Morita Equivalence in Deformation Quantization

by

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Abstract

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In this dissertation we study the notion of Morita equivalence in the realm of formal deformation quantization of Poisson manifolds. We show that Rieffel's construction of induced representations of C^*-algebras and the notion of strong Morita equivalence can be extended to a wider class of *-algebras, including those arising in deformation quantization. In order to study strong Morita equivalence in the framework of deformation quantization, we consider finitely generated projective inner-product modules over hermitian star-product algebras. These objects are quantum analogs of hermitian vector bundles and are obtained from them through a deformation quantization procedure. In this work, we define and study deformation quantization of hermitian vector bundles and show how they produce examples of strongly Morita equivalent deformed *-algebras. Finally, we present a study of the classification of star products on a Poisson manifold M up to Morita equivalence. We show how deformation quantization of line bundles over M produces a canonical action Φ of the Picard group $\mathsf{Pic}(M) \cong H^2(M, \mathbb{Z})$ on the moduli space of equivalence classes of differential star products on M; the orbits of this action characterize Morita equivalent star products on M. We describe the semiclassical limit of Φ explicitly in terms of the characteristic classes of star products by studying contravariant connections arising in the semiclassical limit of line bundle deformations.

Professor Alan D. Weinstein
Dissertation Committee Chair

To the memory of José Antônio Tellechea de Azevedo

Contents

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Chapter 1

Introduction

The notion of "representation equivalence" of unital rings was first made precise and studied by Morita in [50]: two unital rings are Morita equivalent if they have equivalent categories of left modules. Since then, the concept of Morita equivalence has been adapted to many other categories, such as topological and symplectic groupoids [51, 75], C^* and von Neumann algebras [62, 61], and Poisson manifolds [74] (see [46] for a unified approach).

In applications of noncommutative geometry to M-theory [20, 65], Morita equivalence has been shown to be related to physical duality. This motivated the study of the classification of quantum tori up to Morita equivalence [64]. The algebra of functions on quantum tori is obtained from functions on ordinary tori, equipped with a constant Poisson structure, by means of strict deformation quantization [63]; the classification problem is to understand which Poisson structures give rise to Morita equivalent quantum tori.

An analogous question can be studied in the framework of formal deformation quantization, a quantization scheme based on Gerstenhaber's deformation theory [32, 33] and introduced in [4]. In this approach to quantization, algebras of quantum observables are defined by formal deformations of algebras of classical observables. These deformed algebra structures are called *star products*, and they arise in many examples (but not always) as asymptotic expansions of strict quantizations (see [36, Sec. 4] and references therein). It follows from Kontsevich's formality theorem that star products exist on arbitrary Poisson manifolds [44].

Question 1. How are star products classified up to Morita equivalence?

The star-product approach to quantization is related to the formulation of quantum mechanics through operators on Hilbert spaces. In fact, many examples of star products arise in connection with standard quantization procedures based on (pseudo-differential) operator representations of functions. For instance, the usual *Weyl-Moyal* star product in $\mathbb{R}^{2n}$ can be obtained by the asymptotic expansion in $\hbar$ of the integral formula for the product of symbols of operators in the so-called *Weyl quantization* procedure (see, for instance, [17, Sect. 20]). In this specific example one can find convergent star products (or strict quantizations) on subalgebras of $C^\infty(\mathbb{R}^{2n})$ (e.g. on Schwartz functions) which admit natural representations as operators on $L^2(\mathbb{R}^n)$.

Many authors have dealt with the problem of finding domains of convergence for star products and corresponding representations on Hilbert spaces [36, Sect. 4]. In [29, Chp. 7], Fedosov describes necessary conditions for a star product to admit "asymptotic operator representations". However, it is not clear how to consider representations of star products on Hilbert spaces in general.

A new approach to representations of algebras in deformation quantization was introduced in [11]. Using the notion of "asymptotic positivity" in the ring $\mathbb{C}[[\lambda]]$, the authors define a *formal* (pre-) Hilbert space to be a module over $\mathbb{C}[[\lambda]]$ with a $\mathbb{C}[[\lambda]]$-valued inner product. They suggest that these are the natural objects for a general theory of representations of star-product algebras. The deformations in this setting are taken to be *hermitian* ($\overline{f \star g} = \overline{g} \star \overline{f}$), providing the deformed algebras with a natural *-involution. Convergence issues can be handled in specific examples: it is shown in [8, 11] that the standard operator representations (Schrödinger, Bargmann) can be recovered from these "formal" representations. Much of the theory in [11] is developed following the usual representation theory of C^*-algebras. In fact, *positive* linear functionals (states) play a crucial role in this approach. The main examples of representations of star-product algebras on formal (pre-)Hilbert spaces are obtained by means of an algebraic analog of the GNS construction.

For a hermitian star-product algebra $\boldsymbol{\mathcal{A}}$, there is thus a subcategory of the category of $\boldsymbol{\mathcal{A}}$-modules of special importance in physical applications, namely *-representations of $\boldsymbol{\mathcal{A}}$ on formal pre-Hilbert spaces. It is then natural to search for a suitable notion of Morita equivalence for star-product algebras concerned with this subcategory. An analogous situation occurs in the theory of C^*-algebras. In this context, the notion of Morita equivalence was first introduced by M. Rieffel [60], motivated by the study of locally compact groups and Mackey's imprimitivity theorem. The natural modules over a C^*-algebra $\mathcal{A}$ are Hilbert spaces where $\mathcal{A}$ acts by bounded operators; these form the category of hermitian modules, denoted by $\mathsf{Her}(\mathcal{A})$. Rieffel defined the notion of *strong Morita equivalence* of C^*-algebras [60] and proved that if $\mathcal{A}$ and $\mathcal{B}$ are strongly Morita equivalent, then $\mathsf{Her}(\mathcal{A}) \cong \mathsf{Her}(\mathcal{B})$. The construction of equivalence functors for hermitian modules is based on the notion of *Rieffel induction* of representations.

Question 2. Can one carry out Rieffel induction of representations for star-product algebras?

Or even more generally.

Question 3. Can we extend the notion of strong Morita equivalence to a wider class of *-algebras, including star product and C^*-algebras as particular cases?

With such an algebraic notion of strong Morita equivalence, we can consider the classification of star products up to strong Morita equivalence as well.

This dissertation presents results concerning all these questions. We study algebraic analogs of Rieffel induction and strong Morita equivalence for classes of *-algebras over rings of the form $\mathsf{C} = \mathsf{R} \oplus \mathsf{iR}$, where R is ordered. Both C^* and star-product algebras are of this form. The involution and the notion of positivity on the ground ring allow the definitions of positive linear functionals, positive algebra elements and *-representations of this algebra on pre-Hilbert spaces over C. For C^*-algebras, these notions are just the usual ones; in the case of hermitian star products, they coincide with the ones introduced in [11]. The procedure of Rieffel induction of representations can be carried out in this purely algebraic setting analogously to C^*-algebras, except for difficulties in guaranteeing that the induced inner product is positive definite: this is indeed the only point where analytical properties of C^*-algebras come into play. We discuss algebraic conditions avoiding this difficulty and show that these conditions automatically hold for algebras arising from deformation quantization. This provides a positive answer to Question 2.

For *-algebras where algebraic Rieffel induction can be done, there is a natural notion of algebraic strong Morita equivalence, given in terms of certain bimodules equipped with algebra-valued inner products, which play the role of imprimitivity bimodules in C^*-algebras. For algebraically strongly Morita equivalent *-algebras over C, the representation theories on pre-Hilbert spaces over C are equivalent. Furthermore, algebraic strong Morita equivalence recovers Rieffel's strong Morita equivalence: two C^*-algebras are strongly Morita equivalent if and only if their Pedersen ideals[1] are algebraically strongly

[1]The idea of considering Pedersen ideals to recover strong Morita equivalence of nonunital C^*-algebras

Morita equivalent. Several results, such as the equivalence of categories of hermitian modules of strongly Morita equivalent C^*-algebras, follow from the algebraic results for Pedersen ideals. This gives a positive answer to Question 3.

In order to study strong Morita equivalence in the deformation quantization framework, one is naturally led to study finitely generated projective (f.g.p.) inner-product modules over hermitian star-product algebras $\boldsymbol{\mathcal{A}} = (C^\infty(M)[[\lambda]], \star)$. For undeformed algebras, these objects correspond (due to Serre-Swan's theorem) to (smooth sections of) hermitian vector bundles E over M. Hence f.g.p. inner-product modules over star-product algebras can be thought of as quantum hermitian vector bundles, obtained from ordinary hermitian vector bundles through a deformation quantization procedure. In this work we define deformation quantization of (hermitian) vector bundles (with respect to a hermitian star product). We show that these deformations exist and are unique up to a natural notion of equivalence. This result implies the existence of a canonical bijection Φ between differential (hermitian) deformations of $C^\infty(M)$ and $\Gamma^\infty(\mathsf{End}(E))$, in such a way that that corresponding hermitian deformed algebras are strongly Morita equivalent. This discussion extends the result in [48] and clarifies some of the questions and constructions in [59].

Finally, we present a study of the classification of star products on Poisson manifolds up to Morita equivalence (Question 1). As we observed, strong Morita equivalence reduces to Morita equivalence in the context of star products. (An analogous result holds for C^*-algebras [5].) We use deformation quantization of line bundles to show that there is a canonical action

$$\Phi : \mathsf{Pic}(M) \times \mathsf{Def}_{\mathrm{diff}}(M, \pi) \longrightarrow \mathsf{Def}_{\mathrm{diff}}(M, \pi)$$

of the Picard group of M (the set of isomorphism class of complex line bundles over M, with group structure given by tensor product) on $\mathsf{Def}_{\mathrm{diff}}(M, \pi)$, the moduli space of equivalence classes of star products on the Poisson manifold (M, π), in such a way that two star products $\star$ and $\star'$ on M are Morita equivalent if and only if there exists a Poisson diffeomorphism $\psi : M \longrightarrow M$ such that $[\psi^*(\star')]$ and $[\star]$ lie in the same orbit of this action.

We use the well-known descriptions of the set $\mathsf{Def}_{\mathrm{diff}}(M, \pi)$ (in terms of Fedosov-Deligne's characteristic classes in the symplectic case, and in terms of Kontsevich's classes of formal Poisson structures in the Poisson case) to compute the semiclassical limit of the action Φ explicitly. This involves the study of the semiclassical geometry of line bundle

was first suggested to us by P. Ara. [1, 2]

deformations. Just as the semiclassical limit of deformations of the associative algebra structure of $C^\infty(M)$ gives rise to Poisson structures on M, the semiclassical limit of quantum line bundles defines a geometric object on the underlying line bundle: a contravariant connection [30, 67]. Contravariant connections are analogous to ordinary connections, but with cotangent vectors playing the role of tangent ones. They define a characteristic class on line bundles over Poisson manifolds, called the Poisson-Chern class. We show explicitly that the semiclassical limit of Φ "twists" characteristic classes of star products by Poisson-Chern classes. From this analysis, it follows that, when M is symplectic with free $H^2(M,\mathbb{Z})$, the action Φ is faithful, and one gets a parametrization of classes of star products Morita equivalent to a fixed one (up to isomorphism). The discussion also provides an integrality obstruction for Morita equivalence of star products in general.

The dissertation is organized as follows.

In Chapter 2, we have some background material: deformation quantization and star products, Morita equivalence of unital rings, and Rieffel induction and strong Morita equivalence of C^*-algebras.

Chapter 3 presents the study of algebraic strong Morita equivalence, its basic properties and examples, and the connection with the usual notion of strong Morita equivalence of C^*-algebras.

In Chapter 4 we discuss deformation quantization of hermitian vector bundles and the semiclassical geometry related to these deformations.

In Chapter 5 we describe how Picard groups act on deformations, and compute the semiclassical limit of Morita equivalent star products.

We have included two appendices: Appendix A presents some basic definitions and results on Poisson geometry; Appendix B contains a discussion of Gerstenhaber's deformation theory and Kontsevich's formality theorem.

Chapter 2

Preliminaries

2.1 Deformation quantization

In this section we will briefly recall some definitions and results on deformation quantization. The reader is referred to [36, 66, 72] for surveys on the subject. More references and information about Gerstenhaber's deformation theory and star products can be found in Appendix B.

2.1.1 Star products: definition and examples

Let $C^\infty(M)$ be the algebra of complex-valued smooth functions on a smooth manifold M; $C^\infty(M)[[\lambda]]$ will denote the space of formal power series with coefficients in $C^\infty(M)$.

Definition 2.1 ([4]). *A star product on M is a $\mathbb{C}[[\lambda]]$-bilinear associative multiplication $\star : C^\infty(M)[[\lambda]] \times C^\infty(M)[[\lambda]] \longrightarrow C^\infty(M)[[\lambda]]$ of the form*

$$f \star g = \sum_{r=0}^{\infty} C_r(f,g)\lambda^r, \quad f, g \in C^\infty(M), \tag{2.1}$$

where each $C_r : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$ is $\mathbb{C}$-bilinear and $C_0(f,g) = f \cdot g$ (pointwise multiplication)[1].

Definition 2.2. *A star product is called* differential *if each C_r is a bidifferential operator* [2].

[1] The product in (2.1) is extended to $C^\infty(M)[[\lambda]]$ by λ-adic continuity.

[2] Some authors have considered different classes of star products, with C_r's being, for instance, local or continuous. As discussed in Appendix B, there is no substantial difference between these notions – see Proposition B.16.

Star products will be assumed differential, unless otherwise stated.

The associativity of $\star$ in (2.1) implies that the skew symmetric part of C_1 defines a Poisson structure on M (see Appendix A). Thus Poisson manifolds can be thought of as "first-order approximations" to noncommutative algebras [17]. Given a Poisson manifold $(M, \{\cdot,\cdot\})$, the deformation quantization problem is to construct and study star products on M satisfying

$$\{f,g\} = \frac{1}{\mathrm{i}}(C_1(f,g) - C_1(g,f)) = \text{``}\lim_{\lambda\to 0}\text{``}\frac{1}{\mathrm{i}\lambda}[f,g]_\star, \quad f,g \in C^\infty(M), \tag{2.2}$$

where $[f,g]_\star = f\star g - g\star f$ is the $\star$-commutator of f and g. Note that (2.2) illustrates Dirac's correspondence principle [24].

Example 2.3 (Weyl-Moyal star product). *Let V be a vector space equipped with a constant Poisson structure $\pi = \sum_{i,j}\pi^{ij}\partial_{x_i}\wedge\partial_{x_j}$, $\pi^{ij} = -\pi^{ji} \in \mathbb{R}$, where x_i are coordinates on V, $i = 1\dots n$. The Weyl-Moyal star product on V is given by*

$$f\star g(z) = \exp\left(\frac{\mathrm{i}\lambda}{2}\sum_{i,j}\pi^{ij}\partial_{x_i}\partial_{y_j}\right) f(x)g(y)\Big|_{x=y=z}, \quad f,g\in C^\infty(V). \tag{2.3}$$

When π is symplectic, the so-called Weyl quantization procedure establishes a correspondence between certain classes of functions on $V \cong \mathbb{R}^{2n}$ and operators on $L^2(\mathbb{R}^n)$; in this case, the product in (2.3) is the one induced by the composition of operators [31, 17].

When (V,π) is a symplectic vector space, the space of formal power series of polynomials equipped with the Weyl-Moyal star product $W(V) := (S(V)[[\lambda]], \star)$ is called the *Weyl algebra* of V.

Example 2.4 (CBH star product). *Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie algebra, and consider its dual $\mathfrak{g}^*$ equipped with the Lie-Poisson structure*

$$\{f,g\}(\mu) = \langle [df(\mu), dg(\mu)], \mu\rangle, \; f,g \in C^\infty(\mathfrak{g}^*), \; \mu\in\mathfrak{g}^*. \tag{2.4}$$

Let $\mathfrak{g}_\lambda = (\mathfrak{g}[[\lambda]], \lambda[\cdot,\cdot])$, and let $U_{\mathfrak{g}_\lambda}$ be the universal enveloping algebra of $\mathfrak{g}_\lambda$. Let $S(\mathfrak{g})$ be the symmetric tensor algebra of $\mathfrak{g}$, which can be naturally identified with the space of polynomial functions on $\mathfrak{g}^*$, $Pol(\mathfrak{g}^*)$. The Poincare-Birkhoff-Witt theorem [17] establishes an isomorphism (of vector spaces) $\rho : Pol(\mathfrak{g}^*)[[\lambda]] \longrightarrow U_{\mathfrak{g}_\lambda}$, given by symmetrization of monomials. For $p, q \in Pol(\mathfrak{g}^*)[[\lambda]]$, we define

$$p\star q = \rho^{-1}(\rho(p)\circ\rho(q)),$$

where $\circ$ is the multiplication on $U_{\mathfrak{g}_\lambda}$. This product can be extended to $C^\infty(\mathfrak{g}^*)[[\lambda]]$ and gives rise to a differential star product on $\mathfrak{g}^*$ [34].

The existence of differential star products on any symplectic manifold was first proven by De Wilde and Lecomte in [23]; in this paper, the authors explore the Hochshild cohomology of the Poisson algebra $(C^\infty(M), \{\cdot,\cdot\})$ and show that one can avoid the possible obstructions in the stepwise construction of star products.

A different approach to the existence of star products on symplectic manifolds was presented by Fedosov in [28]. For a symplectic manifold (M,ω), Fedosov considered the *Weyl bundle* $\mathcal{W} \to M$, where $W_x = W(T_xM), x \in M$ is the Weyl algebra of T_xM. Starting with a symplectic torsion-free connection on M, the author builds a flat connection ∂ on $\Gamma(\mathcal{W})$ and identifies $C^\infty(M)[[\lambda]]$ with the space of flat sections of ∂. This defines a star product on (M,ω) [3]; Weyl bundles were introduced independently by Omori, Maeda and Yoshioka in their alternative proof of existence of star products on arbitrary symplectic manifolds [54].

In a remarkable paper [44], Kontsevich proved his formality theorem and established, as a consequence, the existence of star products on arbitrary Poisson manifolds. As a first step, Kontsevich constructed a deformation quantization for any open domain $U \subseteq \mathbb{R}^m$, equipped with an arbitrary Poisson structure $\pi = \sum_{i,j} \pi^{ij}\partial_{x_i}\partial_{x_j}$. For each integer $r \geq 0$, Kontsevich considered a special class of labeled graphs G_r. For $\Gamma \in G_r$, one can define a bidifferential operator $B_{\Gamma,\pi}$ on U (depending upon the Poisson structure π) and a weight ω_Γ so that the formula

$$f \star g = \sum_{r=0}^{\infty} \lambda^r \sum_{\Gamma \in G_r} \omega_\Gamma B_{\Gamma,\pi}(f,g) \tag{2.5}$$

is a star product. The expression of Kontsevich's star product is, modulo third order terms, given by

$$\begin{aligned} f \star g \quad = \quad & f.g + \lambda \sum_{i,j} \pi^{ij}\partial_{x_i}(f)\partial_{x_j}(g) + \lambda^2\Big(\frac{1}{2}\sum_{i,j,k,l} \pi^{ij}\pi^{kl}\partial_{x_i}\partial_{x_k}(f)\partial_{x_j}\partial_{x_l}(g) \\ & + \frac{1}{3}\sum_{i,j,k,l} \pi^{i,j}\partial_{x_j}(\pi^{kl})(\partial_{x_i}\partial_{x_k}(f)\partial_{x_l}(g) - \partial_{x_k}(f)\partial_{x_i}\partial_{x_l}(g)) + S(f,g)\Big) + O(\lambda^3), \end{aligned}$$

[3] In fact, both Fedosov's and De Wilde, Lecomte's approaches can be extended to regular Poisson manifolds. A comparison between the two approaches can be found in [22].

where S is symmetric. Using ideas from formal differential geometry, Kontsevich extended the results for domains in $\mathbb{R}^d$ to arbitrary manifolds. An explicit construction of global star products can be found in [18].

2.1.2 Equivalence of star products and characteristic classes

Let G be the group of automorphisms of $C^\infty(M)[[\lambda]]$ (as a $\mathbb{C}[[\lambda]]$-module) deforming the identity, i.e.

$$G = \{\mathsf{id} + \sum_{r=1}^{\infty} T_r \lambda^r,\ \ T_r : C^\infty(M) \longrightarrow C^\infty(M)\ \mathbb{C}\text{-linear operator}\}.$$

This group acts on the set of star products on M by

$$\boldsymbol{T}(\star) = \star' \iff f \star' g = \boldsymbol{T}^{-1}(\boldsymbol{T}(f) \star \boldsymbol{T}(g)),\ \ f, g \in C^\infty(M),\ \boldsymbol{T} \in G.$$

Such a $\boldsymbol{T}$ is called an *equivalence transformation* between $\star$ and $\star'$.

Definition 2.5. *Two star products $\star$, $\star'$ are equivalent if they belong to the same G-orbit.*

Remark 2.6. *Any star-product algebra is unital and equivalent to one for which the unit is $1 \in C^\infty(M)$ [39].*

If $\star$ and $\star'$ are differential star products on M, we call them *differentially equivalent* if there exists an equivalence transformation $\boldsymbol{T} = \mathsf{id} + \sum_{r=1}^{\infty} T_r \lambda^r$ between them where each T_r is a differential operator. It turns out that there is no difference between these notions of equivalence [4] [22, 37]:

Proposition 2.7. *Let $\star$ and $\star'$ be two differential star products. Suppose $\boldsymbol{T} = \mathsf{id} + \sum_{r=1}^{\infty} T_r \lambda^r$ is an equivalence transformation, $\star' = \boldsymbol{T}(\star)$. Then each T_r is a differential operator.*

We define the moduli space

$$\mathsf{Def}_{\text{diff}}(M) := \{\text{differential star products on } M\}/G, \tag{2.6}$$

We denote the equivalence class of a star product $\star$ by $[\star]$.

A simple computation shows that if two star products are equivalent, then they correspond to the same Poisson bracket in the semiclassical limit. For a Poisson manifold (M, π), we define

$$\mathsf{Def}_{\text{diff}}(M, \pi) := \{[\star],\ [f, g]_\star = \lambda \mathrm{i} \pi(df, dg) \bmod \lambda^2\}. \tag{2.7}$$

[4] A similar result holds for local and continuous cochains.

If (M, ω) is symplectic, we denote

$$\mathsf{Def}_{\mathrm{diff}}(M, \omega) = \{[\star],\ [f,g]_\star = \lambda\omega(X_f, X_g) \bmod \lambda^2\}, \tag{2.8}$$

where X_f, X_g are the hamiltonian vector fields of f and g, respectively.

Remark 2.8. *If $\star$ and $\star'$ are star products on (M, π), we remark that, by Proposition B.3, they are isomorphic if and only if there exists a Poisson diffeomorphism $\psi : M \longrightarrow M$ such that $[\psi^*(\star')] = [\star]$.*

Let $H^2_{dR}(M)$ denote the second de Rham cohomology group of M (with coefficients in $\mathbb{C}$). For a symplectic manifold (M, ω), there is a bijection

$$\mathrm{c} : \mathsf{Def}_{\mathrm{diff}}(M, \omega) \longrightarrow [\omega] \oplus \lambda H^2_{dR}(M)[[\lambda]], \tag{2.9}$$

and $\mathrm{c}(\star) = [\omega] + \sum_{r=1}^{\infty}[\omega_r]\lambda^r$ is called the *characteristic class* of $\star$ [52, 6, 73]. A deformation quantization corresponding to the characteristic class $\mathrm{c}(\star) = [\omega]$ is called *canonical*. For star products $\star$ and $\star'$, the difference $t(\star, \star') = c(\star) - c(\star')$ is their *relative Deligne class* [22, 37].

