Finite groups of symplectic birational transformations of IHSM of $OG10$ type

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Building Blocks

Theorem (Beauville-Bogomolov)

Let X be a smooth, compact, Kähler manifold with $c_1(X) = 0$. Then there exists a finite étale cover $\pi : \tilde{X} \to X$ such that:

$$\tilde{X} = T \times \prod C_i \times \prod X_j.$$

where:

1. T is a complex torus
2. C_i is a strict Calabi-Yau manifold, i.e $h^{p,0}(C_i) = 0$ for $p \neq 0, \dim C_i$.
3. X_j is an **irreducible holomorphic symplectic manifold.**

The basics

Definition (IHSM)

An **hyperkähler manifold** X is a compact, kähler manifold that is simply connected and has a unique holomorphic symplectic form, i.e $H^{2,0}(X) = \mathbb{C}\langle \sigma_X \rangle$.

These are also called **irreducible holomorphic symplectic manifolds** (IHSMs). Examples:

- $\dim = 2$: $K3$ surfaces
- $\dim = 2n$: $K3^{[n]}$, Kum_n;
- Sporadic examples in $\dim = 6$, $\dim = 10$ called $OG6$, $OG10$-type respectively.

Big Question: Are there any other (non-equivalent) examples?

A Possible Strategy

Definition

Let X be a hyperkähler manifold. Then $f \in \mathrm{Aut}(X)$ is **symplectic** if $f^*\sigma_X = \sigma_X$.

Let $G \subset \mathrm{Aut}(X)$ a finite group of symplectic automorphism of X.

- Both X/G and $Fix(G) \subset X$ have an induced holomorphic symplectic form.
- One could hope that if we can resolve any singularities symplectically, we would obtain an example of an hyperkähler manifold - which one?

Two natural goals:

1. Classify possible groups of symplectic automorphisms by classifying possible induced actions on

$$f^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z}).$$

2. Study them geometrically.

What is known?

Kamenova, Mongardi, & Oblomkov ('22, '23:

- Studied fixed locus for large class of groups of symplectic automorphisms for X of $K3^{[n]}, Kum_n$ type.
- $Fix(G) =$ union of manifolds of $K3^{[m]}$ type.
- explicit formulas for number of components and the dimension.

Grossi, Onorati & Veniani:('23):

- Classified prime order symplectic automorphisms of $OG6$.

Mongardi & Wandel ('15)

- Fixed locus for certain groups on $OG6$ again $K3^{[m]}$ and finite number of points.

Giovenzana, Grossi, Onorati & Veniani ('22):

- No non-trivial symplectic automorphisms of $OG10$ type manifolds.

Relax notion: $\mathrm{Bir}_S(X)$

Idea: maybe we have more hope if we consider:

$\mathrm{Bir}_S(X) :=$ group of finite order symplectic birational transformations.

Evidence:

- **Markushevich & Tikhomirov ('07):** looked at birational symplectic involution ι of a manifold of $K3^{[3]}$. They showed $Fix(\iota)$ was a new example of IHS variety - mild singularities.
- **Mongardi, Rapagnetta, Saccà: ('16)** realise $OG6$ as the quotient of $K3^{[3]}$ by a birational symplectic involution.

We focus on manifolds of $OG10$ type - why?

$OG10$ from a cubic fourfold

Let $V \subset \mathbb{P}^5$ be a smooth cubic fourfold, i.e. defined by $f_3(x_0, \dots x_5) = 0$.

- One can attach to a smooth hyperplane section $Y = H \cap V$ an abelian variety of dimension 5, $J(Y)$.
  Let $U \subset (\mathbb{P}^5)^\vee$ be the open set parametrising such sections - we obtain a fibration:
  $$\pi_U : J_U \to U.$$
- There exists a 10-dimensional hyperkähler compactification $J_U \subset \mathcal{J}_V$, equipped with a fibration $\pi : \mathcal{J}_V \to (\mathbb{P}^5)^\vee$.
  The hyperkähler manifold $\mathcal{J}_V$ is of $OG10$ type (due to Laza, Saccà, Voisin).

We get interesting birational transformations induced from a cubic fourfold!

The case of $OG10$ type

- Let $\phi : V \to V$ be an automorphism of a cubic fourfold. Each is induced by an involution of the ambient $\mathbb{P}^5$, sending a hyperplane to a hyperplane.
- The automorphism ϕ acts on hyperplane sections of V, and thus on the fibration $J_U \to U$.
- We obtain a birational automorphism $f : \mathcal{J}_V \dashrightarrow \mathcal{J}_V$.

