$. Hence we get

$$\frac{\partial}{\partial z} = \frac{-1}{n+2} \frac{t_1^{n+3}}{t_{n+2}} \frac{\partial}{\partial t_1},$$

and $g^*\eta = t_1\omega_1$ which, by abuse of notation, will just be written as $\eta = t_1\omega_1$. From these two equalities we obtain the base change matrix $\tilde{S} = \tilde{S}(t_1, t_{n+2})$ such that

$$\left(\eta \quad \frac{\partial\eta}{\partial z} \quad \dots \quad \frac{\partial^n\eta}{\partial z^n} \right)^{\text{tr}} = \tilde{S}^{-1} \left(\omega_1 \quad \omega_2 \quad \dots \quad \omega_{n+1} \right)^{\text{tr}}.$$

Thus we find the Gauss-Manin connection in the basis ω_i , $i = 1, 2, \dots, n+1$ as follows:

$$\mathbf{B} = \left(d\tilde{S} + \tilde{S} \cdot A\left(\frac{t_{n+2}}{t_1^{n+2}}\right) \cdot d\left(\frac{t_{n+2}}{t_1^{n+2}}\right) \right) \cdot \tilde{S}^{-1}.$$

Let $B[i, j]$ be the (i, j) -th entry of the Gauss-Manin connection matrix B . We get

$$(6.1) \quad B[i, i] = -\frac{i}{(n+2)t_{n+2}} dt_{n+2}, \quad 1 \leq i \leq n,$$

$$(6.2) \quad B[i, i+1] = dt_1 - \frac{t_1}{(n+2)t_{n+2}} dt_{n+2}, \quad 1 \leq i \leq n,$$

$$B[n+1, j] = \frac{-S_2(n+2, j)t_1^j}{t_1^{n+2} - t_{n+2}} dt_1 + \frac{S_2(n+2, j)t_1^{j+1}}{(n+2)t_{n+2}(t_1^{n+2} - t_{n+2})} dt_{n+2},$$

$$B[n+1, n+1] = \frac{-S_2(n+2, n+1)t_1^{n+1}}{t_1^{n+2} - t_{n+2}} dt_1 + \frac{\frac{n(n+1)}{2}t_1^{n+2} + (n+1)t_{n+2}}{(n+2)t_{n+2}(t_1^{n+2} - t_{n+2})} dt_{n+2},$$

where $1 \leq j \leq n$ and $S_2(r, s)$ is the Stirling number of the second kind defined in (4.9), and the rest of the entries of B are zero. The equalities (6.1) and (6.2) are easy to check, and those with Stirling numbers are checked for $n = 1, 2, 3, 4$. It would be interesting to prove this statement for arbitrary n . We will not need such explicit expressions for the proof of our main theorem. The Gauss-Manin connection matrix B for $n = 1, 2$ are respectively given as follows:

$$B_1 = \begin{pmatrix} -\frac{1}{3t_3} dt_3 & dt_1 - \frac{t_1}{3t_3} dt_3 \\ -\frac{t_1}{t_1^3 - t_3} dt_1 + \frac{t_1^2}{3t_3(t_1^3 - t_3)} dt_3 & -\frac{3t_1^2}{t_1^3 - t_3} dt_1 + \frac{t_1^3 + 2t_3}{3t_3(t_1^3 - t_3)} dt_3 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} -\frac{1}{4t_4} dt_4 & dt_1 - \frac{t_1}{4t_4} dt_4 \\ 0 & -\frac{2}{4t_4} dt_4 \\ -\frac{t_1}{t_1^4 - t_4} dt_1 + \frac{t_1^2}{4t_4(t_1^4 - t_4)} dt_4 & -\frac{7t_1^2}{t_1^4 - t_4} dt_1 + \frac{7t_1^3}{4t_4(t_1^4 - t_4)} dt_4 \\ & 0 \\ & dt_1 - \frac{t_1}{4t_4} dt_4 \\ & -\frac{6t_1^3}{t_1^4 - t_4} dt_1 + \frac{3t_1^4 + 3t_4}{4t_4(t_1^4 - t_4)} dt_4 \end{pmatrix}.$$

Let A to be the Gauss-Manin connection matrix of the family X/T written in the basis α_i , $i = 1, 2, \dots, \alpha_{n+1}$, i.e., $\nabla \alpha = A\alpha$. Then we calculate A as follows:

$$(6.3) \quad A = (dS + S \cdot B) \cdot S^{-1},$$

where S is the base change matrix $\alpha = S\omega$.

