L-series associated with harmonic Maass forms

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Joint work with M. Lee, W. Raji and L. Rolen

Notation

- $k \in \mathbb{Z}$
- $\mathbb{H} = \{z \in \mathbb{C}; \mathrm{Im}(z) > 0\}$.
- Weight k hyperbolic Laplacian on $\mathbb{H}$:

$$\Delta_k := -4y^2 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} + 2iky \frac{\partial}{\partial \bar{z}}$$

where $z = x + iy$ with $x, y \in \mathbb{R}$.

- The action $|_k$ of $\mathrm{SL}_2(\mathbb{R})$ on smooth functions $f : \mathbb{H} \to \mathbb{C}$:

$$(f|_k\gamma)(z) := (cz+d)^{-k} f(\gamma z), \qquad \text{for } \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

- $W_M := \begin{pmatrix} 0 & -1/\sqrt{M} \\ \sqrt{M} & 0 \end{pmatrix}$.

Harmonic Maass forms

Definition. Let $N \in \mathbb{N}$. Let ψ be a Dirichlet character modulo N. A *harmonic Maass form of weight k and character ψ for* $\Gamma_0(N)$ is a smooth function $f : \mathbb{H} \to \mathbb{C}$ such that:

i) For all $\gamma = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma_0(N)$, $f|_k\gamma = \psi(d)f$

ii) $\Delta_k(f) = 0$.

iii) For each $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, there is $P(z) \in \mathbb{C}[q^{-1}]$, such that

$$f(\gamma z)(cz+d)^{-k} - P(z) = O(e^{-\epsilon y}), \qquad \text{as } y \to \infty, \text{ for some } \epsilon > 0.$$

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The study of $H_k(N, \psi)$ has led to important results such as the proof of the Andrews-Dragonette conjecture, insight into Dyson's ranks and applications to L-functions of cusp forms and to mathematical physics.

Fourier expansion of $f \in H_k(N, \psi)$

There is a $n_0 \in \mathbb{N}$ such that

$$f(z) = \sum_{n \geq -n_0} a(n) e^{2\pi i n z} + \sum_{n<0} b(n) \Gamma(1-k, -4\pi n y) e^{2\pi i n z}$$

for some $a(n), b(n) \in \mathbb{C}$. Here $\Gamma(r, z)$ denotes the *incomplete Gamma function* given by

$$\Gamma(r, z) := \int_z^\infty e^{-t} t^r \, \frac{dt}{t}. \tag{1}$$

for $\mathrm{Re}(r) > 0$ and extended to an entire function of r, when $z \neq 0$.
Analogous expansions hold at the other cusps.
Notation: $S_k^!(N, \psi) = \{f \in H_k(N, \psi)$; holomorphic with vanishing constant terms at all cusps.$\}$ (*weakly holomorphic cusp forms*)

Key property

In contrast to the classical holomorphic cusp forms, the Fourier coefficients of $f \in H_k(N, \psi)$ do not have polynomial growth. Instead they satisfy:

$$a(n) = O(e^{C\sqrt{n}}),\ b(-n) = O(e^{C\sqrt{n}}) \qquad \text{as } n \to \infty \text{ for some } C > 0.$$

As a consequence, L-series on elements of $H_k(N, \psi)$ cannot be defined by the usual Dirichlet series, e.g. as

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

A first definition of L-series for weakly holomorphic cusp forms

Special case of weakly holomorphic cusp forms: Let

$$f(z) = \sum_{\substack{n \geq -n_0 \\ n \neq 0}} a(n) e^{2\pi i n z} \in S_k^!(N).$$

Then, for any fixed $t_0 > 0$, set

$$L(s, f) := \sum_{\substack{n \geq -n_0 \\ n \neq 0}} \frac{a(n)\Gamma(s, 2\pi n t_0)}{(2\pi n)^s} + i^k \sum_{\substack{n \geq -n_0 \\ n \neq 0}} \frac{a(n)\Gamma\left(k - s, \frac{2\pi n}{t_0}\right)}{(2\pi n)^{k-s}}$$

for all $s \in \mathbb{C}$. $L(s, f)$ is independent of t_0. [Bringmann, Fricke, Kent]

L-series

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for all $s \in \mathbb{C}$. $L(s, f)$ is independent of t_0. [Bringmann, Fricke, Kent] Since the functional equation of $L(s, f)$ is "built into" its definition it is hard to formulate a converse theorem.

