On K3 Surfaces of High Picard Rank

based on work C. - Malmendier, C. - Doran

1 General Setting

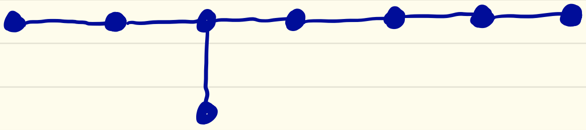
X smooth compact complex surface
$K_X \simeq \mathcal{O}_X$, $H^1(X, \mathcal{O}_X) = 0$
admitting polarization

X algebraic K3 surface / $\mathbb{C}$

$$H^2(X, \mathbb{Z}) = L \simeq H \oplus H \oplus H \oplus E_8 \oplus E_8 \quad \leftarrow \text{K3 Lattice}$$

unimodular, even lattice, rank = 22, signature = (3, 19)

$H : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\qquad E_8 :$



$$c_1 : H^1(X, \mathcal{O}_X^*) \hookrightarrow H^2(X, \mathbb{Z}) \qquad NS(X) \stackrel{\text{def}}{=} \operatorname{Im}(c_1)$$

Neron-Severi Lattice

$\rho_X = \operatorname{rank} NS(X)$ Picard rank

$$\underbrace{1 \leq \rho_X \leq 20}$$

Signature of $NS(X)$: $(1, \rho_X - 1)$

$$H^2(X, \mathbb{C}) = \underbrace{H^{2,0}(X)}_{\underbrace{\langle \omega \rangle}} \oplus \underbrace{H^{1,1}(X)}_{\{\omega, \bar{\omega}\}^{\perp}} \oplus \underbrace{H^{0,2}(X)}_{\langle \bar{\omega} \rangle}$$

Hodge Decomposition

$\omega \in \mathbb{P}H^2(X, \mathbb{C})$ period line

$$\underbrace{(\omega, \omega) = 0 \ , \ (\omega, \bar{\omega}) > 0}$$

Hodge-Riemann bilinear relations

$NS(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Z})$

via Lefschetz theorem on $(1,1)$ classes.

Polarizations

(a) quasi-ample, $\mathcal{L} = \mathcal{O}_X(D)$ with D divisor big and nef

Theorem [Mayer] If $\mathcal{L} = \mathcal{O}_X(D)$ is quasi-ample, there exists $n \geq 1$ such that $|nD|$ is base-point free and $\varphi_{|nD|} : X \longrightarrow \mathbb{P}^N$ consists of contracting all rational curves C with $D \cdot C = 0$. Moreover, all singularities in $\mathrm{Im}(\varphi_{|nD|})$ are rational double points.

(b) lattice polarization (Beauville, Dolgachev)

Let M even lattice of rank r and signature $(1, r-1)$

$i : M \hookrightarrow NS(X)$
primitive lattice embedding
$i(M)$ contains a quasi-ample class

M-polarization on X

K3 surfaces with $\rho_X = 20$

"singular" K3's
Shioda-Inose terminology

$$\underbrace{T_X}_{} \overset{\text{def}}{=\!=} NS(X)^{\perp} \text{ in } H^2(X, \mathbb{Z})$$

transcendental lattice

- In this case T_X is even, positive definite, rank = 2

$$\underbrace{\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}}_{} \qquad \begin{array}{l} a, b, c \in \mathbb{Z} \\ a > 0, \; b > 0, \quad 4ac - b^2 > 0 \end{array}$$

$SL(2, \mathbb{Z})$-conjugacy class classifies X

- **Theorem [Shioda-Inose]**

Given X, Y "singular" K3 surfaces, one has:

$$X \simeq Y \iff T_X \simeq T_Y$$

isometry of lattices

Moduli Spaces – Torelli Theorems

$r \leq 19$

Fix : $\begin{cases} - \ M \text{ even lattice, rank} = r, \text{ signature } (1, r-1) \\ - \text{ primitive lattice embedding } \alpha : M \hookrightarrow L \end{cases}$

Then define : $T = \alpha(M)^{\perp} \subset L$

One has : T even lattice, rank $= 22 - r$, signature $(2, 20-r)$

Marked M-polarized K3 (X, i, ϕ)

X K3 surface

$i : M \to H^2(X, \mathbb{Z})$ polarization

$\phi : H^2(X, \mathbb{Z}) \xrightarrow{\simeq} L$ latice isomorphism

such that $\phi \circ i = \alpha$

Given (X, i, ϕ) :

T inherits a Hodge structure of weight 2

$$T \otimes \mathbb{C} = \underbrace{T^{2,0}}_{\mathbb{C}\omega} \oplus \underbrace{T^{1,1}}_{\{\omega, \bar{\omega}\}^{\perp}} \oplus \underbrace{T^{0,2}}_{\mathbb{C}\bar{\omega}}$$

which, in turn, defines a "period" point in :

$$\underbrace{\Omega_M}_{\text{Period Domain}} = \{ [\omega] \in \mathbb{P}^1(T \otimes \mathbb{C}) \mid (\omega, \omega) = 0, \ (\omega, \bar{\omega}) > 0 \}$$

