30/03/2020 Rio de Janeiro. Let us consider the curve $y^{m_2} = P(x)$, where $P(x) = x^{m_1} + t_1 x^{m_1 - 1} + \dots + t_{m_1}$ is a polynomial dep. on parameters $t_1, \dots, t_{m_1}$. We consider a basis $\omega_1, \omega_2, \dots \omega_\mu$, $\delta_1, \delta_2, \dots \delta_\mu$ of $H^1_{dR}(C_t)$ and $H_1(C_t, \mathbb{Z})$, where C_t is the smooth compact curve after compactification and desing. of $y^{m_2} = P(x)$. We define the period matrix

$$Y(t) = \left[\int_{\delta_i} \omega_j \right]_{\mu \times \mu}$$

there are many polynomial relations among Y_{ij}'s:

1. Let $\Psi = [\langle \delta_i, \delta_j \rangle]_{\mu \times \mu}$ e $\tilde{\Phi} = [\langle \omega_i, \omega_j \rangle]_{\mu \times \mu}$, where $\langle , \rangle$ is the intersection form (resp. $\frac{1}{2\pi i} \int \omega_i \cup \omega_j$) in $H_1(C_t, \mathbb{Z})$ (resp. $H^1_{dR}(C_t)$). The entries of $\tilde{\Phi}$ are rational functions in t with possible poles along the discriminant $\Delta = 0$. We have

$$Y(t) \cdot \Psi^{-tr} Y(t)^{tr} = \Phi$$

This is a quadratic relation between the periods Y_{ij}, see chapter 4 of ModularFormsBeyond Book.

2. $f : C_t \to C_t$ $\quad f(x, y) = (x, \zeta y)$, $\zeta^{m_2} = 1$.
this induces $f_* : H_1(C_t, \mathbb{Z}) \to H_1(C_t, \mathbb{Z})$, and if we choose ω_i's as $\frac{x^{\beta_1} y^{\beta_2} dx \wedge dy}{d(y^{m_2} - P(x))}$, clearly ω_i is an eigen value of f^*: $f^* \omega_i = a_i \omega_i$, $a_i \in \mathbb{Q}(\zeta)$. We have

$$a_i \int_\delta \omega_i = \int_\delta f^* \omega_i = \int_{f_* \delta} \omega_i$$

This gives new linear relations among Y_{ij}'s.

Conj: For $Y_{ij}(t)$ as hol. functions on t, the relations 1 an 2 are the only polynomial equations among Y_{ij} over the field $\mathbb{C}(t)$.

How to approach this conjecture We know

$$dY = Y \cdot A^{tr}$$

where A is the Gauss-Manin connection matrix in the basis ω_i.

See §13.5 of Hodge Theory Book.

Rule: Any polynomial relation between Y_{ij}'s must be seen from the Gauss-Manin connection matrix.

Rule: We know that Y is multivalued and after doing monodrom. it becomes MY, $M \in Mat_{\mu\times\mu}(\mathbb{Z})$. Therefore, if there is a polynomial relation $P(Y)=0$, $P \in \mathbb{C}(t)[Y_{ij}, i,j=1,\dots,\mu]$ then $P(MY)=0$. We migh compute Y at one point for instance $y^{m_2} = x^{m_1}+1$ (let us call it Y_0). and the monodromy group $\Gamma_{\mathbb{Z}} = \langle M_1, \dots M_k \rangle$ and then the Zariski closure of

$$\Gamma_{\mathbb{Z}} \cdot Y_0 \subseteq Mat(r\times r, \mathbb{C}).$$