In order to state the result on classification of star products for general Poisson manifolds [44], we need the following

Definition 2.9. *A formal Poisson structure on a smooth manifold M is a formal series $\pi_\lambda = \pi + \sum_{r=1}^{\infty}\pi_r\lambda^r \in \Gamma^\infty(\bigwedge^2 TM)[[\lambda]]$ with $[\pi_\lambda, \pi_\lambda] = 0$.[5] The corresponding formal Poisson bracket is defined by $\{f, g\}_\lambda = \pi_\lambda(df, dg)$, $f, g \in C^\infty(M)$.*

Let $\mathsf{Der}(C^\infty(M))$ be the space of derivations of $C^\infty(M)$. Let F be the group of formal paths in the diffeomorphism group of M,

$$F := \{\exp(\sum_{r=1}^{\infty} D_r\lambda^r),\ D_r \in \mathsf{Der}(C^\infty(M))\}.$$

The group F acts on the set of formal Poisson structures as follows. We say that $\boldsymbol{T}(\pi_\lambda) = \pi'_\lambda$ if and only if

$$\{f, g\}'_\lambda = \boldsymbol{T}^{-1}(\{\boldsymbol{T}(f), \boldsymbol{T}(g)\}_\lambda),\quad f, g \in C^\infty(M), \tag{2.10}$$

where $\{f, g\}_\lambda = \pi_\lambda(df, dg)$ and $\{f, g\}'_\lambda = \pi'_\lambda(df, dg)$. Kontsevich showed in [44] that there is a bijection

$$\mathrm{c} : \mathsf{Def}_{\mathrm{diff}}(M, \pi) \longrightarrow \{\pi_\lambda = \pi + \sum_{r=1}^{\infty}\pi_r\lambda^r \text{ formal Poisson structure on } M\}/F. \tag{2.11}$$

[5] Here $[\cdot, \cdot]$ is the Schouten bracket on M, extended to $\Gamma(\bigwedge^2 TM)[[\lambda]]$ by linearity with respect to λ.

We denote the equivalence class of a formal Poisson structure π_λ by $[\pi_\lambda]$.

2.1.3 Hermitian star products

The following definition is motivated by the relationship between star products and quantum mechanics.

Definition 2.10. *A star product $\star$ is called hermitian if*

$$\overline{f \star g} = \overline{g} \star \overline{f}, \ f, g \in C^\infty(M). \tag{2.12}$$

(Our convention is that $\overline{\lambda} = \lambda$.)

Hence complex conjugation provides a hermitian star-product algebra with a *-involution – see Section 3.1.2.

Two hermitian star products $\star$ and $\star'$ are *-*equivalent* if there exists an equivalence transformation $\boldsymbol{T} \in G$, $\boldsymbol{T}(\star) = \star'$, satisfying $\boldsymbol{T}(\overline{f}) = \overline{\boldsymbol{T}(f)}$, $f \in C^\infty(M)$. The notion of *-equivalence coincides with the usual notion of equivalence [53, 22]:

Proposition 2.11. *Let $\star$ and $\star'$ be hermitian star products which are equivalent. Then there exists an equivalence transformation $\boldsymbol{T}$ satisfying $\boldsymbol{T}(\overline{f}) = \overline{\boldsymbol{T}(f)}$.*

We define

$$\mathsf{Def}^*_{\mathrm{diff}}(M) := \{\text{hermitian star products on } M\}/G. \tag{2.13}$$

The equivalence class of $\star$ is again denoted by $[\star]$. We define

$$\mathsf{Def}^*_{\mathrm{diff}}(M,\pi) := \{[\star] \in \mathsf{Def}^*_{\mathrm{diff}}(M), \ [f,g]_\star = \lambda\pi(df,dg) \bmod \lambda^2\}. \tag{2.14}$$

By symmetrizing Hochschild cochains [14, Prop. 3.1], one can always produce a hermitian star product out of an arbitrary one. However, these star products might not be equivalent. In fact, there is a description of the characteristic classes of hermitian star products on symplectic manifolds [53]:

Proposition 2.12. *Let (M,ω) be a symplectic manifold and $\star$ a star product on M. Then $\star$ is equivalent to a hermitian star product if and only $\overline{c(\star)} = c(\star)$* [6].

[6] This condition differs slightly from the one in [53] since, in our convetion, $\overline{\lambda} = \lambda$.

Example 2.13. *A simple computation shows that the Weyl-Moyal product (Example 2.3) is hermitian. Generalizations of this star product to cotangent bundles using Fedosov's techniques also yield hermitian star products [7]. In fact, a suitable choice of ingredients in Fedosov's construction produces hermitian star products on arbitrary symplectic manifolds [53]; in particular, canonical Fedosov star products are hermitian [9, Lemma 3.3].*

Example 2.14. *The so-called Wick star product on $\mathbb{C}^n$, given by*

$$f \star g(w, \bar{w}) = \exp\left(2\lambda \sum_k \partial_{z_k} \partial_{\bar{z}'_k}\right) f(z, \bar{z}) g(z', \bar{z}')\Big|_{z=z'=w}, \quad f, g \in C^\infty(\mathbb{C}^n), \tag{2.15}$$

and its generalizations to Kähler manifolds (star products of Wick type) are also hermitian [9].

2.2 Morita equivalence of unital algebras

The reader is referred to [3, 45] for a detailed exposition of Morita theory for unital rings [50].

2.2.1 The definition

The idea of Morita theory is to study unital rings by looking at their representation theory as endomorphisms of abelian groups. For a unital ring $\mathcal{R}$, let ${}_{\mathcal{R}}\mathfrak{M}$ denote the category of left $\mathcal{R}$-modules.

Definition 2.15. *Two unital rings $\mathcal{R}$ and $\mathcal{S}$ are called* Morita equivalent *if ${}_{\mathcal{R}}\mathfrak{M}$ and ${}_{\mathcal{S}}\mathfrak{M}$ are equivalent categories* [7]*.*

Example 2.16. *Let $\mathcal{R}$ be a unital ring and $M_n(\mathcal{R})$ be the ring of $n \times n$ matrices over $\mathcal{R}$. Given a (left) $\mathcal{R}$-module V, we can define a left $M_n(\mathcal{R})$-module $\mathcal{F}(V) = V^n$, with the $M_n(\mathcal{R})$ action given by matrix operating on vectors. The functor $\mathcal{F} : {}_{\mathcal{R}}\mathfrak{M} \longrightarrow {}_{M_n(\mathcal{R})}\mathfrak{M}$ defines an equivalence of categories [45, Thm. 17.20], and $\mathcal{R}$ and $M_n(\mathcal{R})$ are Morita equivalent.*

[7] Two categories $\mathfrak{A}$ and $\mathfrak{B}$ are equivalent if there are (covariant) functors

$$\mathcal{F} : \mathfrak{A} \to \mathfrak{B}, \qquad \mathcal{G} : \mathfrak{B} \to \mathfrak{A}$$

satisfying $\mathcal{F} \circ \mathcal{G} \cong \mathcal{I}_{\mathfrak{B}}$ and $\mathcal{G} \circ \mathcal{F} \cong \mathcal{I}_{\mathfrak{A}}$, where $\cong$ denotes natural equivalence of functors and $\mathcal{I}$ is the identity functor.

Properties of a ring $\mathcal{R}$ which are preserved under Morita equivalence are called *Morita invariants*. Example 2.16 shows that commutativity is *not* a Morita invariant property. However, properties such as $\mathcal{R}$ being semisimple, artinian and noetherian are Morita invariant. Morita equivalent rings have isomorphic K-theory, isomorphic lattice of ideals and isomorphic centers; hence two commutative unital rings are Morita equivalent if and only if they are isomorphic [45, Cor. 18.42].

Let $\mathcal{R}$ and $\mathcal{S}$ be unital rings, and let ${}_{\mathcal{S}}\mathcal{E}_{\mathcal{R}}$ be an $(\mathcal{S},\mathcal{R})$-bimodule. We can construct a functor $\mathcal{F} = ({}_{\mathcal{S}}\mathcal{E}_{\mathcal{R}} \otimes_{\mathcal{R}} \cdot) : {}_{\mathcal{R}}\mathfrak{M} \longrightarrow {}_{\mathcal{S}}\mathfrak{M}$ by

$$\mathcal{F}({}_{\mathcal{R}}V) = {}_{\mathcal{S}}\mathcal{E} \otimes_{\mathcal{R}} V.$$

It is clear that $\mathcal{F}({}_{\mathcal{R}}V)$ has a natural $\mathcal{S}$-module structure determined by $s(x \otimes v) = sx \otimes v$. If $f : {}_{\mathcal{R}}V_1 \longrightarrow {}_{\mathcal{R}}V_2$ is a morphism, then we define $\mathcal{F}(f) : {}_{\mathcal{S}}\mathcal{E} \otimes_{\mathcal{R}} V_1 \longrightarrow {}_{\mathcal{S}}\mathcal{E} \otimes_{\mathcal{R}} V_2$ by $\mathcal{F}(f)(x \otimes v) = x \otimes f(v),\ x \in \mathcal{E},\ v \in V_1$.

It turns out that this way of constructing functors is very general. It follows from a theorem of Eilenberg and Watts [71] that if $\mathcal{F} : {}_{\mathcal{R}}\mathfrak{M} \longrightarrow {}_{\mathcal{S}}\mathfrak{M}$ is an equivalence of categories, then there exists a bimodule ${}_{\mathcal{S}}\mathcal{E}_{\mathcal{R}}$ such that ${}_{\mathcal{S}}\mathcal{E} \otimes_{\mathcal{R}} \cdot \cong \mathcal{F}$.

Example 2.17. *In the case of $\mathcal{R}$ and $M_n(\mathcal{R})$, the functor described in Example 2.16 corresponds to the bimodule ${}_{M_n(\mathcal{R})}\mathcal{R}^n_{\mathcal{R}}$.*

Corollary 2.18. *Two unital rings $\mathcal{R}$ and $\mathcal{S}$ are Morita equivalent if and only there exist bimodules ${}_{\mathcal{S}}\mathcal{E}_{\mathcal{R}}$ and ${}_{\mathcal{R}}\overline{\mathcal{E}}_{\mathcal{S}}$ so that ${}_{\mathcal{R}}\overline{\mathcal{E}} \otimes_{\mathcal{S}} \mathcal{E}_{\mathcal{R}} \cong {}_{\mathcal{R}}\mathcal{R}_{\mathcal{R}}$ and ${}_{\mathcal{S}}\mathcal{E} \otimes_{\mathcal{R}} \overline{\mathcal{E}}_{\mathcal{S}} \cong {}_{\mathcal{S}}\mathcal{S}_{\mathcal{S}}$ (as bimodules).*

Definition 2.19. *A bimodule ${}_{\mathcal{S}}\mathcal{E}_{\mathcal{R}}$ establishing a Morita equivalence is called an* equivalence bimodule*.*

The next section provides a characterization of such bimodules.

2.2.2 Morita's theorem and the Picard group

Definition 2.20. *A right $\mathcal{R}$-module $\mathcal{E}_{\mathcal{R}}$ is called a progenerator if it is finitely generated, projective and a generator*[8]*.*

[8] Recall that a right $\mathcal{R}$-module $\mathcal{E}_{\mathcal{R}}$ is a *generator* if any other right $\mathcal{R}$-module can be obtained as a quotient of a direct sum of copies of $\mathcal{E}_{\mathcal{R}}$.

Theorem 2.21 (Morita). *Two unital rings $\mathcal{R}$ and $\mathcal{S}$ are Morita equivalent if and only if there exists a progenerator right $\mathcal{R}$-module $\mathcal{E}_{\mathcal{R}}$ so that $\mathcal{S} \cong \mathsf{End}_{\mathcal{R}}(\mathcal{E}_{\mathcal{R}})$. Moreover, if ${}_{\mathcal{S}}\mathcal{E}_{\mathcal{R}}$ is an equivalence bimodule, then its inverse is given by ${}_{\mathcal{R}}\overline{\mathcal{E}}_{\mathcal{S}} = \mathsf{Hom}_{\mathcal{R}}(\mathcal{E}_{\mathcal{R}}, \mathcal{R})$* [9].

Definition 2.22. *An idempotent $P \in M_n(\mathcal{R})$ ($P^2 = P$) is called* full *if the span of elements of the form TPS, with $T, S \in M_n(\mathcal{R})$, is $M_n(\mathcal{R})$. (We write $M_n(\mathcal{R})PM_n(\mathcal{R}) = M_n(\mathcal{R})$.)*

A finitely generated projective $\mathcal{R}$-module $P\mathcal{R}^n$ ($P \in M_n(\mathcal{R})$ idempotent) is a generator if and only if P is full [45, 18.10(D)]. We then have an alternative description of Morita equivalent rings.

Theorem 2.23. *$\mathcal{R}$ and $\mathcal{S}$ are Morita equivalent if and only if there exists $n \in \mathbb{N}$ and a full idempotent $P \in M_n(\mathcal{R})$ so that $\mathcal{S} \cong PM_n(\mathcal{R})P$.*

There is a natural group associated to any unital ring $\mathcal{R}$.

Definition 2.24. *We define $\mathsf{Pic}(\mathcal{R})$ as the group of equivalence classes of self-equivalence functors $\mathcal{F} : {}_{\mathcal{R}}\mathfrak{M} \longrightarrow {}_{\mathcal{R}}\mathfrak{M}$, with group operation given by composition of functors. Equivalently, we can define $\mathsf{Pic}(\mathcal{R})$ as the group of isomorphism classes $(\mathcal{R}, \mathcal{R})$-equivalence bimodules ${}_{\mathcal{R}}\mathcal{E}_{\mathcal{R}}$, with group operation given by tensor products (over $\mathcal{R}$).*

The group $\mathsf{Pic}(\mathcal{R})$ is called the *Picard group* of $\mathcal{R}$.

Remark 2.25. *Note that if ${}_{\mathcal{R}}\mathcal{E}_{\mathcal{R}}$ is an equivalence bimodule, then the center of $\mathcal{R}$ need not act the same on the left and right of $\mathcal{E}$. If $\mathcal{R}$ is commutative and $\mathcal{E}$ is an $(\mathcal{R}, \mathcal{R})$-equivalence bimodule, then there exists an $(\mathcal{R}, \mathcal{R})$-equivalence bimodule $\mathcal{E}'$ satisfying $rx = xr$, for all $r \in \mathcal{R}$ and $x \in \mathcal{E}'$, such that $\mathcal{E} \cong \mathcal{E}'$ as right $\mathcal{R}$-modules (pick $\mathcal{E}'$ of the form $P_0\mathcal{R}^n$ as a right $\mathcal{R}$-module and consider the identification $\mathcal{R} \longrightarrow P_0 M_n(\mathcal{R}) P_0$, $r \mapsto rP_0$).*

If $\mathcal{R}$ is commutative, we denote the group of isomorphism classes of $(\mathcal{R}, \mathcal{R})$-equivalence bimodules ${}_{\mathcal{R}}\mathcal{E}_{\mathcal{R}}$ satisfying $rx = xr$, for all $x \in \mathcal{E}$, $r \in \mathcal{R}$, by $\mathsf{Pic}_{\mathcal{R}}(\mathcal{R})$.

Example 2.26. *Let $\mathcal{R} = C^\infty(M)$, M a smooth manifold. As a consequence of Serre-Swan's theorem [3, Chp. XIV] (here used in the smooth category, where the compactness assumption can be dropped), $\mathsf{Pic}_{C^\infty(M)}(C^\infty(M))$ can be identified with $\mathsf{Pic}(M)$, the group of isomorphism classes of complex line bundles over M, with group operation given by fiberwise tensor product. The Chern class map $c_1 : \mathsf{Pic}(M) \longrightarrow H^2(M, \mathbb{Z})$ is a group isomorphism [40, Sec. 3.8], and hence $\mathsf{Pic}_{C^\infty(M)}(C^\infty(M)) \cong H^2(M, \mathbb{Z})$.*

[9] Note that $\mathsf{Hom}_{\mathcal{R}}(\mathcal{E}_{\mathcal{R}}, \mathcal{R})$ has a natural $(\mathcal{R}, \mathcal{S})$-bimodule structure: For $f \in \mathsf{Hom}_{\mathcal{R}}(\mathcal{E}_{\mathcal{R}}, \mathcal{R})$, we define $(r \cdot f)(x) = rf(x)$ and $(f \cdot s)(x) = f(sx)$, $r \in \mathcal{R}, s \in \mathcal{S}, x \in \mathcal{E}$.

2.3 Rieffel induction and strong Morita equivalence of C^*-algebras

In this section we will assume that the reader is familiar with the basics of C^*-algebra theory; a thorough treatment of the subject can be found in [21]. For a comprehensive exposition of the theory of strong Morita equivalence of C^*-algebras, the reader should consult [58, 47].

2.3.1 The category of representations of a C^*-algebra

In order to define the concept of Morita equivalence for C^*-algebras, we need to choose a suitable notion of module in this category[10]. For a C^*-algebra $\mathcal{A}$, we will consider its representation theory as bounded operators on Hilbert spaces.

Definition 2.27. *A* hermitian module *over $\mathcal{A}$ is the Hilbert space $\mathfrak{H}$ of a nondegenerate* [11] *$*$-representation $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathfrak{H})$, together with the left action $A \cdot \phi = \pi(A)\phi$, $A \in \mathcal{A}$ and $\phi \in \mathfrak{H}$.*

We denote the category of hermitian modules over $\mathcal{A}$ by $\mathsf{Her}(\mathcal{A})$, with morphisms given by (bounded) intertwining operators. The GNS construction provides a way to construct many objects in $\mathsf{Her}(\mathcal{A})$ (see Section 3.1.3).

Definition 2.28. *Two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are (categorically)* Morita equivalent *if $\mathsf{Her}(\mathcal{A})$ and $\mathsf{Her}(\mathcal{B})$ are equivalent categories* [12].

It turns out that this notion of Morita equivalence is too weak for most applications in C^*algebra theory (see [62, 61]). We will discuss a stronger notion below.

2.3.2 Hilbert C^*-Modules and Rieffel induction

As in the ring-theoretic setting, we will define functors from $\mathsf{Her}(\mathcal{A})$ to $\mathsf{Her}(\mathcal{B})$ corresponding to bimodules ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$. Since for C^*-algebras we are considering modules which are Hilbert spaces, we need bimodules ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ equipped with more structure in order to carry the Hilbert space structure of one representation to another.

[10] See [46] for a unified treatment of Morita equivalence in different categories.

[11] Recall that a $*$-representation $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathfrak{H})$ is called *nondegenerate* if $\pi(\mathcal{A})\phi = 0 \Rightarrow \phi = 0, \forall \phi \in \mathfrak{H}$.

[12] We require that the equivalence functors preserve the adjoint operation on morphisms (i.e. $\mathcal{F}(f^*) = \mathcal{F}(f)^*$, for $\mathcal{F}$ equivalence functor and f a morphism).

Throughout this section, $\mathcal{A}$ and $\mathcal{B}$ will be C^*-algebras.

Definition 2.29. *An $\mathcal{A}$-module is an algebraic module over $\mathcal{A}$ with a compatible vector space structure over $\mathbb{C}$.*

Definition 2.30. *A right [13] pre-Hilbert $\mathcal{A}$-module is a right $\mathcal{A}$-module $\mathcal{E}_\mathcal{A}$ equipped with a pairing $\langle \cdot, \cdot \rangle_\mathcal{A} : \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{A}$ satisfying*

i.) $\langle x, \lambda y + \beta z \rangle_\mathcal{A} = \lambda \langle x, y \rangle_\mathcal{A} + \beta \langle x, z \rangle_\mathcal{A}$*, for all* $x, y, z \in \mathcal{E}$ *and* $\lambda, \beta \in \mathbb{C}$.

ii.) $\langle x, y \rangle_\mathcal{A} = \langle y, x \rangle_\mathcal{A}^*$*, for all* $x, y \in \mathcal{E}$.

iii.) $\langle x, yA \rangle_\mathcal{A} = \langle x, y \rangle_\mathcal{A} A$*, for all* $A \in \mathcal{A}$ *and* $x, y \in \mathcal{E}$.

iv.) $\langle x, x \rangle_\mathcal{A} \geq 0$*, for all* $x \in \mathcal{E}$ *($\geq$ in $\mathcal{A}$).*

v.) $\langle x, x \rangle_\mathcal{A} = 0 \Rightarrow x = 0$*, for all* $x \in \mathcal{E}$.

It easily follows from the above conditions that $\langle \cdot, \cdot \rangle_\mathcal{A}$ is anti-linear in the first entry and that $\langle Ax, y \rangle_\mathcal{A} = A^* \langle x, y \rangle_\mathcal{A}$. The following version of the Cauchy-Schwarz inequality holds [58].

$$\langle x, y \rangle_\mathcal{A} \langle y, x \rangle_\mathcal{A} \leq \| \langle x, x \rangle_\mathcal{A} \| \langle y, y \rangle_\mathcal{A}, \ x, y \in \mathcal{E}.$$

It follows that $\|x\|_\mathcal{A} := \| \langle x, x \rangle_\mathcal{A} \|^{1/2}$ defines a norm[14] on $\mathcal{E}$.

Definition 2.31. *A (right) pre-Hilbert $\mathcal{A}$-module $\mathcal{E}_\mathcal{A}$ is called a (right) Hilbert $\mathcal{A}$-module if it is complete with respect to $\| \cdot \|_\mathcal{A}$.*

A simple computation shows that if $\mathcal{E}_\mathcal{A}$ is a Hilbert $\mathcal{A}$-module then $\text{span}\{\langle x, y \rangle_\mathcal{A}, \ x, y \in \mathcal{E}\}$ is a two-sided ideal in $\mathcal{A}$.

Definition 2.32. *A Hilbert $\mathcal{A}$-module $\mathcal{E}$ is* full *if $\text{span}\{\langle x, y \rangle_\mathcal{A}, \ x, y \in \mathcal{E}\}$ is dense in $\mathcal{A}$.*

It is not difficult to show that if $\mathcal{E}_\mathcal{A}$ is full, then the action of $\mathcal{A}$ on $\mathcal{E}_\mathcal{A}$ is nondegenerate [15]. Let us consider some examples of Hilbert C^*-modules.

Example 2.33. *Hilbert $\mathbb{C}$-modules are just ordinary Hilbert spaces.*

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[13] The definition of a left Hilbert module is analogous, but with linearity with respect to $\mathcal{A}$ and $\mathbb{C}$ in the first entry of ${}_\mathcal{A}\langle \cdot, \cdot \rangle$.

[14] This norm satisfies $\|xA\|_\mathcal{A} \leq \|x\|_\mathcal{A} \|A\|$, for all $x \in \mathcal{E}$ and $A \in \mathcal{A}$.

[15] This means that $x \cdot A = 0, \ \forall A \in \mathcal{A} \Rightarrow x = 0, x \in \mathcal{E}$.

Example 2.34. *Any C^* algebra $\mathcal{A}$ is a Hilbert module over itself, with module structure given by multiplication on the right and $\mathcal{A}$-valued inner product given by $\langle A, B\rangle_{\mathcal{A}} = A^*B$, $A, B \in \mathcal{A}$. The existence of an approximate identity in $\mathcal{A}$ implies that $\mathcal{A}_{\mathcal{A}}$ is full.*

Example 2.35. *Let $\mathcal{A} = C(X)$, where X is a compact Hausdorff space. If $E \to X$ is a complex hermitian vector bundle, then the space of continuous sections $\Gamma(E)$ is a Hilbert $C(X)$-module: $C(X)$ acts on $\Gamma(E)$ by pointwise multiplication, and we define a $C(X)$-valued inner product on $\Gamma(E)$ by setting $\langle s, t\rangle_{C(X)}(x) = \langle s(x), t(x)\rangle_x$, for $s, t \in \Gamma(E)$ and $x \in X$, where $\langle \cdot, \cdot\rangle_x$ denotes the hermitian inner product on the fiber over x* [16].

