# A twistor space for moduli of logarithmic connections of rank 2 on an open curve

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# Introduction

In this talk, we'll look at the construction of twistor families of moduli spaces of local systems, after Hitchin and Deligne.

Motivated by the classical case of local systems on a compact curve, the next step was to look at rank 1 local systems on an open curve.

In that case, a weight 2 property for the local monodromy transformations around the punctures came into view, and this was related to parabolic structures.

# Introduction

We would now like to consider how to move to higher rank local systems on an open curve. The objective of this talk is to look at the new issues that arise, but first we'll review the compact and rank 1 cases.

Our discussion will take place for rank 2 bundles with logarithmic connection and quasi-parabolic structure on the compactified curve.

# The twistor $\mathbb{P}^1$

Let $\mathbb{H} = \mathbb{R}\langle 1, I, J, K\rangle$ be the algebra of quaternions. If

$$\kappa = xI + yJ + zK$$

then $\kappa^2 = -1$ if and only if $x^2 + y^2 + z^2 = 1$, so the space of complex structures in $\mathbb{H}$ is identified with the two-sphere $S^2$. This itself is provided with a complex structure making it into $\mathbb{P}^1$.

The complex structure $I$ corresponds to $0 \in \mathbb{P}^1$ and $J$ corresponds to $1 \in \mathbb{P}^1$.

# The twistor space of a quaternionic vector space

Suppose $V$ is an $\mathbb{H}$-module i.e. quaternionic vector space. Then for each $\kappa \in \mathbb{P}^1$ we obtain a complex structure on $V$, hence a $\mathbb{C}$-vector space $V_\kappa$.

On the product $V \times \mathbb{P}^1$, declaring the complex structure in the second variable to be the same as that of $\mathbb{P}^1$, this yields a global complex structure making it into the total space of a vector bundle $\mathcal{V}$ over $\mathbb{P}^1$.

# Preferred sections and involution

We call the sections of $\mathcal{V}$ of the form $\{v\} \times \mathbb{P}^1$ the *preferred sections* .

The antipodal involution $(x, y, z) \mapsto (-x, -y, -z)$ corresponds to $\sigma : \lambda \mapsto -\overline{\lambda}^{-1}$ (composition of the three natural involutions in coordinates on $\mathbb{P}^1$).

This involution may be viewed as a real structure on $\mathbb{P}^1$ although its set of real points is empty.

# Preferred sections and involution

The vector bundle $\mathcal{V}$ is provided with an involution defined by $\sigma(v, \lambda) = (v, \sigma(\lambda))$, antilinear and covering $\sigma$ on the base. In other words, the bundle has a real structure over the real structure of the base.

The preferred sections are the only holomorphic sections compatible with $\sigma$.

In the case $V = \mathbb{H}$ one calculates $\mathcal{V} \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$. Since any $\mathbb{H}$-module is just a direct sum of these, it follows in general that the bundle $\mathcal{V}$ is going to be semistable of slope 1.

# Weight 1 property and quaternionic structure

**Theorem**

*If* $(\mathcal{V}, \sigma)$ *is a vector bundle with* $\sigma$*-compatible involution on* $\mathbb{P}^1$ *such that* $\mathcal{V}$ *is semistable of slope* $1$*, then it comes from the above construction for the quaternionic vector space*

$$V = \Gamma(\mathbb{P}^1, \mathcal{V})^{\sigma}.$$

*For each* $\kappa \in \mathbb{P}^1$*, the projection* $V \to \mathcal{V}_{\kappa}$ *is an isomorphism of real vector spaces and this induces the complex structure* $\kappa$ *on* $V$*.*

# Twistor space of a quaternionic manifold

Suppose now that we are given an integrable quaternionic manifold $M$. Then the *twistor space* is

$$Tw(M) := M \times \mathbb{P}^1$$

with complex structure obtained in the same way.

The horizontal "preferred sections" $\{m\} \times \mathbb{P}^1$ are given the holomorphic structure of $\mathbb{P}^1$, and for any $\kappa \in \mathbb{P}^1$ the fiber $M \times \{\kappa\}$ is given the complex structure $M_\kappa$ determined by the action of $\kappa \in \mathbb{H}$ on the tangent spaces of $M$.