Obs. on Ω_M :

- open variety of dim $= 20 - \rho$
- symmetric homog. space $O(2, 20-\rho) / SO(2) \times O(20-\rho)$
- two connected components, each $\cong$ Herm. Dom. $IV_{20-\rho}$

Theorem [Nikulin, Burns-Rapoport, Todorov]

(a) There is a fine moduli space $\underbrace{\mathcal{K}_M}$ for marked M-polarized K3's

(b) $\underbrace{p : \mathcal{K}_M \longrightarrow \Omega_M}_{\text{period map}}$ is etale and surjective

[Asterisque, 162]

Removing markings:

$\underbrace{\mathcal{M}_M}$: coarse moduli space for M-polarized K3's

$p : \mathcal{M}_M \longrightarrow \underbrace{\Gamma_M \backslash \Omega_M}$

$$\underbrace{\Gamma_M = \{ f \in O(T, \mathbb{Z}) \mid f = id \text{ in } Aut(T^*/T) \}}$$

Theorem [Torelli for M-polarized K3's]

$p : \underbrace{\mathcal{M}_M} \longrightarrow \underbrace{\Gamma_M \backslash \Omega_M}$ isomorphism of quasi-projective varieties

② Special M-polarizations of high rank

$$\boxed{\mathcal{F}_M = \Gamma_M \backslash \Omega_M}$$

quasi-projective variety $\underbrace{\dim = 20 - r}$

≈ bounded symmetric domain factored by a discrete arithmetic group

For high rank, one may:

- understand $\mathcal{F}_M$ well, as dimension is small
- in some cases, $\mathcal{F}_M$ is known to number-theorists
- understand compactifications
- explicit descriptions for K3 families
- connect to appropriate modular forms

③ **Case $M = H$** $\boxed{r=2}$

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \sim \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}}$$

Theorem $\underbrace{H\text{-polarization on } X}$ $\Longleftrightarrow$ $\underbrace{\text{Jacobian elliptic fibration}}$

$\left(\underbrace{\varphi : X \to \mathbb{P}^1}, \ \underbrace{s \hookrightarrow X} \text{ section} \right)$

$\langle f, s \rangle \subset NS(X)$

elliptic fiber

section

$\underbrace{f^2 = 0}, \ \underbrace{f \cdot s = 1}, \ \underbrace{s^2 = -2}$

$$\underbrace{\mathcal{M}_H} \approx \text{coarse moduli space for elliptic K3 surfaces with section } (X, \varphi, s)$$

18-dim. quasi-projective variety

4 Nikulin Lattice

rank = 8

$$N = \langle \underbrace{c_1, c_2, \dots, c_8, \tfrac{1}{2}(c_1+c_2+\dots+c_8)}\rangle$$

$$c_i \cdot c_j = -2\delta_{ij}$$

Case $M = H \oplus N$ $\quad$ $r = 10$

Theorem (Nikulin, Van Geemen, Sarti)

$\boxed{H \oplus N\text{-polarization on } X \iff \text{Jacobian elliptic fibration with an order-two section}}$

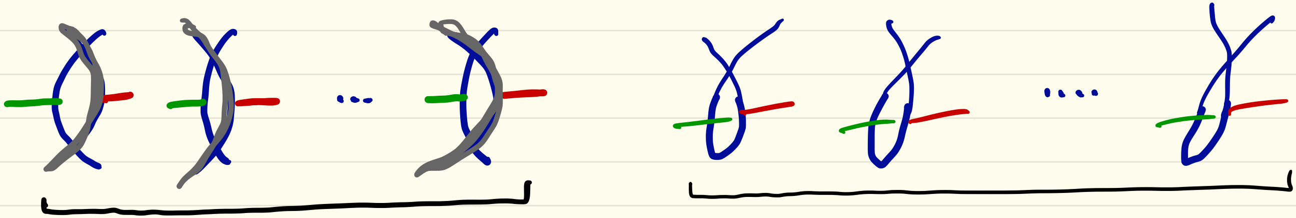
$\varphi : X \longrightarrow \mathbb{P}^1$ elliptic fibration $\qquad y^2 = x\,(x^2 + a(t)x + b(t))$