Let $\mathcal{L}$ be the Laplace transform mapping $\varphi\colon \mathbb{R}_+ \to \mathbb{C}$ to

$$(\mathcal{L}\varphi)(s) = \int_0^\infty e^{-st}\varphi(t)dt$$

for each $s \in \mathbb{C}$ for which the integral converges absolutely.

For $s \in \mathbb{C}$, we define

$$\varphi_s(x) := \varphi(x)x^{s-1}.$$

Note that $\varphi_1 = \varphi$.

L-series of harmonic Maass forms (cont.)

Definition. Let

$$f(z) = \sum_{n \geq -n_0} a(n) e^{2\pi i n z} + \sum_{n<0} b(n) \Gamma\left(1-k, -4\pi n y\right) e^{2\pi i n z},$$

be an element of $H_k(N, \psi)$. The L-series $L_f(\varphi)$ of f is defined by

$$\sum_{n \geq -n_0} a(n)(\mathcal{L}\varphi)(2\pi n) + \sum_{n<0} b(n)(-4\pi n)^{1-k} \int_0^\infty \frac{(\mathcal{L}\varphi_{2-k})(-2\pi n(2t+1))}{(1+t)^k} dt$$

for each piece-wise smooth $\varphi : \mathbb{R} \to \mathbb{C}$ for which the RHS converges.

L-series of harmonic Maass forms (cont.)

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3. The usual definition of L-series of classical cusp forms f can be retrieved by setting, for $\mathrm{Re}(s) > (k-1)/2$,

$$\varphi(x) = \varphi_s(x) := (2\pi)^s x^{s-1} \frac{1}{\Gamma(s)}.$$

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4. The L-function for weakly holomorphic forms given earlier can be obtained, in a non-symmetrised form, upon setting $\phi = \phi_s \mathbf{1}_{t_0}$, where $\mathbf{1}_{t_0} = 1$, if $x > t_0$ and 0 otherwise.

Functional equations

Theorem (D., Lee, Raji, Rolen). For $k \in \mathbb{Z}$, $N \in \mathbb{N}$, ψ a Dirichlet character mod. N let $f \in H_k(N, \psi)$ with expansion as above. For χ mod D ($(D, N) = 1$) set

$$L_f(\chi, \varphi) = \sum_{n \geq -n_0} \tau_{\bar{\chi}}(n) a(n) (\mathcal{L}\varphi)(2\pi n/D)$$

$$+ \sum_{n<0} \tau_{\bar{\chi}}(n) b(n) (-4\pi n/D)^{1-k} \int_0^{\infty} \frac{\mathcal{L}(\varphi_{2-k})(-2\pi n(2t+1)/D)}{(1+t)^k} dt$$

and

$$L'_f(\chi, \varphi) = L_f(\chi, \varphi) - \frac{2\pi}{D} \sum_{n \geq -n_0} n \tau_{\bar{\chi}}(n) a(n) (\mathcal{L}\varphi_2)(2\pi n/D)$$

$$- \frac{2\pi}{D} \sum_{n<0}^{\infty} n \tau_{\bar{\chi}}(n) b(n) (-4\pi n/D)^{1-k} \int_0^{\infty} \frac{(\mathcal{L}\varphi_{3-k})(-2\pi n(2t+1)/D)}{(1+t)^k} dt.$$

Here, $\tau_\chi(n) := \sum_{u \bmod D} \chi(u) e^{2\pi i n \frac{u}{D}}$.

Functional equations (cont.)

Theorem (cont.). Set

$$g := f|_k W_N \tag{2}$$

Then, for each φ such that all L-functions involved are defined, we have

$$L_f(\chi, \varphi) = i^k \frac{\chi(-N)\psi(D)}{N^{k/2-1}} L_g(\bar{\chi}, \varphi|_{2-k} W_N)$$
$$L'_f(\chi, \varphi) = -i^k \frac{\chi(-N)\psi(D)}{N^{k/2-1}} L'_g(\bar{\chi}, \varphi|_{2-k} W_N)$$

where $(\varphi|_{2-k} W_N)(x) := (Nx)^{k-2} \varphi\left(\frac{1}{Nx}\right) \qquad$ for all $x > 0$.

