"INTERMEDIATE JACOBIANS & AUTOMORPHISMS OF SMOOTH HYPERSURFACES OF THE PROJECTIVE SPACE" / $\mathbb{C}$

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§ 0. Motivation

"Classical" fact: Let $(A, \Theta) \in \mathcal{A}_g$ ppav. If $p = 2g+1$ is prime $\Rightarrow$ The order of $\varphi \in \mathrm{Aut}(A, \Theta)$ is $\leq p$.

$\boxed{\text{Torelli}}$: The same is true for $\varphi \in \mathrm{Aut}(C)$, where $C \in \mathcal{M}_g$

? Can we do this in higher dimension?

Adler (1978): The Klein cubic 3-fold

$$X_K = \{x_0^2x_1 + x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_0 = 0\} \subseteq \mathbb{P}^4$$

has $\mathrm{Aut}(X_K) \simeq PSL_2(\mathbb{F}_{11})$, with $|PSL_2(\mathbb{F}_{11})| = 2^2 \cdot 3 \cdot 5 \cdot 11$.

⚠ Clemens - Griffiths (1972): Let $X \subseteq \mathbb{P}^4$ smooth cubic 3-fold. Then, X is determined by its INTERMEDIATE JACOBIAN

$$J(X) := H^{2,1}(X)^*/H_3(X,\mathbb{Z}) \in \mathcal{A}_5$$

and $J(X) \simeq \mathrm{Alb}(S)$, where $S = F_1(X)$ Fano scheme of lines on X.

Roulleau (2009): Proved that $\mathrm{Aut}(X) \simeq \mathrm{Aut}(S) \subseteq \mathrm{Aut}(\mathrm{Alb}(S), \Theta)$ and thus $|\mathrm{Aut}(X)|$ divides $2^{23} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11$.

Moreover:

1° $\nexists\, \varphi \in \mathrm{Aut}(X)$ of order 7.

2° $\exists\, \varphi \in \mathrm{Aut}(X)$ of order 11 $\Leftrightarrow X \simeq X_K$ Klein cubic 3-fold.

Today: How to extend this to other hypersurfaces?
What about $X \subseteq \mathbb{P}^6$ smooth cubic 5-fold and its Intermediate Jacobian $J(X) \in \mathcal{A}_{21}$?

§1. Notation and History

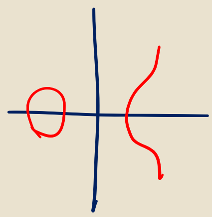
Let us consider $F \in \mathbb{C}[x_0, \dots, x_{n+1}]$ homogeneous $\neq 0$ of degree d and let

$$X := \{ x \in \mathbb{P}^{n+1} \text{ st } F(x) = 0 \} \subseteq \mathbb{P}^{n+1}$$

be a SMOOTH hypersurface of $\begin{cases} \dim(X) = n \geq 1 \\ \deg(X) = d \geq 3 \end{cases}$

Examples (that will appear later):

a) ELLIPTIC CURVES: Let $a, b \in \mathbb{C}$



$$C := \{ [x,y,z] \in \mathbb{P}^2 \text{ st } y^2 z = x^3 + axz^2 + bz^3 \} \subseteq \mathbb{P}^2$$

is smooth $\iff \Delta = 4a^3 + 27b^2 \neq 0 \rightsquigarrow C$ is an elliptic curve.

b) QUARTIC K3 SURFACES: $S \subseteq \mathbb{P}^3$ smooth surface of degree 4 ($\Rightarrow \pi_1(S) \simeq \{1\}$ and $\omega_S \cong \mathcal{O}_S$, i.e., S is K3).

Construction: Consider 4 bilinear equations of the form

$$H_k = \{ \sum_{i,j=0}^{3} a_{ij}^k x_i y_j = 0 \} \subseteq \mathbb{P}^3_x \times \mathbb{P}^3_y \quad (k = 1, \dots, 4)$$

and let $S := H_1 \cap H_2 \cap H_3 \cap H_4 \subseteq \mathbb{P}^3 \times \mathbb{P}^3$.

What is $pr_1(S) \subseteq \mathbb{P}^3_x$?