In order to construct functors between categories of hermitian modules over C^*-algebras, we need

Definition 2.36. *Let $\mathcal{E}_{\mathcal{A}}$ be a Hilbert $\mathcal{A}$-module. A map $T : \mathcal{E} \longrightarrow \mathcal{E}$ is called* adjointable *if there exists a map $T^* : \mathcal{E} \longrightarrow \mathcal{E}$ satisfying $\langle Tx, y\rangle_{\mathcal{A}} = \langle x, T^*y\rangle_{\mathcal{A}}$, for all $x, y \in \mathcal{E}$. The set of all adjointable* [17] *operators on $\mathcal{E}$ is denoted by $\mathcal{L}(\mathcal{E})$.*

The space $\mathcal{L}(\mathcal{E})$ has a natural C^*-algebra structure (with respect to the operator norm).

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras, and suppose ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is a bimodule which is a Hilbert $\mathcal{A}$-module and such that the $\mathcal{B}$-action preserves adjoints ($\langle Bx, y\rangle_{\mathcal{A}} = \langle x, B^*y\rangle_{\mathcal{A}}$, for all $x, y \in \mathcal{E}$). We will discuss a way to define a functor $\mathcal{R}_{\mathcal{E}} : \mathsf{Her}(\mathcal{A}) \longrightarrow \mathsf{Her}(\mathcal{B})$ corresponding to ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$. This procedure is called *Rieffel induction* of representations [60].

Let $\pi : \mathcal{A} \longrightarrow \mathfrak{B}(\mathfrak{H})$ be a *-representation of $\mathcal{A}$ on $\mathfrak{H}$. In order to define a new Hilbert space $\mathfrak{K}$, with a corresponding representation $\rho : \mathcal{B} \longrightarrow \mathfrak{B}(\mathfrak{K})$, we proceed as follows:

- Consider the space $\widetilde{\mathfrak{K}} = \mathcal{E} \otimes_{\mathcal{A}} \mathfrak{H}$ [18]. It is clear that B naturally acts on $\tilde{\mathfrak{K}}$ on the left.

- Define on $\widetilde{\mathfrak{K}}$ the form $\langle \cdot, \cdot\rangle_{\widetilde{\mathfrak{K}}} : \widetilde{\mathfrak{K}} \times \widetilde{\mathfrak{K}} \longrightarrow \mathbb{C}$ by

$$\langle x_1 \otimes \phi_1, x_2 \otimes \phi_2\rangle_{\widetilde{\mathfrak{K}}} := \langle \phi_1, \pi(\langle x_1, x_2\rangle_{\mathcal{A}})\phi_2\rangle_{\mathfrak{H}}, \ \ x_1, x_2 \in \mathcal{E}, \ \ \phi_1, \phi_2 \in \mathfrak{H}.$$

Proposition 2.37. *$\langle \cdot, \cdot\rangle_{\widetilde{\mathfrak{K}}}$ uniquely defines a positive semi-definite inner product on $\widetilde{\mathfrak{K}} = \mathcal{E} \otimes_{\mathcal{A}} \mathfrak{H}$.*

[16] See [19] for a discussion on Hilbert bundles, generalizing this example.

[17] Any adjointable map is a bounded, linear, $\mathcal{A}$-module map (this is a consequence of the definition). But we may have $T : \mathcal{E} \longrightarrow \mathcal{E}$ $\mathcal{A}$-linear and bounded (with respect to $\| \cdot \|_{\mathcal{A}}$) but still with no adjoints (see [58]).

[18] Note that we are tensoring over $\mathcal{A}$, i.e. on this set we have $xA \otimes \phi = x \otimes \pi(A)\phi$, $A \in \mathcal{A}$, $\phi \in \mathfrak{H}$.

Proof. The only non-trivial property to be checked is positivity. Let us first assume that the representation $\pi : \mathcal{A} \longrightarrow \mathfrak{B}(\mathfrak{H})$ is cyclic. Let $\sum_{i=1}^{m} x_i \otimes \phi_i \in \widetilde{\mathfrak{K}}$. Then there exists $\psi \in \mathfrak{H}$ and $A_i,\ i = 1 \dots m$, so that $\phi_i = \pi(A_i)\psi$, and hence

$$\sum_{i=1}^{m} x_i \otimes \phi_i = \sum_{i=1}^{m} x_i \otimes \pi(A_i) = \sum_{i=1}^{m} x_i A_i \otimes \psi.$$

Therefore $\langle \sum_{i=1}^{m} x_i \otimes \phi_i, \sum_{i=1}^{m} x_i \otimes \phi_i \rangle_{\widetilde{\mathfrak{K}}} = \left\langle \psi, \pi(\langle \sum_{i=1}^{m} x_i A_i, \sum_{i=1}^{m} x_i A_i \rangle_{\mathcal{A}})\psi \right\rangle_{\mathfrak{H}} \geq 0$. The general result follows from the fact that a general representation π can be decomposed into a orthogonal direct sum of cyclic ones [47, Ch. I,Prop. 1.5.2]. See [58, Prop. 2.64] for a different proof. □

- Let $\mathcal{N} = \{\psi \in \widetilde{\mathfrak{K}} |\ \langle \psi, \psi \rangle_{\widetilde{\mathfrak{K}}} = 0\}$. Then $\mathfrak{K} := \overline{(\widetilde{\mathfrak{K}}/\mathcal{N})}$, the complection of $\widetilde{\mathfrak{K}}/\mathcal{N}$ with respect to the norm induced by the inner product, has a natural Hilbert space structure.

- The formula $B \cdot ([x \otimes \phi]) := [Bx \otimes \phi],\ B \in \mathcal{B},\ x \in \mathcal{E},\ \phi \in \mathfrak{H}$, determines a *-representation $\rho : \mathcal{B} \longrightarrow \mathfrak{B}(\mathfrak{K})$, where $[\,\cdot\,]$ denotes the image of elements in $\widetilde{\mathfrak{K}}$ in the quotient space $\mathfrak{K}$. We note that if the action of $\mathcal{B}$ on ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is nondegenerate[19], then the induced representation ρ is also nondegenerate.

Finally, this construction is functorial: If $\mathfrak{H}$ is an object in $\mathsf{Her}(\mathcal{A})$, let $\mathcal{R}_{\mathcal{E}}(\mathfrak{H}) = \mathfrak{K}$ be the corresponding object in $\mathsf{Her}(\mathcal{B})$ defined by Rieffel induction. Suppose that $(\pi_1, \mathfrak{H}_1)$ and $(\pi_2, \mathfrak{H}_2)$ are two *-representations of $\mathcal{A}$, and let $T : \mathfrak{H}_1 \longrightarrow \mathfrak{H}_2$ be an intertwining operator. Then there is a well-defined (bounded) operator $\mathcal{R}_{\mathcal{E}}(T) : \mathfrak{K}_1 \longrightarrow \mathfrak{K}_2$ uniquely determined by the condition $\mathcal{R}_{\mathcal{E}}(T)[x \otimes \phi_1] = [x \otimes T\phi_1],\ x \in X,\ \phi_1 \in \mathfrak{H}_1$, which intertwines ρ_1, ρ_2, the corresponding induced representations of $\mathcal{B}$.

Theorem 2.38. *Let ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be a bimodule which is a Hilbert $\mathcal{A}$-module. Suppose that the $\mathcal{B}$-action on $\mathcal{E}$ preserves adjoints and is nondegenerate. Then the Rieffel induction procedure defines a functor $\mathcal{R}_{\mathcal{E}} : \mathsf{Her}(\mathcal{A}) \longrightarrow \mathsf{Her}(\mathcal{B})$, preserving the adjoint operation on morphisms.*

2.3.3 Strong Morita equivalence

We will discuss in this section conditions on the bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ so that the corresponding Rieffel induction functor $\mathcal{R}_{\mathcal{E}} : \mathsf{Her}(\mathcal{A}) \longrightarrow \mathsf{Her}(\mathcal{B})$ is an equivalence of categories.

[19] That is, $B \cdot x = 0\ \forall B \in \mathcal{B} \Rightarrow x = 0, x \in \mathcal{E}$.

Definition 2.39. *A* $(\mathcal{B},\mathcal{A})$-imprimitivity bimodule[20] ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ *is a* $(\mathcal{B},\mathcal{A})$*-bimodule such that:*

i.) ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ *is a full right Hilbert* $\mathcal{A}$*-module and a full left Hilbert* $\mathcal{B}$*-module.*

ii.) For all $x, y \in \mathcal{E}$, $A \in \mathcal{A}$, $B \in \mathcal{B}$,

$$\langle Bx, y\rangle_{\mathcal{A}} = \langle x, B^* y\rangle_{\mathcal{A}}, \ B \in \mathcal{B}, \qquad {}_{\mathcal{B}}\langle xA, y\rangle = {}_{\mathcal{B}}\langle x, yA^*\rangle, \ A \in \mathcal{A}.$$

iii.) For all $x, y, z \in \mathcal{E}$*, we have*

$${}_{\mathcal{B}}\langle x, y\rangle\, z = x\,\langle y, z\rangle_{\mathcal{A}}.$$

Definition 2.40. *We say that two* C^**-algebras* $\mathcal{A}$ *and* $\mathcal{B}$ *are strongly Morita equivalent if there exists a* $(\mathcal{B},\mathcal{A})$*-imprimitivity bimodule* ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$*.*

To see that strong Morita equivalence is a symmetric relation, note that, if ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is a $(\mathcal{B},\mathcal{A})$-imprimitivity bimodule, then we can define an $(\mathcal{A},\mathcal{B})$-imprimitivity bimodule ${}_{\mathcal{A}}\overline{\mathcal{E}}_{\mathcal{B}}$ as follows. Let $\overline{\mathcal{E}}$ be the vector space conjugate[21] to $\mathcal{E}$. The C^*-algebra $\mathcal{A}$ acts on $\overline{\mathcal{E}}$ on the left by adjoint elements: $A\overline{x} = \overline{xA^*}$. Similarly, $\mathcal{B}$ acts on $\overline{\mathcal{E}}$ on the right. As for the inner products, simply set $\langle \overline{x}, \overline{y}\rangle_{\mathcal{B}} = {}_{\mathcal{B}}\langle x, y\rangle$ and ${}_{\mathcal{A}}\langle \overline{x}, \overline{y}\rangle = \langle x, y\rangle_{\mathcal{A}}$. This structure makes $\overline{\mathcal{E}}$ into an $(\mathcal{A},\mathcal{B})$-imprimitivity bimodule.

Example 2.41. *Consider a* C^**-algebra* $\mathcal{A}$ *as an* $(\mathcal{A},\mathcal{A})$*-bimodule in the natural way. Endow it with* $\mathcal{A}$*-valued inner products*

$$\langle A, B\rangle_{\mathcal{A}} = A^* B, \qquad {}_{\mathcal{A}}\langle A, B\rangle = AB^*,$$

$A, B \in \mathcal{A}$*. It is simple to check that this structure makes* ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ *into an* $(\mathcal{A},\mathcal{A})$*-imprimitivity bimodule. Thus any* C^**-algebra is strongly Morita equivalent to itself. In fact, a similar argument shows that any* **-isomorphic* C^**-algebras are strongly Morita equivalent.*

A suitable notion of tensor products of imprimitivity bimodules shows that strong Morita equivalence is a transitive relation [58]. It then follows

Theorem 2.42. *Strong Morita equivalence defines an equivalence relation in the category of* C^* *algebras.*

[20] These bimodules are also called *equivalence bimodules*. The terminology *imprimitivity* is due to applications of such bimodules to prove Mackey's imprimitivity theorem [60, 58].

[21] That is $\lambda\overline{x} = \overline{(\overline{\lambda}x)}$, $\lambda \in \mathbb{C}$.

Furthermore, strong Morita equivalence implies[22] Morita equivalence in the categorical sense [61].

Theorem 2.43. *Suppose* ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ *is an imprimitivity bimodule. Then the corresponding functor* $\mathcal{R}_{\mathcal{E}} : \mathsf{Her}(\mathcal{A}) \longrightarrow \mathsf{Her}(\mathcal{B})$[23] *determines an equivalence of categories. The inverse functor corresponds to the conjugate bimodule* ${}_{\mathcal{A}}\overline{\mathcal{E}}_{\mathcal{B}}$.

Remark 2.44. *It is shown in [5] that if* $\mathcal{A}$ *and* $\mathcal{B}$ *are unital* C^**algebras, then they are strongly Morita equivalent if and only if they are Morita equivalent as rings.*

We will end this section with another characterization of strongly Morita equivalent C^*-algebras. Let $\mathcal{E}_{\mathcal{A}}$ be a right Hilbert $\mathcal{A}$-module. We will consider the analogue of compact operators on Hilbert spaces[24]. We define the operators $\Theta_{x,y} : \mathcal{E} \longrightarrow \mathcal{E}$ by

$$\Theta_{x,y} z = x \langle y, z \rangle_{\mathcal{A}}, \qquad x, y, z \in \mathcal{E}.$$

Note that $\Theta^*_{x,y} = \Theta_{y,x}$ and hence $\Theta_{x,y} \subseteq \mathcal{L}(\mathcal{E})$, for all $x, y \in \mathcal{E}$.

Definition 2.45. *For a Hilbert* $\mathcal{A}$*-module* $\mathcal{E}_{\mathcal{A}}$*, we define* $\mathcal{K}(\mathcal{E}) := \overline{span\{\Theta_{x,y}, x, y \in \mathcal{E}\}}$.

It is not hard to check that $\mathcal{K}(\mathcal{E})$ is a closed 2-sided ideal in $\mathcal{L}(\mathcal{E})$. Thus, in particular, it is a C^*-algebra. It is clear that $\mathcal{E}$ has a $(\mathcal{K}(\mathcal{E}), \mathcal{A})$-bimodule structure. We can define a (full) $\mathcal{K}(\mathcal{E})$-valued inner product on $\mathcal{E}$ by ${}_{\mathcal{K}(\mathcal{E}_{\mathcal{A}})}\langle x, y \rangle = \Theta_{x,y}$.

Theorem 2.46. *Two* C^**-algebras* $\mathcal{A}$ *and* $\mathcal{B}$ *are strongly Morita equivalent if and only if there exists a full Hilbert* $\mathcal{A}$*-module* $\mathcal{E}_{\mathcal{A}}$ *with* $\mathcal{B} \cong \mathcal{K}(\mathcal{E})$.

Example 2.47. *Any Hilbert space* $\mathcal{H}$ *defines a* $(\mathcal{K}(\mathfrak{H}), \mathbb{C})$*-imprimitivity bimodule* ${}_{\mathcal{K}(\mathfrak{H})}\mathfrak{H}_{\mathbb{C}}$. *Hence* $\mathbb{C}$ *and* $\mathcal{K}(\mathcal{H})$ *are strongly Morita equivalent.*

Example 2.48. *Let* $\mathcal{A}$ *be a* C^**-algebra. Then* $\mathcal{A}^n$ *is a* $(M_n(\mathcal{A}), \mathcal{A})$*-imprimitivity bimodule. For* $u, v \in \mathcal{A}^n$*, we have the inner products*

$$\langle u, v \rangle_{\mathcal{A}} = \sum_i u_i^* v_i, \qquad ({}_{M_n(\mathcal{A})}\langle u, v \rangle)_{ij} = u_i v_j^*.$$

[22] In fact, it can be shown that strong Morita equivalence is strictly stronger than categorical Morita equivalence [62, 5].

[23] The fullness condition in the definition of imprimitivity bimodules guarantees that the actions of $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{E}$ (and $\overline{\mathcal{E}}$) are nondegenerate. Hence the corresponding functors carry nondegenerate representations to nondegenerate representations.

[24] Recall that in an ordinary Hilbert space $\mathfrak{H}$, we have $\mathcal{K}(\mathfrak{H}) = \overline{span\{\phi \otimes \overline{\psi},\ \phi, \psi \in \mathfrak{H}\}}$, where $\phi \otimes \overline{\psi}(\eta) = \phi \langle \psi, \eta \rangle_{\mathfrak{H}}$, for $\eta \in \mathfrak{H}$.

More generally, let $P \in M_n(\mathcal{A})$ *be a full projection, i.e.* $P = P^* = P^2$ *and* $M_n(\mathcal{A})PM_n(\mathcal{A}) = M_n(A)$ *(Definition 2.22). In this case,* $\mathcal{E} = P\mathcal{A}^n$ *is a* $(\mathcal{B}, \mathcal{A})$*-imprimitivity bimodule for* $\mathcal{B} = PM_n(\mathcal{A})P$ *(with inner products as before, but restricted to* $P\mathcal{A}^n \subseteq \mathcal{A}^n$*).*

Example 2.49. *Let* $\mathcal{A} = C(X)$ *be the algebra of complex-valued continuous functions on a compact Hausdorff space* X*. Let* $E \to X$ *be an* n*-dimensional complex vector bundle. It then follows from the previous example (and Serre-Swan's theorem [3, Chp. XIV]) that* $\mathcal{E} = \Gamma(E)$ *is an* $(\Gamma(\mathsf{End}(E)), C(X))$*-imprimitivity bimodule.*

We finally remark that strongly Morita equivalent C^*-algebras have many properties in common: they have the same K-theory, isomorphic lattices of (closed, two-sided) ideals and, in the unital case, *-isomorphic centers.

Chapter 3

Algebraic strong Morita equivalence

In this chapter we will extend the notion of strong Morita equivalence to a wider class of *-algebras, including hermitian star-product algebras.

3.1 Algebraic preliminaries

In this section we will lay the groundwork for the description of purely algebraic analogs of Rieffel induction and strong Morita equivalence.

3.1.1 Pre-Hilbert spaces over ordered rings

Definition 3.1. *Let* R *be an associative, commutative and unital ring (with* $1 \neq 0$*). We call it* ordered *if there exists* $\mathsf{P} \subset \mathsf{R}$ *so that* R *is the disjoint union* $\mathsf{R} = -\mathsf{P} \cup \{0\} \cup \mathsf{P}$*, and for all* $a, b \in \mathsf{P}$ *one has* $a + b, ab \in \mathsf{P}$*.*

Elements in P are called *positive* and endow R with a natural order structure: we say that $a > b$ if and only if $a - b \in \mathsf{P}$ (similarly for '$<$', '$\geq$', and '$\leq$'). Note that $a^2 > 0$ for all $a \neq 0$ and, in particular, $1 > 0$. It is easy to check that R has characteristic zero and no zero divisors.

For an ordered ring R, let $\mathsf{C} := \mathsf{R} \oplus \mathrm{i}\mathsf{R} = \mathsf{R}(\mathrm{i})$, where $\mathrm{i}^2 = -1$. This quadratic ring extension has again characteristic zero and no zero divisors. We write elements in C as

$z = a+ib$ with $a, b \in \mathsf{R}$. We define the complex conjugation in C by $z = a+ib \mapsto \overline{z} := a-ib$. Note that $\overline{z}z \geq 0$ and $\overline{z}z = 0$ if and only if $z = 0$.

Example 3.2. $\mathbb{R}$ *is an ordered ring, and the corresponding quadratic ring extension is* $\mathbb{C}$.

Example 3.3. *Let* R *be an ordered ring. The ring of formal power series with coefficients in* R*, denoted by* $\mathsf{R}[[\lambda]]$*, has a natural order structure: we define* $\sum_{r=0}^{\infty} a_r \lambda^r > 0$ *if* $a_{r_0} > 0$*, where* r_0 *is the first index with non-zero coefficient*[1]*. In this case the corresponding ring extension is just* $\mathsf{C}[[\lambda]]$.

Let R be an ordered ring, with corresponding quadratic ring extension C. Let $\mathfrak{H}$ be a C-module.

Definition 3.4. *A* semi-definite Hermitian product *on* $\mathfrak{H}$ *is a map* $\langle \cdot, \cdot \rangle : \mathfrak{H} \times \mathfrak{H} \to \mathsf{C}$ *satisfying*

$$\langle \phi, a\psi + b\eta \rangle = a\langle \phi, \psi \rangle + b\langle \phi, \eta \rangle, \quad \overline{\langle \phi, \psi \rangle} = \langle \psi, \phi \rangle, \quad \text{and} \quad \langle \phi, \phi \rangle \geq 0, \tag{3.1}$$

for all $\phi, \psi, \eta \in \mathfrak{H}$ *and* $a, b \in \mathsf{C}$*. If* $\langle \cdot, \cdot \rangle$ *is in addition nondegenerate*[2]*,* $\langle \phi, \phi \rangle = 0 \implies \phi = 0$*, then* $\langle \cdot, \cdot \rangle$ *is called a* Hermitian product *and* $(\mathfrak{H}, \langle \cdot, \cdot \rangle)$ *is called a* pre-Hilbert space *over* C.

Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be C-modules with semi-definite Hermitian products. As usual, a linear map $U : \mathfrak{H}_1 \to \mathfrak{H}_2$ is called *isometric* if $\langle U\phi, U\psi \rangle_2 = \langle \phi, \psi \rangle_1$ for all $\phi, \psi \in \mathfrak{H}_1$, and *unitary* if U is isometric and bijective. An isometric map is automatically injective if $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are pre-Hilbert spaces.

Let $\mathfrak{H}$ be a C-module with semi-definite Hermitian product. Analogously to complex Hilbert spaces, we observe that the Cauchy-Schwarz inequality

$$\langle \phi, \psi \rangle \overline{\langle \phi, \psi \rangle} \leq \langle \phi, \phi \rangle \langle \psi, \psi \rangle \tag{3.2}$$

holds for all $\phi, \psi \in \mathfrak{H}$. Thus the space $\{\phi \in \mathfrak{H} \mid \langle \phi, \phi \rangle = 0\}$ coincides with $\mathfrak{H}^\perp := \{\phi \in \mathfrak{H} \mid \forall \psi \in \mathfrak{H} : \langle \phi, \psi \rangle = 0\}$, which is a C-submodule of $\mathfrak{H}$. The quotient $\mathfrak{H}/\mathfrak{H}^\perp$ endowed with the Hermitian product $\langle [\phi], [\psi] \rangle := \langle \phi, \psi \rangle$ is a pre-Hilbert space over C. The proofs are essentially analogous to the case of complex numbers, and can be found in [11, 70].

Let $A \in \mathsf{End}_{\mathsf{C}}(\mathfrak{H})$. We say that $A^* \in \mathsf{End}_{\mathsf{C}}(\mathfrak{H})$ is an *adjoint* of A if $\langle \phi, A\psi \rangle = \langle A^*\phi, \psi \rangle$, for all $\phi, \psi \in \mathfrak{H}$. We set

$$\mathfrak{B}(\mathfrak{H}) := \{A \in \mathsf{End}_{\mathsf{C}}(\mathfrak{H}) \mid A \text{ has an adjoint }\}. \tag{3.3}$$

[1] This ordering is non-Archimedian, since $0 < n\lambda < 1$ for all $n \in \mathbb{N}$.

[2] Note that nondegeneracy implies that $\mathfrak{H}$ is torsion-free.

All the standard properties of adjoints hold in this context [15, Lem. 2.2]; in particular, adjoints are unique on pre-Hilbert spaces over C. Analogous definitions and results hold for $A \in \mathsf{End}_{\mathsf{C}}(\mathfrak{H}_1, \mathfrak{H}_2)$ for two C-modules $\mathfrak{H}_1, \mathfrak{H}_2$ with positive semi-definite Hermitian product.

3.1.2 *-Algebras, states and positive elements

Let $\mathcal{A}$ be an associative algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$, where R is an ordered ring. (Algebras in this section will always be over C, unless otherwise stated.)