7. Proof of Theorem 1.1

As we saw in (6.3), the Gauss-Manin connection matrix of the family X/T in the basis α is given by

$$(7.1) \quad A = dS \cdot S^{-1} + S \cdot B \cdot S^{-1}.$$

For a moment, let us consider the entries s_{ij} , $j \leq i$, $(i, j) \neq (1, 1)$ of S as independent parameters with only the following relation:

$$(7.2) \quad s_{(n+1)(n+1)} + s_{nn}s_{22} = 0.$$

We denote by \tilde{T} and $\tilde{\alpha}$ the corresponding family of varieties and a basis of differential forms. The existence of a vector field R in \tilde{T} with the desired property in relation with the Gauss-Manin connection is equivalent to solve the equation

$$(7.3) \quad \dot{S} = YS - S \cdot B(R),$$

where $\dot{x} := dx(R)$ is the derivation of the function x along the vector field R in \tilde{T} . The equalities corresponding to the entries (i, j) , $j \leq i$, $(i, j) \neq (1, 1)$ serves as the definition of \dot{s}_{ij} . The equality corresponding to $(1, 1)$ -th and $(1, 2)$ -th entries give us respectively

$$\dot{t}_1 = t_3 - t_1 t_2, \quad \dot{t}_{n+2} = -(n+2)t_2 t_{n+2}.$$

Recall that $t_2 = s_{21}$ and $t_3 = s_{22}$. The equalities corresponding to $(i, i+1)$ -th, $i = 2, \dots, n-1$, entries compute the quantities Y_i 's:

$$(7.4) \quad Y_{i-1} = \frac{t_3 s_{ii}}{s_{(i+1)(i+1)}}, \quad i = 2, 3, \dots, n-1.$$

Finally the equality corresponding to the $(n, n+1)$ -th entry is given by (7.2) which is already implemented in the definition of \tilde{T} . All the rest are trivial equalities $0 = 0$. We conclude the statement of Theorem 1.1 for the moduli space \tilde{T} .

Now, let us prove the main theorem for the moduli space T . First, note that we have a map

$$(7.5) \quad \tilde{T} \rightarrow \text{Mat}_{(n+1) \times (n+1)}(\mathbb{C}), \quad (t_1, t_{n+2}, S) \mapsto S\Omega S^{\text{tr}}$$

and T is the fiber of this map over the point Φ . We prove that the vector field R is tangent to the fiber of the above map over Φ . This follows from

$$\begin{aligned} \overbrace{(S\Omega S^{\text{tr}})} &= \dot{S}\Omega S^{\text{tr}} + S\dot{\Omega}S^{\text{tr}} + S\Omega\dot{S}^{\text{tr}} \\ &= (YS - SB)\Omega S^{\text{tr}} + S(B\Omega + \Omega B^{\text{tr}})S^{\text{tr}} + S\Omega(S^{\text{tr}}Y^{\text{tr}} - B^{\text{tr}}S^{\text{tr}}) \\ &= Y\Phi + \Phi Y^{\text{tr}} \\ &= 0, \end{aligned}$$

where $\dot{x} := dx(R)$ is the derivation of the function x along the vector field R in T . The last equality follows from (7.4) and Proposition 5.1.

It follows that if n is an even integer then $Y_{i-1} = -Y_{n-i}$, $i = 2, \dots, \frac{n}{2}$ and if n is an odd integer then $Y_{i-1} = -Y_{n-i}$, $i = 2, \dots, \frac{n-1}{2}$ and

$$Y_{\frac{n-1}{2}} = (-1)^{\frac{3n+3}{2}} c_n (n+2)^n \frac{t_3 s_{\frac{n+1}{2}, \frac{n+1}{2}}^2}{t_1^{n+2} - t_{n+2}}.$$

