02/04/2020 Breaking integrals

Recall Proposition 13.9 of HodgeTheoryBook. The periods of $f=g$ (which are $n+m+1$-dimensional) are reduced to $(n+1)$- and n-dimensional periods

$$\int_{\delta}\omega = \underbrace{\sum_{i} a_i \int_{\delta_i}\omega_1}_{1^{\circ}\text{ Part of Prop 13.9}} + \underbrace{\int_{\delta}\omega_2}_{2^{\circ}\text{ part of Prop 13.9}} = A+B$$

where for fixed ω, we have only two integrands ω_1 and ω_2.

We take f a tame polynomial depending on all parameters (see Example 10.9). In this way a_i's are constant

$\int_{\delta_i}\omega_1, \int_{\delta}\omega_2$ are holomorphic functions in $t=(t_1,t_2,\dots)$

- The variety $z^q=f$ has the automorphism $z \mapsto \zeta_q z$ which induces $\mathbb{C}$-linear relations between $\int_{H_{n+1}(z^q=f,\mathbb{Q})}\omega_1$ we take a basis of $\mathbb{C}\left\langle \int_{H_{n+1}(\overline{z^q=f},\mathbb{Q})}\omega_1 \right\rangle$ modulo these relations. For $n=0$ $\int_{\delta}\omega_2 \in \overline{\mathbb{C}(t)}$ and

Assumption: The integrals

$$\int_{\delta_i}\omega_1,\ i=1,2,\dots \quad \text{and } 1$$

are linearly independent over $\overline{\mathbb{C}(t)}$.

This implies that if $\int_{\delta}\omega=0$ then $a_i=0$ $i=1,2,\dots$, $\int_{\delta}\omega_2=0$

Prop*: If δ is a cycle at ∞ and ω has no residue at ∞ then $a_i=0$ and $\int_{\delta}\omega_2=0$.

Proof: we have $\int_{\delta}\omega=0$ and if the above assumption is true then ■

Cor: Cycles at ∞ are generic Hodge cycles.

Jorge Duque's breaking method

$$\int_{\sum n_{\alpha,\beta}\delta_1\otimes\delta_2}\omega_{\beta_1,\beta_2} \xlongequal[\text{case in Prop 13.9}]{\text{only the first}} \underbrace{\sum n_{\alpha,\beta} P(\beta_2,\delta_2)}_{A} \quad \underbrace{\frac{\int \frac{x^{\beta_1} z^{\beta_3}\cdots}{\partial(f-z^{n})}}{P(\beta_3,\delta_3)}}_{B}$$

The sum runs over different choices of $\delta_1=\delta_{1,\beta}$, $\delta_2=\delta_{2,\alpha}$

A depends only on $\delta_2=\delta_{2,\alpha}$

B " " $\delta_1=\delta_{1,\beta}$

Therefore this is zero if

$$\sum_{\alpha} n_{\alpha,\beta} P(\beta_2,\delta_2)=0 \quad \forall \beta.$$

Note that this different from the previous page. We don't care that B are $\overline{\mathbb{C}(t)}$ - linearly independent functions

Different formulation: We write $\delta=\sum_{\delta_1}\delta_1\otimes\delta_2$, where δ_1 runs through a basis of vanishing cycles but δ_2 is an arbitrary cycle

$$\sum_{\delta_1}\int_{\delta_1*\delta_2}\omega_{\beta_1,\beta_2}=\sum_{\delta_1}\left(\int_{\delta_2}\omega_{\beta_2}\right)\cdot \underbrace{f(\beta_1,\delta_1)}$$

expression depending only on β_1,δ_1.

$$\forall\delta_1 \quad \int_{\delta_2}\omega_{\beta_2}=0 \implies \int_{\delta}\omega_{\beta_1,\beta_2}=0.$$

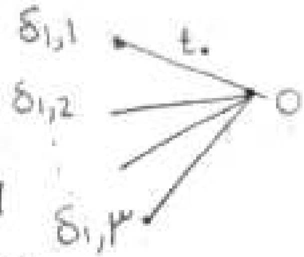
The above discussion is valid for $\omega_\beta=\frac{x^\beta dx}{f}$. We want to see this for higher order poles.