S_0 "base" section $\qquad S_0 : \; x = \infty$

S_1 section, has order 2 in $MW(\varphi, S_0)$ $\qquad S_1 : \; x = 0$

$\mathcal{M}_{H\oplus N}$ - 10-dim. quasi-projective variety

Generic member: elliptic fibration has 8 I_1's and 8 I_0's



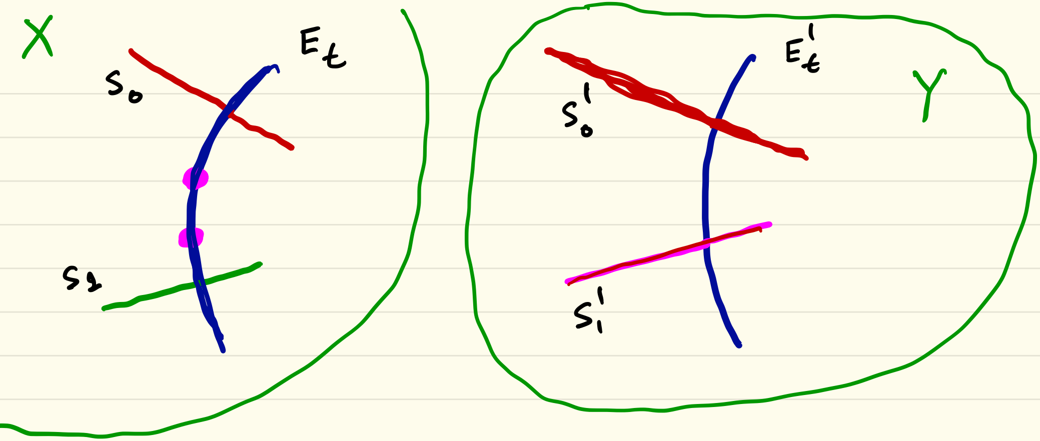
8 I_1 fibers

8 I_0 fibers

$\alpha^*(\omega) = \omega$

<u>**Theorem**</u> (Van Geemen, Sarti)

An $H\oplus N$-polarized K3 surface carries a canonical symplectic involution $\alpha_X : X \longrightarrow X$ given by fiber-wise translations by the order-two section.



$$
\begin{tikzcd}
\alpha_X \circlearrowright X \arrow[rr, dashed, bend left] \arrow[dr] & & Y \circlearrowleft \alpha_Y \arrow[ll, dashed, bend left] \arrow[dl] \\
& \mathbb{P}^1 &
\end{tikzcd}
$$

rational double cover maps

Van Geemen - Sarti - Nikulin Duality (VSN)

We shall consider polarizing lattices M with

$$\underbrace{H \oplus N}_{} \subset \underbrace{M}_{} \qquad \text{rank } M = r$$

M-polarized K3 surfaces $\longleftrightarrow$ VSN dual K3 surfaces

degree = 4, $\underbrace{\text{quartics in } \mathbb{P}^3}_{}$, easier to describe/classify in terms of modular forms

$\underbrace{\text{degree} = 2}_{}$, $\underbrace{\text{double sextics}}_{}$, richer geometry,

(5) $M = H \oplus E_8 \oplus E_8 \oplus (-2n)$, $n \geq 1$ $\qquad r = 19$

$$\mathcal{F}_M = X_0(n)^+ = \Gamma_0(n)^+ \backslash \mathcal{H}_1$$

Fricke Modular Curve of level n

$\mathcal{H}_1$: usual complex upper half-plane

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod{n} \right\}$$

$$\Gamma_0(n)^+ = \left\langle \Gamma_0(n), \begin{pmatrix} 0, & -\frac{1}{\sqrt{n}} \\ \sqrt{n}, & 0 \end{pmatrix} \right\rangle \leq PSL(2,\mathbb{R})$$

VSN dual : $Km(E_1 \times E_2)$

E_1, E_2 dual n-isogeneous elliptic curves

$X_0(n)^+$ is course moduli space for $E_1 \times E_2$

(6) $M = H \oplus E_8 \oplus E_8$ $\qquad r = 18$

Shioda - Inose normal form

Theorem [C-Doran]

Let $(a,b,d) \in \mathbb{C}^3$, with $d \neq 0$. Consider the quartic in $\mathbb{P}^3$:

$$\boxed{y^2 z w - 4x^3 z + a\, x z w^2 + b\, z w^3 - \frac{1}{2}\left(d\, z^2 w^2 + w^4\right) = 0}$$

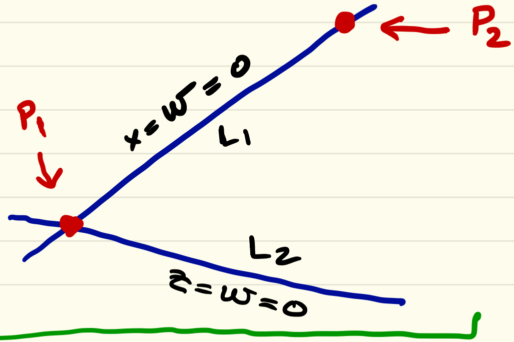
Singularities are rational double points and the min. resolution is a K3 surface $X(a,b,d)$ carrying a canonical $H \oplus E_8 \oplus E_8$ polarization.