To prove the uniqueness, first notice that (7.4) guaranties the uniqueness of Y_i 's. Suppose that there are two vector fields R and \hat{R} such that $\nabla_R \alpha = Y\alpha$ and $\nabla_{\hat{R}} \alpha = Y\alpha$. If we set $H := R - \hat{R}$, then

$$(7.6) \quad \nabla_H \alpha = 0.$$

We need to prove that $H = 0$, and to do this it is enough to verify that any integral curve of H is a constant point. Assume that γ is an integral curve of H given as follows

$$\gamma : (\mathbb{C}, 0) \rightarrow \mathbb{T}; \quad x \mapsto \gamma(x).$$

Let us denote by $\mathcal{C} := \gamma(\mathbb{C}, 0) \subset \mathbb{T}$ the trajectory of γ in \mathbb{T} . We know that the points of \mathbb{T} are pairs $(X, [\alpha_1, \alpha_2, \dots, \alpha_{n+1}])$, in which X is an n -fold mirror variety and $\{\alpha_1, \alpha_2, \dots, \alpha_{n+1}\}$ is a basis of $H_{\text{dR}}^n(X)$ compatible with its Hodge filtration and has constant intersection form matrix Φ . Thus, we can parameterize γ in such a way that for any $x \in (\mathbb{C}, 0)$ the vector field H on \mathcal{C} reduces to $\frac{\partial}{\partial x}$, and so, we have $\gamma(x) = (X(x), [\alpha_1(x), \alpha_2(x), \dots, \alpha_{n+1}(x)])$. We know that $X(x)$ is a member of mirror family that depends only on the parameter z , hence x holomorphically depends to z . From this we obtain a holomorphic function f such that $x = f(z)$. We now proceed to prove that f is constant. Otherwise, by contradiction suppose that $f' \neq 0$. Then we get

$$(7.7) \quad \nabla_{\frac{\partial}{\partial x}} \alpha_1 = \frac{\partial z}{\partial x} \nabla_{\frac{\partial}{\partial z}} \alpha_1.$$

Equation (7.6) gives that $\nabla_{\frac{\partial}{\partial x}} \alpha_1 = 0$, but since $\alpha_1 = \omega_1$, it follows that the right hand side of (7.7) is not zero, which is a contradiction. Thus f is constant and $X(x)$ does not depend on the parameter x . Since $X(x) = X$ does not depend on x , we can write the Taylor series of $\alpha_i(x)$, $i = 1, 2, 3, \dots, n+1$, in x at some point x_0 as $\alpha_i(x) = \sum_j (x - x_0)^j \alpha_{i,j}$, where $\alpha_{i,j}$'s are elements in $H_{\text{dR}}^n(X)$ independent of x . In this way the action of $\nabla_{\frac{\partial}{\partial x}}$ on α_i is just the usual derivation $\frac{\partial}{\partial x}$. Again according to (7.6) we get $\nabla_{\frac{\partial}{\partial x}} \alpha_i = 0$, and we conclude that α_i 's also do not depend on x . Therefore, the image of γ is a point. □

The modular vector field R for $n = 1, 2, 3, 4$, are given as follows:
 $n = 1$:

$$(7.8) \quad R_1 : \begin{cases} \dot{t}_1 = \frac{1}{3c_1}(-3c_1 t_1 t_2 - (t_1^3 - t_3)) \\ \dot{t}_2 = \frac{1}{9c_1^2}(t_1(t_1^3 - t_3) - 9c_1^2 t_2^2) \\ \dot{t}_3 = -3t_2 t_3 \end{cases} .$$

$n = 2$: We know that $\dim T_2 = 3$, hence the modular vector field R_2 should have three components, but to avoid the second root of \check{t}_2 that comes from (5.4) we add one more variable $t_3 := \check{t}_2$. Thus we find R_2 as follows:

$$(7.9) \quad R_2 : \begin{cases} \dot{t}_1 = -t_1 t_2 + t_3 \\ \dot{t}_2 = -\frac{1}{32c_2}(t_1^2 + 16c_2 t_2^2) \\ \dot{t}_3 = -\frac{1}{8c_2}(16c_2 t_2 t_3 + t_1^3) \\ \dot{t}_4 = -4t_2 t_4 \end{cases},$$

such that the following equation holds among t_i 's

$$(7.10) \quad t_3^2 = -\frac{1}{16c_2}(t_1^4 - t_4).$$

$n = 3$: The vector field R_3 has been calculated in [Mov15], but in a different chart. R_3 in the chart chosen in this paper is as follows:

$$(7.11) \quad R_3 : \begin{cases} \dot{t}_1 = t_3 - t_1 t_2 \\ \dot{t}_2 = \frac{t_3^3 t_4 - 5^4 t_2^2 (t_1^5 - t_5)}{5^4 (t_1^5 - t_5)} \\ \dot{t}_3 = \frac{t_3^3 t_6 - 3 \times 5^4 t_2 t_3 (t_1^5 - t_5)}{5^4 (t_1^5 - t_5)} \\ \dot{t}_4 = -t_2 t_4 - t_7 \\ \dot{t}_5 = -5t_2 t_5 \\ \dot{t}_6 = 5^5 t_1^3 - t_2 t_6 - 2t_3 t_4 \\ \dot{t}_7 = -5^4 t_1 t_3 - t_2 t_7 \end{cases}.$$

