The degree of algebraic cycles on hypersurfaces / $\mathbb{C}$

Conj.: (Griffiths-Harris 1985)

Let $X \subset \mathbb{P}^4$ very general hypersurface of degree $d \geq 6$.

$\Rightarrow$ $d \mid \deg C$ $\forall$ curves $C \subset X$

Remarks:
- Lefschetz hyperplane thm.:

$$H^2(X, \mathbb{Z}) = \mathbb{Z} \cdot \alpha, \qquad \alpha \in H^{1,1}(X, \mathbb{Z}) \text{ hyperplane class } (\alpha^3 = d)$$

- Poincaré duality:

$$H^{2,2}(X,\mathbb{Z}) = H^4(X, \mathbb{Z}) = \mathbb{Z} \cdot \tfrac{1}{d} \alpha^2 .$$

- $$Z^4(X) = \frac{H^{2,2}(X, \mathbb{Z})}{\langle \text{alg. classes} \rangle} = \frac{\mathbb{Z} \cdot \frac{1}{d} \alpha^2}{\langle \deg C \cdot \frac{1}{d} \alpha^2 \mid C \subset X \text{ curve} \rangle}$$

$$= \mathbb{Z} / f(d) \cdot \mathbb{Z}$$

where $f(d) = \gcd \{ \deg C \mid C \subset X \text{ curve} \}$

$$\text{IHC} \iff f(d) = 1 \iff Z^4(X) = 0$$

$$\text{GHC} \iff f(d) = d \iff Z^4(X) = \mathbb{Z}/d$$

Thm.: (Kollár 1991) $d \mid 6 \cdot f(d^3)$ $\forall d \geq 1$

Thm.: (Kollár 1991) $d \mid 6 \cdot f(3d^2)$ $\forall d \geq 4$

Thm.: (van Geemen 1991) $3d \mid 2 \cdot f(54d)$ $\forall d \geq 2$

Thm.: (Debarre-Hulek-Spandaw 1994) $d \mid 2 \cdot f(6d)$ $\forall d \geq 9$

Thm.: (P. 2021)
- The set $\{ d \in \mathbb{Z}_{>0} \mid f(d) = d \}$ has positive density, i.p. its infinite.

$$f(\underbrace{5 \cdot 7 \cdot 11 \cdot 13}_{5005}) = 5005$$

- The set $\{ d \in \mathbb{Z}_{>0} \mid f(d) \neq 1 \}$ has density 1.

More generally: • For every $n \geq 3$, there exist degrees d with positive density s.t. for v.g. $X \subset \mathbb{P}^{n+1}$ of deg. d we have $\operatorname{coker} \left(CH^i(X) \xrightarrow{\deg} \mathbb{Z} \right) = \mathbb{Z}/d$ for all $i < n$.

- For every $n \geq 3$, there exist degrees d of density 1 s.t. for v.g. $X \subset \mathbb{P}^{n+1}$ of deg. d

$$Z^{2c}(X) \neq 0 \quad \forall \tfrac{n}{2} < c < n.$$

<u>Lemma:</u> $X \subset \mathbb{P}^{n+1}$ v.g. of deg. d, $C \subset X$ curve

$\Rightarrow \forall X_0 \subset \mathbb{P}^{n+1}$ of deg. d $\exists C_0 \subset X_0$ curve with $\deg C_0 = \deg C$.