The converse theorem

Theorem. (D., Lee, Raji, Rolen) For $N \in \mathbb{N}$, let ψ be a Dirichlet character modulo N. Let $(a_1(n))_{n \geq -n_0}$ and $(b_1(n))_{n<0}$ be sequences of complex numbers such that $a_1(n), b_1(n) = O(e^{C\sqrt{|n|}})$ as $|n| \to \infty$ for some constant $C > 0$. Let $(a_2(n))_{n \geq -n_0}$ and $(b_2(n))_{n<0}$ be another pair of sequences with the smae property. For $j = 1, 2$, define $f_j : \mathbb{H} \to \mathbb{C}$ by the following Fourier expansions:

$$f_j(z) = \sum_{n \geq -n_0} a_j(n) e^{2\pi i n z} + \sum_{n<0} b_j(n) \Gamma\left(1-k, -4\pi n y\right) e^{2\pi i n z}.$$

The converse theorem (cont.)

Theorem (cont.) For all $D \in \{1, 2, \ldots, N^2 - 1\}$, $(D, N) = 1$, let χ be a Dirichlet character modulo D. For each D, χ and any *compactly supported* $\varphi : \mathbb{R}^+ \to \mathbb{C}$, assume that, for each $s \in \mathbb{C}$,

$$L_{f_1}(\chi, \varphi_s) = i^k \frac{\chi(-N)\psi(D)}{N^{\frac{k}{2}-1}} L_{f_2}(\overline{\chi}, \varphi_s|_{2-k} W_N)$$

and

$$L'_{f_1}(\chi, \varphi_s) = -i^k \frac{\chi(-N)\psi(D)}{N^{\frac{k}{2}-1}} L'_{f_2}(\overline{\chi}, \varphi_s|_{2-k} W_N).$$

Then, the function f_1 is a harmonic Maass form with weight k and Nebentypus character ψ for $\Gamma_0(N)$ and $f_2 = f_1|_k W_N$.

Remarks

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5. The test function based method we employ has appeared in a different context in investigations by Booker, Miller-Schmidt, Miyazaki-Sato-Sugiyama-Ueno and others.

A summation formula

We now assume that the level is 1 and the character trivial. Let

$$\xi_{2-k} f := 2iy^{2-k} \overline{\frac{\partial f}{\partial \bar{z}}}.$$

Theorem. (D., Lee, Raji, Rolen) Let $k \in 2\mathbb{N}$ and let $f \in S_k$ with Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}.$$

Suppose that g is an element of H_{2-k} such that $\xi_{2-k} g = f$ for with Fourier expansion

$$g(z) = \sum_{n \geq -n_0} c^{+}(n) e^{2\pi i n z} + \sum_{n<0} c^{-}(n) \Gamma(k-1, -4\pi n y) e^{2\pi i n z}.$$

A summation formula (cont.)

Theorem (cont.) Then, for every piecewise smooth, compactly supported $\varphi : \mathbb{R} \to \mathbb{R}$, we have

$$\sum_{n \geq -n_0} c^+(n) \int_0^\infty \varphi(y) \left(e^{-2\pi ny} - (-iy)^{k-2} e^{-2\pi n/y} \right) dy$$

$$= \sum_{l=0}^{k-2} \sum_{n>0} \overline{a(n)} \Big(\frac{(k-2)!}{l!} (4\pi n)^{1-k+l} \int_0^\infty e^{-2\pi ny} y^l \varphi(y) dy$$

$$+ \frac{2^{l+1}(-1)^k}{(k-1)} (8\pi n)^{-\frac{k}{2}} \int_0^\infty e^{-\pi ny} y^{\frac{k}{2}-1} \varphi(y) M_{1-\frac{k}{2}+l, \frac{k-1}{2}}(2\pi ny) dy \Big)$$

where $M_{\kappa,\mu}(z)$ is the Whittaker hypergeometric function.