We use Proposition 13.10 of Hodge theory book

$$\sum_{\delta_1}\int_{\delta_1*\delta_2}\frac{x^{\beta_1}y^{\beta_2}dx\wedge dy}{(f-g)^k}=\sum_{\delta_1}\left(\int_{\delta_2}\omega_{\beta_2}\right)\cdot f(\beta_1,\delta_1)$$

Proposition 13.9 of HodgeTheoryBook: (03/04/2020)

$$\int_{\delta_1 *_t \delta_2} \frac{\omega_{\beta_1,\beta_2}}{f-g} = \left(\int_{\delta_2} \frac{\omega_{\beta_2}}{g-1} \right) \cdot I(\beta_1, \delta_1)$$



where $\omega_{\beta_1,\beta_2} = x^{\beta_1} y^{\beta_2} dx \wedge dy$

$\omega_{\beta_2} = y^{\beta_2} dy.$

$I(\beta_1, \delta_1)$ is a $(n+1)$- or n-dimensional integral respectively fo $z^q - f = 0$ and $f = 0$.

It only depends on δ_1, β_1 and A_{β_2}. In fact depending on $A_{\beta_2} \in \mathbb{N}$ or $\notin \mathbb{N}$ the two cases are distinguished

Obs1. In the above formula we don't need to assume that δ_2 is a vanishing cycle.

obs2: If f depends on parameters then $\int_{\delta_2} \frac{\omega_{\beta_2}}{g-1}$ is a constant and $I(\beta_1, \delta_1)$ depends on these parameters

Now we rewrite Proposition 13.10:

$$\int_{\delta_1 * \delta_2} \frac{\omega_{\beta_1,\beta_2}}{(f-g)^k} = \int_{\delta_2} \frac{\omega_{\beta_2}}{g-1} \cdot I(\beta_1, \delta_1, k)$$

where I is a $(n+1)$- or n-dim integral depending only on β_1, δ_1, k.

Now assume that $n+m+1$ is even and so $\delta = \sum n_i \delta_{1,i} * \delta_{2,i}$

$\delta_{1,i}$ basis of vanishing cycles, $\delta_{2,i} \in H_m(\{g=1\}, \mathbb{Z})$ arbitrary.

δ is Hodge if

$$\int \frac{\omega_{\beta_1,\beta_2}}{(f-g)^{\frac{n+m+1}{2}}} = 0 \quad \forall (\beta_1,\beta_2) \quad A_{\beta_1} + A_{\beta_2} < \frac{n+m+1}{2}$$

In particular δ is a GHC in the sense of Jorge's thesis

$$\forall (\beta_1,\beta_2) \quad A_{\beta_1}+A_{\beta_2} < \frac{n+m+1}{2} \Rightarrow \int_{\delta_{2,i}} \frac{\omega_{\beta_2}}{g-1} = 0$$

This is equivalent to

$$\forall \beta_2 \qquad A_{\beta_2} < \frac{n+m+1}{2} - \sum_{i=1}^{n+1} \frac{1}{m_i} \Longrightarrow \int_{\delta_{2,i}} \frac{\omega_{\beta_2}}{g-1} = 0.$$

where $f = x_1^{m_1} + x_2^{m_2} + \cdots x_{n+1}^{m_{n+1}} + \cdots$

Proof: $\Downarrow$ put $\beta_1 = (0,0,\ldots 0)$

$\Uparrow$ $\quad A_{\beta_2} + A_{(0,\ldots,0)} \leqslant A_{\beta_1} + A_{\beta_2} < \frac{n+m+1}{2}$

This implies $\quad A_{\beta_2} < \frac{n+m+1}{2} - \sum_{i=1}^{n+1} \frac{1}{m_i}$

Jorge's case $(n=0)$ $\quad A_{\beta_2} < \frac{m+1}{2} - \frac{1}{m_1} \longrightarrow$ In Jorge's notation m_{n+1}

Def:

$$GHC(\underbrace{y_1^{\tilde{m}_1} + y_2^{\tilde{m}_2} + \cdots + y_{m+1}^{\tilde{m}_{m+1}}}_{g} - f(x) = 0)$$

$$\left(\frac{\left\{ \delta \in H_m(g=1, \mathbb{Z}) \;\middle|\; \int_\delta \omega_{\beta_2} = 0 \;\forall\, \beta_2 \;\; A_{\beta_2} < \frac{m+1}{2} - \sum_{i=1}^{n+1} \frac{1}{m_i} \right\}}{\left\{ \;\text{''}\; \;\middle|\; \text{''} \;\; \text{''} \;\; \text{''} \;\; A_{\beta_2} < n+m+2 - \sum_{i=1}^{n+1} \frac{1}{m_i} \right\}} \right)$$

Obs:

Note that there is no assumption on $A_{\beta_2} \notin \mathbb{N}$ or $A_{\beta_2} \in \mathbb{N}$.

This is not strange. Some of $\tilde{I}(\beta_1, \delta_1, A_{\beta_2})$ might be zero

Remember

$$\int \frac{\omega_{\beta_2}}{(g-1)^k} = 0 \quad A_{\beta_2} < k, \; A_{\beta_2} \in \mathbb{N}$$

see §11.4.