The quadratic Euler characteristic of a smooth projective same-degree complete intersection

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The Grothendieck-Witt ring

Notation

Fix a field k s.t. $\mathrm{char}(k) \neq 2$.

Definition

The *Grothendieck-Witt ring* $\mathrm{GW}(k)$ of k is the group completion of the monoid of isometry classes of nondegenerate quadratic forms.

$\mathrm{GW}(k)$ is generated by $\langle a \rangle : x \mapsto ax^2$ for $a \in k^*$ modulo

- $\langle a \rangle \langle b \rangle = \langle ab \rangle$ for $a, b \in k^*$,
- $\langle a \rangle + \langle b \rangle = \langle a + b \rangle + \langle ab(a + b) \rangle$ for $a, b, a + b \in k^*$,
- $\langle ab^2 \rangle = \langle a \rangle$ for all $a, b \in k^*$,
- $\langle a \rangle + \langle -a \rangle = H$ for any $a \in k^*$. Here, $H = \langle 1 \rangle + \langle -1 \rangle$ is the *hyperbolic form*.

Some examples

Example

$\mathrm{GW}(\mathbb{C}) \cong \mathbb{Z}$ via the rank. The same holds for other algebraically closed fields k.

Example

$\mathrm{GW}(\mathbb{R}) \cong \mathbb{Z}[C_2]$ where C_2 is a cyclic group of order two.

Example

$\mathrm{GW}(\mathbb{F}) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ for $\mathbb{F}$ a finite field, $\mathrm{char}(\mathbb{F}) \neq 2$.

Motivic stable homotopy category

Dold-Puppe

Can define a *categorical Euler characteristic* for any strongly dualizable object of a symmetric monoidal category, lives in endomorphism ring of the unit.

Morel and Voevodsky: *motivic stable homotopy category* $\mathrm{SH}(k)$.

Some facts:

- $\mathrm{SH}(k)$ is symmetric monoidal,
- X smooth projective scheme$/k$ has a strongly dualizable image in $\mathrm{SH}(k)$,
- Morel: $\mathrm{End}(1_{\mathrm{SH}(k)}) \cong \mathrm{GW}(k)$.

Quadratic Euler characteristics

Definition

The *quadratic Euler characteristic* $\chi(X) \in \mathrm{GW}(k)$ of a smooth projective scheme X over k is the categorical Euler characteristic of X in $\mathrm{SH}(k)$.

Remark

If $Z \subset X$ is a smooth closed embedding of codimension c and $U \subset X$ is the open complement of Z, then

$$\chi(X) = \langle -1 \rangle^c \chi(Z) + \chi(U).$$

Examples

Example

We have that $\chi(\mathbb{A}^n) = \langle 1 \rangle$.

Example

We have that $\chi(\mathbb{P}^n) = \sum_{i=0}^{n} \langle -1 \rangle^i$.

If $k \subset \mathbb{R}$, then:

- $\mathrm{rank}(\chi(X)) = \chi^{top}(X(\mathbb{C}))$,
- $\mathrm{sgn}(\chi(X)) = \chi^{top}(X(\mathbb{R}))$.

Remark

If $V \to X$ is a rank $r+1$ vector bundle and $\mathbb{P}(V)$ is its projectivization, then $\chi(\mathbb{P}(V)) = \chi(X) \cdot \chi(\mathbb{P}^r)$.

The Motivic Gauss-Bonnet Theorem

Theorem (Levine-Raksit)

Let X be a smooth projective scheme over k. Then:

- *If $\dim(X)$ is odd, then $\chi(X) = a \cdot H$ for some $a \in \mathbb{Z}$.*
- *If $\dim(X) = 2n$ is even, then $\chi(X) = a \cdot H + Q$ for some $a \in \mathbb{Z}$, where Q is given by*

$$H^n(X, \Omega_X^n) \times H^n(X, \Omega_X^n) \xrightarrow{\cup} H^{2n}(X, \Omega_X^{2n}) \xrightarrow{\text{Trace}} k.$$

Here, Ω_X denotes the sheaf of differential forms on X and $\Omega_X^q = \wedge^q \Omega_X$.

