ENS DE LYON

Mahler measures, special values and exactness: a periodic journey

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The plan

1. Mahler measure: definition and basic properties

2. Exact polynomials: from Darboux to Lalín

3. Our contribution: a geometric approach

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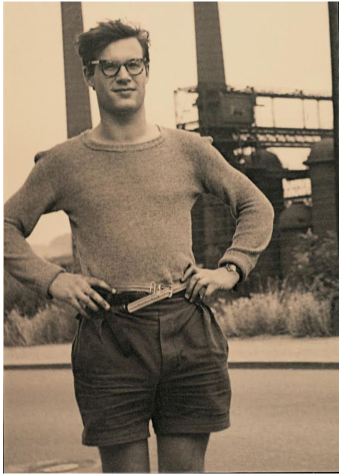
So many periods...



Alexander Grothendieck

Periods $z \in \mathbb{C}$ admit the following equivalent characterizations:

- **Elementary** $|\Re(z)|, |\Im(z)|$ are volumes of $\mathbb{Q}$-semi-algebraic sets;
- **K.-Z.** $|\Re(z)|, |\Im(z)|$ are of the form $\int_{g \leq 0} f$, for $f, g \in \mathbb{Q}(t)$;
- **Motivic** $z = \langle \eta, \gamma \rangle_{(X,D)}$, where $\eta \in H^n_{\mathrm{dR}}(X, D)$ and $\gamma \in H^{\mathrm{B}}_n(X, D)$, for some smooth variety $X_{/\mathbb{Q}}$ and some divisor $D \hookrightarrow X$ which can be taken to have simple normal crossings.

Example: $\Im(2\pi i)/2 = \mathrm{Vol}(x^2 + y^2 \leq 1) = \int_{-\infty}^{+\infty} \frac{dt}{t^2+1}$, and $2\pi i = \left\langle \left[\frac{dz}{z} \right], [\circlearrowleft] \right\rangle$

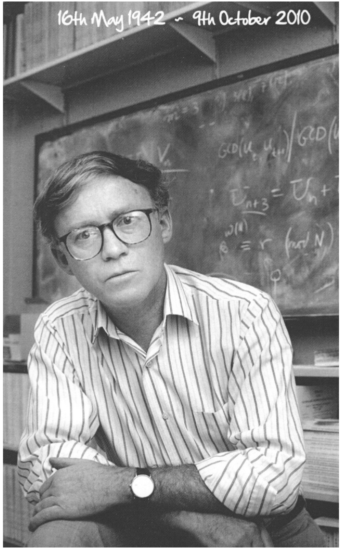
Problem: Given a period, express it as a motivic one. This allows to:

- make predictions about transcendence, via the *period conjecture*;
- place our period in various filtrations (*e.g.* study its *weight*).

Today: We are going to see this for the *Mahler measure* of a polynomial.

$\underline{z}_n = (z_1, \dots, z_n)$, $\underline{z}_n^{\underline{\nu}} = z_1^{\nu_1} \cdots z_n^{\nu_n}$

A huge problem about small Mahler measures



Kurt Mahler

Mahler (1962): For $P \in \mathbb{C}[\underline{z}_n^{\pm 1}] \setminus \{0\}$, let $m(P) := \int_{\mathbb{T}^n} \log|P| d\mu_n$, where $\mathbb{T}^n := (S^1)^n$ and $\mu_n = \frac{1}{(2\pi i)^n}(\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n})$ is the Haar probability measure.

We have $m(P) = \log(\lim_{p \to 0} \|P\|_{p,\mu_n})$, but also $m(PQ) = m(P) + m(Q)$.

Moreover, $m(P) \asymp \log(\ell(P))$, where $\ell(\sum_{\underline{\nu} \in \mathbb{Z}^n} a_{\underline{\nu}} \cdot \underline{z}_n^{\underline{\nu}}) = \sum_{\underline{\nu} \in \mathbb{Z}^n} |a_{\underline{\nu}}|$ is the *length*.

In particular, $m(P) \geq 0$ if $P \in \mathbb{Z}[\underline{z}_n^{\pm 1}] \setminus \{0\}$.