For a manifold X of $OG10$ type and a smooth cubic fourfold V, we have that

$$(H^2(X,\mathbb{Z}), q_X) \cong H^4(V,\mathbb{Z})_{prim}(-1) \oplus U$$

as lattices.
Knowing the action of ϕ on $H^4(V,\mathbb{Z})_{prim}$ helps classify possible transformations of manifolds X.

The Results

Let X be a manifold of $OG10$ type, $G := \mathrm{Bir}_s(X)$.
We let $\Lambda \cong (H^2(X, \mathbb{Z}), q_X)$, and define:

$$\Lambda^G := \{v \in \Lambda | g(v) = v \text{ for all } g \in G\};$$
$$\Lambda_G := (\Lambda^G)^\perp.$$

We classify all possible groups G by classifying their action on Λ.

Theorem (M., Muller)

There are 379 birational conjugacy classes of pairs (X, G) consisting of an IHS manifold X of $OG10$ type and a saturated, finite group $G = \mathrm{Bir}_s(X)$ of symplectic birational transformations.

The classification is up to deformation and monodromy conjugation.

Remarks:

Theorem (M., Muller)

There are 379 birational conjugacy classes of pairs (X, G) consisting of an IHS manifold X of $OG10$ type and a saturated, finite group $G = \mathrm{Bir}_s(X)$ of symplectic birational transformations.

- The action of G on Λ is explicitly determined in each case, i.e Λ_G, Λ^G and the gluing type.
- There are 209 distinct groups, of which 64 have a unique action on Λ.

Theorem (M., Muller)

There are six classes of pairs $(X, \mathbb{Z}/2\mathbb{Z})$, and the action is determined by:

$$\Lambda_G = E_8(2), \quad E_6(2), \quad G_{12}$$
$$D_{12}^+, \quad M, \quad G_{16}.$$

Strategy: Torelli Theorem

Let X be a manifold of $OG10$ type, $\eta : H^2(X, \mathbb{Z}) \to \Lambda := A_2 \oplus E_8^2 \oplus U^3$ a marking.

Definition

An isometry $g \in O(\Lambda)$ is induced by a birational transformation if there exists a $f \in \mathrm{Bir}(X)$ such that $\eta \circ f^* \circ \eta^{-1} = g$.

We want to classify isometries that are induced.

$$G \subset \mathrm{Bir}(X) \Rightarrow G \subset O(\Lambda) \Rightarrow \Lambda^G, \Lambda_G$$

Denote by $\mathcal{W} = \{v \in L : v^2 = -2\} \cup \{v \in L : v^2 = -6,\ div_L(v) = 3\}$.

Corollary (of Markman's Hodge Theoretic Torelli)

A group $G \subset O(\Lambda)$ is induced by a finite group of symplectic birational transformaions if and only if Λ_G is negative definite and

$$\Lambda_G \cap \mathcal{W} = \varnothing.$$

In other words,

$$G \in O(\Lambda) \text{ with } \Lambda_G \text{ negative defn and } \Lambda_G \cap \mathcal{W} = \varnothing$$

$$\Leftrightarrow \exists X \text{ of } OG10 \text{ type }, G = \mathrm{Bir}_s(X) \text{ with } H^2(X, \mathbb{Z})_G \cong \Lambda_G.$$

We have turned the problem into a lattice theoretic problem. There is one more invariant to help us:

$$D_\Lambda := \Lambda^*/\Lambda \cong \mathbb{Z}/3\mathbb{Z}.$$

For $f \in \mathrm{Bir}_s(X)$, $f^*|_{D_\Lambda} = \pm id_{D_\Lambda}$.
Let's start by classifying the groups $G = \mathrm{Bir}_s(X)$ that act trivially on D_Λ.

The Leech Pair Trick

- The Leech lattice $\mathbb{L}$ and it's automorphism group $O(\mathbb{L}) = Co_0$ have been well studied, and prime order automorphisms are well understood.
- Kondō's approach for $K3$ surfaces: embed Λ_G into the Leech lattice.
- This was also used by Laza & Zheng to classify groups of automorphisms for cubic fourfolds.

Lemma

Assume that the induced action of G on D_Λ is trivial. Then (G, Λ_G) is a Leech pair, i.e there exists a primitive embedding of $\Lambda_G \hookrightarrow \mathbb{L}$, and hence an embedding of H into Co_0 with image avoiding $-id$.

- One can extend the action of G to an isometry group of $\mathbb{L}$, and use the classification.
- For $G = \mathbb{Z}/2\mathbb{Z}$, this recovers the $E_8(2)$ and $D_{12}^+(2)$ involutions.