Levine, Lehalleur and Srinivas: compute Q for X a hypersurface in $\mathbb{P}^n$. Inspiration from: Carlson-Griffiths.

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Setup

Consider a smooth hypersurface $X = V(F) \subset \mathbb{P}^n$ where $F \in k[X_0, \cdots, X_n]$ is homogeneous of degree $m \in \mathbb{Z}_{\geq 2}$ s.t. $\mathrm{char}(k) \not| m$.

Definition

The *Jacobian ring* of X is

$$J = k[X_0, \cdots, X_n] / \left(\frac{\partial F}{\partial X_0}, \cdots, \frac{\partial F}{\partial X_n} \right).$$

Note that:

- J has a natural grading.
- Top degree: $J^{(m-2)(n+1)} \cong k$ generated by the *Scheja-Storch element* e_F.

The Scheja-Storch generator

Remark

Formula for e_F: write $\frac{\partial F}{\partial X_i} = \sum_{j=0}^{n} a_{ij} X_j$ then $e_F = \det(a_{ij})$.

Example (Generalized Fermat hypersurface)

Let $a_0, \cdots, a_n \in k^*$ and set $F = \sum_{i=0}^{n} a_i X_i^m$. Then we have that $\frac{\partial F}{\partial X_i} = m a_i X_i^{m-1}$. We have that $J^{(m-2)(n+1)}$ is generated by

$$e_F = m^{n+1} \prod_{i=0}^{n} a_i X_i^{m-2}.$$

Primitive cohomology

Let $i : X \to \mathbb{P}^n$ be the inclusion. This induces a pushforward map

$$i_* : H^q(X, \Omega^p) \to H^{q+1}(\mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^n})$$

for all $p, q \in \mathbb{Z}_{\geq 0}$.

Definition

The *primitive cohomology of* X *with respect to* $p, q \in \mathbb{Z}_{\geq 0}$ is defined by $H^q(X, \Omega^p)_{prim} = \ker(i_*)$.

Have $H^q(X, \Omega^p)_{prim} = H^q(X, \Omega^p)$ whenever $p \neq q$.

Result for a hypersurface

Levine, Lehalleur, Srinivas: For each $q \geq 0$, there is an isomorphism

$$\psi_q : J^{(q+1)m-n-1} \to H^q(X, \Omega^{n-1-q})_{prim}$$

which behaves well with the cup product.

Theorem (Levine, Lehalleur, Srinivas)

Let $p, q \in \mathbb{Z}_{\geq 0}$ satisfy $p + q = n - 1$ and let $A \in J^{(q+1)m-n-1}$ and $B \in J^{(p+1)m-n-1}$. Suppose that $AB = \lambda e_F$ in $J^{(m-2)(n+1)}$, for some $\lambda \in k^*$. Then

$$Tr(\psi_q(A) \cup \psi_p(B)) = -m\lambda.$$

Generalized Fermat hypersurface

Let $X = V(F)$ with $F = \sum_{i=0}^{n} a_i X_i^m$ with $a_i \in k^*$. If $n = 2p+1$ is odd, then

$$H^p(X, \Omega^p) = H^p(X, \Omega^p)_{prim} \oplus c_1(\mathcal{O}(1))^p.$$

Result:

One can prove that

$$\chi(X) = \begin{cases} A_{n,m} \cdot H & \text{if } n \text{ even} \\ A_{n,m} \cdot H + \langle m \rangle & \text{if } n, m \text{ odd} \\ A_{n,m} \cdot H + \langle m \rangle + \langle -m \prod_{i=0}^{n} a_i \rangle & \text{otherwise} \end{cases}$$

for integers $A_{n,m} \in \mathbb{Z}$.

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The complete intersection case

Over $\mathbb{C}$: Konno and Terasoma. Toric geometry: Cox-Batyrev and Villaflor.