Lawton (1977): If $P \in \mathbb{Z}[\underline{z}_n^{\pm 1}] \setminus \{0\}$, then $m(P) = 0 \Leftrightarrow P = \underline{z}_n^{\underline{w}} \cdot \prod_{j \geq 1} \Phi_j(\underline{z}_n^{\underline{\nu}_j})^{a_j}$.

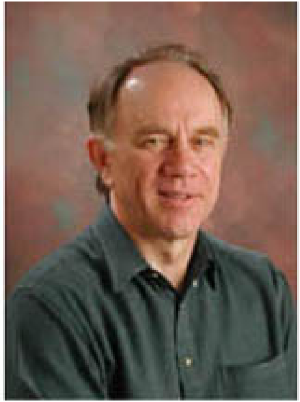
This generalizes Kronecker (1884). Other proofs by Boyd (1981), Smyth (1981).

Pierce (1917) If $P(z_1) = \prod_j (z_1 - \alpha_j) \in \mathbb{Z}[z_1]$, then $\Delta_n(P) := \prod_j \alpha_j^n - 1$ is easier to factor than a random integer. Often, $\Delta_n(P)/\Delta_1(P)$ is prime, if n is prime.

Lehmer (1933) We have $\Delta_{n+1}(P)/\Delta_n(P) \to \exp(m(P))$. Thus, we want the smallest $m(P) > 0$. Does it exist? If so, is it achieved by:

$$P(z_1) = z_1^{10} + z_1^9 - z_1^7 - z_1^6 - z_1^5 - z_1^4 - z_1^3 + z_1 + 1 \ ?$$

The multivariate aspects of Lehmer's problem



David William Boyd

Let $\mathscr{M}_n := m(\mathbb{Z}[\underline{z}_n^{\pm 1}] \setminus \{0\}) \subseteq \mathbb{R}_{\geq 0}$, and $\mathscr{M}_\infty := \varinjlim_{n \geq 1} \mathscr{M}_n(\mathbb{Z}) \subseteq \mathbb{R}_{\geq 0}$.

Boyd (1981) $m(P) = \lim_{d \to +\infty} m(P(z_1, z_1^d, z_1^{d^2}, \dots, z_1^{d^{\overset{n-1}{n}}}))$, if $P \in \mathbb{C}[\underline{z}_n^{\pm 1}] \setminus \{0\}$. Hence, we have $\mathscr{M}_1 \subseteq \mathscr{M}_\infty \subseteq \overline{\mathscr{M}_1}$, and $\overline{\mathscr{M}_1} = \mathbb{R}_{\geq 0}$ if $\inf(\mathscr{M}_1 \setminus \{0\}) = 0$. Thus, if $\mathscr{M}_\infty$ is closed, then Lehmer's question has a positive answer.

For $P \in \mathbb{C}[\underline{z}_n^{\pm 1}]$ and $A \in \mathbb{Z}^{m \times n}$, let $P_A(\underline{z}_m) := P(z_1^{a_{1,1}} \cdots z_m^{a_{m,1}}, \dots, z_1^{a_{1,n}} \cdots z_m^{a_{m,n}})$.

Smyth (2018) For $P \in \mathbb{C}[\underline{z}_n^{\pm 1}]$, the set $\mathscr{M}(P) := \{m(P_A) : A \in \mathbb{Z}^{* \times n}\}$ is closed. Moreover, $\mathscr{M}_\infty$ is filtered by the sets $\mathscr{M}(Q_d)$, where $Q_d := \sum_{j=1}^d (z_{2j-1} - z_{2j})$.

Brunault, Guilloux, Mehrabdollahei, P. (2021) For $P \in \mathbb{C}[\underline{z}_n^{\pm 1}] \setminus \{0\}$, we have that $m(P) = \lim_{\rho(A) \to +\infty} m(P_A)$, where $\rho(A) := \min\{\|v\|_\infty : v \in \ker(A) \setminus \{0\}\}$. This gives us new limit points inside $\mathscr{M}_\infty$, and generalizes Lawton (1983).

As a special case, we recover an identity of Mehrabdollahei (2020), concerning the limit of $m(P_d)$, where $P_d(z_1, z_2) = \sum_{0 \leq a+b \leq d} z_1^a z_2^b$.