The "integrability" condition says that these are integrable complex structures, and the general Penrose theory yields an integrable complex structure on the total space $Tw(M)$.

# Hitchin's hyperkähler structure

We now recall that Hitchin defined a quaternionic structure on the moduli space $M$ of local systems on a smooth compact Riemann surface $X$.

The twistor space $Tw(M)$ was considered at length in Hitchin's paper.

He also promoted it to a hyperkähler structure by defining a Riemannian metric and Kähler forms.

# Dolbeault and de Rham

For the complex structure $\kappa = I$ corresponding to $\lambda = 0$ in $\mathbb{P}^1$, the complex moduli space $M_0$ is the moduli space of Hitchin pairs or "Higgs bundles" $(E, \varphi)$ on $X$. We may call this the *Dolbeault moduli space* denoted $M_{\mathrm{Dol}}$ in view of the analogy with Dolbeault cohomology.

For the complex structure $\kappa = J$ corresponding to $\lambda = 1$ in $\mathbb{P}^1$, the complex moduli space $M_1$ is the moduli space of vector bundles with integrable connection $(E, \nabla)$ on $X$. We call this the *de Rham moduli space* denoted $M_{\mathrm{dR}}$ in view of the analogy with de Rham cohomology.

# Trivialization

Furthermore, for all the complex structures $\kappa$ corresponding to $\lambda \neq 0, \infty$ the moduli spaces $M_\kappa$ are naturally isomorphic, so they are all isomorphic to $M_{\mathrm{dR}} = M_1$ the moduli space of vector bundles with integrable connection, which is in turn isomorphic to the moduli space of local systems or representations of the fundamental group.

# Deligne's interpretation

Deligne, via Witten, gave a reinterpretation of this space as follows. Each $M_\lambda$ is the moduli space of *vector bundles with* $\lambda$*-connection* $(E, \nabla)$. For $\lambda \neq 0$ the rescaling $\lambda^{-1}\nabla$ is just a connection, yielding the isomorphisms refered to above, whereas for $\lambda = 0$ a $\lambda$-connection is the same thing as a Higgs field $\varphi$.

# Deligne's interpretation

We may make an algebraic geometry construction of the family of moduli spaces over $\mathbb{A}^1 \subset \mathbb{P}^1$, which for reasons of analogy with the Dolbeault and de Rham terminology, we call $M_{\mathrm{Hod}}$ for Hodge.

This space together with its $\mathbb{C}^*$-action is viewed as the "Hodge filtration" relating de Rham to Dolbeault.

# Deligne's interpretation

Deligne observes that the isomorphisms between different nonzero $\lambda \in \mathbb{A}^1 - \{0\} = \mathbb{G}_m$ fit together to give an analytic trivialization

$$M_{\mathrm{Hod}}|_{\mathbb{G}_m} \cong M_B \times \mathbb{G}_m$$

where $M_B$ (for "Betti") is the moduli space of representations of the fundamental group.

# Deligne glueing

Then, the condition of existence of an antipodal involution covering $\sigma$ motivated Deligne to define a *glueing* between $M_{\mathrm{Hod}}(X)$ and $M_{\mathrm{Hod}}(\overline{X})$ using the isomorphism

$$\pi_1(X) \cong \pi_1(\overline{X}) \quad \text{whence} \quad M_B(X) \cong M_B(\overline{X})$$

and applying the involution $\lambda \mapsto -\lambda^{-1}$ on $\mathbb{G}_m$.

# Deligne glueing

Glueing the two pieces together yields a space

$$M_{\mathrm{Hod}}(X) \cup M_{\mathrm{Hod}}(\overline{X}) \quad =: \quad M_{\mathrm{DH}} \to \mathbb{P}^1$$

and one can define an antipodal involution using that $\overline{X}$ is the complex conjugate variety to $X$ and the fact that the moduli space $M_{\mathrm{Hod}}$ is a canonical algebraic geometry construction so it supports a complex conjugation operation.

(I won't try to give the formula here as that would probably just induce a sign error.)

