04/May/2021 Theta series of Hodge lattices.

By lattice Λ we mean a $\mathbb{Z}$-module with a sym. product. $x \cdot y \in \mathbb{Z}$, $x, y \in \Lambda$.

$$\Theta_\Lambda := \sum_{a \in \Lambda} q^{\frac{1}{2} a \cdot a}$$

$$= \sum_{n=-\infty}^{+\infty} \left(\# \{ a \mid a \cdot a = n \} \right) q^n$$

$$= \sum_{a \in \mathbb{Z}^\mu} q^{a A a^{tr}}$$

where $A = [e_i \cdot e_j]$ is the Gram matrix and e_i's form a basis of Λ.

Proposition: Let Λ be a unimodular lattice of rank μ then

$$\Theta_\Lambda \left(-\frac{1}{\tau} \right) = \left(\frac{\tau}{i} \right)^\mu \Theta_{\Lambda^*}(\tau).$$

One has to adapt the proof in Ebelings book page 40 to this context. (Main ingr. Poisson summation formula)

Note that we choose a basis of Λ and $\Lambda \otimes \mathbb{R} = \mathbb{R}^\mu$. In this way $\Lambda = \mathbb{Z}^\mu$ with the intersection matrix A.

Dual of the lattice $\Lambda \subseteq \mathbb{R}^\mu$

$$\mathbb{R}^\mu \xrightarrow{\sim} (\mathbb{R}^\mu)^*$$

$$a \longmapsto (b \to a \cdot b)$$

$$\Lambda^* \longleftarrow \Lambda^* = \{ \Lambda \to \mathbb{Z}, \ \mathbb{Z}\text{-linear} \}$$

$$\| \quad \{ a \in \mathbb{R}^\mu \mid a \cdot \Lambda \subseteq \mathbb{Z} \}$$

we have $\Lambda \subseteq \Lambda^*$.

credeal

Λ^* is no more integral lattice and the pairing has ~~rational values~~

A basis of Λ^*: $\Lambda = \oplus \mathbb{Z} e_i$ $\quad e_i = (0, \ldots 0, \underset{\uparrow}{1}, 0, \ldots 0)$
i-th place

$$a \cdot A e_i \in \mathbb{Z} \ \forall i \iff aA \text{ has entries in } \mathbb{Z}$$

Take a_i such that

$$a_i A = e_i \implies a_i = e_i A^{-1}$$

$\implies$ the rows of A^{-1} form a basis of Λ^*.

The intersection form in Λ^*

$$[a_i \cdot a_j^{tr}] = (A^{-1}) \cdot A \, (A^{-1})^{tr} = (A^{-1})^{tr}.$$

Problem: In the Poisson summation formula we use the function

$$f : \mathbb{R}^{\mu} \to \mathbb{R}$$
$$f(x) = e^{-\pi (\frac{1}{t}) x \cdot x}$$

Maybe we have to use other functions to get more functional equation of Θ_Λ & Θ_{Λ^*}.

Problem: $\Lambda_2 \subseteq \Lambda_1$ $\quad \Lambda_1$ - unimodular $\# \Lambda_1 / \Lambda_2 < \infty$
It follows that Θ_{Λ_2} is convergent. Is Θ_{Λ_2} is algebraic over a field of modular forms containg Θ_{Λ_1}.

The only evidence for this conjecture.

$$\Lambda_1 = \mathbb{Z} e_1 \quad e_1 \cdot e_1 = 1, \quad \Lambda_2 = \mathbb{Z} N e_1 \implies$$

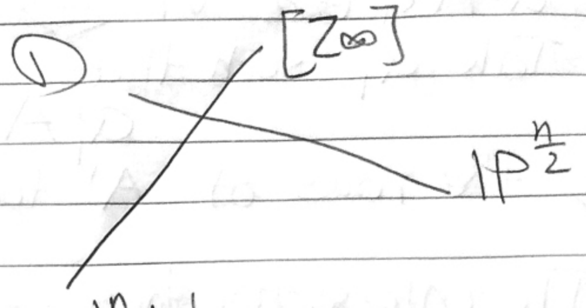
$$\Theta_{\Lambda_2} = \Theta_{\Lambda_1}(q^{N^2})$$

Fact: If $f(q)$ is a modular form for $SL(2,\mathbb{Z})$, $f(q^N)$ is a modular form for $\Gamma_0(N)$ and it is algebraic over the field $\mathbb{Q}(E_4, E_6)$. For this we use Hecke operators.

Problem: Does Θ_Λ satisfy a polynomial diff. equ.?

Examples of rank 2, non-unimodular lattices coming from geometry

$$\begin{array}{c|cc} & \mathbb{P}^{\frac{n}{2}} & [Z_\infty] \\ \hline \mathbb{P}^{\frac{n}{2}} & \bullet & 1 \\ Z_\infty & 1 & d \end{array}$$

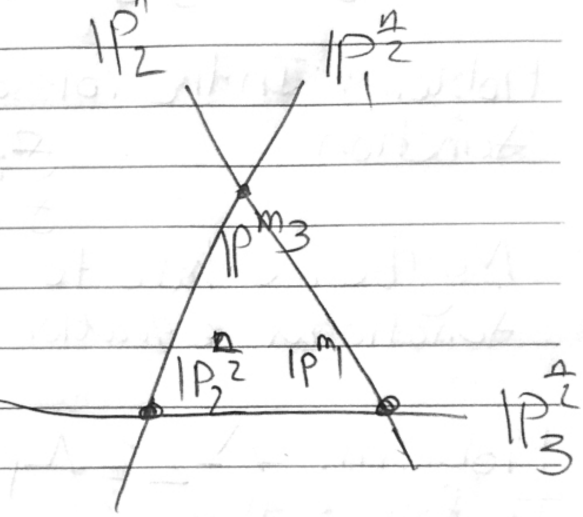


$$\bullet = \frac{1-(-d+1)^{\frac{n}{2}+1}}{d}$$

$$\det(A) = -(-d+1)^{\frac{n}{2}+1}$$

$$\delta_1 = \mathbb{P}_1^{\frac{n}{2}} - \mathbb{P}_3^{\frac{n}{2}}$$

$$\delta_2 = \mathbb{P}_2^{\frac{n}{2}} - \mathbb{P}_3^{\frac{n}{2}}$$



Recall that

$$\mathbb{P}_1^{\frac{n}{2}} \cdot \mathbb{P}_2^{\frac{n}{2}} = \frac{1-(-d+1)^{m+1}}{d}$$

$$\mathbb{P}_1^{\frac{n}{2}} \cap \mathbb{P}_2^{\frac{n}{2}} = \mathbb{P}^m$$

See also my headaches, chapter 16, quadric hypersurfaces for some useful stuff