$n = 4$: Similar to the case $n = 2$ and in order to avoid the second root of \check{t}_3 given in (5.5), we add the variable $t_8 := \check{t}_3$ and we find:

$$(7.12) \quad R_4 : \begin{cases} \dot{t}_1 = t_3 - t_1 t_2 \\ \dot{t}_2 = \frac{1296c_4 t_3^2 t_4 t_8 - t_1^6 t_2^2 + t_2^2 t_6}{t_1^6 - t_6} \\ \dot{t}_3 = \frac{1296c_4 t_3^2 t_5 t_8 - 3t_1^6 t_2 t_3 + 3t_2 t_3 t_6}{t_1^6 - t_6} \\ \dot{t}_4 = \frac{-1296c_4 t_3^2 t_7 t_8 - t_1^6 t_2 t_4 + t_2 t_4 t_6}{t_1^6 - t_6} \\ \dot{t}_5 = \frac{1296c_4 t_3 t_5^2 t_8 - 4t_1^6 t_2 t_5 - 2t_1^6 t_3 t_4 + 5t_1^4 t_3 t_8 + 4t_2 t_5 t_6 + 2t_3 t_4 t_6}{2(t_1^6 - t_6)} \\ \dot{t}_6 = -6t_2 t_6 \\ \dot{t}_7 = \frac{1296c_4 t_4^2 - t_1^2}{2592c_4} \\ \dot{t}_8 = \frac{-3t_1^6 t_2 t_8 + 3t_1^5 t_3 t_8 + 3t_2 t_6 t_8}{t_1^6 - t_6} \end{cases},$$

where

$$(7.13) \quad t_8^2 = \frac{1}{1296c_4}(t_1^6 - t_6).$$

In this case the functions Y_1 and Y_2 are given by

$$(7.14) \quad Y_1^2 = (-Y_2)^2 = \frac{1296c_4t_3^4}{t_1^6 - t_6}.$$

8. Enumerative properties of q -expansions

In order to find the q -expansion of a solution of R , we follow the process given in [Mov17, § 5.2] for the case $n = 3$. Consider the vector field R as follows

$$(8.1) \quad R : \begin{cases} \dot{t}_1 = f_1(t_1, t_2, \dots, t_d) \\ \dot{t}_2 = f_2(t_1, t_2, \dots, t_d) \\ \vdots \\ \dot{t}_d = f_d(t_1, t_2, \dots, t_d) \end{cases},$$

where for $1 \leq j \leq d$,

$$f_j \in \mathbb{C}[t_1, t_2, \dots, t_d, \frac{1}{t_{n+2}(t_{n+2} - t_1^{n+2})\tilde{t}}],$$

and \tilde{t} is the same as in Theorem 1.1. Let us assume that

$$t_j = \sum_{k=0}^{\infty} t_{j,k} q^k, \quad j = 1, 2, \dots, d,$$

form a solution of R , where $t_{j,k}$'s are subject to be constants, and $\dot{*} = a \cdot q \cdot \frac{\partial *}{\partial q}$, where a is an unknown constant. By comparing the coefficients of q^k , $k \geq 2$ in both sides of (8.1) we find recursions for $t_{j,k}$'s. Let

$$p_k := (t_{1,k}, t_{2,k}, \dots, t_{d,k}), \quad k = 1, 2, 3, \dots$$

By comparing the coefficients of q^0 we get that p_0 is a singularity of R . The same for q^1 , gives us some constrains on $t_{j,1}$. Therefore, some of the coefficients $t_{j,k}$, for finite number of j and k , are free initial parameters of the recursion and we have to fix them by other means.