Definition 3.5. *An* involution *on $\mathcal{A}$ is an anti-linear map $^* : \mathcal{A} \longrightarrow \mathcal{A}$ satisfying $(A^*)^* = A$ and $(AB)^* = B^*A^*$, for $A, B \in \mathcal{A}$. An associative algebra over C with an involution * is called a* *-algebra *over C.*

Let $\mathcal{A}$ be a *-algebra. We define *hermitian*, *normal*, *isometric* and *unitary* elements of $\mathcal{A}$ in the usual way.

Definition 3.6. *A linear functional $\omega : \mathcal{A} \longrightarrow \mathsf{C}$ is called* positive *if*

$$\omega(A^*A) \geq 0, \text{ for all } A \in \mathcal{A}. \tag{3.4}$$

If $\mathcal{A}$ is unital, then ω is called a state *if ω is positive and $\omega(1) = 1$.*

As in the case of C^*-algebras, every positive linear functional ω satisfies $\omega(A^*B) = \overline{\omega(B^*A)}$ and the Cauchy-Schwarz inequality [11, Lem. 5]

$$\omega(A^*B)\overline{\omega(A^*B)} \leq \omega(A^*A)\omega(B^*B). \tag{3.5}$$

Definition 3.7. *Let $\mathcal{A}$ be a *-algebra over C. A Hermitian element $A \in \mathcal{A}$ is called* positive *if $\omega(A) \geq 0$ for every positive linear functional $\omega : \mathcal{A} \to \mathsf{C}$. We denote the set of positive elements by $\mathcal{A}^+$.*

Elements of the form $A = b_1 B_1^* B_1 + \ldots + b_n B_n^* B_n$, where $B_i \in \mathcal{A}$ and $b_i \in \mathsf{P} \subseteq \mathsf{R}$, are called *algebraically positive*. We denote the set of algebraically positive elements by $\mathcal{A}^{++}$. It is clear that $\mathcal{A}^{++} \subseteq \mathcal{A}^+$.

Example 3.8 ([15, App. A]). *Let $\mathcal{A} = M_n(\mathsf{C})$, where $\mathsf{C} = \mathsf{R}(\mathrm{i})$, R ordered. This algebra has a natural *-involution $(A_{ij}) \mapsto (\overline{A_{ji}})$. In this case, a hermitian element $A \in \mathcal{A}$ is positive if and only if $\langle v, Av \rangle \geq 0$ for all $v \in \mathsf{C}^n$, where $\langle v, u \rangle = \sum_{i=1}^n \overline{v_i} u_i$.*

Example 3.9. *If $\mathcal{A}$ is a C^*-algebra, then the previous definitions coincide with the usual notions of positive linear functionals and positive algebra elements. In this case, $\mathcal{A}^{++} = \mathcal{A}^+$.*

Example 3.10 ([15, App. B]). *Let $\mathcal{A} = C^\infty(M)$ be the algebra of complex-valued smooth functions on a manifold M, endowed with the natural involution given by complex conjugation. In this case positive linear functionals correspond to positive measures with compact support, and $f \in C^\infty(M)^+$ if and only if $f(x) \geq 0$, for all $x \in M$. Note that if $\mathcal{A} = C^\infty([0,1])$, then $f(x) = x$ is positive but $f \notin \mathcal{A}^{++}$.*

Example 3.11. *Let $\boldsymbol{\mathcal{A}}$ be the *-algebra $(C^\infty(\mathbb{C}^n)[[\lambda]], \star)$, where $\star$ is the Wick product, and the involution is complex conjugation (see Example 2.14). In this case, one can show [14, Lem. 4.4] that the λ-linear extension of any positive linear functional in $C^\infty(\mathbb{C}^n)$ is positive for $\boldsymbol{\mathcal{A}}$.*

Remark 3.12. *More generally, any hermitian star product $\star$ on a manifold M defines a *-algebra $(C^\infty(M)[[\lambda]], \star)$ over $\mathbb{C}[[\lambda]]$. A natural question is which positive linear functionals ω on $C^\infty(M)$ can be deformed into positive linear functionals*

$$\boldsymbol{\omega} = \omega + \lambda\omega_1 + O(\lambda^2), \;\; \omega_i : C^\infty(M) \longrightarrow \mathbb{C},$$

on $(C^\infty(M)[[\lambda]], \star)$. This matter was discussed in [14, Prop. 5.1], where it is shown that if M is a symplectic manifold and $\star$ is hermitian, then all positive linear functionals on $C^\infty(M)$ can be deformed into positive linear functionals on $(C^\infty(M)[[\lambda]], \star)$.

It is often useful to have substitutes for the identity element in nonunital algebras. Motivated by C^*-algebras, we consider the following

Definition 3.13. *Let $\mathcal{A}$ be a *-algebra and I a directed set. Let $\{E_\alpha\}_{\alpha \in I}$ be a set of elements $E_\alpha = E_\alpha^* \in \mathcal{A}$ such that for all $\alpha < \beta$ we have $E_\alpha = E_\alpha E_\beta = E_\beta E_\alpha$. Suppose that there are subspaces $\mathcal{A}_\alpha$, $\alpha \in I$, so that $\mathcal{A}_\alpha \subseteq \mathcal{A}_\beta$, for $\alpha \leq \beta$, and $\mathcal{A} = \bigcup_{\alpha \in I} \mathcal{A}_\alpha$. Finally assume that, for all $A \in \mathcal{A}_\alpha$, $A = E_\alpha A = A E_\alpha$. We call $\{\mathcal{A}_\alpha, E_\alpha\}_{\alpha \in I}$ an* (algebraic) approximate identity *for $\mathcal{A}$.*

It is clear that if $\mathcal{A}$ has a unit element then $\{\mathcal{A}, 1\}$ is an approximate identity. A less trivial example is the following.

Example 3.14. *Let $\mathcal{A} = C_c^\infty(M)$, the algebra of complex-valued smooth functions with compact support on a non-compact manifold M. It is easy to see that $\mathcal{A}$ admits an algebraic approximate identity.*

The following weaker definitions are also important in nonunital settings.

Definition 3.15. *Let $\mathcal{R}$ be an arbitrary ring. We call $\mathcal{R}$* nondegenerate *if $r \cdot \mathcal{R} = 0$ or $\mathcal{R} \cdot r = 0$, then $r = 0$; we say that $\mathcal{R}$ is* idempotent *if elements of the form $r_1 r_2$ span $\mathcal{R}$.*

Example 3.16. *Any C^*-algebra $\mathcal{A}$ is nondegenerate and idempotent: nondegeneracy comes from the fact that $xx^* = 0$ implies that $x = 0$; idempotency holds since any element of a C^*-algebra can be decomposed into its real and imaginary part, and each of them can be written as differences of positive elements, which are squares. Pedersen ideals of C^*-algebras are also nondegenerate and idempotent (we will come back to this later).*

It is clear that any *-algebra with an algebraic approximate identity is nondegenerate and idempotent. The converse is not true in general.

3.1.3 *-Representations and the algebraic GNS construction

Let $\mathcal{A}$ be an associative algebra, and let $\mathcal{E}$ be an $\mathcal{A}$-module. We denote the action of $\mathcal{A}$ on $\mathcal{E}$ by $\Psi : \mathcal{A} \longrightarrow \mathsf{End}(\mathcal{E})$. We call this action *faithful* if the map Ψ is injective, and *nondegenerate* if $\Psi(A)x = 0$, for all $A \in \mathcal{A}$, implies $x = 0$.

Definition 3.17. *The action Ψ is called* strongly nondegenerate *if the C-linear span of all vectors of the form $A \cdot x$, with $A \in \mathcal{A}$ and $x \in \mathcal{E}$, is $\mathcal{E}$.*

Definition 3.18. *The action Ψ is called* cyclic*, with* cyclic vector *$\Omega \in \mathcal{E}$, if for all $x \in \mathcal{E}$ there is a $A \in \mathcal{A}$ such that $x = A \cdot \Omega$.*

Recall that a *filtration* of a C-module $\mathcal{E}$ is a collection of subspaces $\{\mathcal{E}_\alpha\}$, $\alpha \in I$ for some directed set I, so that $\mathcal{E}_\alpha \subseteq \mathcal{E}_\beta$ if $\alpha \leq \beta$ and $\mathcal{E} = \bigcup_{\alpha \in I} \mathcal{E}_\alpha$.

Definition 3.19. *The action Ψ of $\mathcal{A}$ on $\mathcal{E}$ is called* pseudo-cyclic *if $\mathcal{E}$ is filtered and each subspace $\mathcal{E}_\alpha$ of the filtration is cyclic for Ψ with cyclic vector Ω_α. In this case $\{\Omega_\alpha\}_{\alpha \in I}$ are called the* pseudo-cyclic vectors *of Ψ.*

Let $\mathfrak{H}$ be a pre-Hilbert space over C, and let $\mathcal{A}$ be a *-algebra over C.

Definition 3.20. *A* *-representation *of $\mathcal{A}$ on $\mathfrak{H}$ is a *-homomorphism $\pi : \mathcal{A} \longrightarrow \mathfrak{B}(\mathfrak{H})$.*

The representation π is called *faithful* (resp. *nondegenerate*, resp. *strongly nondegenerate*) if it is faithful (resp. nondegenerate, resp. strongly nondegenerate) as above. One can check that, if $\mathcal{A}$ is unital, then π is nondegenerate if and only if $\pi(1) = \mathsf{id}$. In this case, a simple computation shows that nondegeneracy and strong nondegeneracy are equivalent. In general, strong nondegeneracy implies nondegeneracy [3]. Note that if π is a direct orthogonal sum of pseudo-cyclic *-representations of $\mathcal{A}$, then π is strongly nondegenerate.

Let π_1 and π_2 be two *-representations of $\mathcal{A}$ on $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively, and let $T : \mathfrak{H}_1 \to \mathfrak{H}_2$ be a linear map. We recall that T is called an *intertwiner* if $\pi_2(A)T = T\pi_1(A)$, for all $A \in \mathcal{A}$.

Definition 3.21. *We define* $\mathsf{rep}(\mathcal{A})$ *to be the category of* **-representations of* $\mathcal{A}$ *on pre-Hilbert spaces over* C*, with isometric intertwiners as morphisms. We define* $\mathsf{Rep}(\mathcal{A})$ *to be the subcategory of strongly nondegenerate* **-representations of* $\mathcal{A}$*.*

Objects in $\mathsf{rep}(\mathcal{A})$ can be constructed out of positive linear functionals by means of the so-called algebraic GNS construction [11, 10]: Let $\omega : \mathcal{A} \to \mathsf{C}$ be a positive linear functional on the *-algebra $\mathcal{A}$. We define the left ideal $\mathcal{J}_\omega := \{A \in \mathcal{A} \mid \omega(A^*A) = 0\} \subseteq \mathcal{A}$. The quotient $\mathfrak{H}_\omega := \mathcal{A}/\mathcal{J}_\omega$ carries a hermitian product defined by $\langle \psi_A, \psi_B \rangle := \omega(A^*B)$, where $\psi_A, \psi_B \in \mathfrak{H}_\omega$ denote the equivalence classes of $A, B \in \mathcal{A}$, respectively. Since $\mathcal{J}_\omega$ is a left ideal, $\mathfrak{H}_\omega$ is an $\mathcal{A}$-left module. The *GNS representation* of $\mathcal{A}$ is defined by $\pi_\omega(A)\psi_B := \psi_{AB}$. A straightforward computation shows that $\pi_\omega(A) \in \mathfrak{B}(\mathfrak{H}_\omega)$ and $\pi_\omega(A^*) = \pi_\omega(A)^*$.

Example 3.22 (Schrödinger representation). *Consider* $\mathbb{R}^{2n} = \{(q^i, p_i)\}$ *with its standard symplectic form, and let* $\mathcal{A} = \{f \in C^\infty(\mathbb{R}^{2n}) \mid \operatorname{supp}(f) \cap i(\mathbb{R}^n)$ *is compact*$\}$*, where* $i : \mathbb{R}^n \hookrightarrow \mathbb{R}^{2n} \cong T^*\mathbb{R}^n$ *is the embedding of* $\mathbb{R}^n$ *as the zero section. Let* $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$*, where* $\star$ *is the Weyl product (Example 2.3), and consider the linear functional* $\omega : \boldsymbol{\mathcal{A}} \longrightarrow \mathbb{C}[[\lambda]]$*,*

$$\omega(f) := \int_{\mathbb{R}^n} i^* f \, dq.$$

One can show that ω *is positive, and that the corresponding GNS pre-Hilbert space* $\mathfrak{H}_\omega$ *is isometrically isomorphic to* $C_c^\infty(\mathbb{R}^n)[[\lambda]]$*, equipped with its natural* L^2*-inner product [11]. The GNS representation of* $\boldsymbol{\mathcal{A}}$ *on* $\mathfrak{H}_\omega$ *extends to a* **-representation* $\rho : C^\infty(\mathbb{R}^{2n})[[\lambda]] \longrightarrow \mathfrak{B}(\mathfrak{H}_\omega)$ *so that*

$$\rho(q^i) = m_{q^i} \text{ (multiplication by } q_i\text{)}, \qquad \rho(p_i) = \frac{\lambda}{\mathrm{i}} \frac{\partial}{\partial q^i},$$

and polynomials are mapped into the corresponding operators following Weyl's symmetrization rule. Convergence properties of this representation are discussed in [11], where it is

[3] Note that in the case of a *-representation of a C^*-algebra, nondegeneracy implies that the span of all $\pi(A)\phi$ is dense in the Hilbert space $\mathfrak{H}$.

shown that, for certain classes of observables, one recovers Schrödinger's representation on $L^2(\mathbb{R}^n)$, with λ replaced by $\hbar$. A generalization of this construction to cotangent bundles of arbitrary configuration spaces can be found in [8, 7]. An analogous construction can be carried out for Kähler manifolds equipped with star products of Wick type: in this case, the GNS representation corresponding to delta functionals produces a formal Bargmann representation [11].

The crucial fact in constructing faithful representations of C^*-algebras on Hilbert spaces (Gelfand-Naimark theorem, [21]) using "GNS blocks" is that C^*-algebras have "many" states, or, more precisely, that if $\omega(A) = 0$ for all states ω, then $A = 0$. This motivates the following

Definition 3.23. *Let $\mathcal{A}$ be a *-algebra over C. Then $\mathcal{A}$ has* sufficiently many positive linear functionals *if for any non-zero hermitian element H there exists a positive linear functional of $\mathcal{A}$ such that $\omega(H) \neq 0$.*

The proof of the following proposition can be found in [15, Prop. 3.8].

Proposition 3.24. *Let $\mathcal{A}$ be a *-algebra over C with an approximate identity. Then $\mathcal{A}$ has sufficiently many positive linear functionals if and only if there exists a faithful *-representation of $\mathcal{A}$.*

As a result, *-algebras with sufficiently many positive linear functionals have nice algebraic properties. For instance, if $A \in \mathcal{A}$ and $A^*A = 0$, then $A = 0$; there are no non-zero nilpotent normal elements in $\mathcal{A}$; and $\mathcal{A}$ is torsion-free, i.e. $zA = 0$, for $0 \neq z \in \mathsf{C}$ and $A \in \mathcal{A}$, implies $A = 0$.

Example 3.25. *C^*-algebras have sufficiently many positive linear functionals.*

Example 3.26. *The *-algebra $C^\infty(M)$ has sufficiently many positive linear functionals, since Dirac functionals are positive.*

Example 3.27. *Let M be an arbitrary Poisson manifold and $\star$ a hermitian star product on M. For any compact set $K \subseteq M$ with non-empty interior and such that $K = \overline{intK}$ (where int denotes interior), we can find a positive Borel measure μ with supp$(\mu) = K$ and $\mu(\partial K) = 0$ (where ∂ denotes boundary). A long but simple computation shows that the linear extension of the functional $f \mapsto \int_M f d\mu$ to $C^\infty(M)[[\lambda]]$ defines a positive functional. This shows that $(C^\infty(M)[[\lambda]], \star)$ has sufficiently many positive linear functionals [14, Prop.5.3].*

Example 3.28 (A non-example). *Let $\mathcal{A} = \bigwedge(\mathsf{C}^n)$ be the Grassmann algebra of C^n. Define a *-involution on $\bigwedge(\mathsf{C}^n)$ by setting $1^* = 1$ and $e_i^* = e_i$ for all $i = 1 \dots n$, where $e_1, \dots, e_n$ is the canonical basis of C^n [14, Sect. 2]). It is easy to see that $\omega : \bigwedge(\mathsf{C}^n) \longrightarrow \mathsf{C}$ is positive if and only if $\omega(1) \geq 0$ and $\omega(e_{i_1} \wedge \dots \wedge e_{i_r}) = 0$ for all $i_1, \dots, i_r$. Thus there is, up to normalization, only one positive linear functional on $\bigwedge(\mathsf{C}^n)$, and it is clear that this *-algebra does not have sufficiently many positive linear functionals.*

3.2 Algebraic Rieffel induction

We will define in this section algebraic analogues of Hilbert modules and discuss Rieffel induction of representations in the context of *-algebras over C.

3.2.1 Inner-product modules

Let $\mathcal{R}$ be an arbitrary *-ring[4]. Let $\mathcal{E}_{\mathcal{R}}$ be a right module over $\mathcal{R}$.

Definition 3.29. *An $\mathcal{R}$-valued inner product on $\mathcal{E}_{\mathcal{R}}$ is a map $\langle \cdot, \cdot \rangle_{\mathcal{R}} : \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{R}$ satisfying:*

i) $\langle x, yr \rangle_{\mathcal{R}} = \langle x, y \rangle_{\mathcal{R}} r$, *for all* $x, y \in \mathcal{E},\ r \in \mathcal{R}$.

ii) $\langle x, y \rangle_{\mathcal{R}} = \langle y, x \rangle_{\mathcal{R}}^*,\ \ x, y \in \mathcal{E}$.

If $\langle x, y \rangle_{\mathcal{R}} = 0$ for all $x \in \mathcal{E}$ implies that $y = 0$, then we say that $\langle \cdot, \cdot \rangle_{\mathcal{R}}$ is nondegenerate. *In this case we call $(\mathcal{E}_{\mathcal{R}}, \langle \cdot, \cdot \rangle_{\mathcal{R}})$ a* nondegenerate (right) inner product $\mathcal{R}$-module.

Left inner product $\mathcal{R}$-modules are defined analogously, but with the $\mathcal{R}$-linearity property in the first entry.

Definition 3.30. *We call $\langle \cdot, \cdot \rangle_{\mathcal{R}}$* full *if the span of elements of the form $\langle x, y \rangle_{\mathcal{R}}$ is $\mathcal{R}$.*[5]

A module homomorphism $T : \mathcal{E}_{\mathcal{R}} \longrightarrow \mathcal{E}_{\mathcal{R}}$ is called *adjointable* if there exists a module homomorphism T^* satisfying $\langle x, Ty \rangle_{\mathcal{R}} = \langle T^* x, y \rangle_{\mathcal{R}}$ for all $x, y \in \mathcal{E}$.

Example 3.31. *If $\mathcal{R}$ is a *-ring, then $\mathcal{R}_{\mathcal{R}}$ has a natural $\mathcal{R}$-valued inner product given by $\langle r_1, r_2 \rangle_{\mathcal{R}} = r_1^* r_2$. This inner product is full if and only if $\mathcal{R}$ is idempotent, and nondegenerate if and only if $\mathcal{R}$ is nondegenerate.*

[4] That is, $^* : \mathcal{R} \longrightarrow \mathcal{R}$ is a homomorphism of abelian groups, $(r^*)^* = r$ and $(r_1 r_2)^* = r_2^* r_1^*$.

[5] Recall that, for C^*-algebras, fullness meant that this subset was dense.

Definition 3.32. *We denote the set of all adjointable operators on* $\mathcal{E}_\mathcal{R}$ *by* $\mathcal{L}(\mathcal{E})$. *The span of operators of the form* $\Theta_{x,y}$, *where* $\Theta_{x,y}z = x\langle y, z\rangle_\mathcal{R}$, $x, y, z \in \mathcal{E}$, *is denoted by* $\mathfrak{F}(\mathcal{E})$.

Let $\mathcal{A}$ be a *-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$, R ordered, and let $(\mathcal{E}_\mathcal{A}, \langle\cdot,\cdot\rangle_\mathcal{A})$ be an inner product $\mathcal{A}$-module.

Definition 3.33. *An* $\mathcal{A}$*-valued inner product* $\langle\cdot,\cdot\rangle_\mathcal{A}$ *is called* positive semi-definite *if, for all* $x \in \mathcal{E}$, $\langle x, x\rangle_\mathcal{A} \in \mathcal{A}^+$. *It is called* positive definite *if, in addition,* $\langle x, x\rangle_\mathcal{A} = 0$ *implies* $x = 0$.

Example 3.34. *It is clear that, if* $\mathcal{R} = \mathcal{A}$ *in Example 3.31, then the inner product is positive semi-definite.*

Example 3.35. *Let* $E \to M$ *be a complex* k*-dimensional smooth vector bundle over a manifold* M. *The space of smooth sections* $\Gamma^\infty(E)$ *can be regarded as a right* $C^\infty(M)$*-module. If* h *is a hermitian structure on* E, *then there is a naturally induced* $C^\infty(M)$*-valued inner product on* $\Gamma^\infty(E)$ *(analogous to Example 2.35), which is full and positive definite.*

3.2.2 Rieffel induction and inner products

Let $\mathcal{A}, \mathcal{B}$ be *-algebras over C, and let ${}_\mathcal{B}\mathcal{E}_\mathcal{A}$ be a $(\mathcal{B}, \mathcal{A})$-bimodule. Motivated by the ingredients of Rieffel induction of C^*-algebras, we have

Definition 3.36. *We call* ${}_\mathcal{B}\mathcal{E}_\mathcal{A}$ *a* right Rieffel bimodule *if* ${}_\mathcal{B}\mathcal{E}_\mathcal{A}$ *has a positive semi-definite* $\mathcal{A}$*-valued inner product* $\langle\cdot,\cdot\rangle_\mathcal{A}$, *and the action of* $\mathcal{B}$ *on* $\mathcal{E}_\mathcal{A}$ *satisfies* $\langle Bx, y\rangle_\mathcal{A} = \langle x, B^*y\rangle_\mathcal{A}$, *for* $x, y \in \mathcal{E}$ *and* $B \in \mathcal{B}$.

Let $\pi : \mathcal{A} \longrightarrow \mathfrak{B}(\mathfrak{H})$ be a *-representation of $\mathcal{A}$ on a pre-Hilbert space $\mathfrak{H}$. In order to construct a new pre-Hilbert space where $\mathcal{B}$ acts, we proceed as in the C^*-algebra case. We consider the space $\widetilde{\mathfrak{K}} = \mathcal{E} \otimes_\mathcal{A} \mathfrak{H}$, where $\mathcal{B}$ acts on the left in a natural way. On $\widetilde{\mathfrak{K}}$ we define the form $\langle\cdot,\cdot\rangle_{\widetilde{\mathfrak{K}}} : \widetilde{\mathfrak{K}} \times \widetilde{\mathfrak{K}} \longrightarrow \mathsf{C}$ by

$$\langle x_1 \otimes \phi_1, x_2 \otimes \phi_2\rangle_{\widetilde{\mathfrak{K}}} := \langle \phi_1, \pi(\langle x_1, x_2\rangle_\mathcal{A})\phi_2\rangle_\mathfrak{H}, \quad x_1, x_2 \in \mathcal{E}, \;\; \phi_1, \phi_2 \in \mathfrak{H}. \tag{3.6}$$

Unfortunately, the proof of Proposition 2.37, that guarantees that this inner product is positive semi-definite, no longer works in this purely algebraic setting[6]. We will come back to this issue later, but for now we will simply define

[6]Note that for all $\psi \in \mathfrak{H}$ the linear functional $A \mapsto \langle \psi, \pi_\mathcal{A}(A)\psi\rangle_\mathfrak{H}$ is positive. Thus for elementary tensors $x \otimes \psi \in \widetilde{\mathfrak{K}}$ we have $\langle x \otimes \psi, x \otimes \psi\rangle_{\widetilde{\mathfrak{K}}} = \langle \psi, \pi_\mathcal{A}(\langle x, x\rangle_\mathcal{A})\psi\rangle_\mathfrak{H} \geq 0$, since the algebra element $\langle x, x\rangle_\mathcal{A} \in \mathcal{A}$ is positive. This is, however, not enough to guarantee the positivity of arbitrary elements of the form $x_1 \otimes \psi_1 + \cdots + x_n \otimes \psi_n \in \widetilde{\mathfrak{K}}$.