Note that $p \in \mathbb{P}^3_x$ belongs to $pr_1(S)$ if and only if $B(p)\,y = 0$ has a solution $y_0 \neq 0 \iff \det B(p) = 0$

$\hookrightarrow B = (b_{kj})$ with $b_{kj} = \sum_i a^k_{ij} x_i$ linear form

$\rightsquigarrow pr_1(S) = \{x \in \mathbb{P}^3 \text{ st } \det B(x) = 0\} =: S_1 \leftarrow$ degree 4 surface

Similarly: $pr_2(S) = \{y \in \mathbb{P}^3 \text{ st } \det C(y) = 0\} =: S_2$

$\hookrightarrow C = (c_{ki}) = (\sum_j a^k_{ij} y_j)$

FACT: If the coefficients a^k_{ij} are GENERAL then:
- i) $S \subseteq \mathbb{P}^3 \times \mathbb{P}^3$ is smooth ✓
- ii) $S \cong S_1 \subseteq \mathbb{P}^3$ and $S \cong S_2 \subseteq \mathbb{P}^3$ via the projections ✓

c) FERMAT HYPERSURFACE:

$$X_F = \{x \in \mathbb{P}^{n+1} \text{ st } x_0^d + x_1^d + \dots + x_{n+1}^d = 0\} \text{ smooth } ✓$$

d) KLEIN HYPERSURFACE:

$$X_K = \{x \in \mathbb{P}^{n+1} \text{ st } x_0^{d-1}x_1 + x_1^{d-1}x_2 + \dots + x_n^{d-1}x_{n+1} + x_{n+1}^{d-1}x_0 = 0\}$$

smooth (since $d \geq 3$) ✓

$\boxed{\text{Question}}$: Given two smooth hypersurfaces $X_1, X_2 \subseteq \mathbb{P}^{n+1}$, when $X_1 \cong X_2$ as abstract/embedded varieties?

If $X_1 = X_2 = X$ one should look at

$$\mathrm{Aut}(X) := \{\varphi : X \xrightarrow{\sim} X \text{ isomorphism (as abstract variety)}\}$$

GROUP OF (REGULAR) AUTOMORPHISMS.

Examples:

a) $\mathrm{Aut}(\mathbb{P}^{n+1}) \cong PGL_{n+2}(\mathbb{C})$

b) Given $X \subseteq \mathbb{P}^{n+1}$ hypersurface, we define

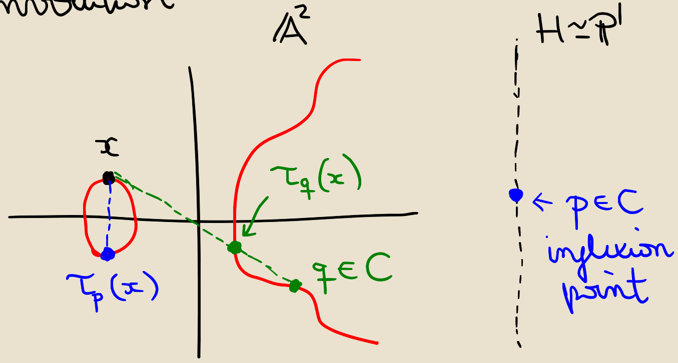
$$\mathrm{Lin}(X) := \{\varphi \in \mathrm{Aut}(X) \text{ st } \exists\, \Phi \in PGL_{n+2}(\mathbb{C}) \text{ st } \varphi = \Phi|_X\}$$

the group of LINEAR automorphisms.
(e.g. permutation matrices act on $X_F = \{x_0^d + \dots + x_{n+1}^d = 0\}$)

c) <u>ELLIPTIC CURVES</u>: Fix $p_0 \in C$ and let

$\tau_{p_0} : C \longrightarrow C$, $x \longmapsto$ 3rd intersection point of $\langle p_0, x\rangle$ with C

Involution



$\tau_{p_0} \in \mathrm{Lin}(C)$ if and only if p_0 is an inflexion point

$\Longrightarrow \mathrm{Lin}(C) \subsetneq \mathrm{Aut}(C)$!

d) <u>QUARTIC K3 SURFACES</u>: With the same notation as before, we have

$$S \cong \underbrace{\{\det B = 0\} \subseteq \mathbb{P}^3}_{= S_1} \quad \text{and} \quad S \cong \underbrace{\{\det C = 0\} \subseteq \mathbb{P}^3}_{= S_2}$$