Setup

Let $n, r \in \mathbb{Z}_{\geq 1}$ s.t. $n \geq r + 2$. Let $F_0, \cdots, F_r \in k[X_0, \cdots, X_n]$ be homogeneous of the same degree $m \geq 2$. Assume m is coprime to $\mathrm{char}(k)$. Let $X = V(F_0, \cdots, F_r) \subset \mathbb{P}^n$ and assume X is a smooth complete intersection. Consider the smooth hypersurface

$$\mathcal{X} = V(F) \subset \mathbb{P}^r \times \mathbb{P}^n$$

where $F = Y_0F_0 + \cdots + Y_rF_r$.

Notation: Write $\bar{F}_j = \frac{\partial F}{\partial X_j}$ for $j \in \{0, \cdots, n\}$.

Relating $\chi(X)$ and $\chi(\mathcal{X})$

Lemma

We have that $\chi(\mathcal{X}) = \chi(\mathbb{P}^{r-1})\chi(\mathbb{P}^n) + \langle -1\rangle^r \chi(X)$.

Proof.

Let $U = \mathbb{P}^n \setminus X$ and let $\pi : \mathcal{X} \to \mathbb{P}^n$ be the projection. Then $\pi^{-1}(X) = \mathbb{P}^r \times X$ and $\pi^{-1}(U) \to U$ is a Zariski locally trivial $\mathbb{P}^{r-1}$-bundle. We have that $\chi(\mathbb{P}^n) = \chi(U) + \langle -1\rangle^{r+1}\chi(X)$. This yields

$$\begin{aligned}\chi(\mathcal{X}) &= \chi(\mathbb{P}^{r-1})\chi(U) + \langle -1\rangle^r \chi(\mathbb{P}^r)\chi(X)\\ &= \chi(\mathbb{P}^{r-1})\chi(\mathbb{P}^n) + \langle -1\rangle^r \chi(X)\end{aligned}$$

as desired. □

The Jacobian ring

Definition

The *Jacobian ring* is

$$J = k[Y_0, \cdots, Y_r, X_0, \cdots, X_n] / \left(F_0, \cdots, F_r, \bar{F}_0, \cdots, \bar{F}_n\right).$$

Note: J is bigraded and infinite dimensional over k.

Proposition (Konno and Terasoma over $\mathbb{C}$, V. for general case)

For $q \geq r$, there are isomorphisms

$$\psi_q : J^{q-r,(q+1)m-(n+1)} \to H^q(\mathcal{X}, \Omega_{\mathcal{X}}^{n+r-1-q})_{prim}.$$

Compatibility with the cup product

Proposition (Konno and Terasoma over $\mathbb{C}$, V. for general case)

Let $\rho = (n-r-1, (n+r+1)m - 2(n+1))$. There exists a surjective morphism ϕ, such that the diagram

$$\begin{array}{ccc} H^q(\mathcal{X}, \Omega^p_{\mathcal{X}})_{prim} \otimes H^p(\mathcal{X}, \Omega^q_{\mathcal{X}})_{prim} & \xrightarrow{i_* \circ \cup} & H^{n+r}(\mathbb{P}^r \times \mathbb{P}^n, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n}) \\ \uparrow{\scriptstyle \psi_p \otimes \psi_q} & & \uparrow{\scriptstyle \phi} \\ J^{q-r,(q+1)m-(n+1)} \otimes J^{p-r,(p+1)m-(n+1)} & \longrightarrow & J^{\rho} \end{array}$$

commutes for all $p, q \in \mathbb{Z}_{\geq 0}$ such that $p + q = n + r - 1$.

Proof uses: cover of $\mathbb{P}^r \times \mathbb{P}^n$ by

$$\mathcal{U} = \{\{F_0 \neq 0\}, \cdots, \{F_r \neq 0\}, \{\bar{F}_0 \neq 0\}, \cdots, \{\bar{F}_n \neq 0\}\}.$$

Note:

- Not all elements are of the same bidegree
- The cover is too big

Elements of the Čech cohomology group $C^{n+r}(\mathcal{U}, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n})$ look like cycles $\{s_0, \cdots, s_r, \bar{s}_0, \cdots, \bar{s}_n\}$ where s_i lives on the intersection of everything except $\{F_i \neq 0\}$.