Definition 3.37. *We say that a Rieffel bimodule* ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ *satisfies* Property (P) *if the expression in (3.6) defines a semi-definite hermitian product on* $\widetilde{\mathfrak{K}}$ *for every* **-representation* $\pi : \mathcal{A} \longrightarrow \mathfrak{B}(\mathfrak{H})$.

Let us assume that ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ satisfies property (P). We denote the action of $\mathcal{B}$ on $\widetilde{\mathfrak{K}}$ by $\widetilde{\pi}_{\mathcal{B}}$. A simple computation using that the action of $\mathcal{B}$ on ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ preserves involution shows that $\langle x \otimes \psi, \widetilde{\pi}_{\mathcal{B}}(B) y \otimes \phi \rangle_{\widetilde{\mathfrak{K}}} = \langle \widetilde{\pi}_{\mathcal{B}}(B^*) x \otimes \psi, y \otimes \phi \rangle_{\widetilde{\mathfrak{K}}}$. By linearity it follows that $\widetilde{\pi}_{\mathcal{B}}(B^*)$ is an adjoint of $\widetilde{\pi}_{\mathcal{B}}(B)$. We define

$$\mathfrak{K} := \widetilde{\mathfrak{K}} / \widetilde{\mathfrak{K}}^{\perp}, \tag{3.7}$$

which is now a pre-Hilbert space over C. It is not hard to check that the representation $\widetilde{\pi}_{\mathcal{B}}$ of $\mathcal{B}$ on $\widetilde{\mathfrak{K}}$ passes to the quotient $\mathfrak{K}$, defining a *-representation $\pi_{\mathcal{B}}$ of $\mathcal{B}$ on $\mathfrak{K}$. This representation is defined by

$$\pi_{\mathcal{B}}(B)[x \otimes \psi] := [\widetilde{\pi}_{\mathcal{B}}(B)(x \otimes \psi)] = [B \cdot x \otimes \psi], \tag{3.8}$$

for $B \in \mathcal{B}$ and $x \otimes \psi \in \widetilde{\mathfrak{K}}$. We call $\pi_{\mathcal{B}}$ the representation of $\mathcal{B}$ induced by $\pi_{\mathcal{A}}$ through ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$. We call this construction *algebraic Rieffel induction*, in analogy to the Rieffel induction in the theory of C^*-algebras.

Algebraic Rieffel induction is functorial and the proof is just the same as for C^*-algebras: Let $(\mathfrak{H}_1, \pi_{\mathcal{A}}^{(1)})$ and $(\mathfrak{H}_2, \pi_{\mathcal{A}}^{(2)})$ be two *-representations of $\mathcal{A}$ and let $U : \mathfrak{H}_1 \to \mathfrak{H}_2$ be an intertwiner. We define $\widetilde{V} : \widetilde{\mathfrak{K}}_1 \to \widetilde{\mathfrak{K}}_2$ by

$$\widetilde{V}(x \otimes \psi) := x \otimes U\psi \tag{3.9}$$

for $x \otimes \psi \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathfrak{H}_1$ and extend it by linearity. We have

$$\widetilde{V}\left(\widetilde{\pi}_{\mathcal{B}}^{(1)}(B)(x \otimes \psi)\right) = \widetilde{\pi}_{\mathcal{B}}^{(2)}(B)\left(\widetilde{V}(x \otimes \psi)\right), \tag{3.10}$$

for all $B \in \mathcal{B}$. Hence $\widetilde{V}$ is an intertwiner from $\widetilde{\pi}_{\mathcal{B}}^{(1)}$ to $\widetilde{\pi}_{\mathcal{B}}^{(2)}$. If we assume in addition that $\widetilde{V}$ is an isometric intertwiner, then a simple computation shows that $\widetilde{V} : \widetilde{\mathfrak{K}}_1 \to \widetilde{\mathfrak{K}}_2$ is also isometric. Thus $\widetilde{V}$ passes to the quotients and yields an isometric map $V : \mathfrak{K}_1 \to \mathfrak{K}_2$, which intertwines $\pi_{\mathcal{B}}^{(1)}$ and $\pi_{\mathcal{B}}^{(2)}$. We summarize the discussion in the next

Theorem 3.38. *Let* $\mathcal{A}$, $\mathcal{B}$ *be* **-algebras over* C. *A Rieffel bimodule* ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ *satisfying Property* (P) *yields a functor* $\mathcal{R}_{\mathcal{E}} : \mathsf{rep}(\mathcal{A}) \to \mathsf{rep}(\mathcal{B})$ *defined through algebraic Rieffel induction.*

We also observe the following simple fact.

Proposition 3.39. *Let $\mathcal{A}$, $\mathcal{B}$ be *-algebras over C, and let ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be a right Rieffel bimodule satisfying (P). If the action of $\mathcal{B}$ on ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is strongly nondegenerate, then the functor $\mathcal{R}_{\mathcal{E}}$ maps* $\mathsf{rep}(\mathcal{A})$ *into* $\mathsf{Rep}(\mathcal{B})$.

3.2.3 Positivity conditions and examples

We discuss in this section examples of Rieffel bimodules as well as conditions guaranteeing property (P).

Example 3.40. *Let $\mathcal{A}$ be a *-algebra over C and $\omega : \mathcal{A} \longrightarrow \mathsf{C}$ a positive linear functional. Consider the bimodule ${}_{\mathcal{A}}\mathcal{A}_{\mathsf{C}}$ equipped with the C-valued inner product $\langle A, B\rangle_\omega = \omega(A^*B)$: this defines a Rieffel bimodule satisfying (P) [15, Lem. 4.6]. Moreover, if we consider the representation of C on itself by left multiplication (with the Hermitian product in C given by $\langle z, w\rangle = \overline{z}w$), the induced representation of $\mathcal{A}$ through ${}_{\mathcal{A}}\mathcal{A}_{\mathsf{C}}$ is just the GNS representation of $\mathcal{A}$ corresponding to ω [15, Prop. 4.7]. Hence algebraic Rieffel induction generalizes the algebraic GNS construction.*

Let ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be a Rieffel bimodule and suppose the action of $\mathcal{A}$ on $\mathcal{E}$ decomposes into an orthogonal sum of pseudocyclic actions (see Definition 3.19), i.e.

(P1) ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} = \bigoplus_{i\in I} \mathcal{E}^{(i)}$ and $\mathcal{E}^{(i)} \perp \mathcal{E}^{(j)}$ for all $i \neq j \in I$ with respect to $\langle \cdot, \cdot\rangle_{\mathcal{A}}$.

(P2) The action of $\mathcal{A}$ on $\mathcal{E}$ preserves this direct sum.

(P3) Each $\mathcal{E}^{(i)}$ has directed filtered submodules $\mathcal{E}^{(i)} = \bigcup_{\alpha\in I^{(i)}} \mathcal{E}^{(i)}_\alpha$ and pseudo-cyclic vectors $\Omega^{(i)}_\alpha$.

Lemma 3.41. *Let ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be a Rieffel bimodule so that the action of $\mathcal{A}$ on $\mathcal{E}$ satisfies conditions $(P1), (P2)$ and $(P3)$. Then it satisfies (P).*

Proof. Let $\pi_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathfrak{B}(\mathfrak{H})$ be a *-representation of $\mathcal{A}$ on the pre-Hilbert space $\mathfrak{H}$. Let $\widetilde{\mathfrak{K}} = \mathcal{E} \otimes_{\mathcal{A}} \mathfrak{H}$, and note that this space can be decomposed into a direct sum $\bigoplus_{i\in I} \mathcal{E}^{(i)} \otimes_{\mathcal{A}} \mathfrak{H}$, for the decomposition of $\mathcal{E}$ is preserved by $\mathcal{A}$. Since $\langle k_i, k_j\rangle_{\widetilde{\mathfrak{K}}} = 0$ if $k_r \in \mathcal{E}^{(r)} \otimes_{\mathcal{A}} \mathfrak{H}, r = i, j$ and $i \neq j$, it suffices to prove the positivity of $\langle \cdot, \cdot\rangle_{\widetilde{\mathfrak{K}}}$ restricted to each submodule $\mathcal{E}^{(i)} \otimes_{\mathcal{A}} \mathfrak{H}$. By $(P3)$ we can find $A_1, \ldots, A_n \in \mathcal{A}$ such that $x_1 = \Omega^{(i)}_\alpha \cdot A_1, \ldots, x_n = \Omega^{(i)}_\alpha \cdot A_n$, where $\alpha \in I^{(i)}$ is large enough. Then $x_1 \otimes \psi_1 + \cdots + x_n \otimes \psi_n = \Omega^{(i)}_\alpha \otimes \phi$ with $\phi = \pi_{\mathcal{A}}(A_1)\psi_1 + \cdots + \pi_{\mathcal{A}}(A_n)\psi_n$. Thus any vector in $\mathcal{E}^{(i)} \otimes_{\mathcal{A}} \mathfrak{H}$ can be written as an elementary tensor and the result follows. ∎

Example 3.42. *Let $\mathcal{A}$ be a *-algebra over C. Note that $\mathcal{E} = \mathcal{A}^n$, $n = 1, 2 \ldots$, is naturally a $(M_n(\mathcal{A}), \mathcal{A})$-bimodule, and it is a Rieffel bimodule with respect to $\langle x, y \rangle_{\mathcal{A}} = \sum_{i=1}^n x_i^* y_i$, where $x = (x_i), y = (y_i)$. It is easy to see that if $\mathcal{A}$ has an algebraic approximate identity, then the action of $\mathcal{A}$ on $\mathcal{E}$ satisfies $(P1), (P2)$ and $(P3)$. Thus $\mathcal{E}$ is a Rieffel bimodule satisfying (P).*

3.3 Algebraic strong Morita equivalence

We saw previously how to construct a functor $\mathcal{R}_{\mathcal{E}} : \mathsf{rep}(\mathcal{A}) \longrightarrow \mathsf{rep}(\mathcal{B})$, starting with a Rieffel bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ satisfying (P) and proceeding by Rieffel induction. In this section we will be concerned with an algebraic version of strong Morita equivalence.

3.3.1 Equivalence bimodules

Let $\mathcal{E}$ be a C-module, and let $\overline{\mathcal{E}}$ be its conjugate: as an additive group, $\overline{\mathcal{E}} = \mathcal{E}$; however, the identity map (of groups) $^- : \mathcal{E} \longrightarrow \overline{\mathcal{E}}$ is C-antilinear, $\alpha\overline{x} = \overline{\overline{\alpha}x}$, $\alpha \in \mathsf{C}$. If ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is a $(\mathcal{B}, \mathcal{A})$-bimodule, we endow $\overline{\mathcal{E}}$ with the structure of an $(\mathcal{A}, \mathcal{B})$-bimodule by considering the actions of $\mathcal{A}$ and $\mathcal{B}$ by adjoints: $A\overline{x} = \overline{xA^*}$ and $\overline{x}B = \overline{B^*x}$. Note that a (left) $\mathcal{B}$-valued inner product on ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is equivalent to a (right) $\mathcal{B}$-valued inner product on ${}_{\mathcal{A}}\overline{\mathcal{E}}_{\mathcal{B}}$ by

$$
{}_{\mathcal{B}}\langle x, y \rangle = \langle \bar{x}, \bar{y} \rangle_{\mathcal{B}}, \; x, y \in \mathcal{E}.
$$

Definition 3.43. *A left Rieffel bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ satisfies property (Q) if the conjugate right Rieffel bimodule ${}_{\mathcal{A}}\overline{\mathcal{E}}_{\mathcal{B}}$ satisfies (P).*

Definition 3.44. *Let ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be a bimodule with inner products $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ and ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle$. We say that these inner products are* compatible *if ${}_{\mathcal{B}}\langle x, y \rangle \cdot z = x \cdot \langle y, z \rangle_{\mathcal{A}}$, $x, y, z \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$.*

We now define a purely algebraic version of an imprimitivity bimodule.

Definition 3.45. *A bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is called an* equivalence bimodule *if it is a right full Rieffel bimodule, a left full Rieffel bimodule[7], with compatible $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ and ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle$ and satisfying (P) and (Q).*

We have the following algebraic notion corresponding to strong Morita equivalence.

[7]This is, of course, equivalent to requiring that both ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ and ${}_{\mathcal{A}}\overline{\mathcal{E}}_{\mathcal{B}}$ are full right Rieffel bimodules.

Definition 3.46. *Let $\mathcal{A}$ and $\mathcal{B}$ be *-algebras over* C*. We say that $\mathcal{A}$ and $\mathcal{B}$ are* algebraically strongly Morita equivalent[8] *if there exists an equivalence bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$.*

We will omit the word "algebraically" whenever there is no confusion, in anticipation to the fact that this notion will reduce to strong Morita equivalence when applied to C^*-algebras.

It is clear from the symmetry in the definitions that if ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is an equivalence bimodule, then so is ${}_{\mathcal{A}}\overline{\mathcal{E}}_{\mathcal{B}}$. Thus strong Morita equivalence is a symmetric relation. A simple computation shows that if $\mathcal{A}$ and $\mathcal{B}$ are strongly Morita equivalent, and if $\mathcal{A}'$ is *-isomorphic to $\mathcal{A}$, then $\mathcal{A}$ and $\mathcal{A}'$ are strongly Morita equivalent.

Example 3.47. *Let $\mathcal{A}$ be a *-algebra with approximate identity. Then it is easy to check that $\mathcal{A}$ is strongly Morita equivalent to itself (with equivalence bimodule ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$, and ${}_{\mathcal{A}}\langle A, B\rangle = AB^*, \langle A, B\rangle_{\mathcal{A}} = A^*B$). It follows that if $\mathcal{A}$ and $\mathcal{B}$ are *-isomorphic, then they are strongly Morita equivalent.*

Lemma 3.48. *Let $\mathcal{B}$ be a nondegenerate *-algebra. Let ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be an equivalence bimodule. Then $\mathcal{B}$ is *-isomorphic to $\mathfrak{F}(\mathcal{E})$.*

Proof. The action of $\mathcal{B}$ on $\mathcal{E}$ defines a *-homomorphism $\mathsf{L}_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{L}(\mathcal{E})$. By the compatibility of inner products in ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ and the fullness of ${}_{\mathcal{B}}\langle\cdot,\cdot\rangle$, it follows that $\mathsf{L}_{\mathcal{B}}(\mathcal{B}) = \mathfrak{F}(\mathcal{E}_{\mathcal{A}})$. Suppose now $\mathsf{L}_{\mathcal{B}}(B) = 0$, i.e. $Bx = 0$ for all $x \in \mathcal{E}$. Let B' be any element in $\mathcal{B}$. By fullness of ${}_{\mathcal{B}}\langle\cdot,\cdot\rangle$, we can write $B' = \sum_i {}_{\mathcal{B}}\langle x_i, y_i\rangle$, for some $x_i, y_i \in \mathcal{E}$. Thus $BB' = B(\sum_i {}_{\mathcal{B}}\langle x_i, y_i\rangle) = \sum_i {}_{\mathcal{B}}\langle Bx_i, y_i\rangle = 0$. Since B' is arbitrary and $\mathcal{B}$ is nondegenerate, $B = 0$. Hence $\mathsf{L}_{\mathcal{B}}$ is a *-isomorphism. □

Under the identification $\mathcal{B} \cong \mathfrak{F}(\mathcal{E})$, the inner product ${}_{\mathcal{B}}\langle\cdot,\cdot\rangle$ corresponds to $(x, y) \mapsto \Theta_{x,y}$; in order to show that ${}_{\mathfrak{F}(\mathcal{E})}\mathcal{E}_{\mathcal{A}}$ is an equivalence bimodule, one must in general check that $\Theta_{x,x} \in \mathfrak{F}(\mathcal{E})^+$ and property (Q).

There is also a natural notion of Morita equivalence for *-rings, introduced by Ara in [1], without positivity requirements.

Definition 3.49. *Let $\mathcal{R}$ and $\mathcal{S}$ be nondegenerate and idempotent *-rings. We say that they are* Morita *-equivalent *if there exists a right $\mathcal{R}$-module $\mathcal{E}_{\mathcal{R}}$ equipped with a full $\mathcal{R}$-valued inner product so that $\mathcal{S} \cong \mathfrak{F}(\mathcal{E}_{\mathcal{R}})$*[9]*.*

[8] This notion was introduced in [15] under the name of *formal Morita equivalence*.

[9] In Ara's original definition [1, 2], the module $\mathcal{E}_{\mathcal{R}}$ satisfied 3 further conditions: *i*) $\langle\cdot,\cdot\rangle_{\mathcal{R}}$ should be nondegenerate; *ii*) $\mathcal{E}\mathcal{R} = \mathcal{E}$ (strong nondegeneracy of $\mathcal{R}$-action); *iii*) $x\mathcal{R} = 0$ implies $x = 0$. We showed in [12, Lem. 3.1] that if $\mathcal{R}, \mathcal{S}$ are nondegenerate and idempotent, and if we find $\mathcal{E}_{\mathcal{R}}$ as in Definition 3.49, then there also exists a module with the extra requirements.

Morita *-equivalence is an equivalence relation in the category of nondegenerate idempotent *-rings [1].

Example 3.50. *Let $\mathcal{R}$ be a unital *-ring, and let $P \in M_n(\mathcal{R})$ be a full projection (Definition 2.22). The right $\mathcal{R}$-module $\mathcal{E}_\mathcal{R} = P\mathcal{R}^n$ has a natural full $\mathcal{R}$-valued inner product (induced from the canonical one on $\mathcal{R}^n$). This shows that $\mathcal{R}$ and $PM_n(\mathcal{R})P$ are Morita *-equivalent.*

It is clear from the definitions that strong Morita equivalence implies Morita *-equivalence. We also observe the connection with Morita equivalence of unital rings.

Proposition 3.51. *Let $\mathcal{R}$ and $\mathcal{S}$ be unital *-rings. If they are Morita *-equivalent, then they are automatically Morita equivalent (in the classical sense).*

Proof. Let $\mathcal{E}_\mathcal{R}$ be the module defining Morita *-equivalence, with $\mathcal{S} \cong \mathfrak{F}(\mathcal{E}_\mathcal{R})$. Consider the bimodule homomorphisms

$$f: {}_\mathcal{R}\overline{\mathcal{E}}_\mathcal{S} \otimes_\mathcal{S} {}_\mathcal{S}\mathcal{E}_\mathcal{R} \longrightarrow \mathcal{R},\ \overline{x} \otimes y \mapsto \langle x, y\rangle_\mathcal{R} \ \text{ and } g: {}_\mathcal{S}\mathcal{E}_\mathcal{R} \otimes_\mathcal{R} {}_\mathcal{R}\overline{\mathcal{E}}_\mathcal{S} \longrightarrow \mathcal{S},\ x \otimes \overline{y} \mapsto {}_\mathcal{S}\langle x, y\rangle .$$

It is easy to check that $(\mathcal{R}, \mathcal{S}, {}_\mathcal{R}\overline{\mathcal{E}}_\mathcal{S}, {}_\mathcal{S}\mathcal{E}_\mathcal{R}, f, g)$ defines a set of equivalence data in the sense of [3]. As a result, $\mathcal{R}$ and $\mathcal{S}$ are Morita equivalent. □

We can use this observation to show the following [15, Prop. 7.6]

Proposition 3.52. *Let $\mathcal{R}$ and $\mathcal{S}$ be Morita *-equivalent unital *-rings. Then they have *-isomorphic centers*[10].

Proof. Let $\mathcal{E}_\mathcal{R}$ be a module defining Morita *-equivalence. Since this module also establishes Morita equivalence of $\mathcal{R}$ and $\mathcal{S} \cong \mathfrak{F}(\mathcal{E}_\mathcal{R})$, there is an isomorphism $\phi : \text{center}(\mathcal{R}) \longrightarrow \text{center}(\mathcal{S})$ defined by $x \cdot r = \phi(r) \cdot x$, for all $x \in \mathcal{E}$ [Chp. 2,Thm. 3.5][3]. Let $r \in \text{center}(\mathcal{R})$, and let ${}_\mathcal{S}\langle x, y\rangle := \theta_{x,y}$. Then

$${}_\mathcal{S}\langle xr, y\rangle = {}_\mathcal{S}\langle x, yr^*\rangle = {}_\mathcal{S}\langle x, \phi(r^*)y\rangle = {}_\mathcal{S}\langle x, y\rangle\,(\phi(r^*))^*, \text{ for all } x, y \in \mathcal{E}.$$

On the other hand ${}_\mathcal{S}\langle xr, y\rangle = \phi(r){}_\mathcal{S}\langle x, y\rangle$, $x, y \in \mathcal{E}$. Since ${}_\mathcal{S}\langle \cdot, \cdot\rangle$ is full and $\mathcal{S}$ is unital, it follows that $(\phi(r^*))^* = \phi(r)$ and hence ϕ preserves involution. □

Corollary 3.53. *Two commutative unital *-algebras are strongly Morita equivalent if and only if they are *-isomorphic.*

[10] A generalization of this result to nondegenerate and idempotent rings was proven in [1, Thm. 4.2].

3.3.2 Equivalence of representation theory

We will show in this section that strongly Morita equivalent *-algebras have the same representation theory on pre-Hilbert spaces. The next two lemmas are completely analogous to results in C^*-algebras.