8.1. The case $n = 1$. Considering the modular vector field R_1 given in (7.8), we find $Sing(R_1) = Sing_1 \cup Sing_2$, where

$$Sing_1 : t_2 = t_1^3 - t_3 = 0,$$

$$Sing_2 : t_3 = t_1^2 + 3c_1 t_2 = 0.$$

Thus we get

$$p_0 = (t_{1,0}, -\frac{1}{3c_1} t_{1,0}^2, 0) \in Sing_2.$$

The comparison of the coefficients of q^1 gives us $a = \frac{1}{c_1} t_{1,0}^2$ and

$$p_1 = \left(\frac{2 t_{3,1}}{9 t_{1,0}^2}, -\frac{1}{27 c_1} \frac{t_{3,1}}{t_{1,0}}, t_{3,1} \right).$$

If we choose $c_1 = 3^{-3}$, $t_{1,0} = \frac{1}{3}$ and $t_{3,1} = 1$, then we find the solution given in (1.11) for R_1 .

The coefficient of q^k , $k = 0, 1, 2, 3, \dots$, in $\theta_3(q^{2r})\theta_3(q^{2s})$, $r, s \in \mathbb{N}$, gives the number of integer solutions of the equation $rx^2 + sy^2 = k$, where x and y are unknown variables. Therefore

Proposition 8.1. *The coefficient of q^k , $k = 0, 1, 2, 3, \dots$, in*

$$\theta_3(q^2)\theta_3(q^6),$$

gives the number of integer solutions of equation $x^2 + 3y^2 = k$.

For more information about the number of integer solutions of equation $x^2 + 3y^2 = k$ see [OEI64, A033716] and the references therein. As we saw in (1.11), $t_1(q) = \frac{1}{3}(2\theta_3(q^2)\theta_3(q^6) - \theta_3(-q^2)\theta_3(-q^6))$. If we denote by $t_1(q) := \sum_{k=0}^{\infty} t_{1,k}q^k$, then in the following proposition we state enumerative properties of $t_{1,k}$.

Proposition 8.2. *Let k be a non-negative integer. If $k = 4m$ for some $m \in \mathbb{Z}$, then the equation $x^2 + 3y^2 = k$ has $3t_{1,k}$ integer solutions, otherwise the equation has $t_{1,k}$ integer solutions.*

Proof. Suppose that $\theta_3(q^2)\theta_3(q^6) = \sum_{k=0}^{\infty} a_k q^k$ and $\theta_3(-q^2)\theta_3(-q^6) = \sum_{k=0}^{\infty} b_k q^k$. Fix a non-negative integer k . If $k = 4m$ for some $m \in \mathbb{Z}$, then $a_k = b_k$, otherwise $a_k = -b_k$. This fact together with Proposition 8.1 complete the proof. q.e.d.

Y. Martin in [Mar96] studied a more general class of η -quotients. By definition an η -quotient is a function $f(q)$ of the form

$$f(q) = \prod_{j=1}^s \eta^{r_j}(q^{t_j}),$$

where t_j 's are positive integers and r_j 's are arbitrary integers. He gives an explicit finite classification of modular forms of this type which is listed in [Mar96, Table I]. In (1.11) we found

$$(8.2) \quad t_3(q) = \frac{\eta^9(q^3)}{\eta^3(q)},$$

which is the multiplicative η -quotient $\sharp 3$ presented by Y. Martin in Table I of [Mar96]. For more details and references about this η -quotient the reader is referred to [OEI64, A106402]. Finally, note that if we define $\sum_{k=0}^{\infty} a_k q^k := t_2(q) = \frac{1}{8}(E_2(q^2) - 9E_2(q^6))$, then one can see that $3 \mid a_k$ for integers $k \geq 1$.

Table 1. Coefficients of q^k , $0 \leq k \leq 15$, in the q -expansion of the solutions of R_1 and R_2

R_1	t_1	t_2	t_3	R_2	$\frac{10}{6}t_1(\frac{q}{10})$	$\frac{10}{4}t_2(\frac{q}{10})$	$10^4t_4(\frac{q}{10})$
q^0	1/3	-1	0	q^0	1/24	1/8	0
q^1	2	-3	1	q^1	1	-1	1
q^2	0	-9	3	q^2	1	-5	-8
q^3	2	15	9	q^3	4	-4	12
q^4	2	-21	13	q^4	1	-13	64
q^5	0	-18	24	q^5	6	-6	-210
q^6	0	45	27	q^6	4	-20	-96
q^7	4	-24	50	q^7	8	-8	1016
q^8	0	-45	51	q^8	1	-29	-512
q^9	2	69	81	q^9	13	-13	-2043
q^{10}	0	-54	72	q^{10}	6	-30	1680
q^{11}	0	-36	120	q^{11}	12	-12	1092
q^{12}	2	105	117	q^{12}	4	-52	768
q^{13}	4	-42	170	q^{13}	14	-14	1382
q^{14}	0	-72	150	q^{14}	8	-40	-8128
q^{15}	0	90	216	q^{15}	24	-24	-2520