$\log|z_1|\, d\arg(z_2) -$
$- \log|z_2|\, d\arg(z_1)$

Mahler measures and special values of L-functions

Christopher Deninger

Boyd (1998) looked for small numbers inside $\mathscr{M}_2$, and found numerically:

$$m\Big(\overbrace{z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + k}^{P_k = 0}\Big) \overset{?}{\sim}_{\mathbb{Q}^\times} L'(E_k, 0)$$

whenever $k^2 \in \mathbb{Z}$. Today, this has been proven for:

$$k \in \{-4\sqrt{2}, -2\sqrt{2}, 1, 2, 3, 2\sqrt{2}, 3\sqrt{2}, 5, 8, 12, 16, i, 2i, 3i, 4i, \sqrt{2}i\}$$

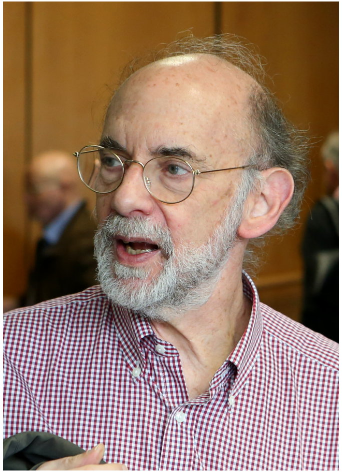
by Rodriguez-Villegas (1999), Rogers & Zudilin (2014), Brunault (2016), etc...

These identities can be related to the conjectures of Beilinson (1984) on special values of L-functions. Indeed, Deninger (1997) proved that:

$$m(P) = m(P(\underline{z}_{n-1}, 0)) + \langle r^{\infty}_{V_P}(\{z_1, \dots, z_n\}), [\gamma_P] \otimes (2\pi\sqrt{-1})^{1-n} \rangle_{(V_P, \partial\gamma_P)}$$

where $V_P := \{P = 0\} \hookrightarrow \mathbb{G}_m^n$ and $\gamma_P := V_P(\mathbb{C}) \cap \{|z_1| = \dots |z_{n-1}| = 1, |z_n| \leq 1\}$. Hence, if V_P *is smooth*, P *is tempered*, and $\partial\gamma_P = \emptyset$, Beilinson's conjectures predict $m(P) \leftrightsquigarrow L^*(\underline{H}^{n-1}(\overline{V}_P), 0)$, for a smooth compactification $\overline{V}_P$ of V_P.

Some little steps...



Spencer Janney Bloch

In particular, Bornhorn (1999), following the ideas of Deninger (1997), proves that Boyd's conjecture:

$$m\left(z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + k\right) \overset{?}{\sim}_{\mathbb{Q}^\times} L'(E_k, 0)$$

holds under Beilinson's conjectures. This can be generalized to the family:

$$P(z_1, z_2) = z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + \frac{z_1}{z_2} + \frac{z_2}{z_1} + k$$

which was treated in Theorem 4.4.3 of P. (2020).

One can also use a weak form of Beilinson's conjectures for CM elliptic curves $E_{/\mathbb{Q}}$, proved by Bloch (1978) (see also Rohrlich (1987)), to show that:

$$m(P) = rL'(E, 0) + \log|s|$$

for some $P \in \mathbb{Z}[z_1, z_2]$, and two numbers $r \in \mathbb{Q}$ and $s \in \overline{\mathbb{Q}}^\times$.
This was done in Theorem 9.2.4 of P. (2020).

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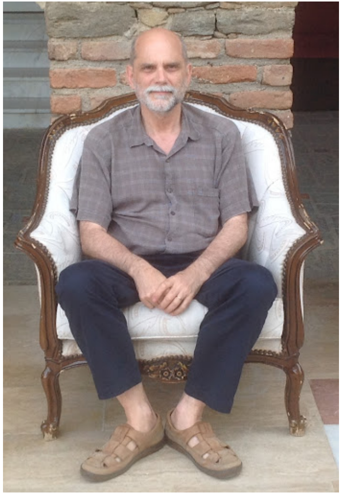
A gallery of explicit relations



Fernando Rodriguez-Villegas

Some identities between Mahler measures and zero-dimensional L-functions:

$\mathbb{Q}(\sqrt{-3})$ $j=2$ $\Delta=3$

Smyth (1981) $m(L_2) = L'(\chi_{-3}, -1)$ and $m(L_3) = -14 \cdot \zeta'(-2)$, with $L_n = \sum_{j=0}^{n} z_j + 1$

Ray (1987), Boyd & Rodriguez-Villegas (2002) Many more $L'(\chi_\Delta, -1)$ for $\Delta < 0$.