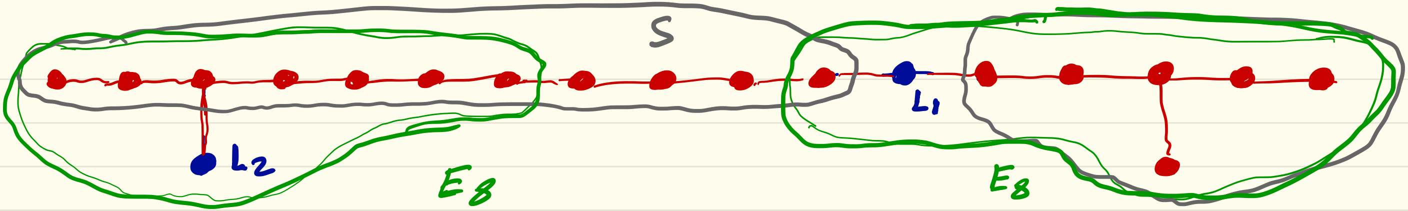
Singular Points

(generic case)

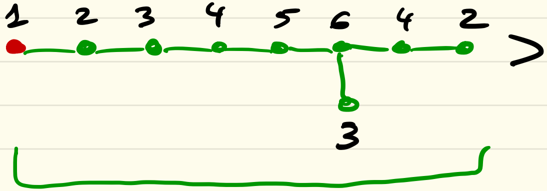
P_1 : $[0\ 1\ 0\ 0]$ $\leftarrow A_{11}$

P_2 : $[0\ 0\ 1\ 0]$ $\leftarrow E_6$





$$M = E_8 \oplus E_8 \oplus \langle S,$$



Theorem [C-Doran]

(a) All $H \oplus E_8 \oplus E_8$-polarized K3 surfaces, up to izomorphism, can be realized as $X(a,b,d)$

(b) $X(a,b,d) \simeq X(\tilde{a},\tilde{b},\tilde{d})$ if and only if

$$\tilde{a} = t^2 \cdot a, \quad \tilde{b} = t^3 \cdot b, \quad \tilde{d} = t^6 \cdot d$$

for some $t \in \mathbb{C}^*$

Obs: $\mathcal{M}_{H\oplus E_8\oplus E_8} \overset{def}{=} \{[a,b,d]\in \mathbb{WP}(2,3,6) \mid d\neq 0\}$

is a coarse moduli space for $H\oplus E_8\oplus E_8$ - polarized K3 surfaces.

Lemma $\mathcal{F}_{H\oplus E_8\oplus E_8} = \Pi \backslash \mathcal{H}_1\times\mathcal{H}_1$

Hilbert Modular Surface

$$\Pi = [PSL(2,\mathbb{Z})\times PSL(2,\mathbb{Z})]\rtimes \mathbb{Z}/2\mathbb{Z}$$

Theorem [C-Doran] Inverse period map is given by:

$$\boxed{[a,b,d] = [g_2(\tau_1)g_2(\tau_2),\ 27\, g_3(\tau_1)g_3(\tau_2),\ \Delta(\tau_1)\Delta(\tau_2)]}$$

where:

$$g_2(\tau)=60\cdot E_4(\tau),\quad g_3(\tau)=140\cdot E_6(\tau),\quad \Delta(\tau)=g_2^3(\tau)-27\cdot g_3^2(\tau)$$

Observations :

- $\mathcal{F}_{H \oplus E_8 \oplus E_8}$ is also coarse moduli space for $A = \underbrace{E_1 \times E_2}$ with E_1, E_2 elliptic curves

- VSN duals : $\underbrace{Km(E_1 \times E_2)}$

- In fact, via the J-invariant function $J = \dfrac{g_2^3}{\Delta}$:

$$\boxed{J(E_1) + J(E_2) = \frac{a^3 - b^2 + d}{d}, \quad J(E_1) \cdot J(E_2) = \frac{a^3}{d}}$$

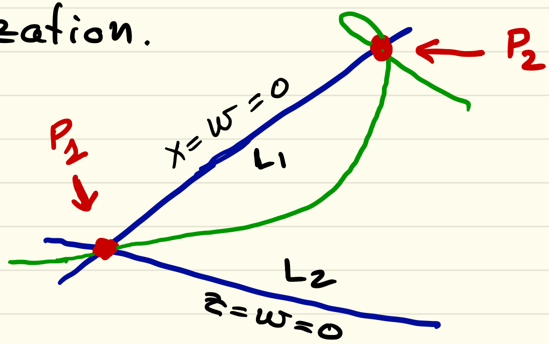
⑦ $M = \underbrace{H \oplus E_8 \oplus E_7}$, $\quad r = 17$

Theorem [C-Doran]

Let $(a,b,c,d) \in \mathbb{C}^4$ with $(c,d) \neq (0,0)$. Consider the quartic in $\mathbb{P}^3$:

$$\boxed{y^2zw - 4x^3z + \underline{a}\,xzw^2 + \underline{b}\,zw^3 + \underline{c}\,xz^2w - \frac{1}{2}(\underline{d}\,z^2w^2 + w^4) = 0}$$

Singularities are rational double points and the min. resolution is a K3 surface $X(a,b,c,d)$ carrying a canonical $H \oplus E_8 \oplus E_7$ polarization.