The Cramer's rule from Linear Algebra allows us to construct an isomorphism $\varphi : S_1 \xrightarrow{\sim} S_2$ given by degree 3 polynomials (3×3 minors !) $\underset{(\dots)}{\rightsquigarrow}$ $\mathrm{Lin}(S_1) \subsetneq \mathrm{Aut}(S_1)$

$$\boxed{\begin{array}{l}\textbf{Thm} \text{ (Matsumura–Monsky 1964): } X \subseteq \mathbb{P}^{n+1} \text{ smooth hypersurface of dim } n \geq 1 \text{ and degree } d \geq 3. \text{ Then:}\\ \text{(1) } \mathrm{Lin}(X) \text{ is a finite group (cf. Jordan 1880)}\\ \text{(2) If } X \text{ is general (in its moduli space)} \Rightarrow \mathrm{Aut}(X) = \{\mathrm{Id}_X\}\\ \text{(3) If } (n,d) \notin \{(1,3),(2,4)\} \Rightarrow \mathrm{Lin}(X) = \mathrm{Aut}(X).\end{array}}$$

Even better: By means of "classical" results (Noether, Lefschetz, Matsusaka, Mumford, ...) together with recent ones (Oguiso 2016, Shimada - Shioda 2017) we have:

$\boxed{\begin{array}{l}\textbf{Thm:} \text{ Let } X_1, X_2 \subseteq \mathbb{P}^{n+1} \text{ be smooth hypersurfaces of degrees} \\ d_1, d_2 \geq 3 \text{ respectively. Suppose that } \varphi : X_1 \xrightarrow{\sim} X_2 \text{ is an} \\ \text{isomorphism, Then } \exists\, \Phi \in PGL_{n+2}(\mathbb{C}) \text{ st } \varphi = \Phi|_{X_1} \text{ except (maybe)} \\ \text{in the following cases:} \\ \quad 1^\circ \ n=1 \text{ and } d_1 = d_2 = 3 \text{ (elliptic curves)} \\ \quad 2^\circ \ n=2 \text{ and } d_1 = d_2 = 4 \text{ (quartic K3 surfaces).}\end{array}}$

§2. Recent works: Let $n, d \in \mathbb{N}^{\geq 1}$ be fixed.

$N+1 = \binom{n+d+1}{d}$

Recall that $\mathcal{H}_n(d) := \mathbb{P}H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d)) \cong \mathbb{P}^N$

parametrizes hypersurfaces $X \subseteq \mathbb{P}^{n+1}$ of degree d, and that

$$\mathcal{U}_n(d) := \{ \underline{\text{smooth}} \text{ hypersurfaces } X \subseteq \mathbb{P}^{n+1} \text{ of deg } d\} \subseteq \mathcal{H}_n(d)$$

is a Zariski open subset st $\mathcal{H}_n(d) \setminus \mathcal{U}_n(d) = \mathcal{D} = \{\Delta = 0\}$ is a divisor.

"discriminant"

$\boxed{\textbf{Assumption: } n \geq 1,\ d \geq 3 \text{ and } (n,d) \notin \{(1,3),(2,4)\}}$

↳ In particular, for every $X \in \mathcal{U}_n(d)$ we have that $\mathrm{Aut}(X) \subseteq PGL_{n+2}(\mathbb{C})$ is $\underline{\text{finite}}$.

Ehresmann (1951): All the $X \in \mathcal{U}_n(d)$ are diffeomorphic. In part, $H^n(X, \mathbb{Z}) \cong \mathbb{Z}^{\oplus b_n}$ are constant