# Isomorphism

## Theorem

*The Deligne-Hitchin moduli space constructed by Deligne's glueing is isomorphic to the twistor space for Hitchin's quaternionic structure:*

$$\begin{tikzcd} M_{\mathrm{DH}} \arrow[dr] & \cong & Tw(M) \arrow[dl] \\ & \mathbb{P}^1 & \end{tikzcd}$$

# Preferred sections from harmonic bundles

The preferred sections of the twistor space correspond to sections of the fibration $M_{\mathrm{DH}} \to \mathbb{P}^1$ that we also call "preferred sections". These are maps $\mathbb{P}^1 \to M_{\mathrm{DH}}$ that are obtained whenever we have a *harmonic bundle*

$$(E, \partial, \overline{\partial}, \varphi, \varphi^{\dagger})$$

corresponding to a solution of Hitchin's equations.

# Preferred sections from harmonic bundles

For $\lambda \in \mathbb{A}^1$ the point in the moduli space of holomorphic vector bundles with $\lambda$-connections is

$$\left(\mathcal{E}_\lambda := (E, \overline{\partial} + \lambda\varphi^\dagger),\ \ \nabla_\lambda := \lambda\partial + \varphi\right).$$

If one gets the formulas right, then these preferred sections are compatible with the antipodal involution.

# Weight 1 property

The fact that this construction gives the twistor space of a quaternionic manifold comes from the following property:

## Proposition

*Suppose $\rho : \mathbb{P}^1 \to M_{\mathrm{DH}}$ is a preferred section defined as coming from a harmonic bundle in the above way. Let*

$$\mathcal{V} = \rho^* T(M_{\mathrm{DH}}/\mathbb{P}^1)$$

*be the pullback of the relative tangent bundle, or equivalently the normal bundle of $M_{\mathrm{DH}}$ to the section. Then $\mathcal{V}$ is a semistable vector bundle of slope 1 over $\mathbb{P}^1$.*

# Weight 1 property

An analogy with Hodge structures motivates us to call the property of being semistable of slope 1, a property of *weight* 1.

Thus, the fact that we have a quaternionic structure on the moduli space comes from a weght 1 property for the tangent bundle to the preferred section.

We recall that the tangent bundle to a moduli space is calculated as an $H^1$ of an appropriate complex, so we are saying here that the $H^1$ has weight 1. This may be viewed as a purity statement for nonabelian Hodge structures.

# Weight 1 property

Fundamentally, the calculation going into the proof uses the observation that the tangent space of the moduli space is an $H^1$, calculated by some kind of harmonic forms. Then, the fact that they are 1-forms means that the transition functions needed to pass from the $\mathbb{A}^1$ neighborhood of $\lambda = 0$ to the $\mathbb{A}^1$ neighborhood of $\lambda = \infty$ involve $\lambda^{-1}$ leading to the semistable of slope 1 property.

This weight 1 property is the nonabelian cohomology analogue of the statement in usual Hodge theory that $H^1_{\mathrm{dR}}(X)$ has a weight 1 Hodge structure, and similarly for a variety over $\mathbb{F}_q$ that the étale cohomology $H^1_{\mathrm{et}}(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell)$ has weight 1 in the sense that the eigenvalues of Frobenius have norm $q^{1/2}$.

# Weight 2

We may naturally suspect that for certain cohomological aspects, this weight 1 property would become a weight 2 property.

Recall from arithmetic geometry that the inertia group has the form of a Tate twist, so it has weight $-2$ and the space of representations of the inertia group should be thought of as having weight 2.

# Weight 2

One finds that this is indeed the case, upon examining the case of moduli spaces of local systems of rank 1 over an open curve. This was done in the paper

"A weight two phenomenon for the moduli of rank one local systems on open varieties" in *From Hodge theory to integrability and TQFT tt*-geometry*, Proc. Sympos. Pure Math (2008).

# The weight 2 property in the case of rank 1

Let's look at some of the details in this rank 1 case.

For simplicity let's consider $X = \mathbb{G}_m$ where the only data of a local system is a single local monodromy at a puncture.

A logarithmic $\lambda$-connection will be over a trivial bundle on $X$, having the form

$$\nabla(\lambda, a) = \lambda d + a\frac{dz}{z}.$$

# The case of rank 1

We note that there is an action of change of the trivialization making $(\lambda, a)$ equivalent to $(\lambda, a + k\lambda)$ for any $k \in \mathbb{Z}$. The singularity of this action at $\lambda = 0$ is one of the difficulties of the open curve situation.