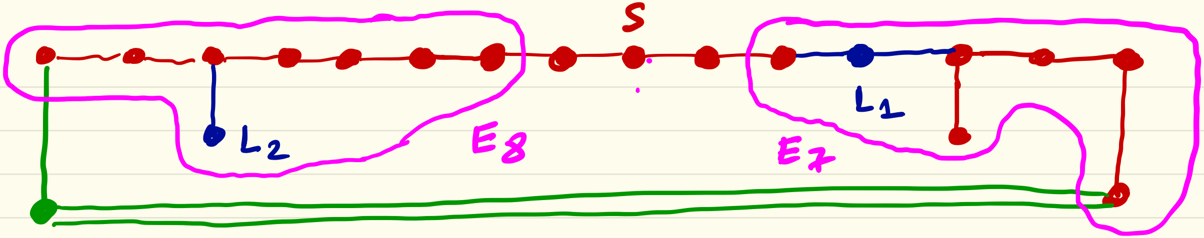


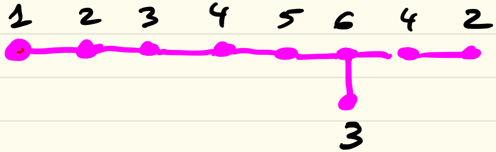
Singular Points
(generic case)

$P_1 : [0\ 1\ 0\ 0] \leftarrow A_{11}$

$P_2 : [0\ 0\ 1\ 0] \leftarrow A_5 \quad (c \neq 0)$

$$\boxed{2cx - dw = [3ac^2d + 2bc^3 - d^3]\, zw^2 - c^3w^3 + 2c^3y^2z = 0}$$



$M = E_8 \oplus E_7 \oplus \langle S,$  $\rangle$

Theorem [C-Doran]

(a) All $H \oplus E_8 \oplus E_7$-polarized K3 surfaces, up to izomorphism, can be realized as $X(a, b, c, d)$

(b) $X(a, b, c, d) \simeq X(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ if and only if:

$$\boxed{\begin{array}{c} \tilde{a} = t^2 \cdot a, \quad \tilde{b} = t^3 \cdot b, \quad \tilde{c} = t^5 \cdot c, \quad \tilde{d} = t^6 \cdot d \\ \text{for some } t \in \mathbb{C}^* \end{array}}$$

$$\mathcal{M}_{H\oplus E_8\oplus E_7} \stackrel{def}{=} \{ [a,b,c,d] \in WP(2,3,5,6) \mid (c,d) \neq (0,0) \}$$

is a coarse moduli space for $H\oplus E_8 \oplus E_7$ - polarized K3 surfaces.

Lemma $$\mathcal{F}_{H\oplus E_8\oplus E_7} = \underbrace{Sp(4,\mathbb{Z}) \backslash \mathcal{H}_2}_{\text{Siegel Modular Threefold}}$$

$$\mathcal{H}_2 = \left\{ \tau = \begin{pmatrix} x & y \\ y & z \end{pmatrix} \in M(2\times 2, \mathbb{C}) \;\middle|\; \mathrm{Im}(x)\cdot\mathrm{Im}(z) > \mathrm{Im}(y)^2,\ \mathrm{Im}(x) > 0 \right\}$$

$$Sp(4,\mathbb{Z}) = \left\{ m = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(4,\mathbb{Z}) \;\middle|\; M^t J M = J \text{ where } J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \right\}$$

$$m \cdot \tau = (A\tau + B)(C\tau + D)^{-1}$$

Obs:

- $$Sp(4,\mathbb{Z})\backslash \mathcal{H}_2 \simeq \underbrace{\mathcal{A}_2}_{\text{Moduli space of P.P. abelian surfaces}}$$

$E_1 \times E_2$

$\mathrm{Jac}(C)$

C genus-two curve

- Siegel modular forms in genus two

Theorem [Igusa]

$$\underbrace{A^*(Sp(4,\mathbb{Z}),\mathbb{C})}_{\text{graded ring of modular forms for } Sp(4,\mathbb{Z})} \simeq \mathbb{C}[\underbrace{\mathcal{E}_4,\mathcal{E}_6}_{\text{Eisenstein Series}},\underbrace{C_{10},C_{12},C_{35}}_{\text{Cusp Forms}}] \Big/ C_{35}^2 = P_{70}(\mathcal{E}_4,\mathcal{E}_6,C_{10},C_{12})$$

$$C_{10} = (\cdots)\left[\mathcal{E}_4\mathcal{E}_6 - \mathcal{E}_{10}\right], \quad C_{12} = (\cdots)\left[3^2\cdot 7^2\,\mathcal{E}_4^3 + 2\cdot 5^3\cdot\mathcal{E}_6^2 - 691\cdot\mathcal{E}_{12}\right]$$