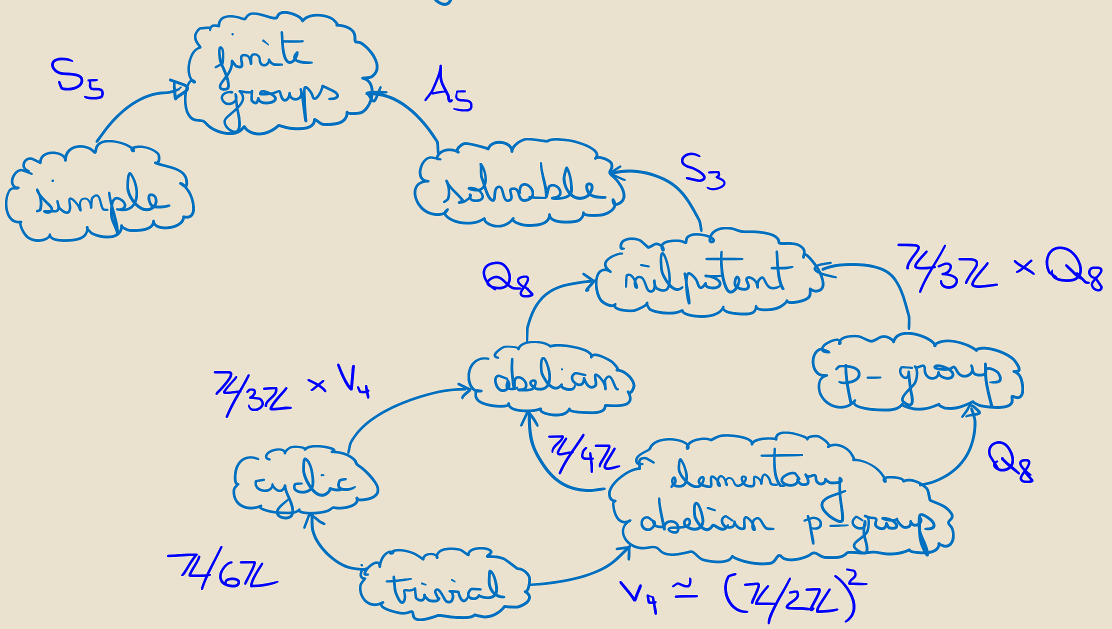
$$b_n = \underline{(d-1)^{n+2} + (-1)^n (d-1)}$$
$$d$$

Moreover, $\mathrm{Aut}(X) \curvearrowright H^n(X, \mathbb{Z})$ faithfully (def. theory)

$\xrightarrow{\text{Minkowski (1887)}}$ $\exists\, C(n,d)$ st $|\mathrm{Aut}(X)| \leq C(n,d)$ for all $X \in \mathcal{U}_n(d)$

$\boxed{\begin{array}{l}\textbf{Main Question: } \text{Which finite groups } G \text{ are ADMISSIBLE in} \\ \mathcal{U}_n(d) \text{ (i.e., } \exists\, X_d^n \subseteq \mathbb{P}^{n+1} \text{ smooth st } G \subseteq \mathrm{Aut}(X)) \ ?\end{array}}$

Hierarchy of Finite Groups:



Some results:

✓ Dolgachev & Iskovskikh (2009): $S \subseteq \mathbb{P}^3$ smooth cubic surface
⇝ Det. all admissible G (11 non-isomorphic of max. order).

✓ González-Aguilera & Liendo (2011, 2013):
Let $q = p^\alpha$ for $\alpha \in \mathbb{N}^{\geq 1}$ and p prime. Assume that q is relatively prime to d and $d-1$. Then:

$G \simeq \mathbb{Z}/q\mathbb{Z}$ admissible in $\mathcal{U}_n(d)$ $\iff$ $\exists\, l \in \{1, \dots, n+2\}$ st. $(1-d)^l \equiv 1 \pmod{q}$.

✓ Pambianco (2014) and Harui (2019): $C \subseteq \mathbb{P}^2$ smooth curve of degree $d \geq 4$. Then, $|\mathrm{Aut}(C)| \leq 6d^2$ with "=" if and only if $C \simeq \{x_0^d + x_1^d + x_2^d = 0\}$ FERMAT. Unless:

① $d = 4$ ⇝ $C_{max} \simeq \{x_0^3 x_1 + x_1^3 x_2 + x_2^3 x_0 = 0\}$ KLEIN ⇝ $PSL_2(\mathbb{F}_7)$

② $d = 6$ ⇝ $C_{max} \simeq \{10x_0^3x_1^3 + 9x_0^5x_2 + 9x_1^5x_2 - 45x_0^2x_1^2x_2^2 - 135x_0x_1x_2^4 + 27x_2^6 = 0\}$
↳ "WIMAN sextic" ⇝ A_6

✓ Oguiso & Yu (2019): $X \subseteq \mathbb{P}^4$ quintic threefold
⇝ Det. all admissible G (22 non-isomorphic of max. order)

✓ Adler (1978) + Wei & Yu (2020): $X \subseteq \mathbb{P}^4$ cubic threefold
⇝ Det. all admissible G (6 non-isomorphic of max. order)

✓ Zheng (arXiv 2020): Det. ALL $G \cong \mathbb{Z}/q\mathbb{Z}$ admissible in $\mathcal{U}_n(d)$

§3. A Liftability result

Dolgachev & Iskovskikh $\rightsquigarrow$ Study $\mathrm{Aut}(S) \curvearrowright H^2(S,\mathbb{Z}) \cong \mathbb{Z}^7$

{ Oguiso & Wei & Yu $\rightsquigarrow$ Study whether or not a group
{ $G \subseteq \mathrm{Aut}(X) \subseteq PGL_5(\mathbb{C})$ admits a LIFTING $\tilde{G} \subseteq GL_5(\mathbb{C})$

our starting point!