Let $\mathscr{G} \cong \mathbb{Z}$ be this "gauge group" acting.

# The case of rank 1

For the moduli space we may write

$$M_{\mathrm{Hod}} := \mathbb{A}^1 \times \mathbb{C}/\mathscr{G}$$

using $\mathbb{A}^1$ for the $\lambda$ variable and $\mathbb{C}$ for the coefficient $a$.

Note that the group acts discretely over $\lambda \neq 0$ but the stabilizer group of the fiber over $\lambda = 0$ is the full $\mathscr{G} = \mathbb{Z}$.

It is therefore not completely clear what kind of structure best to accord to the quotient. More on this aspect later.

# Riemann-Hilbert and period integrals

The Riemann-Hilbert corresponce over $\lambda \neq 0$ is the exponential

$$\mathbb{G}_m \times \mathbb{C}/\mathscr{G} \xrightarrow{\cong} \mathbb{G}_m \times \mathbb{C}^*$$

sending $(\lambda, a)$ to $(\lambda, \exp(2\pi i a/\lambda))$. This exponential of the period integral

$$2\pi i a/\lambda = \oint \lambda^{-1} a \frac{dz}{z}$$

is the monodromy of the connection $\lambda^{-1}\nabla(\lambda, a)$ around the loop generating $\pi_1(X)$.

# Deligne glueing in this case

We would like to glue $M_{\mathrm{Hod}}$ to the other piece in the Deligne glueing. For that, let $\mu$ denote the coordinate of the other chart $\mathbb{A}^1 \subset \mathbb{P}^1$ with $\mu = -\lambda^{-1}$. A point

$$(\mu, b) \in M_{\mathrm{Hod}}(\overline{X})$$

has monodromy transformation $\exp(2\pi ib/\mu)$ along the generating loop for $\pi_1(\overline{X})$.

# Deligne glueing in this case

The topological isomorphism $X^{\mathrm{top}} \cong \overline{X}^{\mathrm{top}}$ will take the generator to minus the generator, so the Deligne glueing should associate $(\lambda, a)$ with $(\mu, b)$ (up to the $\mathscr{G}$ action) when

$$\exp(2\pi i a/\lambda) = \exp(-2\pi i b/\mu).$$

# Deligne glueing in this case: getting weight 2

Lift over the action of the gauge group to say that we would like

$$a/\lambda = -b/\mu.$$

Recalling that $\mu = -\lambda^{-1}$, this associates $(\lambda, a)$ with $(-\lambda^{-1}, b)$ when

$$a/\lambda = -b/(-\lambda^{-1}) = \lambda b \quad \Leftrightarrow \quad a = \lambda^2 b.$$

This is the glueing condition for the line bundle $\mathcal{O}_{\mathbb{P}^1}(2)$ over $\mathbb{P}^1$.

# The weight 2 twistor space

From the above discussion we get

$$M_{\mathrm{DH}} = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(2))/\mathscr{G}.$$

There is also a natural antipodal involution, and the *preferred sections* are the sections that are compatible with $\sigma$. These are going to lift over the action of the gauge group to $\sigma$-equivariant sections of $\mathcal{O}_{\mathbb{P}^1}(2)$, so in what follows we'll sometimes ignore the $\mathscr{G}$-action.

# Antipode-invariant sections

Recall from the compact case that we wanted to look at the space of $\sigma$-equivariant sections of $M_{\mathrm{DH}}/\mathbb{P}^1$. Here let's lift and look at the space of $\sigma$-equivariant sections of $\mathcal{O}_{\mathbb{P}^1}(2)$. Recall that before asking for $\sigma$-equivariance we have

$$\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) \cong \mathbb{C}^3.$$

# Complex structures

## Lemma

$$\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))^{\mathrm{sigma}} \cong \mathbb{R}^3.$$

*For any $\kappa \in \mathbb{P}^1$ the restriction morphism from these sections to the fiber $\mathbb{C}_\kappa = \mathcal{O}_{\mathbb{P}^1}(2)_\kappa$ is a surjection*

$$\mathbb{R}^3 \to \mathbb{C}_\kappa.$$

*There is a natural splitting as $\mathbb{R}^3 \cong \mathbb{R} \times \mathbb{C}_\kappa$ such that the generator of the gauge group has the form $(1, \lambda)$.*

# Parabolic weight parameter

The fascinating thing that happens here is that the extra real parameter, kernel of the restriction map, may be viewed as the *parabolic weight parameter*.