Generators

Let $\omega = \sum_{i=0}^r (-1)^i Y_i dY^i$ (a generator of $\Omega^r_{\mathbb{P}^r}(r+1)$) and $\bar{\omega} = \sum_{j=0}^n (-1)^j X_j dX^j$ (a generator of $\Omega^n_{\mathbb{P}^n}(n+1)$). Then $\omega \wedge \bar{\omega}$ is a generator of $\Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n}(r+1, n+1)$.

One computes: i_* of a cup product of two images from J on Čech cohomology.

Then: The map ϕ is constructed from the morphism

$$\begin{aligned}\tilde{\phi} : k[Y_0, \cdots, Y_r, X_0, \cdots, X_n]^\rho &\to C^{n+r}(\mathcal{U}, \Omega^{n+r}_{\mathbb{P}^r \times \mathbb{P}^n}),\\ D &\mapsto \{s_0, \cdots, s_r, \bar{s}_0, \cdots, \bar{s}_n\}\end{aligned}$$

where for $v \in \{0, \cdots, r\}$ and $w \in \{0, \cdots, n\}$, we have

$$s_v = \frac{(-1)^{v+r+1} m D Y_v F_v \omega \wedge \bar{\omega}}{\prod_{i=0}^r F_i \prod_{j=0}^n \bar{F}_j} \text{ and } \bar{s}_w = \frac{(-1)^{w+1} D X_w \bar{F}_w \omega \wedge \bar{\omega}}{\prod_{i=0}^r F_i \prod_{j=0}^n \bar{F}_j}.$$

One dimensionality results

In fact: J^ρ is one dimensional, unless $\mathcal{X}$ is odd dimensional, $r = 1$ and $m = 2$.

Let

$$\tilde{J} = k[Y_0, \cdots, Y_r, X_0, \cdots, X_n]/(Y_0 F_0, \cdots, Y_r F_r, X_0 \bar{F}_0, \cdots, X_n \bar{F}_n).$$

Then: $\tilde{J}^{\rho+(r+1,n+1)}$ is one dimensional.

Lemma (V.)

If we do not have that $\dim(\mathcal{X})$ *is odd,* $r = 1$ *and* $m = 2$ *then*

$$\psi : J^\rho \to \tilde{J}^{\rho+(r+1,n+1)}, D \mapsto D \prod_{i=0}^{r} Y_i \prod_{j=0}^{n} X_j$$

is an isomorphism.

Towards the trace

Extra assumptions

1. $m+1$ is invertible in k.
2. $V(F_i)$ is smooth for all $i \in \{0, \cdots, r\}$ and $V(F_0, \cdots, F_r)$ is smooth and of codimension $r+1$.
3. The assumption (2) remains true after setting any subset of the X_i or Y_i equal to zero.

These assumptions mean that we can cover $\mathbb{P}^r \times \mathbb{P}^n$ by

$$\{\{Y_0F_0 \neq 0\}, \cdots, \{Y_rF_r \neq 0\}, \{X_0\bar{F}_0 \neq 0\}, \cdots, \{X_n\bar{F}_n \neq 0\}\}.$$

But

$$m\sum_{i=0}^{r} Y_i F_i - \sum_{j=0}^{n} X_j \bar{F}_j = mF - mF = 0.$$

Better cover

This means we have the cover

$$\mathcal{W} = \{\{Y_1 F_1 \neq 0\}, \cdots, \{Y_r F_r \neq 0\}, \{X_0 \bar{F}_0 \neq 0\}, \cdots, \{X_n \bar{F}_n \neq 0\}\}$$

Note that:

- All elements have bidegree $(1, m)$.
- This cover has the right amount of elements.
- This is a refinement of $\mathcal{U}$.