8.2. The case $n = 2$. From (7.9) we get

$$Sing(R_2) = \{(t_1, t_2, t_3, t_4) \mid t_4 = t_3 - t_1t_2 = t_1^2 + 16c_2t_2^2 = 0\},$$

hence we find

$$p_0 = (t_{1,0}, \frac{1}{4}k_0t_{1,0}, \frac{1}{4}k_0t_{1,0}^2, 0) \in Sing(R_2),$$

where $k_0 = \frac{1}{\sqrt{-c_2}}$. Comparing the coefficients of q^1 , we get $a = -t_{1,0}k_0$ and

$$p_1 = (-\frac{6}{5}\frac{t_{3,1}}{k_0t_{1,0}}, \frac{1}{10}\frac{t_{3,1}}{t_{1,0}}, t_{3,1}, -\frac{64}{5}\frac{t_{1,0}^2t_{3,1}}{k_0}),$$

where the equality $t_{4,1} = -\frac{64}{5}\frac{t_{1,0}^2t_{3,1}}{k_0}$ follows from (7.10). We set $c_2 = -\frac{1}{64}$, $t_{1,0} = \frac{1}{40}$ and $t_{3,1} = -1$ and find the solution given in (1.12) for R_2 .

The sum of positive odd divisors of a positive integer k , which is also known as *odd divisor function*, was introduced by Glaisher [Gla06] in 1906. Let k be a positive integer. We denote the sum of divisors, the sum of odd divisors and the sum of even divisors of k , by $\sigma(k)$, $\sigma^o(k)$ and $\sigma^e(k)$, respectively, i.e.,

$$\sigma(k) = \sum_{d|k} d \quad \& \quad \sigma^o(k) = \sum_{\substack{d|k \\ d \text{ is odd}}} d \quad \& \quad \sigma^e(k) = \sum_{\substack{d|k \\ d \text{ is even}}} d.$$

We have $\sigma(k) = \sigma^o(k) + \sigma^e(k)$ and $\sigma^o(k) = \sigma(k) - 2\sigma(k/2)$, where $\sigma(k/2) := 0$ if k is an odd integer. It follows from (1.12) that t_1 is the generating function of the odd divisor function:

$$\frac{10}{6}t_1\left(\frac{q}{10}\right) = \sum_{k=0}^{\infty} \sigma^o(k)q^k = \frac{1}{24}(\theta_3^4(q^2) + \theta_2^4(q^2)),$$

where by definition $\sigma^o(0) = 1/24$. For more details about the odd divisor function see [OEI64, A000593].

Comparing the coefficients of t_2 presented in Table 1 with the integers sequence given in the [OEI64, A215947] we find that

$$\sum_{k=0}^{\infty} (\sigma^o(2k) - \sigma^e(2k))q^k = \frac{10}{4}t_2\left(\frac{q}{10}\right) = \frac{1}{24}(E_2(q^2) + 2E_2(q^4)),$$

where we define $\sigma^o(0) - \sigma^e(0) := 1/8$.

Another nice observation is about $10^4 t_4(\frac{q}{10}) = \eta^8(q)\eta^8(q^2)$. The same as t_3 in the case of elliptic curve, see (8.2), we see that t_4 is the η -quotient #2 classified by Y. Martin in Table I of [Mar96], see also [OEI64, A002288] and the references therein. It is worth to point out that this η -quotient appears in the work of Heekyoung Hahn [Hah07]. She proved that $3 \mid \mu_{3k}$, $k = 0, 1, 2, \dots$, where μ_k is defined as follows

$$\sum_{k=0}^{\infty} \mu_k q^k := \eta^8(q)\eta^8(q^2).$$

She also found some partition congruences by using the notion of colored partitions, for more details see [Hah07, §6].