Representation Theory!

Let $X = \{x \in \mathbb{P}^{n+1} \text{ st } F(x) = 0\}$ as before, and let $G \subseteq \mathrm{Aut}(X) \subseteq PGL_{n+2}(\mathbb{C})$ be a subgroup. We say that a subgroup $\tilde{G} \subseteq GL_{n+2}(\mathbb{C})$ is a LIFTING of G if:

1° $\pi|_{\tilde{G}} : \tilde{G} \subseteq GL_{n+2}(\mathbb{C}) \xrightarrow{\sim} G \subseteq PGL_{n+2}(\mathbb{C})$ isomorphism.

2° For every $g \in \tilde{G}$ we have $g \cdot F = F$

Examples:

a) $X_F = \{x_0^d + \dots + x_{n+1}^d = 0\} \subseteq \mathbb{P}^{n+1} \Rightarrow \mathrm{Aut}(X_F) \cong S_{n+2} \ltimes (\mathbb{Z}/d\mathbb{Z})^{n+1}$ liftable ✓

b) $X_K = \{x_0^{d-1}x_1 + x_1^{d-1}x_2 + \dots + x_{n+1}^{d-1}x_0 = 0\} \subseteq \mathbb{P}^{n+1}$, and assume that $\gcd(d, n+2) > 1$. Given p prime st $p \mid d$ & $p \mid (n+2)$, we consider

$$\tilde{g} := \mathrm{diag}(\overbrace{1, \xi, \xi^2, \dots, \xi^{p-1}, \dots, 1, \xi, \xi^2, \dots, \xi^{p-1}}^{(n+2)/p \text{ -times}}) \qquad \xi = e^{\frac{2\pi i}{p}}$$

$\Rightarrow \tilde{g}$ induces $g \in \mathrm{Aut}(X_K)$. However, $\tilde{g} \cdot K = \xi K$ is not liftable!

Thm A (GA. L. M. 2020): $X \subseteq \mathbb{P}^{n+1}$ smooth hypersurface of degree $d \geq 3$ st $(n,d) \notin \{(1,3),(2,4)\}$. Then:

$$\mathrm{Aut}(X) \text{ liftable} \iff \gcd(d, n+2) = 1.$$

$\Uparrow$

Thm B (GA. L. M. 2020): Let $n \geq 1$ and $d \geq 3$ st $(n,d) \notin \{(1,3),(2,4)\}$ and let $q = p^r$ with $r \in \mathbb{N}^{\geq 1}$ and p prime. Then, q is the order of some liftable automorphism of some $X \in \mathcal{U}_n(d)$ iff

a) $p \mid (d-1)$ and $r \leq k(n+1)$, where $d-1 = p^k e$ & $p \nmid e$; or

b) $p \mid d$ and $\exists \ell \in \{1, \dots, n+1\}$ st $(1-d)^\ell \equiv 1 \pmod{q}$; or

c) $p \nmid d(d-1)$ and $\exists \ell \in \{1, \dots, n+2\}$ st $(1-d)^\ell \equiv 1 \pmod{q}$ ← [GA. L. 2011, 2013]

↳ An independent work of Z. Zheng (2020) generalizes this!

Cubic examples: Let $X \subseteq \mathbb{P}^{n+1}$ smooth cubic. Then, all the possible $\mathbb{Z}/p^r\mathbb{Z}$ which are admissible and liftable are:

- **Surfaces**: 2^{r_2} $(r_2 \leq 3)$, 3^{r_3} $(r_3 \leq 2)$ or 5.
- **Threefolds**: 2^{r_2} $(r_2 \leq 4)$, 3^{r_3} $(r_3 \leq 2)$, 5 or 11.
- **Fourfolds**: 2^{r_2} $(r_2 \leq 5)$, 3^{r_3} $(r_3 \leq 2)$, 5, 7 or 11.
- **Fivefolds**: 2^{r_2} $(r_2 \leq 6)$, 3^{r_3} $(r_3 \leq 2)$, 5, 7, 11 or 43.