Want: Represent something of which we know the trace on this cover and compare with cup products.
Let M be the Jacobian matrix of $Y_0F_0, \cdots, Y_rF_r, X_0\bar{F}_0, \cdots, X_n\bar{F}_n$.

Lemma (V.)

There exists a unique $\tilde{C} \in k[Y_0, \cdots, Y_r, X_0, \cdots, X_n]^{\rho+(r+1,n+1)}$ *such that*

$$(m+1)Y_iX_j\tilde{C} = (-1)^j \det(M_{0|j+r+1})Y_i + (-1)^{r+i}\det(M_{0|i})X_j$$

for $i \in \{0, \cdots, r\}$ *and* $j \in \{0, \cdots, n\}$. *Moreover,*

$$\frac{\tilde{C}\omega \wedge \bar{\omega}}{\prod_{i=1}^r Y_iF_i \prod_{j=0}^n X_j\bar{F}_j} \in C^{n+r}(\mathcal{W}, \Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n})$$

represents $c_1(\mathcal{O}(1,m))^{n+r}$.

We know: $Tr(c_1(\mathcal{O}(1,m))^{n+r}) = m^n\binom{n+r}{r}$.
And: This is represented by

$$\frac{\tilde{C}\omega \wedge \bar{\omega}}{\prod_{i=1}^{r} Y_i F_i \prod_{j=0}^{n} X_j \bar{F}_j} \in C^{n+r}(\mathcal{W}, \Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}).$$

Also: For $p+q = n+r-1, A \in J^{q-r,(q+1)m-(n+1)}$ and $B \in J^{p-r,(p+1)m-(n+1)}$, have that $\phi(AB) = i_*(\psi_q(A) \cup \psi_p(B))$ is represented by

$$\frac{(-1)^{r+1} mAB\omega \wedge \bar{\omega}}{\prod_{i=1}^{r} F_i \prod_{j=0}^{n} \bar{F}_j} \in C^{n+r}(\mathcal{W}, \Omega^{n+r}_{\mathbb{P}^r\times\mathbb{P}^n}).$$

Finally: If we don't have $r = 1, m = 2$ and $\dim(\mathcal{X})$ odd then $\tilde{C} = \psi(C)$ for a unique $C \in J^\rho$. And C has to be a generator.

Theorem (V.)

Assume that we do not have $\dim(\mathcal{X})$ *is odd,* $r = 1$ *and* $m = 2$*. Let* $p, q \in \mathbb{Z}_{\geq 0}$ *s.t.* $p + q = n + r - 1$*. For* $A \in J^{q-r,(q+1)m-(n+1)}$ *and* $B \in J^{p-r,(p+1)m-(n+1)}$*, write* $AB = \lambda C$ *in* J^{ρ} *for some* $\lambda \in k$*. Then*

$$\begin{aligned} Tr(\psi_q(A) \cup \psi_p(B)) &= Tr(i_*(\psi_q(A) \cup \psi_p(B))) \\ &= (-1)^{r+1} m^{n+1} \binom{n+r}{r} \lambda. \end{aligned}$$

Intersecting two generalized Fermat hypersurfaces

Theorem (V.)

Let $a_0, \cdots, a_n, b_0, \cdots, b_n \in k^*$ *s.t.* $a_i b_j - a_j b_i \neq 0$ *for all* $i \neq j$. *Let* $F_0 = \sum_{i=0}^n a_i X_i^m, F_1 = \sum_{i=0}^n b_i X_i^m$. *Let* $X = V(F_0, F_1) \subset \mathbb{P}^n$. *Then*

$$\chi(X) = \begin{cases} B_{n,m} \cdot H & \text{if } n \text{ is odd} \\ B_{n,m} \cdot H + \langle 1 \rangle & \text{if } n \text{ is even, } m \text{ odd} \\ B_{n,m} \cdot H + \langle 1 \rangle & \text{if } n, m \text{ are even} \\ + \sum_{i=0}^n \langle \prod_{j \neq i} (a_i b_j - a_j b_i) \rangle & \end{cases}$$

for some $B_{n,m} \in \mathbb{Z}$.