If $(E, \partial, \overline{\partial}, \varphi, \varphi^{\dagger}, h)$ is a tame harmonic bundle over $X$ then it yields a $\sigma$-invariant section, and for $\lambda \in \mathbb{A}^1$ the corresponding point in $\mathbb{R} \times \mathbb{C}$ is $(p, e)$ where $p$ is the parabolic weight and $e$ the eigenvalue of the residue of the $\lambda$-connection.

The parabolic weight expresses the growth rate of the harmonic metric $h$ near a puncture.

# Parabolic weight parameter

The expression of $\mathbb{R} \times \mathbb{C}$, depending on $\lambda$, as corresponding to a unique $\mathbb{R}^3$ independent of $\lambda$, allows to recover the formulas (see below) for the relationship between parabolic weights and eigenvalues of the residue between Higgs bundles and flat bundles.

The fact that we have an extra real parameter for the parabolic weight, may therefore be seen as a manifestation of the fact that the monodromy transformations around punctures lie in a space whose Hodge weight is 2.

# Formulas

Sabbah and Mochizuki gave formulas for the variation of parabolic weight $\mathfrak{p}$ and eigenvalue $\mathfrak{e}$ as a function of $\lambda$, generalizing my formulas for $\lambda = 1$. Starting with $(a, \alpha) \in \mathbb{R} \times \mathbb{C}$ at $\lambda = 0$, we have:

$$\mathfrak{p}(\lambda, (a, \alpha)) = a + 2\mathrm{Re}(\lambda\overline{\alpha})$$

$$\mathfrak{e}(\lambda, (a, \alpha)) = \alpha - a\lambda - \overline{\alpha}\lambda^2.$$

These enter into Mochizuki's discussion of KMS-structure and what he calls "difficulty (b)".

These formulas are also a consequence of the weight 2 twistor space interpretation.

# Analogy with MHS and $\mathbb{Q}_\ell$ cohomology

For comparison let us recall for

$$X = \mathbb{G}_m = \mathbb{P}^1 - \{0, \infty\}$$

that the mixed Hodge structure on $H^1(X)$ is one-dimensional, of weight 2, and in the arithmetic setting

$$H^1_{\mathrm{et}}(X, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(1)$$

is a Tate twist having weight two.

# Going towards the higher-rank case: challenges

We would like to extend this picture to higher rank local systems on an open curve. Let's look at some potential difficulties in light of the previous discussion.

A first observation is that the action of the gauge group becomes singular over $\lambda = 0$, indeed the entire $\mathscr{G} = \mathbb{Z}$ stabilizes the full fiber over $\lambda = 0$.

For this reason, we'll tend not to really look at the quotient space, but to retain just the action groupoid instead. One possible solution here would be to invoke the notion of *diffeological space*.

# Going towards the higher-rank case: challenges

A next observation is that we have sidestepped any discussion of stability. From the compact case recall that the construction of $M_{\mathrm{Dol}}$ requires the notion of stability of a Higgs bundle, so this information is needed in the fiber over $\lambda = 0$ for the construction of $M_{\mathrm{Hod}}$.

However, in the quasiprojective case, defining stability requires knowing the parabolic weights, but we are trying to recover the parabolic weights from the twistor space construction itself as happened in rank 1.

Without a notion of stability, we are going to be getting moduli spaces that are not of finite type but only locally of finite type. This should be accepted.

# Going towards the higher-rank case: challenges

A third difficulty for making the construction is that, from the formulas for the variation of eigenvalue of the residue as a function of $\lambda$, a preferred section is always going to have some points where the eigenvalues are resonant, so we are not able just to impose a non-resonance condition on the residues.

We do however impose some conditions on the fiber over 0 so as to improve somewhat the moduli problem. This makes it so that the discussion will work for most but not all "preferred sections".