Lalín (2006) $m(z_0 S_{2k}^- + S_{2k}^+) \in \langle \zeta'(-2), \ldots, \zeta'(-2k) \rangle_{\mathbb{Q}}$, for $S_m^{\pm} = \prod_{j=1}^{m} (1 \pm z_j)$.

D'Andrea & Lalín (2007) $m((1-z_1)(1-z_2)-(1-z_3)(1-z_4)) = -18 \cdot \zeta'(-2)$.

What about higher-dimensional L-functions?

Rodriguez-Villegas (2004) $m(L_4) \stackrel{?}{=} -L'(f,-1)$ and $m(L_5) \stackrel{?}{=} -8 \cdot L'(g,-1)$, for two modular forms $f \in S_3(15)$ and $g \in S_4(6)$. How to go on?

Note that $m(L_n)$ is related to the probability density of a random walk with n-steps, as studied by Borwein & Straub & Wan & Zudilin (2012).

Finally, some elliptic curves may appear, such as:

$$m(z_1 - (1-z_2)(1-z_3)) \stackrel{?}{=} -2 \cdot L'(X_1(15), -1)$$

studied by Boyd & Rodriguez-Villegas (2004) and Lalín (2013).

Looking for answers: the notion of exactness

Matilde Noemí Lalín

In all the previous examples, either $\partial\gamma_P \neq \emptyset$ or V_P is not smooth.

Maillot (2004) We should look at $W_P := V_P \cap V_{P^*}$, where $P^*(z_n) := \overline{P(\overline{z}_n^{-1})}$.

Why this? Suppose P is *exact*, *i.e.* $r^\infty_{V_P}(\{z_1, \dots, z_n\}) = 0$ inside $H^{n-1}_{\mathrm{dR}}(V_P)$. Then, using Stokes, we can write $m(P) = \int_{\partial\gamma_P} \omega$ for some $\omega \in \Omega^{n-2}(W_P)$.

This already explains Smyth's identity $m(L_2) = L'(\chi_{-3}, -1)$, because $W_{L_2} = \{(\zeta_3, -\zeta_3 - 1), (-\zeta_3, \zeta_3 - 1)\}$. What about $m(L_3) = -14 \cdot \zeta'(-2)$?

Lalín (2007) Some polynomials are *successively exact*, so we can apply Stokes's theorem multiple times. Since $\partial \circ \partial = 0$, this can't be done directly.

However, if W_P is singular, the pullback of ω to the desingularization $\widetilde{W}_P$ of W_P may become exact, while $\partial\gamma_P$ might acquire a boundary!

For instance, if $P \in \mathbb{C}[z_3^{\pm 1}]$, one can take $\widetilde{W}_P = \{\mathrm{Res}_{z_3}(P, P^*) = 0\}$.

The plan

The first step: symmetrizing the Deninger cycle

Jean Gaston Darboux

Brunault & P. (2021): If $P \in \mathbb{C}[\underline{z}_n^{\pm 1}] \setminus \{0\}$ and $V_0 \colon \{P \cdot P^* = 0\} \hookrightarrow \mathbb{G}_m^n$, then:

$H^{n-1}_{dR}(V_0)$
(1)

$$m(P) - m(P(\underline{z}_{n-1}, 0)) = \langle \eta_0, \gamma_0 \rangle_{V_0}$$

where $\eta_0 = r^{\infty}_{V_0}(\{z_1, \dots, z_n\})$ and $\gamma_0 \in H^{\mathrm{B}}_{n-1}(V_0)$ is a symmetrized version of γ_P.