8.3. The case $n = 4$. The set of the singularities of R_4 contains the set of (t_1, t_2, \dots, t_8) 's that satisfy

$$(8.3) \quad t_6 = t_3 - t_1 t_2 = 6^4 c_4 t_4^2 - t_1^2 = t_8 - 6^4 c_4 t_4^3 = t_5 - 3t_1 t_4 = -t_4^2 - t_2 t_7 = 0.$$

Table 2. Coefficients of q^k , $0 \leq k \leq 5$, in the q -expansion of a solution of R_4

R_4	q^0	q^1	q^2	q^3	q^4	q^5
$\frac{1}{20}t_1$	$\frac{1}{720}$	1	4131	51734044	918902851011	19562918469120126
$\frac{1}{216}t_2$	$-\frac{1}{216}$	9	110703	2248267748	55181044614231	1498877559908208054
$\frac{1}{14}t_3$	$-\frac{1}{504}$	11	115137	2265573692	54820079452449	1477052190387154386
$\frac{1}{24}t_4$	$-\frac{1}{144}$	16	193131	3904146832	95619949713765	2594164605185043648
$\frac{1}{2}t_5$	$-\frac{1}{144}$	45	469872	9215455916	222628516313454	5992746995783064438
$-6^6 t_6$	0	-1	1944	10066356	139857401664	2615615263199250
$-\frac{1}{2}t_7$	$-\frac{1}{72}$	7	32859	414746092	7395891627375	157811370338782458
$\frac{18}{7}t_8$	$-\frac{1}{3024}$	7	54855	1034706148	24546181658391	653902684588247058

Hence if we fix $t_{1,0}$ and $t_{2,0}$, then from (8.3) we get

$$p_0 = (t_{1,0}, t_{2,0}, t_{1,0}t_{2,0}, -\frac{1}{36k_0}t_{1,0}, -\frac{1}{12k_0}t_{1,0}^2, 0, -\frac{1}{1296c_4}\frac{t_{1,0}^2}{t_{2,0}}, -\frac{1}{36k_0}t_{1,0}^3),$$

where $c_4 = k_0^2$. By comparing coefficients of q^1 we find

$$a = -6t_{2,0},$$

and

$$p_1 = \left(\frac{60k_0t_{8,1}}{49t_{1,0}^2}, \frac{-162k_0t_{2,0}t_{8,1}}{49t_{1,0}^3}, \frac{-66k_0t_{2,0}t_{8,1}}{7t_{1,0}^2}, \frac{16t_{8,1}}{147t_{1,0}^2}, \frac{45t_{8,1}}{49t_{1,0}}, \right. \\ \left. \frac{3888k_0t_{1,0}^3t_{8,1}}{49}, \frac{t_{8,1}}{1512k_0t_{1,0}t_{2,0}}, t_{8,1} \right).$$

After fixing $k_0 = 6^{-3}$, $t_{1,0} = \frac{1}{36}$, $t_{2,0} = -1$ and $t_{8,1} = \frac{49}{18}$ we find the q -expansion of a solution of \mathbf{R}_4 . We list the first six coefficients of q^k 's in Table 2. As it was expected, after multiplying t_j 's by a constant, all the coefficients are integers. If we compute the q -expansion of \mathbf{Y}_1^2 given in (7.14), then we find

$$\frac{1}{6}\mathbf{Y}_1^2 = 6 + 120960q + 4136832000q^2 + 148146924602880q^3 \\ + 5420219848911544320q^4 + 200623934537137119778560q^5 \\ + 7478994517395643259712737280q^6 \\ + 280135301818357004749298146851840q^7 \\ + 10528167289356385699173014219946393600q^8 \\ + 396658819202496234945300681212382224722560q^9 \\ + 14972930462574202465673643937107499992165427200q^{10} + \dots$$

which is the 4-point function discussed in [GMP95, Table 1, $d = 4$]. We have also computed the q -expansion of the modular coordinate z

$$(8.4) \quad \frac{z}{6^6} = \frac{t_6}{(6t_1)^6} = q - 6264q^2 - 8627796q^3 - 237290958144q^4 \\ - 4523787606611250q^5 - 101677347076292728992q^6 - \dots$$

which coincides with the one computed in [KP08, §6.1]. The computation of genus 1 topological string partition function \mathcal{F}_1 in [KP08, §6.1] may offer further evidences that our computation of \mathbf{R}_4 is correct. For the computer codes used in this article see the first author's webpage or the bottom of the tex file of the present article in arxiv.org.

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