As an application, we can get information about certain Sylow p-subgroups of $\mathrm{Aut}(X)$ (ie, of order p^r with r maximal)

Notation: Let p be a prime number st $p \nmid d(d-1)$ and let $r \in \mathbb{N}^{\geq 1}$. We define

$$\ell(p^r) := \min\{k \in \mathbb{N}^{\geq 1} \text{ st } (1-d)^k \equiv 1 \pmod{p^r}\}$$

Prop C (G.A.L.M. 2020): Let $X \subseteq \mathbb{P}^{n+1}$ smooth hypersurface of degree $d \geqslant 3$ st $\gcd(d, n+2) = 1$, and let p be a prime st $p \nmid d(d-1)$:

If $\ell(p^2) > n+2$ & $2\ell(p) > n+2 \implies p^2 \nmid |\mathrm{Aut}(X)|$.

In part, if $G \subseteq \mathrm{Aut}(X)$ p-Sylow then $G \simeq \{1\}$ or $\mathbb{Z}/p\mathbb{Z}$.

Cubic examples: Let $X \subseteq \mathbb{P}^{n+1}$ smooth cubic. Then:

$\boxed{n=2}$: $|\mathrm{Aut}(X)| = 2^{r_2} 3^{r_3} 5^{r_5}$ with $r_5 \leq 1$.

$\boxed{n=3}$: $|\mathrm{Aut}(X)| = 2^{r_2} 3^{r_3} 5^{r_5} 11^{r_{11}}$ with $r_5, r_{11} \leq 1$.

$\boxed{n=4}^{*}$: $|\mathrm{Aut}(X)| = 2^{r_2} 3^{r_3} 5^{r_5} 7^{r_7} 11^{r_{11}}$ with $r_5, r_7, r_{11} \leq 1$

$\boxed{n=5}$: $|\mathrm{Aut}(X)| = 2^{r_2} 3^{r_3} 5^{r_5} 7^{r_7} 11^{r_{11}} 43^{r_{43}}$ with $r_5, r_7, r_{11}, r_{43} \leq 1$

Idea: Enough to analyze the cases

$G \simeq \mathbb{Z}/p^2\mathbb{Z}$ ← impossible by assumption or $G \simeq (\mathbb{Z}/p\mathbb{Z})^2$ ← "not enough space"

§4. Sketch of Proof:

Let $X = \{F = 0\} \subseteq \mathbb{P}^{n+1} = \mathbb{P}(V)$ smooth hypersurface, $F \in S^d(V^*)$ with $(n,d) \notin \{(1,3),(2,4)\}$.

$\boxed{\text{Step 1}}$ Consider $\varphi \in \mathrm{Aut}(X) \subseteq PGL(V)$ and assume $\mathrm{ord}(\varphi) = q$.
Let $\tilde{\varphi} \in GL(V)$ st $\pi(\tilde{\varphi}) = \varphi$ and $\tilde{\varphi}^q = \mathrm{Id}_V$.

Key Rmk: Let p prime st $p \mid d$ and suppose $\mathrm{ord}(\varphi) = p$.
If φ is not liftable $\Rightarrow p \mid (n+2)$.

↳ Idea: $\tilde{\varphi} \cdot F = \xi F$, $\xi^p = 1$ and $\xi \neq 1$. (★)
Let $V(i) \subseteq V$ be the eigenspace asso. to ξ^i
(★) + X smooth $\Rightarrow \dim V(0) = \dim V(i)$ $\forall i \Rightarrow n+2 = p \dim V(0)$ ✓

Lemma: $\mathrm{ord}(\varphi) = q$ and $\tilde{\varphi} \cdot F = \xi^c F$ with $\xi^q = 1$ primitive and $c \in \mathbb{Z}$. Then, φ liftable $\iff \gcd(d,q) \mid c$.

In part., if φ is not liftable: $\exists\, p$ prime factor of $\gcd(d,q)$ st $p \nmid c$. Write $q = pr$ and $\psi := \varphi^r$ of order p.
$\rightsquigarrow \varphi$ not liftable $\Rightarrow \psi$ not liftable $\Rightarrow \gcd(d, n+2) > 1$.