Thus, looking at the Mayer-Vietoris long exact sequence:

$$\dots \to H^{n-2}_{\mathrm{dR}}(W_P) \xrightarrow{\delta} H^{n-1}_{\mathrm{dR}}(V_0) \to H^{n-1}_{\mathrm{dR}}(V_P) \oplus H^{n-1}_{\mathrm{dR}}(V_{P^*}) \to \dots$$

we get a class $\eta_1 \in H^{n-2}_{\mathrm{dR}}(W_P)$ if $\eta_0|_{V_P} = 0$. Hence, we get:

$$m(P) - m(P(\underline{z}_{n-1}, 0)) = \langle \eta_1, \gamma_1 \rangle_{V_1} \; V_P$$

where $\gamma_1 = \partial(\gamma_0)$ is obtained by looking at the adjoint Mayer-Vietoris long exact sequence in homology. Thus, say that *P is exact* if $\eta_0|_{V_P} = 0$, as before.

Historical note: Maillot points out that the relation between the involution $\underline{z}_n \mapsto \underline{z}_n^{-1}$ and the intersection $V_P(\mathbb{C}) \cap \mathbb{T}^n$ might go back to Darboux (1875).

$W_j := V_j^{\text{sing}}$

Heisuke Hironaka

How to go on: successive desingularization

There exist exact reciprocal polynomials, such as $P = z_1 + z_1^{-1} + z_2 + z_2^{-1} + 4$. In this case, $W_P = V_P$ is not the good variety. How to deal with them?

Brunault & P. (2021): Generically, $W_P = V_0^{\text{sing}}$. So, one can look at:

$$\begin{tikzcd}
\cdots \arrow[r] & \widetilde{V}_{k+1} \arrow[r] \arrow[d, two heads] & \widetilde{V}_k \arrow[r] \arrow[d, two heads] & \cdots \arrow[r] & \widetilde{V}_2 \arrow[r] \arrow[d, two heads] & \widetilde{V}_1 \arrow[r] \arrow[d, two heads] & \widetilde{V}_0 \arrow[r] \arrow[d, two heads] & \mathbb{G}_m^n \arrow[d, two heads, dotted] \\
\cdots \arrow[r] & V_{k+1} \arrow[r] \arrow[d, two heads] \arrow[ur, hook] & V_k \arrow[r] \arrow[d, two heads] & \cdots \arrow[r] & V_2 \arrow[r] \arrow[d, two heads] \arrow[ur, hook] & V_1 \arrow[r] \arrow[d, two heads] \arrow[ur, hook] & V_0 \arrow[r, dotted] \arrow[d, two heads, dotted] \arrow[ur, hook] & ?? \\
\cdots \arrow[r] & W_{k+1} \arrow[r] \arrow[ur, hook] & W_k \arrow[r] & \cdots \arrow[r] & W_2 \arrow[r] \arrow[ur, hook] & W_1 \arrow[r, dotted] \arrow[ur, hook] & ?? \arrow[ur, hook, dotted] &
\end{tikzcd}$$

where each $\widetilde{V}_k$ is smooth. Here, a polynomial is *exact* if $\eta_0|_{\widetilde{V}_0} = 0$.

By induction, P is *k-exact* if it is $(k-1)$-exact and $\eta_{k-1}|_{\widetilde{V}_{k-1}} = 0$. We get:

$$m(P) - m(P(\underline{z}_{n-1}, 0)) = \langle \eta_k, \gamma_k \rangle_{V_k}$$

for $\eta_k \in H^{n-1-k}_{\mathrm{dR}}(V_k)$ and $\gamma_k = \partial(\gamma_{k-1}) \in H^{\mathrm{B}}_{n-1-k}(V_k)$.

$H^{n-k}_{P}(D^{(k)}, \mathbb{R}(n))$

$P(z_1) = z_1 - \lambda$

A more canonical approach: the exactness filtration

Let $X := \widetilde{V}_0$ (smooth), and $D := V_1 \hookrightarrow X$ (simple normal crossings divisor). Write $D = D_1 \cup \cdots \cup D_r$ and $D^{(i)} = \bigsqcup_{|I|=i} D_I$, where $D_I = \bigcap_{i \in I} D_i$ (smooth).