Lemma 3.54. *Suppose $\mathcal{A}, \mathcal{B}$ are *-algebras over C and let ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be an equivalence bimodule. Let $(\mathfrak{H}, \pi_{\mathcal{A}})$ be a strongly nondegenerate *-representation of $\mathcal{A}$. Then $\mathcal{R}_{\overline{\mathcal{E}}} \circ \mathcal{R}_{\mathcal{E}}(\mathfrak{H}, \pi_{\mathcal{A}})$ is unitarily equivalent to $(\mathfrak{H}, \pi_{\mathcal{A}})$. Analogously, if $(\mathfrak{K}, \pi_{\mathcal{B}})$ is a strongly nondegenerate *-representation of $\mathcal{B}$, then $\mathcal{R}_{\mathcal{E}} \circ \mathcal{R}_{\overline{\mathcal{E}}}(\mathfrak{K}, \pi_{\mathcal{B}})$ is unitarily equivalent to $(\mathfrak{K}, \pi_{\mathcal{B}})$.*

Proof. The proof basically follows [58, Sect. 3.3]. Let $\widetilde{\mathfrak{K}} = {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathfrak{H}$ and $\mathfrak{K} = \widetilde{\mathfrak{K}}/(\widetilde{\mathfrak{K}})^{\perp}$. Let $\tilde{\mathfrak{H}}' = {}_{\mathcal{A}}\overline{\mathcal{E}}_{\mathcal{B}} \otimes_{\mathcal{B}} \mathfrak{K}$ and $\mathfrak{H}' = \tilde{\mathfrak{H}}'/(\mathfrak{H}')^{\perp}$. There is a linear map $U : \mathfrak{H} \longrightarrow \mathfrak{H}'$ uniquely defined by

$$U([\bar{x} \otimes [y \otimes \psi]\,]) = \pi_{\mathcal{A}}(\langle x, y \rangle_{\mathcal{A}})\psi, \text{ for } x, y \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}},\ \psi \in \mathfrak{H}.$$

Since $\pi_{\mathcal{A}}$ is strongly nondegenerate and $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ is full, it immediately follows that U is onto. A simple computation using the definitions shows that U preserves the hermitian products, and therefore it is unitary. It is also easy to check that U intertwines $\pi_{\mathcal{A}}$ and $\mathcal{R}_{\overline{\mathcal{E}}} \circ \mathcal{R}_{\mathcal{E}}(\pi_{\mathcal{A}})$. Thus the conclusion follows. The same argument holds for $\mathcal{B}$. □

The previous construction is natural in the following sense:

Lemma 3.55. *Suppose we have two strongly nondegenerate *-representations $(\mathfrak{H}_1, \pi^1_{\mathcal{A}})$ and $(\mathfrak{H}_2, \pi^2_{\mathcal{A}})$ of $\mathcal{A}$, and let $T : \mathfrak{H}_1 \longrightarrow \mathfrak{H}_2$ be an isometric intertwiner operator. Let $U_1 : \mathcal{R}_{\overline{\mathcal{E}}} \circ \mathcal{R}_{\mathcal{E}}(\mathfrak{H}_1) \longrightarrow \mathfrak{H}_1$ and $U_2 : \mathcal{R}_{\overline{\mathcal{E}}} \circ \mathcal{R}_{\mathcal{E}}(\mathfrak{H}_2) \longrightarrow \mathfrak{H}_2$ be the two unitary equivalences as in Lemma 3.54. Then $U_2 \circ (\mathcal{R}_{\overline{\mathcal{E}}} \circ \mathcal{R}_{\mathcal{E}}(T)) = T \circ U_1$. An analogous statement holds for $\mathcal{B}$.*

Proof. The computation is just as for C^*-algebras (see [58, Sect. 3.3]). □

Definition 3.56. *We call an equivalence bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ nondegenerate if the actions of $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{E}$ are strongly nondegenerate.*

In this case, $\mathcal{R}_{\mathcal{E}}(\mathsf{rep}(\mathcal{A})) \subseteq \mathsf{Rep}(\mathcal{B})$ and $\mathcal{R}_{\overline{\mathcal{E}}}(\mathsf{rep}(\mathcal{B})) \subseteq \mathsf{Rep}(\mathcal{A})$ (Proposition 3.39). We can then consider

$$\mathcal{R}_{\mathcal{E}} \circ \mathcal{R}_{\overline{\mathcal{E}}} : \mathsf{Rep}(\mathcal{B}) \longrightarrow \mathsf{Rep}(\mathcal{A}), \qquad \mathcal{R}_{\overline{\mathcal{E}}} \circ \mathcal{R}_{\mathcal{E}} : \mathsf{Rep}(\mathcal{A}) \longrightarrow \mathsf{Rep}(\mathcal{B}). \tag{3.11}$$

Theorem 3.57. *Let $\mathcal{A}$ and $\mathcal{B}$ be *-algebras over C. If ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ is a nondegenerate equivalence bimodule, then $\mathcal{R}_{\mathcal{E}}$ and $\mathcal{R}_{\overline{\mathcal{E}}}$ define an equivalence of categories between $\mathsf{Rep}(\mathcal{A})$ and $\mathsf{Rep}(\mathcal{B})$.*

The proof is a direct consequence of Lemmas 3.54, 3.55.

Lemma 3.58. *Let $\mathcal{A}$ and $\mathcal{B}$ be idempotent *-algebras over C. If they are strongly Morita equivalent, then there exists a nondegenerate equivalence bimodule.*

Proof. Let ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ be an equivalence bimodule. It follows from the compatibility of ${}_{\mathcal{B}}\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ and their fullness that $\mathcal{B}\cdot\mathcal{E}=\mathcal{E}\cdot\mathcal{A}$. If $\hat{\mathcal{E}}=\mathcal{B}\cdot\mathcal{E}=\mathcal{E}\cdot\mathcal{A}$, then $\hat{\mathcal{E}}$ is a $(\mathcal{B},\mathcal{A})$-bimodule with compatible inner products defined by the restrictions of ${}_{\mathcal{B}}\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{\mathcal{A}}$. In order to show that $\hat{\mathcal{E}}$ is a nondegenerate equivalence bimodule, we must check that ${}_{\mathcal{B}}\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ are still full when restricted to $\hat{\mathcal{E}}$. To this end, let $A\in\mathcal{A}$. By idempotency we can write $A=\sum_i A_1^iA_2^iA_3^i$, and by fullness of $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ we can write $A_2^i=\sum_j\left\langle x_j^i,y_j^i\right\rangle_{\mathcal{A}}$. So $A=\sum_{i,j}A_1^i\left\langle x_j^i,y_j^i\right\rangle_{\mathcal{A}}A_3^i=\sum_{i,j}\left\langle x_j^iA_1^{i\,*},y_j^iA_3^i\right\rangle_{\mathcal{A}}$, and this concludes the proof. □

Corollary 3.59. *Let $\mathcal{A}$ and $\mathcal{B}$ be idempotent *-algebras. If they are strongly Morita equivalent, then $\mathsf{Rep}(\mathcal{A})$ and $\mathsf{Rep}(\mathcal{B})$ are equivalent categories.*

We remark that one can have *-algebras with equivalent categories of representations on pre-Hilbert spaces which are not strongly Morita equivalent.

Example 3.60. *Consider the *-algebra $\bigwedge(\mathsf{C}^n)$ as in Example 3.28. We claim that $\mathsf{Rep}(\mathsf{C})$ and $\mathsf{Rep}(\bigwedge(\mathsf{C}^n))$ are equivalent categories. To see that, let $(\mathfrak{H},\pi)$ be a strongly non-degenerate *-representation of $\bigwedge(\mathsf{C}^n)$. Since $\pi(e_i)$ is selfadjoint and nilpotent (for $e_1\wedge e_1=0$), it follows that $\pi(e_i)=0$ for all $i=1\ldots n$. So $\pi(e_{i_1}\wedge\cdots\wedge e_{i_r})=\pi(e_{i_1})\ldots\pi(e_{i_r})=0$ for all $r\geq 1$ and $i_j\in\mathbb{N}$ (and $\pi(1)=\mathsf{id}$ by nondegeneracy). Regarding C as embedded in $\bigwedge(\mathsf{C}^n)$ in the natural way, we conclude that any strongly nondegenerate *-representation of C extends uniquely to a strongly nondegenerate *-representation of $\bigwedge(\mathsf{C}^n)$. It is clear that any representation of $\bigwedge(\mathsf{C}^n)$ restricts to C. This correspondence establishes an equivalence of categories of representations; however, it is shown in [15, Prop. 5.19] that the property of having sufficiently many positive linear functionals is invariant under strong Morita equivalence. As a result, $\bigwedge(\mathsf{C}^n)$ and C are not strongly Morita equivalent[11].*

The matter of *-algebras having equivalent representation theory but not being strongly Morita equivalent is further discussed in [12], where we investigate the ideal structure of strongly Morita equivalent *-algebras.

[11] We do not know whether, for unital *-algebras with sufficiently many positive linear functionals, equivalence of representation theories is equivalent to strong Morita equivalence.

3.3.3 Properties of equivalence bimodules and examples

The goal of this section is to establish a result analogue to Examples 2.48 and 3.50 for strong Morita equivalence; this will be done by imposing an extra condition on the full projection.

In order to show that a full positive semi-definite inner product module $\mathcal{E}_{\mathcal{A}}$ over a *-algebra $\mathcal{A}$ is an $(\mathfrak{F}(\mathcal{E}_{\mathcal{A}}), \mathcal{A})$-equivalence bimodule, we must check 3 conditions:

1. Property (P),
2. $\Theta_{x,x} \in \mathfrak{F}(\mathcal{E}_{\mathcal{A}})^+$,
3. Property (Q).

Concerning 2., we observe the following simple fact.

Lemma 3.61. *Let $\mathcal{E}_{\mathcal{A}}$ be a positive semi-definite inner product module. For $x \in \mathcal{E}$, suppose there are $y_i \in \mathcal{E}$ so that $x \sum_i \langle y_i, y_i \rangle_{\mathcal{A}} = x$. Then $\Theta_{x,x} \in \mathfrak{F}(\mathcal{E}_{\mathcal{A}})^{++}$.*

Proof. Just note that $\sum_i \Theta_{x,y_i}\Theta_{y_i,x} = \Theta_{x,x\sum_i \langle y_i,y_i\rangle_{\mathcal{A}}} = \Theta_{x,x}$. □

Example 3.62. *Let $\mathsf{C} = \mathsf{R}(\mathrm{i})$, R ordered ring. Equip C^n with its natural C-valued inner product $\langle x, y\rangle_{\mathsf{C}} = \sum_i \overline{x_i} y_i$. It is clear that we can find $x \in \mathsf{C}^n$ with $\langle x, x\rangle_{\mathsf{C}} = 1$ and hence (Lemma 3.61) $\Theta_{x,x} \geq 0$ for all $x \in \mathsf{C}^n$; Let $e_i, i = 1 \dots n$, denote the canonical orthonormal basis of C^n. Note that the action of $M_n(\mathsf{C})$ on C^n is cyclic (e_1 is a cyclic vector, for $x = \Theta_{x,e_1} e_1$). Since $\mathsf{C}^n \cong \oplus_i \mathsf{C} e_i$, the C-action decomposes into an orthogonal sum of cyclic ones. It follows from Lemma 3.41 that (P) and (Q) hold. Thus C^n is a $(M_n(\mathsf{C}), \mathsf{C})$-equivalence bimodule*[12].

Example 3.63. *Let $\hat{\mathsf{R}}$ be an ordered field, and let $\hat{\mathsf{C}} = \hat{\mathsf{R}}(\mathrm{i})$. If $\mathfrak{H}$ is a pre-Hilbert space over $\hat{\mathsf{C}}$, then it can be regarded as a right $\hat{\mathsf{C}}$-module with full positive definite $\hat{\mathsf{C}}$-valued inner product (fullness follows since $\hat{\mathsf{C}}$ is a field). To see that $\Theta_{x,x} \geq 0$, choose $y \in \mathfrak{H}$ with $\langle y, y\rangle_{\mathsf{C}} \neq 0$; then $\Theta_{x,x} = \frac{1}{\langle y,y\rangle_{\mathsf{C}}} \Theta_{x,y}\Theta_{x,y}^* \in \mathfrak{F}(\mathfrak{H})^+$. Note that $\mathfrak{F}(\mathfrak{H})$ acts on $\mathfrak{H}$ in a cyclic way (any nonzero vector is a cyclic vector), and this guarantees (Q); Property (P) holds since the tensor product of any two pre-Hilbert spaces over $\hat{\mathsf{C}}$ has a well-defined inner product (see [15, App. A]). Hence $\mathfrak{H}$ is a $(\mathfrak{F}(\mathfrak{H}), \hat{\mathsf{C}})$-equivalence bimodule (compare with Example 2.47).*

[12] The same holds for arbitrary free modules over C, not necessarily finitely generated [15, Prop. 6.7]

Example 3.64. *Let $\mathcal{A}$ be a *-algebra over C with approximate identity. Similar arguments as in Example 3.62 (using the approximate identity instead of the unit) show that $M_n(\mathcal{A})$ and $\mathcal{A}$ are strongly Morita equivalent, with equivalence bimodule $\mathcal{A}^n$ (See Example 2.48)*[13].

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be *-algebras over C. Let $({}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}, \langle\cdot,\cdot\rangle_{\mathcal{A}})$ and $({}_{\mathcal{A}}\mathcal{E}'_{\mathcal{C}}, \langle\cdot,\cdot\rangle_{\mathcal{C}})$ be right Rieffel bimodules, and consider $\mathcal{E}'' := \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}'$, equipped with its natural $(\mathcal{B},\mathcal{C})$-bimodule structure. There is a $\mathcal{C}$-valued inner product on ${}_{\mathcal{B}}\mathcal{E}''_{\mathcal{C}}$ uniquely determined by

$$\langle\!\langle x_1 \otimes x_1', x_2 \otimes x_2' \rangle\!\rangle_{\mathcal{C}} = \langle x_1', \langle x_1, x_2\rangle_{\mathcal{A}}\, x_2' \rangle_{\mathcal{C}}\,,\ x_i \in \mathcal{E},\, x_i' \in \mathcal{E}'. \tag{3.12}$$

Lemma 3.65. *If the action of $\mathcal{A}$ on ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ decomposes into an orthogonal sum of pseudocyclic actions (properties $(P1)$, $(P2)$,$(P3)$), then $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathcal{C}}$ is positive semi-definite.*

Proof. Since the action of $\mathcal{A}$ on ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ satisfies properties $(P1)$, $(P2)$,$(P3)$, any $z \in {}_{\mathcal{B}}\mathcal{E}''_{\mathcal{C}}$ can be written in the form

$$z = \sum_i x_1^{(i)} \otimes x_1' + \ldots + x_n^{(i)} \otimes x_n', \quad x_1^{(i)},\ldots,x_n^{(i)} \in \mathcal{E}^{(i)}.$$

For each i, there exists an α_i such that $x_1^{(i)},\ldots,x_n^{(i)} \in \mathcal{E}^{(i)}_{\alpha_i}$. So there exist $A_1^{(i)},\ldots,A_n^{(i)} \in \mathcal{A}$ so that $x_1^{(i)} = \Omega^{(i)}_{\alpha_i} A_1^{(i)},\ldots,x_n^{(i)} = \Omega^{(i)}_{\alpha_i} A_n^{(i)}$. Hence

$$z = \sum_i \Omega^{(i)}_{\alpha_i} A_1^{(i)} \otimes x_1' + \ldots + \Omega^{(i)}_{\alpha_i} A_n^{(i)} \otimes x_n' = \sum_i \Omega^{(i)}_{\alpha_i} \otimes y_i$$

where $y_i = A_1^{(i)} x_1' + \ldots + A_n^{(i)} x_n'$. Since $\mathcal{E}^{(i)} \perp \mathcal{E}^{(j)}$ for all $i \neq j$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{A}}$, it follows that $\langle\!\langle z,z\rangle\!\rangle_{\mathcal{C}} = \sum_i \langle y_i, \langle \Omega^{(i)}_{\alpha_i}, \Omega^{(i)}_{\alpha_i}\rangle_{\mathcal{A}} \cdot y_i\rangle_{\mathcal{C}}$ and it is easy to check[14] that $\langle\!\langle z,z\rangle\!\rangle_{\mathcal{C}} \in \mathcal{C}^+$. ☐

If $({}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}, {}_{\mathcal{B}}\langle\cdot,\cdot\rangle)$ and $({}_{\mathcal{A}}\mathcal{E}'_{\mathcal{C}}, {}_{\mathcal{A}}\langle\cdot,\cdot\rangle)$ are left Rieffel bimodules and the $\mathcal{A}$-action on ${}_{\mathcal{A}}\mathcal{E}'_{\mathcal{C}}$ decomposes into an orthogonal sum of pseudocyclic actions, then a similar result holds and the inner product

$${}_{\mathcal{B}}\langle\!\langle x_1 \otimes x_1', x_2 \otimes x_2' \rangle\!\rangle = {}_{\mathcal{B}}\langle x_1\, {}_{\mathcal{A}}\langle x_1', x_2' \rangle\,, x_2 \rangle \tag{3.13}$$

is positive semi-definite. Moreover, simple computations show the following [58].

Lemma 3.66. *If $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{C}}$ are full, then so is $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathcal{C}}$ (and similarly for ${}_{\mathcal{A}}\langle\cdot,\cdot\rangle$, ${}_{\mathcal{B}}\langle\cdot,\cdot\rangle$ and ${}_{\mathcal{B}}\langle\!\langle\cdot,\cdot\rangle\!\rangle$); if ${}_{\mathcal{B}}\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{\mathcal{A}}$, and ${}_{\mathcal{A}}\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{\mathcal{C}}$ are compatible, then $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathcal{C}}$ and ${}_{\mathcal{B}}\langle\!\langle\cdot,\cdot\rangle\!\rangle$ are compatible.*

[13] See [15, Prop. 6.10] for details and generalizations.

[14] This follows since, if $A \in \mathcal{A}^+$, then for all $x' \in {}_{\mathcal{A}}\mathcal{E}'_{\mathcal{C}}$ we have $\langle x', Ax'\rangle_{\mathcal{C}} \in \mathcal{C}^+$. Indeed, if $\omega : \mathcal{C} \longrightarrow$ is a positive functional and $x' \in {}_{\mathcal{A}}\mathcal{E}'_{\mathcal{C}}$ is fixed, the linear functional $\hat{\omega} : \mathcal{A} \longrightarrow$, $\hat{\omega}(A) = \omega(\langle x', Ax'\rangle_{\mathcal{C}})$ is clearly positive, and the result follows.

Lemma 3.67. *If ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ and ${}_{\mathcal{A}}\mathcal{E}'_{\mathcal{C}}$ satisfy (P), then so does ${}_{\mathcal{B}}\mathcal{E}''_{\mathcal{C}}$ (similarly for (Q)).*

Proof. The proof follows from the observation that induction of representations through ${}_{\mathcal{B}}\mathcal{E}''_{\mathcal{C}}$ coincides with the composition of inductions through ${}_{\mathcal{A}}\mathcal{E}'_{\mathcal{C}}$, first, and then ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$. □

Let $\mathcal{A}$ be a *-algebra over C. For simplicity, we will assume that $\mathcal{A}$ is unital. Let $P \in M_n(\mathcal{A})$ be a full projection.

Lemma 3.68. *The $(PM_n(\mathcal{A})P, M_n(\mathcal{A}))$-bimodule $\mathcal{E} := PM_n(\mathcal{A})$ has the following properties:*

- *It is a right full Rieffel bimodule, with respect to $\langle PL, PS\rangle_{M_n(\mathcal{A})} = LP^*PS = LPS$, $L, S \in M_n(\mathcal{A})$*[15]*.*
- *It is a left full Rieffel bimodule, with respect to ${}_{PM_n(\mathcal{A})P}\langle PL, PS\rangle = PLS^*P$, $L, S \in M_n(\mathcal{A})$.*
- *The $M_n(\mathcal{A})$ action on $\mathcal{E}$ is cyclic; in particular, $\mathcal{E}_{M_n(\mathcal{A})}$ satisfies (P).*

Lemma 3.69. *The $(M_n(\mathcal{A}), \mathcal{A})$-bimodule $\mathcal{E}' := \mathcal{A}^n$ satisfies the following:*

- *It is a right full Rieffel bimodule, with respect to $\langle x, y\rangle_{\mathcal{A}} = \sum_{i=1}^n x_i^* y_i$;*
- *It is a left full Rieffel bimodule, with respect to ${}_{M_n(\mathcal{A})}\langle x, y\rangle = \Theta_{x,y}$;*
- *The action of $M_n(\mathcal{A})$ on $\mathcal{E}'$ is cyclic; in particular, $\mathcal{E}'$ satisfies (Q);*
- *The action of $\mathcal{A}$ on $\mathcal{E}'$ decomposes into a sum of cyclic actions; in particular, $\mathcal{E}'$ satisfies (P).*

Definition 3.70. *A projection $P \in M_n(\mathcal{A})$ is called* strongly full[16] *if there exists $\tau \in \mathcal{A}$ such that $\mathsf{tr}(P) = (\tau\tau^*)^{-1}$.*

This definition is due to S. Waldmann. We are now ready to show the main result in this section.

Theorem 3.71. *Let $P \in M_n(\mathcal{A})$ be a strongly full projection. Then $PM_n(\mathcal{A})P$ and $\mathcal{A}$ are strongly Morita equivalent* [17]*.*

[15] Fullness follows from the fullness of P.

[16] This notion is stronger than fullness. In fact, let $e_1, \ldots, e_n$ be the canonical basis of $\mathcal{A}^n$. For $x, y \in \mathcal{A}^n$, an easy computation shows that $\Theta_{x,y} = \sum_i \Theta_{x,e_i\tau} P \Theta_{e_i\tau,y}$. Since elements of the form $\Theta_{x,y}$ generate $M_n(\mathcal{A})$, the result follows.

[17] See Example 2.48.

Proof. Consider the $(PM_n(\mathcal{A})P, \mathcal{A})$-bimodule $P\mathcal{A}^n$. A simple computation shows that we have a natural identification $P\mathcal{A}^n \cong \mathcal{E} \otimes_{M_n(\mathcal{A})} \mathcal{E}'$, where $\mathcal{E}$, $\mathcal{E}'$ are as in Lemmas 3.68, 3.69. Under this identification, the inner products (3.12), (3.13) correspond to the restrictions of the inner products ${}_{M_n(\mathcal{A})}\langle \cdot, \cdot \rangle = \Theta_{\cdot,\cdot}$ and $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ on $\mathcal{A}^n$ to $P\mathcal{A}^n \subseteq \mathcal{A}^n$. It is an immediate consequence of Lemmas 3.68, 3.69 that $\mathcal{E}'' = P\mathcal{A}^n$ is a right Rieffel bimodule and a left Rieffel bimodule, with compatible inner-products, and satisfying (P). In order to show that it is an equivalence bimodule, it remains to check that (Q) holds.

We must check that (P) holds for the conjugate bimodule $\overline{\mathcal{E}''}$. Let $(\pi, \mathfrak{H})$ be a *-representation of $PM_n(\mathcal{A})P$, and let $\phi_1, \ldots, \phi_r \in \mathfrak{H}$, $x_1, \ldots, x_r \in \mathcal{E}$. Then

$$\begin{aligned}
\sum_{i,j} \langle x_i \otimes \phi_i, x_j \otimes \phi_j \rangle_{\tilde{\mathfrak{K}}} &= \sum_{i,j} \left\langle \phi_i, \pi\left(\Theta_{x_i,x_j}\right) \phi_j \right\rangle_{\mathfrak{H}} \\
&= \sum_{i,j,k} \left\langle \pi\left(\Theta_{x_i,Pe_k\tau}\right) \phi_i, \pi\left(\Theta_{x_j,Pe_k\tau}\right) \phi_j \right\rangle_{\mathfrak{H}} \\
&= \sum_{k} \left\langle \sum_i \pi\left(\Theta_{x_i,Pe_k\tau}\right) \phi_i, \sum_i \pi\left(\Theta_{x_i,Pe_k\tau}\right) \phi_i \right\rangle_{\mathfrak{H}} \\
&\geq 0,
\end{aligned}$$

where we used that $\Theta_{x_i,x_j} = \sum_k \Theta_{x_i,Pe_k\tau}\Theta_{Pe_k\tau,x_j}$. ∎

Example 3.72. *Let M be a smooth manifold, and let $E \to M$ be an n-dimensional vector bundle over M, with hermitian metric h. Let $\mathcal{A} = C^\infty(M)$, the *-algebra of complex-valued smooth functions on M. The space of smooth sections of E, $\mathcal{E} := \Gamma^\infty(E)$, is a $(\Gamma^\infty(\mathsf{End}(E)), C^\infty(M))$-bimodule, where $\Gamma^\infty(\mathsf{End}(E))$ is the algebra of smooth sections of the endomorphism bundle associated to E, equipped with a *-involution defined with respect to the metric h. A classical result by Serre-Swan asserts that $\mathcal{E}$ is finitely generated and projective over $\mathcal{A}$, and hence we can identify $\mathcal{E}$ with $P\mathcal{A}^n$, for some projection $P \in M_n(\mathcal{A})$. This identification can be chosen to be isometric (with respect to h and $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ on $P\mathcal{A}^n$), and we get a *-isomorphism $\Gamma^\infty(\mathsf{End}(E)) \cong PM_n(\mathcal{A})P$. Since $\mathsf{tr}P = n$, it immediately follows that P is strongly full. It follows from Theorem 3.71 that $C^\infty(M)$ and $\Gamma^\infty(\mathsf{End}(E))$ are strongly Morita equivalent[18] (compare with Example 2.49); in particular, they have equivalent representation theory on complex pre-Hilbert spaces.*

[18] A different argument [12, Sec. 6] shows that the same is true for $C_c^\infty(M)$ and $\Gamma_c^\infty(\mathsf{End}(E))$.