Theorem [C-Doran] Inverse period map for $M = H \oplus E_8 \oplus E_7$:

$$[a, b, c, d] = [\mathcal{E}_4(\tau), \mathcal{E}_6(\tau), 2^{12} \cdot 3^5 \cdot C_{10}, 2^{12} \cdot 3^6 \cdot C_{12}]$$

Observations :

- $\{X, H \oplus E_8 \oplus E_7\text{-polarized}\} \xleftrightarrow{\text{one-to-one corresp.}} \{A, \text{ P.P. abelian surface}\}$
- $C_{10} = 0 \Rightarrow A = E_1 \times E_2 \quad [M = H \oplus E_8 \oplus E_8]$
- $C_{10} \neq 0 \Rightarrow A = \mathrm{Jac}(C)$, C genus-two curve
- VSN duals : $\underbrace{\mathrm{Km}(A)}$

(8) $M = \underline{H \oplus E_7 \oplus E_7}$, $r = 16$

Theorem [C-Doran-Malmendier]

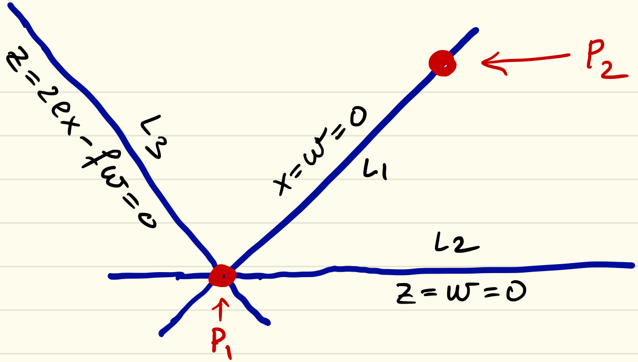
Let $(a,b,c,d,e,f) \in \mathbb{C}^6$ with $(c,d) \neq (0,0)$ and $(e,f) \neq (0,0)$

Consider the following quartic surface in $\mathbb{P}^3$:

$$\boxed{y^2 z w - 4x^3 z + 3a x z w^2 + b z w^3 + c x z^2 w - \frac{1}{2}\left(d z^2 w^2 + f w^4\right) + e x w^3 = 0}$$

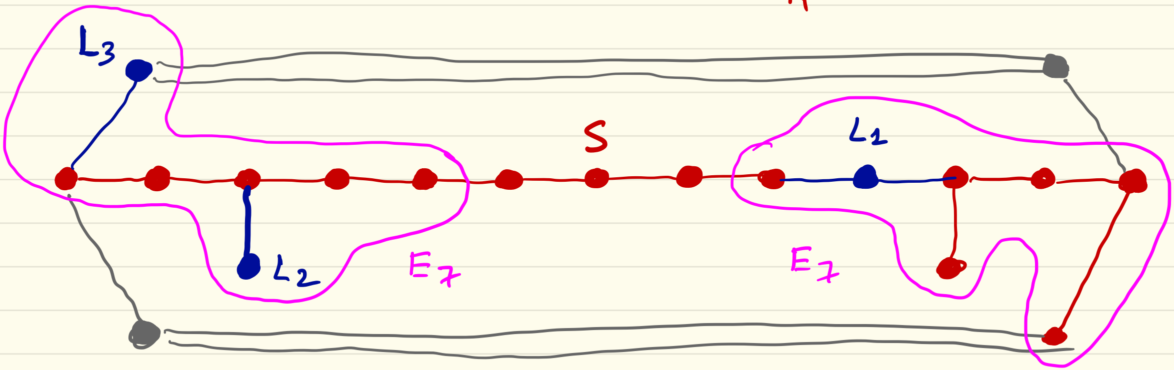
Singularities of above quartic are rational double points and the minimal resolution $X(a,b,c,d,e,f)$ is a K3 surface carrying a canonical $H \oplus E_7 \oplus E_7$ polarization.

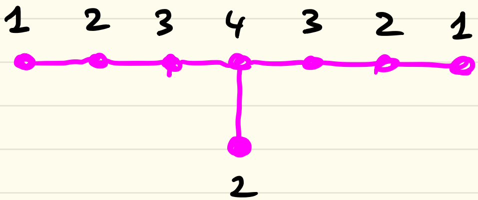
Singular Points
(generic case)



P_1 : [0 1 0 0] $\leftarrow$ A_9

P_2 : [0 0 1 0] $\leftarrow$ A_5



$M = E_7 \oplus E_7 \oplus \langle S,$  $\rangle$

Theorem [C-Doran-Malmendier]

(a) All $H \oplus E_7 \oplus E_7$-polarized K3 surfaces, up to izomorphism, can be realized as $X(a,b,c,d,e,f)$

(b) $X(a,b,c,d,e,f) \simeq X(\tilde{a},\tilde{b},\tilde{c},\tilde{d},\tilde{e},\tilde{f})$

if and only if:

(i) $(\tilde{a},\tilde{b},\tilde{c},\tilde{d},\tilde{e},\tilde{f}) = (a,b,e,f,c,d)$