Take $\widetilde{\eta} \in H^{n-1}_{\mathrm{dR}}(X, D)$ lifting $\eta_0 \in H^{n-1}_{\mathrm{dR}}(V_0)$ through the diagram:

V_0^{sing}

$$\begin{tikzcd}
\cdots \arrow[r] & H^{n-2}_{\mathrm{dR}}(W_P) \arrow[r] \arrow[d] & H^{n-1}_{\mathrm{dR}}(V_0, W_P) \arrow[r] \arrow[d, "\wr"'] & H^{n-1}_{\mathrm{dR}}(V_0) \arrow[r] \arrow[d] & H^{n-1}_{\mathrm{dR}}(W_P) = 0 \\
\cdots \arrow[r] & H^{n-2}_{\mathrm{dR}}(D) \arrow[r] & H^{n-1}_{\mathrm{dR}}(X, D) \arrow[r] & H^{n-1}_{\mathrm{dR}}(X) \arrow[r] & H^{n-1}_{\mathrm{dR}}(D) = 0
\end{tikzcd}$$

and consider the spectral sequence $H^q(D^{(p)}) \Rightarrow H^{p+q}(X, D)$, inducing $\mathrm{Fil}^{\bullet}_{\mathrm{rel}}$.

Brunault & P. (2021): Say that P is k-exact if $\widetilde{\eta} \in \mathrm{Fil}^{k}_{\mathrm{rel}}(H^{n-1}_{\mathrm{dR}}(X, D))$.
Let $\widetilde{\gamma} \in H^{\mathrm{B}}_{n-1}(X, D)$. If $\widetilde{\eta} \notin \mathrm{Fil}^{k+1}_{\mathrm{rel}}$ and $\widetilde{\gamma} \in \mathrm{Fil}^{\mathrm{rel}}_{k} \setminus \mathrm{Fil}^{\mathrm{rel}}_{k-1}$, we have:

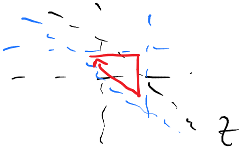
$$m(P) - m(P(\underline{z}_{n-1}, 0)) = \langle \mathrm{gr}^{k}_{\mathrm{rel}}(\widetilde{\eta}), \mathrm{gr}^{\mathrm{rel}}_{k}(\widetilde{\gamma}) \rangle_{(X,D)} = \langle \widetilde{\eta}_k, \widetilde{\gamma}_k \rangle_{D^{(k)}}$$

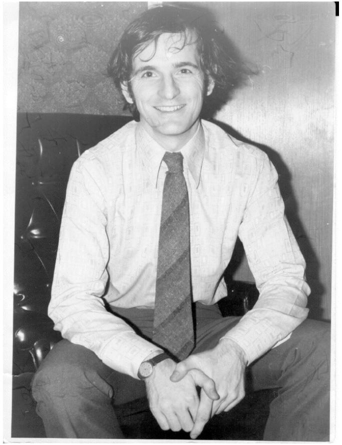
which computes $m(P)$ as an absolute period on the smooth variety $D^{(k)}$.

Pierre René Deligne

$L(H^{n-1-k}(\overline{D}^{(k)}), n)$

An example: the three-variable linear polynomial





Christopher Smyth

Let $P = L_3 = z_1 + z_2 + z_3 + 1$. Recall that $\boxed{m(P) = -14\zeta'(-2)}$ by Smyth (1981).
Let $Z = \{z_1 = 0\} \cup \{z_2 = 0\} \cup \{z_1 + z_2 = 1\}$ and $S = \{z_1 = 1\} \cup \{z_2 = 1\} \cup \{z_1 + z_2 = 0\}$.

We have $V_P \cong \mathbb{A}^2 \setminus Z$, thus $H^2_{\mathrm{dR}}(V_P) \cong H^1_{\mathrm{dR}}(V_P) \cong \mathbb{R}^3$, and $W_P \cong S \setminus (S \cap Z)$.
Hence, $H^1_{\mathrm{dR}}(W_P) \cong \mathbb{R}^4$ and $\mathrm{Im}(H^1_{\mathrm{dR}}(V_P) \oplus H^1_{\mathrm{dR}}(V_{P^*}) \to H^1_{\mathrm{dR}}(W_P)) \cong \mathbb{R}^3$.