# Framed quasi-parabolic logarithmic $\lambda$-connections

Let $Y$ be a compact Riemann surface and $D \subset Y$ a reduced divisor, for today we'll just suppose $D = \{y\}$. Set $X := Y - D$. Choose a base-point $x \in X$.

We look at bundles of rank 2.

# Framed quasi-parabolic logarithmic $\lambda$-connections

A framed quasi-parabolic bundle with logarithmic $\lambda$-connection is

$$(\lambda, E, \nabla, F, \beta)$$

where $\lambda \in \mathbb{A}^1$, $E$ is a rank 2 vector bundle on $Y$, $\nabla$ is a logarithmic $\lambda$-connection i.e.

$$\nabla : E \to E \otimes \Omega^1_Y(\log D), \quad \nabla(ae) = a\nabla(e) + \lambda d(a)e$$

$F = F_y \subset E_y$ is a one-dimensional subspace preserved by $\mathrm{res}_y(\nabla)$, and $\beta : E_x \cong \mathbb{C}^2$ is a framing over the base-point.

# Hypotheses

Those are the objects that our parameter space is going to parametrize. We make the following hypotheses $(*)$:

- The framed object is rigid: there are no automorphisms of $(E, \nabla)$ that preserve the framing $\beta$; and
- If $\lambda = 0$, the spectral curve of $\varphi = \nabla$ (which is a Higgs field in this case) is irreducible.

# Moduli space

**Theorem**

*There exists a smooth algebraic space, locally of finite type, parametrizing the tuples* $(\lambda, E, \nabla, F, \beta)$ *satisfying Hypothesis* $(*)$*,*

$$\widetilde{M}_{\mathrm{Hod}}(X) \xrightarrow{\lambda} \mathbb{A}^1.$$

# Gauge groupoid

Now, instead of a gauge group we'll define a *gauge groupoid* $\mathscr{G}_{\mathrm{Hod}}$ acting on this moduli space. This groupoid consists of partially defined morphisms from the moduli space to itself, generated by the following operations on a tuple $(\lambda, E, \nabla, F, \beta)$

- $(H)$: this is the Hecke operation going to the bundle $\ker(E \to E_y/F_y)$;
- $(T)$: this is tensoring yielding $E \otimes \mathcal{O}_Y(y)$;
- $(P)$: this operation is partially defined on the open set of points where the eigenvalues of $\mathrm{res}_y(\nabla)$ are distinct, yielding $(\lambda, E, \nabla, F^{\perp}, \beta)$ where $F_y^{\perp}$ is the eigenspace different from $F_y$.

# Gauge groupoid: going to the quotient

## Proposition

*The gauge groupoid is étale, that is to say the map*

$$\mathscr{G}_{\mathrm{Hod}} \to \widetilde{M}_{\mathrm{Hod}}(X) \times \widetilde{M}_{\mathrm{Hod}}(X)$$

*when composed with either of the projections is étale. This map is however not proper so the "quotient space", that would be denoted $M_{\mathrm{Hod}}(X)$, would be non-separated.*

The quotient could be viewed as an analytic stack, though not necessarily having very good finiteness properties, or perhaps a complex diffeological space. Our approach will be rather to just consider the groupoid as a whole.

# Morphisms to the groupoid

Suppose $Z$ is a complex analytic space. Then a morphism $f : Z \to \left(\widetilde{M}_{\mathrm{Hod}}(X), \mathscr{G}_{\mathrm{Hod}}\right)$ is by definition the data of an open covering $Z = \bigcup Z_i$ and morphisms $f_i : Z_i \to \widetilde{M}_{\mathrm{Hod}}(X)$ and $g_{ij} : Z_i \cap Z_j \to \mathscr{G}_{\mathrm{Hod}}$ such that $f_i|_{Z_i \cap Z_j}$ and $f_i|_{Z_i \cap Z_j}$ are respectively the first and second projections following $g_{ij}$, and such that the $g_{ij}$ satisfy the cocycle condition with respect to the composition of the groupoid $\mathscr{G}_{\mathrm{Hod}}$.

If $(Z, G_Z)$ is itself a groupoid one can similarly define the notion of a morphism $(Z, G_Z) \to \left(\widetilde{M}_{\mathrm{Hod}}(X), \mathscr{G}_{\mathrm{Hod}}\right)$.