OR

(ii) $(\tilde{a},\tilde{b},\tilde{c},\tilde{d},\tilde{e},\tilde{f}) = (t^2 \cdot a, t^3 \cdot b, t^5 \cdot c, t^6 \cdot d, t^{-1} \cdot e, f)$

for some $t \in \mathbb{C}^*$

One defines then the following invariants :

$$K_4 = a$$

$$K_6 = b$$

$$K_8 = ce$$

$$K_{10} = cf + de$$

$$K_{12} = df$$

Theorem [C-Doran-Malmendier]

$$\mathcal{M}_{H \oplus E_7 \oplus E_7} = \left\{ [K_4, K_6, K_8, K_{10}, K_{12}] \in WP(2,3,4,5,6) \;\middle|\; (K_8, K_{10}, K_{12}) \neq (0\,0\,0) \right\}$$

is a coarse moduli space for

$H \oplus E_7 \oplus E_7$ - polarized K3 surfaces.

Period Moduli Space

Matsumoto - Sasaki - Yoshida

$$\mathcal{F}_{H\oplus E_7\oplus E_7} = \underbrace{G}\backslash D$$

Hermitian Symmetric Domain of Type IV

$$D = \left\{ W \in M(2\times 2, \mathbb{C}) \;\middle|\; i(W^* - W) > 0 \right\}$$

$$W = \begin{pmatrix} x & y \\ z & w \end{pmatrix}, \qquad 4\cdot \mathrm{Im}(x)\cdot \mathrm{Im}(w) > |y - \bar{z}|^2$$

$$G_1 = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(4, \mathbb{Z}[i]) \;\middle|\; g^* J g = J, \; J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \right\}$$

$$*W \overset{\text{def}}{=} W^t \qquad\qquad *g*W = \bar{g}\, W$$

$$G = G_1 \rtimes \langle * \rangle$$

__Theorem__ [C-Doran-Malmendier]

(i) $K_4, K_6, K_8, K_{10}, K_{12}$ are modular forms of weight 4, 6, 8, 10 and 12, respectively, with respect to group G.

(ii) $$\underbrace{A^{2*}(G, \mathbb{C})}_{\text{graded ring of even weight modular forms over } G} = \mathbb{C}[K_4, K_6, K_8, K_{10}, K_{12}]$$

VSN duals : K3 surfaces obtained as minimal resolutions of double covers of $\mathbb{P}^2$ branched over six distinct lines

(9) $P = H \oplus E_8 \oplus (A_1)^4 \simeq H \oplus E_7 \oplus D_4 \oplus A_1$ $\quad r = 14$

<u>Theorem</u> [C-Malmendier]

Let $(a,b,c,d,e,f,g,h,k,l) \in \mathbb{C}^{10}$ such that $(c,d) \neq (0,0)$ $(e\ f) \neq 0$, $(g,h) \neq (0,0)$ and $(k,l) \neq (0,0)$ symultaneously.

Consider the following quartic surface in $\mathbb{P}^3$:

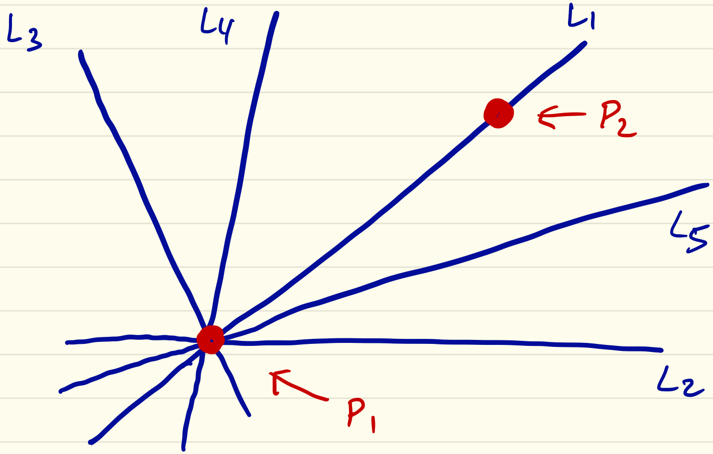
$$y^2 z w - 4x^3 z + 3\underline{a} x z w^2 + \underline{b} z w^3 -$$
$$- \frac{1}{2}(2\underline{c}x - \underline{d}w)(2\underline{g}x - \underline{h}w)z^2 - \frac{1}{2}(2\underline{e}x - \underline{f}w)(2\underline{k}x - \underline{l}w) = 0$$