Therefore, $H^1_{\mathrm{dR}}(V_0) \cong \mathbb{R}^3$ and $H^2_{\mathrm{dR}}(V_0) \cong \mathbb{R}^7$. We get a diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^1_{\mathrm{dR}}(V_0) \arrow[r] \arrow[d] & H^1_{\mathrm{dR}}(W_P) \arrow[r, "\delta"] \arrow[d] & H^2_{\mathrm{dR}}(V_0, W_P) \arrow[r] \arrow[d, "\wr"'] & H^2_{\mathrm{dR}}(V_0) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^1_{\mathrm{dR}}(X) \arrow[r] & H^1_{\mathrm{dR}}(D) \arrow[r, "\delta"] & H^2_{\mathrm{dR}}(X, D) \arrow[r] & H^2_{\mathrm{dR}}(X) \arrow[r] & 0
\end{tikzcd}$$

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whose rows are exact. Note that $D = W_P \sqcup W_P$ and $X = V_P \sqcup V_{P^*}$.

Finally, $\widetilde{\eta} \in \mathrm{Fil}^1_{\mathrm{rel}}(H^2(X,D)) = \mathrm{Fil}^2_{\mathrm{rel}}(H^2(X,D)) \cong \mathbb{R}^2$, and $\boxed{D^{(2)} \cong \mathrm{Spec}(\mathbb{Q})^{\sqcup 6}}$.
Thus, we should indeed expect (and we can prove) $\boxed{m(P) \sim_{\mathbb{Q}^\times} \zeta'(--2)}$.

$\log(\alpha), \ \alpha \in \overline{\mathbb{Q}} \cap \mathbb{R}_{>0}$

$\forall \Delta < 0, \ L'(\chi_\Delta, -1) = \sqrt{(*)} \cdot m(P(z_1, z_2))$, $\in \mathbb{Q}^\times$

Further steps and directions

François Brunault

- Resolve the ambiguity $\mathrm{Im}(H^{n-2}_{\mathrm{dR}}(V_0^{\mathrm{sing}}) \to H^{n-1}_{\mathrm{dR}}(X, D))$ for $\widetilde{\eta}$.
- Write $m(P)$ as a period for $(\overline{X} \setminus A, B \setminus (A \cap B))$, with $\overline{X}$ smooth projective. Relate this to $\mathfrak{X}(\Delta_P)$ (toric variety) and to (successive) *temperedness*.
- Compare with the weight filtration on $H^{n-1}(V_0)$.
- Compare with the degeneration $P \cdot P^* = t$ for $t \to 0$. To do so, study $P \cdot P^* - t \in \mathbb{C}((t))[\underline{z}_n^{\pm 1}]$, maybe via *tropical homology*.
- Make γ_P more canonical, following (perhaps) Besser & Deninger (1999).
- Study the families L_n (1-exact if $n \neq 3$) and $z_0 S_n^- + S_n^+$ ($(n-1)$-exact).
- "Compute" $m(L_4)$ and $m(L_5)$ up to $\mathbb{Q}^\times$, and assuming Beilinson's conj.
- Study the co-exactness filtration on $\mathscr{M}_\infty$.
- Write $m(P)$ as a *motivic period*, following Brown (2017). Maybe it's a *single valued period*, as in Brown & Dupont (2021).
- Study this in fibrations, as in Doran & Kerr (2011).
- Compute $\mathrm{trdeg}(\mathbb{Q}(\pi, m(P_1), \ldots, m(P_r))/\mathbb{Q})$, assuming the period conj.

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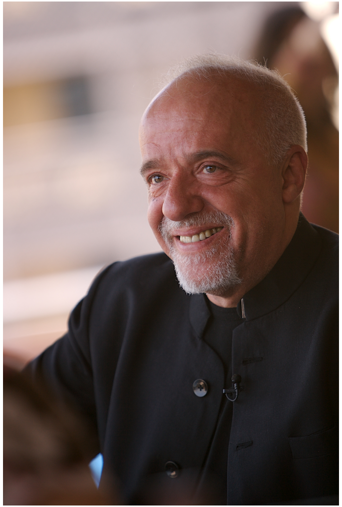
Thank you very much for your attention!



Está preparando seu espírito e sua vontade, porque existe uma grande verdade neste planeta: seja você quem for ou o que faça, quando quer com vontade alguma coisa, é porque esse desejo nasceu na alma do Universo.

Paulo Coelho de Souza, *O Alquimista*

P.S: Did you get curious about Mahler measures? Check out the book:
Many Variations of Mahler Measures: A Lasting Symphony
by Brunault and Zudilin (2020).