# Betti version

Let us define the "Betti" version of this moduli space. Let $\widetilde{M}_B(X)$ denote the moduli space of tuples $(L, F, \beta)$ where $L$ is a rank 2 local system on $X$, $F = \{F_y\}$ is a sub-local system of the restriction of $L$ to a punctured disk around $y$, and $\beta : L_x \cong \mathbb{C}^2$ is a framing.

Let $\widetilde{M}_B(X)$ be the moduli space for such framed quasi-parabolic local systems.

# Betti version of the gauge groupoid

Define the *Betti gauge groupoid* $\mathscr{G}_B$ acting on $\widetilde{M}_B(X)$ to be the groupoid consisting of partially defined morphisms from $\widetilde{M}_B(X)$ to itself, generated by the operation:

- $(P)$: defined on the open set where the eigenvalues of the local monodromy transformation are distinct, sending $(L, F, \beta)$ to $(L, F^{\perp}, \beta)$ where $F^{\perp}_y$ is the eigenspace of the local monodromy that is different from $F_y$.

# Betti quotient space

We again get an etale groupoid. Here the finiteness conditions are better so the quotient $M_B(X) = \widetilde{M}_B(X)/\mathscr{G}_B$ is a non-separated algebraic stack.

We note that because part of the gauge groupoid comes from a partially-defined operation $(P)$, the quotient stack has singular points that exhibit the "bug-eye" property that I first learned from a talk by János Kollár a long time ago.

J. Kollár. Cone theorems and bug-eyed covers. *J. Alg. Geom* (1992)

# Riemann-Hilbert

**Theorem (Riemann-Hilbert correspondence)**

*We have an equivalence of analytic groupoids*

$$\left(\widetilde{M}_{\mathrm{Hod}}(X),\, \mathscr{G}_{\mathrm{Hod}}\right) \times_{\mathbb{P}^1} \mathbb{G}_m \;\cong\; (\widetilde{M}_B(X),\, \mathscr{G}_B) \times \mathbb{G}_m.$$

# Preparation for Deligne glueing

The isomorphism of topological spaces between $X^{\mathrm{top}}$ and $\overline{X}^{\mathrm{top}}$ gives an equivalence

$$(\widetilde{M}_B(X), \mathscr{G}_B) \;\cong\; (\widetilde{M}_B(\overline{X}), \mathscr{G}_B).$$

# Deligne glueing

Using the Riemann-Hilbert correspondence we can then make a Deligne glueing. In terms of groupoids this can be viewed as follows: let

$$\widetilde{M}_{\mathrm{DH}} := \widetilde{M}_{\mathrm{Hod}}(X) \sqcup \widetilde{M}_{\mathrm{Hod}}(\overline{X})$$

with gauge groupoid $\mathscr{G}_{\mathrm{DH}}$ combining the $\mathscr{G}_{\mathrm{Hod}}$ on both pieces, together with pieces identifying points that correspond to elements of the Betti moduli space that are identified under the previous equivalence:

$$M_{\mathrm{DH}} = \left(\widetilde{M}_{\mathrm{DH}}, \mathscr{G}_{\mathrm{DH}}\right).$$

# Preferred section from a harmonic bundle

Suppose given a tame harmonic bundle $(E_X, \partial, \overline{\partial}, \varphi, \varphi^\dagger, h)$ on $X$, satisfying our condition that the spectral curve of the Higgs field is irreducible, and choose a framing $\beta : E_x \cong \mathbb{C}^2$.

For each $\lambda \in \mathbb{A}^1$, there is a well-defined extension of $E_X$ to a parabolic bundle $E^\lambda$ over $Y$ (now including parabolic weights) together with a logarithmic $\lambda$-connection $\nabla_\lambda$.

We recall that the parabolic structure reflects the growth properties of the harmonic metric.

# Preferred section from a harmonic bundle

At our puncture $y$ and for each $\lambda$, we get a pair of elements $(p_1(\lambda), e_1(\lambda))$ and $(p_2(\lambda), e_2(\lambda))$ in $\mathbb{R} \times \mathbb{C}$.