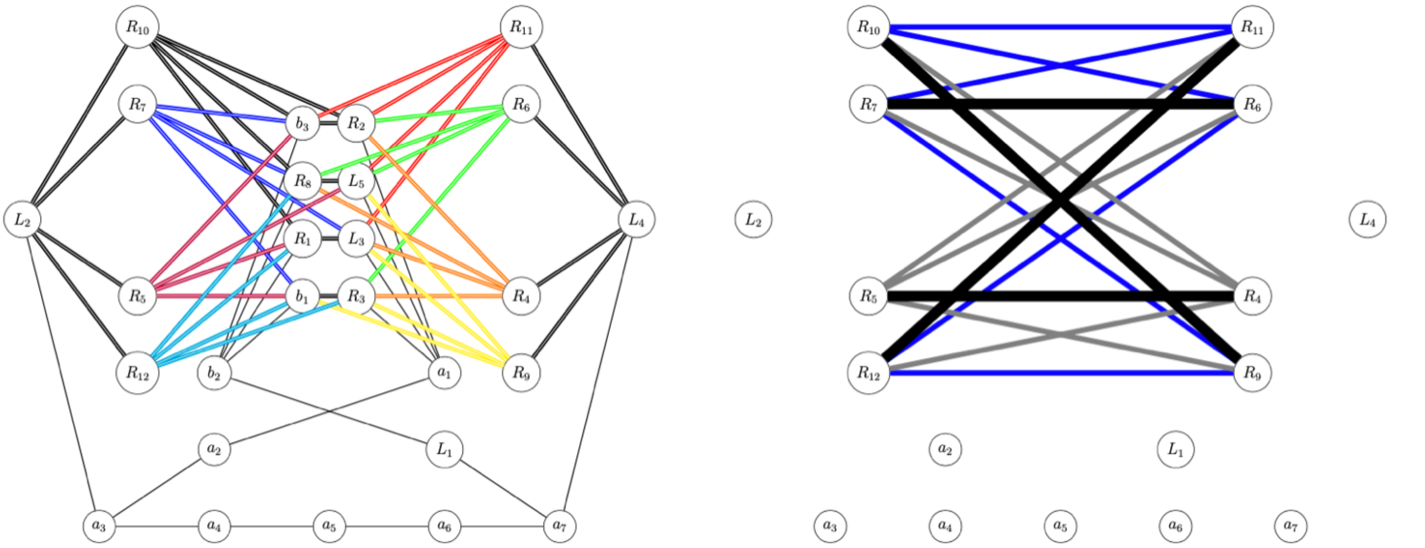
The singularities of the above quartic are rational double points. The minimal resolution $X(a\,b\,c\,d\,e\,f\,g\,h\,k\,l)$ is a K3 surface carrying a canonical P-polarization.

Singular Points
(generic case)

$P_1 : [0\,1\,0\,0] \leftarrow A_7$

$P_2 : [0\,0\,1\,0] \leftarrow A_3$





(A) with double lines and simple lines (B) with 6-fold lines (thick), 4-fold lines (thin)

Theorem [C.-Malmendier]

The quartic coefficients define seven invariants

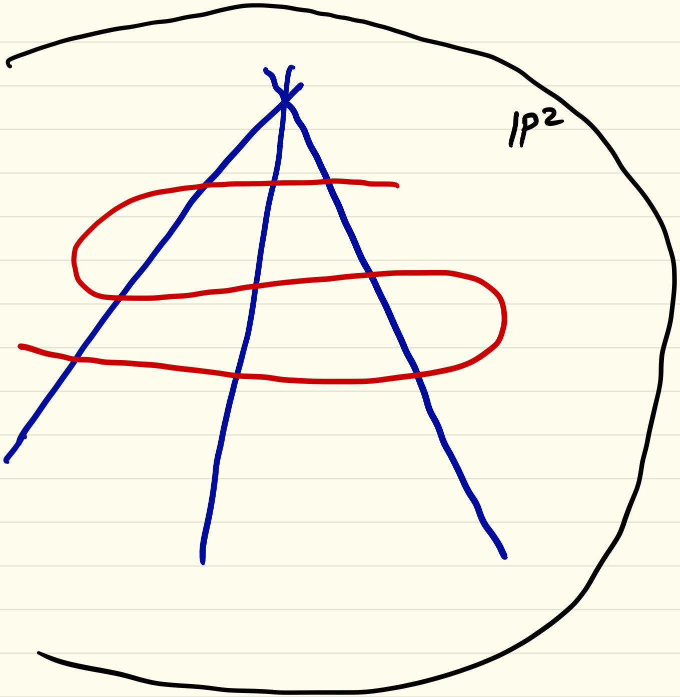
$$(K_4, K_6, K_8, K_{10}, K_{12}, B_6, B_8).$$

The six-dimensional open space:

$$\left\{ [K_4, K_6, K_8, K_{10}, K_{12}, B_6, B_8] \in \mathbb{WP}(2,3,4,5,6,3,4) \,\middle|\, (K_8, K_{10}, K_{12}, B_6, B_8) \neq (0\;0\;0\;0\;0) \right\}$$

is a coarse moduli space for P-polarized K3 surfaces

VSN - duals for P-polarized K3 surfaces



Double covers of $\mathbb{P}^2$ branched over three lines and a conic.