$\boxed{\text{Step 2}}$ Consider φ st $\mathrm{ord}(\varphi) = p^r$ and φ liftable
$\rightsquigarrow$ We prove Thm B by analyzing the eigenspaces of $\tilde{\varphi} = \mathrm{diag}(\xi^{a_0}, \dots, \xi^{a_{n+1}})$
+ Providing explicit examples for the "if" part.

$\boxed{\text{Step 3}}$* Consider φ st $\mathrm{ord}(\varphi) = p^r$ and sup. $p \nmid d(d-1)$ or $p \nmid (n+2)$. Then, $\exists!$ lifting $\tilde{\varphi}$ to $SL(V)$.

↳ Application: X cubic 4-fold $\rightsquigarrow F_1(X) \underset{\text{defs}}{\simeq} K3^{[2]}$
$\varphi \in \mathrm{Aut}(X)$ induces $\hat{\varphi} \in \mathrm{Aut}(F_1(X))$. Then, for $\mathrm{ord}(\varphi) = p^r$ st $p \neq 2,3$ we have that $\hat{\varphi}$ symplectic $\Rightarrow \mathrm{ord}(\varphi) = 5, 7, 11$ (cf. Fu '16)

$\boxed{\text{Step 4}}$* Lifting of Sylow p-subgroups $G_p \leq \mathrm{Aut}(X)$ (to $SL(V)$) in many cases!) → Here: We use $\gcd(d, n+2) = 1$ to simplify and extend some arguments from group cohomology used by Oguiso–Yu (e.g. Hochschild–Serre exact seq.) $\rightsquigarrow$ Get a lifting of $\mathrm{Aut}(X)$ to $GL(V)$ ■

§5. Some words about cubic 5-folds $X \subseteq \mathbb{P}^6$:

To a smooth cubic 5-fold $X \subseteq \mathbb{P}^6 \rightsquigarrow J(X) := H^{3,2}(X)^* / H_5(X,\mathbb{Z}) \in \mathcal{A}_{21}$

Torelli theorem is known for GENERIC $X \subseteq \mathbb{P}^6$ (Donagi 1983).
Similar (Collino 1986): For GENERIC $X \subseteq \mathbb{P}^6$ we have

$J(X) \simeq \mathrm{Alb}(S)$, where $S = F_2(X)$ Fano scheme of planes.
Can be singular for special $X \subseteq \mathbb{P}^6$!

$\rightsquigarrow$ We are able to say something if $J(X) \in \mathcal{A}_{21}$ is an EXTREMAL ppav (ie, $\exists$ automorphism of order $p = 2g+1 = 43$).

(GAL 2011): $\exists\, \varphi \in \mathrm{Aut}(X)$ of order 43 $\iff X \simeq X_K \subseteq \mathbb{P}^6$ Klein cubic 5-fold.
$\Rightarrow J(X_K)$ is extremal and has CM

We show that:

1. $G := \mathrm{Aut}(J(X_K), \Theta) \not\simeq PSL_2(\mathbb{F}_{43})$ and hence (Bennama–Bertin 1997) $G^+ := \mathrm{Aut}(J(X_K), \Theta)/\{\pm \mathrm{Id}\} \simeq (\mathbb{Z}/43\mathbb{Z}) \rtimes \mathbb{Z}/m\mathbb{Z}$ st $m = 7$ or 21.
2. The liftability of $\mathrm{Aut}(X_K)$ implies $\mathrm{Aut}(X_K) \hookrightarrow G^+$
3. * [In progress]: $G^+ \simeq (\mathbb{Z}/43\mathbb{Z}) \rtimes \mathbb{Z}/7\mathbb{Z}$ $(\Rightarrow \mathrm{Aut}(X_K) \simeq G^+)$.

Some (new) evidence: For $n \geq 2$ and $d \geq 4$ with $(n,d) \neq (2,4)$

$$\mathrm{Aut}(X_K) \simeq (\mathbb{Z}/m\mathbb{Z}) \rtimes \mathbb{Z}/(n+2)\mathbb{Z} \quad \text{with} \quad m = [(d-1)^{n+2} - (-1)^{n+2}]/d.$$

By the recent work of Z. Zheng (2020), this is related to the following question:

(?) Abelian subgroups of maximal order: In a work in progress we are looking at abelian p-groups admissible in $\mathcal{U}_n(d)$

↳ 1° Complementary to the work of ZHENG (2020).
↳ 2° MUCH easier than general p-groups (eg. 99,2% of finite groups of order ≤ 2000 have order 2^{10})!

Thanks for your attention !