These all correspond to the same pair of elements of $\mathbb{R}^3 = \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))^\sigma$, modulo the action of the local gauge group $\mathbb{Z}$.

We assume that the two elements are distinct modulo the gauge group—this is our non-resonance assumption.

# Preferred section from a harmonic bundle

At general points $\lambda$, the parabolic weights will then be distinct, so the parabolic structure gives a well-defined quasi-parabolic structure, i.e. a rank 1 subspace $F_y \subset E_y$. This yields a point of $\widetilde{M}_{\mathrm{Hod}}$ well-defined up to $\mathscr{G}_{\mathrm{Hod}}$.

Notice that to get a bundle we need to pick one of the open intervals between parabolic weights; choosing a different interval is going to result in a change using the operations $(H)$ and $(T)$.

There may, on the other hand, be some points at which the parabolic weights coincide in $\mathbb{R}/\mathbb{Z}$. Here, the parabolic structure no longer defines a quasi-parabolic structure, indeed the dimension jumps are by 2 at the unique parabolic weight in each interval $(k, k+1]$.

Our non-resonance assumption however implies, at these points, that the eigenvalues of $\mathrm{res}_y(\nabla)$ are different. Therefore, we may just choose one of the two eigenspaces to be $F_y$. Making the opposite choice yields a point that differs by the operation $(P)$.

Thus, we also get a point of $\widetilde{M}_{\mathrm{Hod}}$ well-defined up to $\mathscr{G}_{\mathrm{Hod}}$.

# Preferred section from a harmonic bundle

Putting these cases together, we see that for any $\lambda \in \mathbb{A}^1$ we obtain a point of $\widetilde{M}_{\mathrm{Hod}}$ well-defined up to $\mathscr{G}_{\mathrm{Hod}}$. One can check that locally these specifications patch together to give a section

$$\mathbb{A}^1 \to \left(\widetilde{M}_{\mathrm{Hod}}(X), \mathscr{G}_{\mathrm{Hod}}\right).$$

It patches with the same construction on the opposite chart to give a section of the Deligne-Hitchin space

$$\rho : \mathbb{P}^1 \to \left(\widetilde{M}_{\mathrm{DH}}(X), \mathscr{G}_{\mathrm{DH}}\right).$$

This is the *preferred section* corresponding to the harmonic bundle.

# Mixed twistor normal bundle of the preferred section

The tangent bundle of the "quotient" $M_{\mathrm{DH}}(X)$ may be defined, since the groupoid is étale.

## Theorem

*Let $\mathcal{V}$ be the pullback by $\rho$ of the tangent bundle of $M_{\mathrm{DH}}(X)$. It has a filtration*

$$0 \subset W_0\mathcal{V} \subset W_1\mathcal{V} \subset W_2\mathcal{V} = \mathcal{V}$$

*where $W_1\mathcal{V}$ is the set of tangent vectors that preserve the eigenvalues of the residues, and $W_0\mathcal{V}$ is the tangent space of the change of framing. Then $(\mathcal{V}, W_\cdot)$ is a* mixed twistor structure, *meaning that $W_k/W_{k-1}$ is a semistable bundle on $\mathbb{P}^1$ of slope $k$ (for $k = 0, 1, 2$).*

# Mixed twistor normal bundle of the preferred section

This theorem is the analogue in our case of the purity property in the compact case, and generalizes to rank 2 the weight 2 property observed for the rank 1 case in the previous paper.

Roughly speaking it should tell us that the moduli space of flat bundles with fixed local monodromy transformations has a hyperkähler structure. That is certainly already known in many cases, either by invoking the theory over compact orbicurves in the case of finite monodromy, or at least stated by physicists in a more general setting.

# Mixed twistor $\mathscr{D}$-modules

For the proof of the theorem, we need a basic 1-dimensional part of Takuro Mochizuki's and Claude Sabbah's theory of mixed twistor $\mathscr{D}$-modules.

They show that pure twistor $\mathscr{D}$-modules are closed under operations such as higher direct image. For the map from a curve to a point this tells us that the cohomology of a pure twistor $\mathscr{D}$-module is again a pure twistor structure.

This special case is of course only a small part of the general theory.

Comparing setups to extract our statement from their general theory is still a work in progress.