Non-vanishing of Fourier coefficients of Siegel cusp forms

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Every Siegel cusp form F of weight k over a Hecke congruence subgroup $\Gamma_0^n(N)$ of the symplectic group $Sp_n(\mathbb{Z}) \subseteq \mathbb{Z}^{2n,2n}$ has a Fourier series representation

$$F(Z) = \sum_{T \in J_n} c(T) \exp\left(2\pi i \; trace(TZ)\right)$$

where $Z \in H_n$, the Siegel space of degree n, and J_n denotes the set of positive-definite, half-integral, symmetric, n by n matrices. It also has a Fourier-Jacobi series representation

$$F(Z) = \sum_{m=1}^{\infty} f_m(Z', w) \exp\left(2\pi i m \tau\right), \quad \text{where} \;\; Z = \begin{pmatrix} Z' & {}^t w \\ w & \tau \end{pmatrix} \in H_n$$

with $Z' \in H_{n-1}$, $\tau \in H_1$, and each f_m is a Jacobi cusp form of weight k and index m over the group $\Gamma_0^{n-1}(N) \ltimes (\mathbb{Z}^{1,n-1} \times \mathbb{Z}^{1,n-1})$.

In this talk we show that $F \neq 0$ implies $f_p \neq 0$ for infinitely many prime numbers p. This is a consequence of a more specific statement on the non-vanishing of Fourier coefficients $c(T)$ indexed by matrices of type

$$T = \begin{pmatrix} T' & 0 \\ 0 & p \end{pmatrix} \in J_n \quad \text{with} \quad T' \in J_{n-1}$$

which we shall also discuss.

Furthermore, as an auxiliary result of independent interest we will see that every primitive, positive-definite, integral quadratic form in n variables represents infinitely many prime numbers. This is a generalization of a 1874 result for binary quadratic forms due to Mertens and Weber.