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Setup

Suppose $X = V(F_0, F_1)$ is the intersection of two Fermat curves $V(F_0)$ and $V(F_1)$ in $\mathbb{P}^2$ with

$$F_0 = a_0X_0^m + a_1X_1^m + a_2X_2^m$$

and

$$F_1 = b_0X_0^m + b_1X_1^m + b_2X_2^m$$

where the $a_i, b_i \in k^*$ satisfy $a_ib_j - a_jb_i \neq 0$ for all $i \neq j$.

Goal:

Compute $\chi(X)$!

Trick

Map induced by field extensions

For a separable field extension $k \subset L$, there is a morphism

$$\pi_* : \mathrm{GW}(L) \to \mathrm{GW}(k).$$

For $\langle u \rangle \in \mathrm{GW}(L)$, we have that $\pi_* \langle u \rangle$ is given by the composition

$$L \times L \xrightarrow{\langle u \rangle} L \xrightarrow{\mathrm{Tr}_{L/k}} k.$$

By a result of Hoyois, we have that $\chi(\mathrm{Spec}(L)) = \pi_*(\langle 1 \rangle)$.

Lemma

Let K be a field of characteristic coprime to $2m$ and let $a \in K^*$. Let $K(\alpha) = K[X]/(X^m + a)$ and let $u \in K(\alpha)^*$. Then

$$Tr_{K(\alpha)/K}(\langle u \rangle) = \begin{cases} \frac{m-1}{2} H + \langle um \rangle & \text{if } m \text{ is odd} \\ \frac{m-2}{2} H + \langle um \rangle + \langle -aum \rangle & \text{if } m \text{ is even} \end{cases}$$

Proof idea: $1, \alpha, \cdots, \alpha^{m-1}$ is a basis of $K(\alpha)$. We have that

$$\mathrm{Tr}_{K(\alpha)/K}(u\alpha^{i+j}) = \begin{cases} um & \text{if } i = j = 0 \\ -aum & \text{if } i + j = m \\ 0 & \text{otherwise} \end{cases}$$

Without loss of generality, assume that $X = V(F_0, F_1)$ lies in $X_2 \neq 0$. With coordinates $x = \frac{X_0}{X_2}$ and $y = \frac{X_1}{X_2}$ on $\mathbb{A}^2$, we have that

$$X = V(a_0x^m + a_1y^m + a_2, b_0x^m + b_1y^m + b_2).$$

Let

$$K = k[x, y]/(a_0x^m + a_1y^m + a_2, b_0x^m + b_1y^m + b_2).$$

Note that

$$a_0x^m + a_1y^m + a_2 = 0 \text{ and } b_0x^m + b_1y^m + b_2 = 0$$

implies that

$$(a_1b_0 - a_0b_1)y^m + a_2b_0 - a_0b_2 = 0 \text{ and } (a_0b_1 - a_1b_0)x^m + a_2b_1 - a_1b_2 = 0.$$

So we have

$$k \subset k(\alpha) = k[t]/(t^m + \frac{a_0b_2 - a_2b_0}{a_1b_0 - a_0b_1}) \subset K = k(\alpha)[s]/(s^m + \frac{a_1b_2 - a_2b_1}{a_0b_1 - a_1b_0}).$$

Final result

In particular: $k \subset K$ is separable.
Applying the lemma twice now gives:

Proposition

The quadratic Euler characteristic of X equals

$$\chi(X) = \begin{cases} \frac{(m+1)(m-1)}{2} H + \langle 1 \rangle & m \text{ odd} \\ \frac{(m+2)(m-2)}{2} H + \langle 1 \rangle + \sum_{i=0}^{2} \langle \prod_{j \neq i} (a_i b_j - a_j b_i) \rangle & m \text{ even} \end{cases}$$