<u>Proof:</u>

$$\begin{array}{c} \bigcup_P \mathcal{H}_P = \{ C \subset X \subset \mathbb{P}^{n+1} \mid C \text{ 1-dim. subscheme of } X \text{ with Hilbert poly. } P \} \\ \downarrow \varrho_P \\ \mathbb{P}^N = \{ X \subset \mathbb{P}^{n+1} \text{ of deg. } d \} \end{array}$$

Take $[X] \in \mathbb{P}^N \setminus \bigcup_{\varrho_P(\mathcal{H}_P) \neq \mathbb{P}^N} \varrho_P(\mathcal{H}_P)$. □

<u>Lemma:</u> (Kollár) Y smooth proj. 3-fold

L very ample l.b. on Y, $L^3 = d$

Assume $k \mid B \cdot L$ $\forall$ curves $B \subset Y$

$\Rightarrow k \mid 6 \cdot f(d)$

<u>Proof:</u>

$$\begin{tikzcd} Y \arrow[r, hook, "|L|"] \arrow[dr, "\pi"'] & \mathbb{P}^r \arrow[d, dashed, "\text{general linear proj.}"] \\ & X_0 \end{tikzcd}$$

$X_0 = \pi(Y) \subset \mathbb{P}^4$ hypersurface of deg. d

<u>fact:</u> all but finitely many fibres of π have ≤ 3 pts.

$\forall C_0 \subset \pi(Y)$: $B := \pi^{-1}(C_0)^{\text{red}} \xrightarrow{\pi} C$ finite cover of deg. ≤ 3

$\Rightarrow k \mid B \cdot L = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \cdot \deg C_0$

$\Rightarrow k \mid 6 \cdot \deg C_0$

$\overset{\text{degeneration}}{\Longrightarrow} k \mid 6 \cdot f(d)$. □

<u>Applications:</u> • $Y = \mathbb{P}^3$, $L = \mathcal{O}_{\mathbb{P}^3}(d)$, $L^3 = d^3$, $d \mid B \cdot L$ $\forall B \subset Y$

$\Rightarrow d \mid 6 \cdot f(d^3)$.

- $Y = S \times \mathbb{P}^1$, $\quad L = pr_1^* \mathcal{O}_S(1) \otimes pr_2^* \mathcal{O}_{\mathbb{P}^1}(d)$

  $S \subset \mathbb{P}^3$ v.g. of deg. $d \geq 4$ $\quad \Rightarrow d \mid B \cdot L \ \forall B \subset Y$

  $L^3 = 3d^2$

$$\Longrightarrow \quad d \mid 6 \cdot f(3d^2)$$

- Y abelian 3-fold, $L =$ l.b. of type $(1,1,d)$, $d \geq 9$

  $\xRightarrow{[DHS]}$ L very ample

  $L^3 = 6d$

$$\Longrightarrow \quad 3d \mid 6 \cdot f(6d)$$
$$\Longrightarrow \quad d \mid 2 \cdot f(6d).$$

<u>Lemma:</u> $d \mid f(d_1)$ and $d \mid f(d_2)$ $\Longrightarrow$ $d \mid f(d_1 + d_2)$

<u>pf.:</u> specialize $X \subset \mathbb{P}^4$ v.g. of deg. $d_1 + d_2$ to a union $X_1 \cup X_2 \subset \mathbb{P}^4$, $\quad X_i \subset \mathbb{P}^4$ v.g. of deg. d_i $\square$

e.g. how to prove $f(5005) = 5005$?

$$5005 = 5 \cdot 7 \cdot 11 \cdot 13$$

<u>Lemma:</u> If $\overset{\text{prime power}}{q} \mid d$, $2q^3 + 3q^2 + 54 \leq d$, $\gcd(6, q) = 1$

$$\Longrightarrow q \mid f(d)$$

<u>pf.:</u> write $d = i \cdot q^3 + j \cdot 3q^2 + k \cdot 6$ with $i \in \{0,1,2\}$, $j \in \{0,1\}$, $k \geq 9$

$$\left.\begin{array}{l} \boxed{q \mid f(q^3)} \\ q \mid f(3q^2) \\ q \mid f(6k) \end{array}\right\} \xRightarrow{\text{Lemma}} q \mid f(d)$$

$\square$

<u>Thm.:</u> $\exists\, X \subset \mathbb{P}^4$ defined over $\mathbb{Q}$ of some deg. $d > 1$ s.t. $\forall\, C \subset X$ curve: $d \mid \deg C$.