$\mathbb{Z}/2\mathbb{Z}$ extensions

Now suppose $G = \mathrm{Bir}_s(X)$ acts nontrivially on D_Λ. We get:

$$0 \to H \to G \to D_\Lambda = \mathbb{Z}/2\mathbb{Z} \to 0.$$

- (H, Λ_H) is a Leech pair - we call this the heart of G.
- Take $g \in G \setminus H$ that generates G/H.
- g restricted to Λ^H is an isometry of Λ^H of order at most 2. The pair (Λ^H, g) is called the head of G. Note that Λ^H may not be unique in its genus.
- For each head (Λ^H, g), we compute all possible equivariant primitive embeddings $\Lambda^H \hookrightarrow \Lambda$, that glue the heart and head together in a compatible way.

We make this process abstract - given a heart H, we compute all possible extensions G and use the Torelli theorem to determine which can occur as $G = \mathrm{Bir}_s(X)$ for an $OG10$ type manifold X.

Extension phenomena explained

Let us return to our cubic fourfold $V \subset \mathbb{P}^5$, and $\pi : \mathcal{J}_V \to \mathbb{P}^5$ it's associated $OG10$.

An $\phi \in \mathrm{Aut}(V)$ is symplectic if it acts trivially on $H^{3,1}(V)$.

- $\phi \in Aut(V)$ symplectic $\Rightarrow \phi \in \mathrm{Bir}(X)$ symplectic.
- $\phi \in Aut(V)$ non-symplectic $\Rightarrow \phi \in \mathrm{Bir}(X)$ non-symplectic.

But!

- $\pi : \mathcal{J} \to B$ has an additional anti-symplectic involution τ acting by $x \mapsto -x$ fiberwise, since its smooth fibers are abelian varieties.
- Taking a non-symplectic involution $\phi \in \mathrm{Aut}(V)$ and composing $\tau \circ \phi \in \mathrm{Bir}_s(X)$ gives you an extra symplectic involution!
- $\tau \circ \phi$ acts non-trivially on D_Λ, giving you the $\mathbb{Z}/2\mathbb{Z}$ extension!

This recovers the $E_6(2), M$ - $\mathbb{Z}/2\mathbb{Z}$ actions - they are both induced from antisymplectic involutions on a cubic fourfold.

Example: The Fermat Cubic

Let $V = \{x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0\} \subset \mathbb{P}^5$.

- $H := \mathrm{Aut}_s(V) = C_3^4 \rtimes A_6$.
- $\mathrm{Aut}(V)/H = \langle \phi H, \psi H \rangle$
- ϕ is an involution $x_0 \leftrightarrow x_1$.
- ψ is order 3 sending $x_0 \mapsto \omega x_0$.

Consider $\pi : \mathcal{J}_V \to B$ with anti-symplectic involution τ acting as $\tau(x) = -x$ on the fibers. We have:

$$G := \langle H, \tau \circ \phi \rangle = \mathrm{Bir}_s(X),$$

where

$$0 \to H \to G \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

One can check that $G \cong C_3^4 \rtimes S_6$.
In fact, many of our $\mathbb{Z}/2\mathbb{Z}$- extensions occuring in the classification come from a cubic fourfold that admits an additional anti-symplectic involution.

Cubic fourfold Criteria

We would like to know whether a pair (X, G) in our classification can be realised from looking at a group $\mathrm{Aut}(V)$ for some cubic fourfold V, and applying this construction.

Theorem (M., Muller)

Let X be an IHS manifold of $OG10$ type, $G \leq \mathrm{Bir}_s(X)$ a finite group. Then the following are equivalent:

1. $U \subseteq \Lambda^G$
2. *there exists a cubic fourfold V with $G \leq \mathrm{Aut}(V)$ acting either purely symplecticly, or whose symplectic subgroup $G_s \leq G$ has index 2.*

In this case, (X, G) can be deformed to a pair that is birationally conjugate to $(\mathcal{J}_V, G)$.

Additional Results

- IHS manifolds of $OG10$ type can also be constructed as desingularised moduli spaces of sheaves $\widetilde{M}_v(S,\theta)$ on a $K3$ surface. We identify which groups in our classification can be realised via this construction, using a criteria of Felsetti, Giovenzana & Grossi.
- We obtain the largest finite group G of birational symplectic transformations (acting non trivially on cohomology) for IHS manifolds of known deformation types. It has order:

$$|G| = \mathbf{6{,}531{,}840} = 2^8 \cdot 3^6 \cdot 5 \cdot 7.$$

- Moreover, we obtain the largest finite group of birational transformations (not necessarily symplectic) acting non-trivially on cohomology for IHS manifolds of known deformation type. It has order:

$$\mathbf{39{,}191{,}040} = 2^9 \cdot 3^7 \cdot 5 \cdot 7.$$

Thank you!