3.4 Back to C^*-algebras

In this section, we will show that one can recover the usual notion of strong Morita equivalence of C^*-algebras from algebraic strong Morita equivalence.

3.4.1 Morita *-equivalence and Pedersen ideals

The reader is referred to [55, Chp. 5] for details on this section.

Definition 3.73. *The* Pedersen ideal *of a C^*-algebra $\mathcal{A}$ is its minimal dense ideal; we denote it by $\mathcal{P}_{\mathcal{A}}$.*

Example 3.74. *Let X be a locally compact topological Hausdorff space and $\mathcal{A} = C_0(X)$, the algebra of complex-valued continuous functions on X vanishing at infinity. In this case $\mathcal{P}_{\mathcal{A}} = C_c(X)$, the ideal of compactly supported functions.*

Example 3.75. *Let $\mathfrak{H}$ be a complex Hilbert space, and let $\mathcal{A} = \mathcal{K}(\mathfrak{H})$ be the algebra of compact operators on $\mathfrak{H}$. Then $\mathcal{P}_{\mathcal{A}} = \mathfrak{F}(\mathfrak{H})$, the ideal of finite rank operators.*

An important property of Pedersen ideals is the following [55, Thm. 5.6.2]

Proposition 3.76. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{P}_{\mathcal{A}}$ its Pedersen ideal. If $A \in \mathcal{P}_{\mathcal{A}}$, then the hereditary C^*-algebra*[19] *generated by A is contained in $\mathcal{P}_{\mathcal{A}}$.*

It was proven by Beer in [5] that two unital C^*-algebras are strongly Morita equivalent if and only if they are Morita equivalent in the classical sense, as unital rings. Pedersen ideals can be used to give a purely algebraic characterization of strong Morita equivalence in the nonunital case [2, Thm. 2.4].

Theorem 3.77. *Two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are strongly Morita equivalent if and only if $\mathcal{P}_{\mathcal{A}}, \mathcal{P}_{\mathcal{B}}$ are Morita *-equivalent.*

We will show in the next section that the same characterization can be given in terms of algebraic strong Morita equivalence.

[19] A subalgebra $\mathcal{H} \subseteq \mathcal{A}$ is called *hereditary* if $0 \leq A \leq H$, $H \in \mathcal{H}^+$ implies $A \in \mathcal{H}$.

3.4.2 Algebraic strong Morita equivalence of C^*-algebras

We start with some general lemmas.

Lemma 3.78. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{P}_{\mathcal{A}}$ its Pedersen ideal. Then $A \in \mathcal{P}_{\mathcal{A}}$ is positive in $\mathcal{A}$ ($A \in \mathcal{A}^+$) if and only if $A \in \mathcal{P}_{\mathcal{A}}{}^+$ (in the sense of Definition 3.7).*

Proof. Let $A \in \mathcal{P}_{\mathcal{A}}$ be positive in $\mathcal{A}$ and let $\omega : \mathcal{P}_{\mathcal{A}} \longrightarrow \mathbb{C}$ be an arbitrary positive linear functional[20]. We must show that $\omega(A) \geq 0$. It follows from Proposition 3.76 that if $A \in \mathcal{P}_{\mathcal{A}}$, then $C^*(A) \subseteq \mathcal{P}_{\mathcal{A}}$, where $C^*(A)$ denotes the C^*-algebra generated by A. But if $A \in \mathcal{A}^+$, then $A = A_1^* A_1$, with $A_1, A_2 \in C^*(A)^+ \subseteq \mathcal{P}_{\mathcal{A}}$. Thus $\omega(A) = \omega(A_1^* A_1) \geq 0$. Conversely, if ω is a positive linear functional in $\mathcal{A}$, then $\omega|_{\mathcal{P}_{\mathcal{A}}}$ is positive in $\mathcal{P}_{\mathcal{A}}$. So if $A \in \mathcal{P}_{\mathcal{A}}^+$, then $\omega(A) \geq 0$. □

We now observe a general property of *-representations of Pedersen ideals on pre-Hilbert spaces.

Lemma 3.79. *Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{P}_{\mathcal{A}}$ its Pedersen ideal and suppose $\pi : \mathcal{P}_{\mathcal{A}} \longrightarrow \mathfrak{B}(\mathfrak{H})$ is a *-representation on a complex pre-Hilbert space $\mathfrak{H}$. Then π extends to a *-representation $\pi^{\mathrm{cl}} : \mathcal{P}_{\mathcal{A}} \longrightarrow \mathfrak{B}(\mathfrak{H}^{\mathrm{cl}})$, where $\mathfrak{H}^{\mathrm{cl}}$ is the completion of $\mathfrak{H}$. Moreover, $\|\pi^{\mathrm{cl}}(A)\| \leq \|A\|$ for all $A \in \mathcal{P}_{\mathcal{A}}$ and hence π^{cl} also extends to a *-representation of $\mathcal{A}$ on $\mathfrak{H}^{\mathrm{cl}}$.*

Proof. [21] First, observe that representations of C^*-algebras on pre-Hilbert spaces are always bounded. Indeed, let $\mathcal{B}$ be a C^*-algebra and let $\pi : \mathcal{B} \longrightarrow \mathfrak{B}(\mathfrak{H})$ be a *-representation of $\mathcal{B}$ on a pre-Hilbert space $\mathfrak{H}$. For any vector $\varphi \in \mathfrak{H}$, we define a positive linear functional ω_φ on $\mathcal{B}$ by $\omega_\varphi(B) = \langle \varphi, \pi(B)\varphi \rangle_{\mathfrak{H}}$, for $B \in \mathcal{B}$. It follows that ω_φ is bounded[22], with $\|\omega_\varphi\| \leq \|\varphi\|^2$. Hence $\|\pi(B)\varphi\|^2 = \|\omega_\varphi(B^*B)\| \leq \|\varphi\|^2 \|B\|^2$, and $\|\pi(B)\| \leq \|B\|$ for all $B \in \mathcal{B}$. So π naturally extends to a *-representation $\pi^{\mathrm{cl}} : \mathcal{B} \longrightarrow \mathfrak{B}(\mathfrak{H}^{\mathrm{cl}})$.

If $\mathcal{P}_{\mathcal{A}}$ is the Pedersen ideal of a C^*-algebra $\mathcal{A}$ and $\pi : \mathcal{P}_{\mathcal{A}} \rightarrow \mathfrak{B}(\mathfrak{H})$ is a *-representation, then if $A \in \mathcal{P}_{\mathcal{A}}$, $C^*(A) \subseteq \mathcal{P}_{\mathcal{A}}$ and $\pi|_{C^*(A)}$ is a *-representation of $C^*(A)$ on $\mathfrak{B}(\mathfrak{H})$. Then, as we just saw, $\pi(A)$ extends to a bounded operator on the Hilbert space

[20] In the sense of Definition 3.4, i.e. $\omega(A^*A) \geq 0$ for all $A \in \mathcal{P}_{\mathcal{A}}$.

[21] As pointed out to us by P. Ara, this lemma follows from a more general fact:
Let $\mathcal{A}$ be an arbitrary complex *-algebra with positive definite involution ([38, pp. 338]), and let $\mathcal{A}_b$ be the *-subalgebra of bounded elements of $\mathcal{A}$ ([38, pp. 338]). Then any *-representation of $\mathcal{A}_b$ on a complex pre-Hilbert space $\mathfrak{H}$ naturally extends to a *-representation of $\overline{\mathcal{A}_b}$ (closure of $\mathcal{A}_b$ with respect to its natural seminorm, as defined in [38, pp. 342]) on $\mathfrak{H}^{\mathrm{cl}}$.

[22] To see that, note that if $\mathcal{B}$ is unital, then $\|\omega_\varphi\| = \omega_\varphi(1) = \|\pi(1)\varphi\|^2 \leq \|\varphi\|^2$, since $\pi(1)$ is a projection in $\mathfrak{B}(\mathfrak{H})$ and $\|\pi(1)\| \leq 1$; if $\mathcal{B}$ is non-unital, one gets the same result by adjoining a unit.

$\mathfrak{H}^{\mathrm{cl}}$ (with $\|\pi^{\mathrm{cl}}(A)\| \leq \|A\|$), for all $A \in \mathcal{P}_{\mathcal{A}}$. Thus π admits a natural extension π^{cl}, defining a *-representation of $\mathcal{P}_{\mathcal{A}}$ on $\mathfrak{B}(\mathfrak{H}^{\mathrm{cl}})$. □

Proposition 3.80. *Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{P}_{\mathcal{A}}$ its Pedersen ideal and $\mathcal{B}$ an arbitrary *-algebra over $\mathbb{C}$. Suppose ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{P}_{\mathcal{A}}}$ is a Rieffel bimodule. Then it satisfies property (P).*

Proof. Let $\mathfrak{H}$ be a complex pre-Hilbert space and $\pi : \mathcal{P}_{\mathcal{A}} \longrightarrow \mathfrak{B}(\mathfrak{H})$ a *-representation of $\mathcal{P}_{\mathcal{A}}$. We must check that the formula $\langle x_1 \otimes \phi_1, x_2 \otimes \phi_2 \rangle_{\widetilde{\mathfrak{K}}} = \left\langle \phi_1, \pi(\langle x_1, x_2 \rangle_{\mathcal{P}_{\mathcal{A}}})\phi_2 \right\rangle_{\mathfrak{H}}$, for $x_1, x_2 \in \mathcal{E}$ and $\phi_1, \phi_2 \in \mathfrak{H}$, defines a positive semi-definite inner product on $\widetilde{\mathfrak{K}} = \mathcal{E} \otimes_{\mathcal{P}_{\mathcal{A}}} \mathfrak{H}$. Since π extends to a *-representation $\pi^{\mathrm{cl}} : \mathcal{P}_{\mathcal{A}} \longrightarrow \mathfrak{B}(\mathfrak{H}^{\mathrm{cl}})$ on the Hilbert space completion of $\mathfrak{H}$, there is a natural isometric embedding $\iota : \mathcal{E} \otimes_{\mathcal{P}_{\mathcal{A}}} \mathfrak{H} \longrightarrow \mathcal{E} \otimes_{\mathcal{P}_{\mathcal{A}}} \mathfrak{H}^{\mathrm{cl}}$, where the $\mathcal{P}_{\mathcal{A}}$-balanced tensor products are defined using π and π^{cl}, respectively. So for $\phi_1, \ldots, \phi_n \in \mathfrak{H} \subseteq \mathfrak{H}^{\mathrm{cl}}$ and $x_1, \ldots, x_n \in \mathcal{E}$, we have $\langle \sum_i x_i \otimes \phi_i, \sum_i x_i \otimes \phi_i \rangle_{\widetilde{\mathfrak{K}}} = \sum_{i,j} \left\langle \phi_i, \pi(\langle x_i, x_j \rangle_{\mathcal{A}})\phi_j \right\rangle_{\mathfrak{H}} = \sum_{i,j} \left\langle \phi_i, \pi^{\mathrm{cl}}(\langle x_i, x_j \rangle_{\mathcal{A}})\phi_j \right\rangle_{\mathfrak{H}^{\mathrm{cl}}}$. But the right-hand side of this equation is positive by exactly the same argument in the proof of Proposition 2.37, and the result follows. □

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras, with Pedersen ideals $\mathcal{P}_{\mathcal{A}}$, $\mathcal{P}_{\mathcal{B}}$.

Lemma 3.81. *If $\mathcal{P}_{\mathcal{A}}$ and $\mathcal{P}_{\mathcal{B}}$ are Morita *-equivalent, then they are automatically algebraically strongly Morita equivalent.*

Proof. Since $\mathcal{P}_{\mathcal{A}}$ and $\mathcal{P}_{\mathcal{B}}$ are Morita *-equivalent, there exists a right $\mathcal{P}_{\mathcal{A}}$-module $\mathcal{E}_{\mathcal{P}_{\mathcal{A}}}$, equipped with a nondegenerate, full $\mathcal{P}_{\mathcal{A}}$-valued inner product $\langle \cdot, \cdot \rangle_{\mathcal{P}_{\mathcal{A}}}$ so that $\mathcal{P}_{\mathcal{B}} \cong \mathfrak{F}(\mathcal{E}_{\mathcal{P}_{\mathcal{A}}})$. Ara shows in [2, Thm. 2.4] that, in this case, it automatically follows that $\langle x, x \rangle_{\mathcal{P}_{\mathcal{A}}} \in \mathcal{A}^+$ and $\Theta_{x,x} \in \mathcal{B}^+$, for all $x \in \mathcal{E}$. Hence, by Lemma 3.78, we have $\langle x, x \rangle \in \mathcal{P}_{\mathcal{A}}^+$ and $\Theta_{x,x} \in \mathcal{P}_{\mathcal{B}}^+$, for all $x \in \mathcal{E}$. Finally, properties (P), (Q) follow from Lemma 3.79. □

It immediately follows from Lemma 3.81 and [2, Thm. 2.4] that

Theorem 3.82. *Two C^*-algebras are strongly Morita equivalent if and only if their Pedersen ideals are algebraically strongly Morita equivalent. In particular, unital C^*-algebras are strongly Morita equivalent if and only if they are algebraically strongly Morita equivalent.*

As a result, we have two theorems on equivalence of representations of strongly Morita equivalent C^*-algebras $\mathcal{A}$ and $\mathcal{B}$: Rieffel's theorem (Theorem 2.43) asserts that $\mathsf{Her}(\mathcal{A})$ and $\mathsf{Her}(\mathcal{B})$ are equivalent categories; Corollary 3.59 shows that $\mathsf{Rep}(\mathcal{P}_{\mathcal{A}})$ and $\mathsf{Rep}(\mathcal{P}_{\mathcal{B}})$ are equivalent. We will briefly discuss how these results are related.

For a C^*-algebra $\mathcal{A}$, let $\mathsf{HerD}(\mathcal{A})$ be the following category: objects will be pairs $(\pi, \mathfrak{L})$, where π is a *-representation of $\mathcal{A}$ on a Hilbert space $\mathfrak{H}$, and $\mathfrak{L} \subseteq \mathfrak{H}$ is a dense subspace invariant under π restricted to $\mathcal{P}_{\mathcal{A}}$, and the action of $\mathcal{P}_{\mathcal{A}}$ on $\mathfrak{L}$ is strongly nondegenerate; morphism are isometric intertwiners $T : \mathfrak{H} \longrightarrow \mathfrak{H}'$ satisfying $T(\mathfrak{L}) \subseteq \mathfrak{L}'$.

Claim. $\mathsf{Rep}(\mathcal{P}_{\mathcal{A}})$ and $\mathsf{HerD}(\mathcal{A})$ are naturally equivalent.

Proof. The extension of representations discussed in Lemma 3.79 gives rise to a functor $E : \mathsf{Rep}(\mathcal{P}_{\mathcal{A}}) \longrightarrow \mathsf{HerD}(\mathcal{A})$; the natural restriction of representations of $\mathcal{P}_{\mathcal{A}}$ on $\mathfrak{H}$ to $\mathfrak{L}$ defines a functor $R : \mathsf{HerD}(\mathcal{A}) \longrightarrow \mathsf{Rep}(\mathcal{P}_{\mathcal{A}})$. These functors define an equivalence of categories. □

There are natural functors relating $\mathsf{Her}(\mathcal{A})$ and $\mathsf{HerD}(\mathcal{A})$: We consider the functor $F : \mathsf{HerD}(\mathcal{A}) \longrightarrow \mathsf{Her}(\mathcal{A})$ that forgets the dense subspaces (and is the identity on morphisms); conversely, we define a functor $G : \mathsf{Her}(\mathcal{A}) \longrightarrow \mathsf{HerD}(\mathcal{A})$ by $G(\pi, \mathfrak{H}) = (\pi, \mathfrak{H}, \mathcal{P}_{\mathcal{A}}\mathfrak{H})$ (and the identity on morphisms). It is not hard to see that these natural functors do not define an equivalence of categories. In fact, F is onto but need not be injective, and G is injective but not onto[23]. One can use the functors F and G, however, to prove that Rieffel's theorem (Theorem 2.43) follows from Corollary 3.59 applied to Pedersen ideals[24].

[23] We mean injective and onto with respect to the skeletons of the categories, i.e., modulo isomorphisms.

[24] We showed in [12] that, similarly, the so-called Rieffel correspondence theorem for closed ideals [58, Thm. 3.22] follows from a purely algebraic result for Pedersen ideals.

Chapter 4

Deformation quantization of hermitian vector bundles

Let $E \to M$ be a smooth vector bundle over a manifold M. We saw in Example 3.72 that the *-algebras $C^\infty(M)$ and $\Gamma^\infty(\mathsf{End}(E))$ are strongly Morita equivalent. We will show in this section that there is a natural bijection between hermitian differential star products on these algebras, and that corresponding deformed algebras are strongly Morita equivalent. Most of the results in this Chapter appeared in [13].

4.1 Deformations of projections

In this section $\mathcal{A}$ will be an associative algebra over a commutative, unital ring k of characteristic zero. A formal deformation of $\mathcal{A}$ will be denoted by $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$. If $\mathcal{A}$ is a *-algebra, we will automatically assume that k is of the form $\mathsf{C} = \mathsf{R}(\mathrm{i})$, where R is an ordered ring; formal deformations of *-algebras will be always assumed to be hermitian. Moreover, we will assume throughout that $\mathbb{Q} \subset k$. The basic results on deformations of associative algebras can be found in Appendix B.

Let us consider a k-algebra $\mathcal{A}$ and a formal deformation $\boldsymbol{\mathcal{A}}$. It is clear that $M_n(\boldsymbol{\mathcal{A}})$ can be identified with $M_n(\mathcal{A})[[\lambda]]$ as a $k[[\lambda]]$-module, and this identification naturally defines a deformation of $M_n(\mathcal{A})$. Note that if $\mathcal{A}$ is a *-algebra, we can define a *-involution on $M_n(\mathcal{A})$ in the usual way. It is simple to check that if $\boldsymbol{\mathcal{A}}$ is a hermitian deformation of $\mathcal{A}$, then $M_n(\boldsymbol{\mathcal{A}})$ defines a hermitian deformation of $M_n(\mathcal{A})$.

Lemma 4.1. *Let $\boldsymbol{\mathcal{A}}$ be a hermitian deformation of a unital *-algebra $\mathcal{A}$ over C. Let $L_0 \in M_n(\mathcal{A})$ be invertible and let $\boldsymbol{S} = \sum_{r=0}^{\infty} S_r\lambda^r \in M_n(\boldsymbol{\mathcal{A}})$ be hermitian with $S_0 = L_0^* L_0$. Then there exist $L_r \in M_n(\mathcal{A}), r \geq 1$, such that $\boldsymbol{L} = \sum_{r=0}^{\infty} L_r\lambda^r$ satisfies $\boldsymbol{S} = \boldsymbol{L}^* \star \boldsymbol{L}$.*

Proof. We define $\boldsymbol{L}$ recursively. Suppose $L_0, L_1, \ldots, L_{k-1} \in M_n(\mathcal{A})$ are such that $\boldsymbol{L}_{k-1} = L_0 + L_1\lambda + \ldots + L_{k-1}\lambda^{k-1}$ satisfies $\boldsymbol{S} - \boldsymbol{L}_{k-1}^* \star \boldsymbol{L}_{k-1} = b_k\lambda^k + o(\lambda^{k+1})$. Note that since $\boldsymbol{S}$ is hermitian, so is b_k. We need to find L_k so that $\boldsymbol{L}_k = \sum_{j=0}^{k} L_j\lambda^j$ satisfies $\boldsymbol{S} = \boldsymbol{L}_k^* \star \boldsymbol{L}_k$ up to order λ^{k+1}. But this happens if and only if $L_k^* L_0 + L_0^* L_k = b_k$. Then $L_k = \frac{1}{2}(b_k L_0^{-1})^*$ is a solution. □

Corollary 4.2. *Let $\mathcal{A}$ be a unital *-algebra over C and $\boldsymbol{\mathcal{A}}$ a hermitian deformation of $\mathcal{A}$. Then any unitary $U_0 \in M_n(\mathcal{A})$ can be deformed into a unitary $\boldsymbol{U} = \sum_{r=0}^{\infty} U_r\lambda^r \in M_n(\boldsymbol{\mathcal{A}})$.*

We recall the following well-known fact.

Lemma 4.3. *Let $P_0 \in M_n(\mathcal{A})$ be an idempotent element ($P_0^2 = P_0$). For any formal deformation $\boldsymbol{\mathcal{A}}$, there exists an idempotent $\boldsymbol{P} \in M_n(\boldsymbol{\mathcal{A}})$ with $\boldsymbol{P} = P_0 + O(\lambda)$. Moreover, if P_0 is a projection ($P_0^2 = P_0 = P^*$), then $\boldsymbol{P}$ can also be chosen to be a projection.*

Proof. The proof of the first assertion can be found, for example, in [27, 29, 33]; It is simple to check that the proof in [27] works for projections as well. One can also check all the results using the explicit formula [29, Eq. (6.1.4)]

$$\boldsymbol{P} = \frac{1}{2} + \left(P_0 - \frac{1}{2}\right) \star \frac{1}{\sqrt[\star]{1 + 4(P_0 \star P_0 - P_0)}}. \tag{4.1}$$

□

We say that $\boldsymbol{P}$ in Lemma 4.3 is a *deformation* of P_0 (with respect to the formal deformation $\boldsymbol{\mathcal{A}}$). Let us discuss the behavior of fullness under deformation.

Lemma 4.4. *Let $\mathcal{A}$ be a unital k-algebra. Let $P_0 \in M_n(\mathcal{A})$ be an idempotent and $\boldsymbol{P} \in M_n(\boldsymbol{\mathcal{A}})$ be a deformation of P_0. Then P_0 is full if and only if $\boldsymbol{P}$ is full.*

Proof. Assume P_0 is full. Let $\boldsymbol{J} \subseteq M_n(\boldsymbol{\mathcal{A}})$ be the ideal spanned by elements of the form $\boldsymbol{T} \star \boldsymbol{P} \star \boldsymbol{S}$, for $\boldsymbol{T}, \boldsymbol{S} \in M_n(\boldsymbol{\mathcal{A}})$. We must show that $\boldsymbol{J} = M_n(\boldsymbol{\mathcal{A}})$. There exist, by the fullness of P_0, $T_i, S_i \in M_n(\mathcal{A})$, $i = 1, \ldots, m$, with $\sum_{i=1}^{m} T_i P_0 S_i = \mathsf{id}$. But then $\sum_{i=1}^{m} T_i \star P_0 \star S_i = \mathsf{id} + O(\lambda) \in \boldsymbol{J}$ is invertible, and the result follows. The converse is a simple computation. □

Lemma 4.5. *Let $\mathcal{A}$ be a *-algebra. Let $P_0 \in M_n(\mathcal{A})$ be a projection, and let $\boldsymbol{P} \in M_n(\boldsymbol{\mathcal{A}})$ be a projection deforming P_0. Then P_0 is strongly full if and only if $\boldsymbol{P}$ is strongly full.*

Proof. If P_0 is strongly full, then $\mathsf{tr}(P_0) = \tau\tau^*$, for some invertible $\tau \in \mathcal{A}$. Hence $\mathsf{tr}(\boldsymbol{P}) = \tau\tau^* + O(\lambda) \in \mathcal{A}[[\lambda]]$, and by Lemma 4.1 it follows that there exists $\boldsymbol{\tau} = \tau + O(\lambda) \in \mathcal{A}[[\lambda]]$ with $\mathsf{tr}(\boldsymbol{P}) = \boldsymbol{\tau} \star \boldsymbol{\tau}^*$. Since τ is invertible, so is $\boldsymbol{\tau}$, and the result follows. The converse is just as simple. □

Lemma 4.6. *Let $\mathcal{A}$ be a k-algebra and suppose $P_0, Q_0 \in M_n(\mathcal{A})$ are idempotents. Let $\boldsymbol{\mathcal{A}}$ be a deformation of $\mathcal{A}$ and $\boldsymbol{P} = \sum_{r=0}^{\infty} P_r\lambda^r, \boldsymbol{Q} = \sum_{r=0}^{\infty} Q_r\lambda^r \in M_n(\boldsymbol{\mathcal{A}})$ be deformations of P_0, Q_0, respectively. Then the map $I : P_0M_n(\mathcal{A})Q_0[[\lambda]] \longrightarrow \boldsymbol{P} \star M_n(\boldsymbol{\mathcal{A}}) \star \boldsymbol{Q}$ given by*

$$I(P_0LQ_0) = \boldsymbol{P} \star (P_0LQ_0) \star \boldsymbol{Q}, \qquad L \in M_n(\mathcal{A})[[\lambda]], \tag{4.2}$$

is a $k[[\lambda]]$-module isomorphism.

Proof. The $k[[\lambda]]$-linearity and the injectivity of I are clear since $\boldsymbol{P}$ and $\boldsymbol{Q}$ are deformations of P_0 and Q_0. To prove surjectivity, let $\boldsymbol{L} = \sum_{r=0}^{\infty} L_r\lambda^r \in \boldsymbol{P} \star M_n(\boldsymbol{\mathcal{A}}) \star \boldsymbol{Q}$ be given. Then $\boldsymbol{P} \star \boldsymbol{L} \star \boldsymbol{Q} = \boldsymbol{L}$, hence $P_0L_0Q_0 = L_0$. Defining $S_0 := L_0 \in P_0M_n(\mathcal{A})Q_0$, we have $I(S_0) = \boldsymbol{L}$ up to order λ^0. Since $I(S_0) - \boldsymbol{L} \in \boldsymbol{P} \star M_n(\boldsymbol{\mathcal{A}}) \star \boldsymbol{Q}$ starts with order λ, we can repeat the argument to find a $S_1 \in P_0M_n(\mathcal{A})Q_0$ such that $I(S_0 + \lambda S_1)$ coincides with $\boldsymbol{L}$ up to order λ. A simple induction proves that I is onto. □

Let us denote the deformed product of $M_n(\boldsymbol{\mathcal{A}})$ by

$$L \star S = \sum_{r=0}^{\infty} C_r(L,S)\lambda^r, \quad \text{for } L, S \in M_n(\mathcal{A}).$$

If we consider $\boldsymbol{P} \star M_n(\boldsymbol{\mathcal{A}}) \star \boldsymbol{Q} \subseteq M_n(\boldsymbol{\mathcal{A}})$, we can write $I : P_0M_n(\mathcal{A})Q_0[[\lambda]] \longrightarrow M_n(\boldsymbol{\mathcal{A}}) = M_n(\mathcal{A})[[\lambda]]$ as $I = \sum_{r=0}^{\infty} I_r\lambda^r$. A simple computation shows that

$$I_r(B) = \sum_{i+j+k+m=r} C_m(C_k(P_i, B), Q_j), \qquad \text{for} \quad B \in P_0M_n(\mathcal{A})Q_0. \tag{4.3}$$

We note that I is just a deformation of the natural inclusion $P_0M_n(\mathcal{A})Q_0 \hookrightarrow M_n(\mathcal{A})$.

If $P_0 \in M_n(\mathcal{A})$ is an idempotent, then $P_0M_n(\mathcal{A})P_0$ is a unital algebra. It is clear that if $\mathcal{A}$ is a *-algebra and P_0 is a projection, then $P_0M_n(\mathcal{A})P_0$ has a natural involution. We observe the following simple consequence of Lemma 4.6.

Corollary 4.7. *The map* $I : P_0M_n(\mathcal{A})P_0[[\lambda]] \longrightarrow \boldsymbol{P} \star M_n(\boldsymbol{\mathcal{A}}) \star \boldsymbol{P}$ *defined in (4.2) induces a formal deformation of* $P_0M_n(\mathcal{A})P_0$. *Moreover, if* $\mathcal{A}$ *is a* **-algebra,* P_0 *is a projection and* $\boldsymbol{\mathcal{A}}$ *is hermitian, then* I *induces a hermitian deformation of* $P_0M_n(\mathcal{A})P_0$.

Proof. Given $B, B' \in P_0M_n(\mathcal{A})P_0$, we set $B \star' B' = I^{-1}(I(B) \star I(B))$, which defines a deformation of $P_0M_n(\mathcal{A})P_0$. In the case $\mathcal{A}$ is a *-algebra and $\boldsymbol{\mathcal{A}}$ is a hermitian deformation, it immediately follows from the expression of I in (4.2) that $I(B^*) = I(B)^*$, and hence the induced deformation is hermitian. □

Remark 4.8. *One can check that* $\boldsymbol{P} \star M_n(\boldsymbol{\mathcal{A}}) \star \boldsymbol{Q} \cong P_0M_n(\mathcal{A})Q_0[[\lambda]]$ *as a* $k[[\lambda]]$*-module by more general arguments; the importance of Lemma 4.6 lies in the explicit expression of the isomorphism* I*, which will be important when we deal with differential star products.*

4.2 Deformations of projective inner-product modules

Throughout this section, we will only deal with unital algebras.

4.2.1 Deformations of projective modules

Let $\mathcal{A}$ be a unital k-algebra and let $\mathcal{E}$ be a right[1] module over $\mathcal{A}$. Let $R_0 : \mathcal{E} \times \mathcal{A} \longrightarrow \mathcal{E}$ denote the right action of $\mathcal{A}$ on $\mathcal{E}$, $R_0(x, A) = x \cdot A$ for $x \in \mathcal{E}$, $A \in \mathcal{A}$. Let $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$ be a formal deformation of $\mathcal{A}$ and suppose there exist k-bilinear maps $R_r : \mathcal{E} \times \mathcal{A} \longrightarrow \mathcal{E}$, for $r \geq 1$, such that the map

$$\boldsymbol{R} = \sum_{r=0}^{\infty} R_r \lambda^r : \mathcal{E}[[\lambda]] \times \boldsymbol{\mathcal{A}} \longrightarrow \mathcal{E}[[\lambda]] \tag{4.4}$$

makes $\mathcal{E}[[\lambda]]$ into a module over $\boldsymbol{\mathcal{A}}$. We will write $\boldsymbol{R}(x, A) = x \bullet A$, for $x \in \mathcal{E}$, $A \in \mathcal{A}$.

Definition 4.9. *We call* $\boldsymbol{\mathcal{E}} = (\mathcal{E}[[\lambda]], \bullet)$ *a* deformation *of the (right)* $\mathcal{A}$*-module* $\mathcal{E}$ *corresponding to* $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$. *Two deformations* $\boldsymbol{\mathcal{E}} = (\mathcal{E}[[\lambda]], \bullet)$, $\boldsymbol{\mathcal{E}}' = (\mathcal{E}[[\lambda]], \bullet')$ *are* equivalent *if there exists an* $\boldsymbol{\mathcal{A}}$*-module isomorphism* $\boldsymbol{T} : \boldsymbol{\mathcal{E}} \longrightarrow \boldsymbol{\mathcal{E}}'$ *of the form* $\boldsymbol{T} = \mathsf{id} + \sum_{r=1}^{\infty} T_r \lambda^r$, *with* k*-linear maps* $T_r : \mathcal{E} \longrightarrow \mathcal{E}$.

Here we will be only interested in finitely generated projective modules (f.g.p.m.).

[1] The reader will have no problem to adapt all the definitions and results to left modules.

Proposition 4.10. *Let $\mathcal{A}$ be a unital k-algebra and $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$ be a deformation of $\mathcal{A}$. Let $\mathcal{E}$ be a (right) f.g.p.m. over $\mathcal{A}$. Then there exists a deformation $\boldsymbol{\mathcal{E}}$ of $\mathcal{E}$ corresponding to $\boldsymbol{\mathcal{A}}$ so that $\boldsymbol{\mathcal{E}}$ is a f.g.p.m. over $\boldsymbol{\mathcal{A}}$. Moreover, this deformation is unique up to equivalence; in particular, every deformation of $\mathcal{E}$ is finitely generated and projective over $\boldsymbol{\mathcal{A}}$.*

Proof. For the existence, note that, since $\mathcal{E}$ is f.g.p.m., we can write $\mathcal{E} = P_0\mathcal{A}^n$, for some $n \geq 1$ and $P_0 \in M_n(\mathcal{A})$ idempotent. Let $\boldsymbol{P} \in M_n(\boldsymbol{\mathcal{A}})$ be an idempotent deforming P_0 and consider the (right) f.g.p.m. over $\boldsymbol{\mathcal{A}}$ given by $\boldsymbol{P}\star\boldsymbol{\mathcal{A}}^n$. By Lemma 4.6 (choosing Q_0 to be 1 in the upper right corner and zero elsewhere), we can use the isomorphism $I : \mathcal{E}[[\lambda]] \longrightarrow \boldsymbol{P}\star\boldsymbol{\mathcal{A}}^n$ to pull this $\boldsymbol{\mathcal{A}}$-module structure back to $\mathcal{E}[[\lambda]]$, i.e. $x \bullet A := I^{-1}(\boldsymbol{P} \star x \star A) = x \cdot A + O(\lambda)$ for $x \in \mathcal{E}$ and $A \in \mathcal{A}$. So $\boldsymbol{\mathcal{E}} = (\mathcal{E}[[\lambda]], \bullet)$ is a finitely generated projective deformation of $\mathcal{E}$.

Assume $\boldsymbol{\mathcal{E}}' = (\mathcal{E}[[\lambda]], \bullet')$ is another deformation of $\mathcal{E}$. Let $\mathfrak{C} : \boldsymbol{\mathcal{E}} \longrightarrow \boldsymbol{\mathcal{E}}/\lambda\boldsymbol{\mathcal{E}} = \mathcal{E}$ and $\mathfrak{C}' : \boldsymbol{\mathcal{E}}' \longrightarrow \boldsymbol{\mathcal{E}}'/\lambda\boldsymbol{\mathcal{E}}' = \mathcal{E}$ be the natural projections, which are surjective $\boldsymbol{\mathcal{A}}$-module homomorphisms. Then it follows by projectivity of $\boldsymbol{\mathcal{E}}$ that there exists an $\boldsymbol{\mathcal{A}}$-module homomorphism $\boldsymbol{T} : \boldsymbol{\mathcal{E}} \longrightarrow \boldsymbol{\mathcal{E}}'$ satisfying $\mathfrak{C}' \circ \boldsymbol{T} = \mathfrak{C}$. Since $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}' = \mathcal{E}[[\lambda]]$ as $k[[\lambda]]$-modules, we can write $\boldsymbol{T} = \sum_{r=0}^{\infty} T_r\lambda^r$, and it is readily seen that $T_0 = \mathsf{id}$. So $\boldsymbol{T}$ is an equivalence. □

Let $\mathsf{Def}(\mathcal{A})$ denote the set of equivalence classes of formal deformation of $\mathcal{A}$ (Definition B.6). The following result is stated in [33, pp. 148].

Proposition 4.11. *Let $P_0 \in M_n(\mathcal{A})$ be a full idempotent. Then there exists a bijection $\Phi : \mathsf{Def}(\mathcal{A}) \longrightarrow \mathsf{Def}(\mathcal{B})$, where $\mathcal{B} = P_0M_n(\mathcal{A})P_0$; formal deformations related by Φ are Morita equivalent (as unital rings).*

Proof. Let us first define Φ. Let $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$ be a formal deformation of $\mathcal{A}$, and let $\boldsymbol{P} \in M_n(\boldsymbol{\mathcal{A}})$ be an idempotent deforming P_0. Then, by Corollary 4.7, $\boldsymbol{P} \star M_n(\boldsymbol{\mathcal{A}}) \star \boldsymbol{P}$ induces a formal deformation $(\mathcal{B}[[\lambda]], \star')$. Note that if $\boldsymbol{P}'$ is another deformation of P_0, then the induced deformation of $\mathcal{B}$ is equivalent to $\star'$: This holds since $\boldsymbol{P} \star M_n(\boldsymbol{\mathcal{A}}) \star \boldsymbol{P} = \mathsf{End}_{\boldsymbol{\mathcal{A}}}(\boldsymbol{P} \star \boldsymbol{\mathcal{A}}^n)$, $\boldsymbol{P}' \star M_n(\boldsymbol{\mathcal{A}}) \star \boldsymbol{P}' = \mathsf{End}_{\boldsymbol{\mathcal{A}}}(\boldsymbol{P}' \star \boldsymbol{\mathcal{A}}^n)$, and $\boldsymbol{P} \star \boldsymbol{\mathcal{A}}^n$ and $\boldsymbol{P}' \star \boldsymbol{\mathcal{A}}^n$ are equivalent (by Proposition 4.10, for they are both deformations of $P_0\mathcal{A}^n$). Moreover, if we change $\star$ by an equivalent deformation, the equivalence class of the deformation of $\mathcal{B}$ is not altered. Hence this procedure defines a canonical map

$$\Phi : \mathsf{Def}(\mathcal{A}) \longrightarrow \mathsf{Def}(\mathcal{B}).$$

We will show that Φ is a bijection. Since P_0 is full, $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent. By symmetry of Morita equivalence, there exists a full idempotent $Q_0 \in M_m(\mathcal{B})$, for some

$m \geq 1$, so that $\mathcal{E} = \mathcal{B}^m Q_0$ ($\mathcal{B}^m$ as *row* vectors) as a left $\mathcal{B}$-module and $\mathcal{A} \cong \mathsf{End}_{\mathcal{B}}(\mathcal{B}^m Q_0) = Q_0 M_n(\mathcal{B}) Q_0$ as a unital algebra. Thus we can define a map $\hat{\Phi} : \mathsf{Def}(\mathcal{B}) \longrightarrow \mathsf{Def}(\mathcal{A})$ just as we did for Φ. Let $[\star] \in \mathsf{Def}(\mathcal{A})$ and $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$. Let $\boldsymbol{P}$ be a deformation of P_0 and $\boldsymbol{\mathcal{B}} = (\mathcal{B}[[\lambda]], \star')$ be the deformation induced by $\boldsymbol{P} \star M_n(\boldsymbol{\mathcal{A}}) \star \boldsymbol{P}$, in such a way that $[\star'] = \Phi([\star])$. Let $\boldsymbol{\mathcal{E}} = \boldsymbol{P} \star \boldsymbol{\mathcal{A}}^n$, which is naturally a $(\boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{A}})$-bimodule. Note that, by Morita theory, we have $\boldsymbol{\mathcal{A}} = \mathsf{End}_{\boldsymbol{\mathcal{B}}}(\boldsymbol{\mathcal{E}})$. Now pick $\boldsymbol{Q} \in M_n(\boldsymbol{\mathcal{B}})$, a deformation of Q_0 and let $\boldsymbol{\mathcal{E}}' = \boldsymbol{\mathcal{B}}^m \star' \boldsymbol{Q}$. Then $[\star''] = \hat{\Phi}([\star']) = \hat{\Phi} \circ \Phi([\star])$ is induced by $\boldsymbol{Q} \star' M_m(\boldsymbol{\mathcal{B}}) \star' \boldsymbol{Q}$, and $\boldsymbol{\mathcal{A}}' = (\mathcal{A}[[\lambda]], \star'')$ can be identified with $\mathsf{End}_{\boldsymbol{\mathcal{B}}}(\boldsymbol{\mathcal{E}}')$. It is not hard to check that $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{E}}'$ are both deformations of $\mathcal{E} = P_0 \mathcal{A}^n = \mathcal{B}^m Q_0$ corresponding to $\boldsymbol{\mathcal{B}}$. Hence, by Proposition 4.10 (for left modules), these deformations are equivalent and so are $\star$ and $\star''$. Therefore $\hat{\Phi} \circ \Phi = \mathsf{id}$ and a similar argument shows that $\Phi \circ \hat{\Phi} = \mathsf{id}$.

It follows from Lemma 4.4 and the very definition of Φ that deformations related by Φ are Morita equivalent. This concludes the proof. □

4.2.2 Deformations of $\mathcal{A}$-valued inner products

Let $\mathcal{A}$ be a *-algebra, and let $\mathcal{E}_{\mathcal{A}}$ be a right f.g.p.m. over $\mathcal{A}$ equipped with a positive definite $\mathcal{A}$-valued inner product. The $\mathcal{A}$-right linear endomorphisms of $\mathcal{E}$ are denoted by $\mathsf{End}_{\mathcal{A}}(\mathcal{E})$; as before, the subalgebra of adjointable endomorphisms is denoted by $\mathcal{L}(\mathcal{E}_{\mathcal{A}})$.

Let $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$ be a hermitian deformation of $\mathcal{A}$. Let $\boldsymbol{\mathcal{E}} = (\mathcal{E}[[\lambda]], \bullet)$ be a corresponding deformation of $\mathcal{E}$ and suppose there exist $h_r : \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{A}$ such that

$$\boldsymbol{h} = \sum_{r=0}^{\infty} h_r \lambda^r \tag{4.5}$$

defines a positive definite, $\boldsymbol{\mathcal{A}}$-valued inner product on $\mathcal{E}[[\lambda]]$.

Definition 4.12. *We call $\boldsymbol{\mathcal{E}} = (\mathcal{E}[[\lambda]], \bullet, \boldsymbol{h})$ a* hermitian deformation *of the inner-product module $(\mathcal{E}, h_0)$ corresponding to $\boldsymbol{\mathcal{A}}$. Two hermitian deformations $\boldsymbol{\mathcal{E}} = (\mathcal{E}[[\lambda]], \bullet, \boldsymbol{h})$, $\boldsymbol{\mathcal{E}} = (\mathcal{E}[[\lambda]], \bullet', \boldsymbol{h}')$ are called* equivalent *if there is an equivalence $\boldsymbol{T} = \mathsf{id} + \sum_{r=0}^{\infty} T_r \lambda^r : \boldsymbol{\mathcal{E}} \longrightarrow \boldsymbol{\mathcal{E}}'$ (as in Definition 4.9) satisfying $\boldsymbol{h}'(\boldsymbol{T}(x), \boldsymbol{T}(y)) = \boldsymbol{h}(x, y)$, $x, y \in \mathcal{E}$.*

Let $P_0 \in M_n(\mathcal{A})$ be a projection. Consider the f.g.p.m. over $\mathcal{A}$ given by $\mathcal{E} = P_0 \mathcal{A}^n$. We observe that $\mathcal{E}$ has a canonical $\mathcal{A}$-valued inner product h_0, namely the restriction to $P_0 \mathcal{A}^n$ of the canonical $\mathcal{A}$-valued inner product on the free-module $\mathcal{A}^n$ given by $\langle x, y \rangle = \sum_{i=1}^{n} x_i^* y_i$. In this case $M_n(\mathcal{A}) = \mathsf{End}_{\mathcal{A}}(\mathcal{A}^n) = \mathcal{L}(\mathcal{A}^n)$, and, similarly, $P_0 M_n(\mathcal{A}) P_0 = \mathsf{End}_{\mathcal{A}}(P_0 \mathcal{A}^n) = \mathcal{L}(P_0 \mathcal{A}^n)$.

Proposition 4.13. *Let $\mathcal{A}$ be a unital *-algebra, and let $P_0 \in M_n(\mathcal{A})$ be a projection. Let $\boldsymbol{\mathcal{A}}$ be a hermitian deformation of $\mathcal{A}$ and consider the $\mathcal{A}$-module $\mathcal{E} = P_0\mathcal{A}^n$, equipped with its canonical $\mathcal{A}$-valued inner product h_0. Then there exists a hermitian deformation of $\mathcal{E}$ corresponding to $\boldsymbol{\mathcal{A}}$, which is unique up to equivalence.*

Proof. As in Proposition 4.10, we choose $\boldsymbol{P} \in M_n(\boldsymbol{\mathcal{A}})$, a projection deforming P_0, and consider the $\boldsymbol{\mathcal{A}}$-module $\boldsymbol{P} \star \boldsymbol{\mathcal{A}}^n$, which we know to define a deformation $\boldsymbol{\mathcal{E}} = (\mathcal{E}[[\lambda]], \bullet)$ of $\mathcal{E}$. Let $\boldsymbol{h}$ be the $\boldsymbol{\mathcal{A}}$-valued inner product on $\boldsymbol{\mathcal{E}}$ obtained from $\langle\cdot,\cdot\rangle$ restricted to $\boldsymbol{P} \star \boldsymbol{\mathcal{A}}^n$. A simple computation shows that $\boldsymbol{h}$ is a deformation of h_0 and hence $\boldsymbol{\mathcal{E}} = (\boldsymbol{P} \star \boldsymbol{\mathcal{A}}^n, \boldsymbol{h})$ is a hermitian deformation of $(\mathcal{E}, h_0)$.

Let $\boldsymbol{\mathcal{E}}' = (\mathcal{E}[[\lambda]], \bullet', \boldsymbol{h}')$ be another hermitian deformation of $(\mathcal{E}, h_0)$. By Proposition 4.10, we may assume that $\boldsymbol{\mathcal{E}}' = \boldsymbol{\mathcal{E}}$ as an $\boldsymbol{\mathcal{A}}$-module, with some $\boldsymbol{\mathcal{A}}$-valued inner product $\boldsymbol{h}'$ deforming h_0. Recall that any $\boldsymbol{\mathcal{A}}$-valued inner product $\langle\cdot,\cdot\rangle'$ on the free $\boldsymbol{\mathcal{A}}$-module $\boldsymbol{\mathcal{A}}^n$ can be written as $\langle\cdot,\cdot\rangle' = \langle\cdot, \boldsymbol{H}\cdot\rangle$ for some hermitian element $\boldsymbol{H} \in M_n(\boldsymbol{\mathcal{A}})$, where $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \sum_{i=1}^n \boldsymbol{x}_i^* \star \boldsymbol{y}_i$, $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{\mathcal{A}}^n$. Since $\boldsymbol{P} \star \boldsymbol{\mathcal{A}}^n \subseteq \boldsymbol{\mathcal{A}}^n$ is projective, one can check that the same holds for this submodule. Hence, there is a hermitian element $\boldsymbol{H} \in \mathsf{End}_{\boldsymbol{\mathcal{A}}}(\boldsymbol{\mathcal{E}})$ so that $\boldsymbol{h}'(\cdot,\cdot) = \boldsymbol{h}(\cdot, \boldsymbol{H}\cdot)$. But since $\boldsymbol{h}$ and $\boldsymbol{h}'$ are deformations of h_0, we can write $\boldsymbol{H} = \sum_{r=0}^\infty H_r \lambda^r$ with $H_0 = \mathsf{id}$. It then follows from Lemma 4.1 that we can find $\boldsymbol{U} = \mathsf{id} + \sum_{r=0}^\infty U_r \lambda^r$ so that $\boldsymbol{H} = \boldsymbol{U}^* \star \boldsymbol{U}$. It is then clear that $\boldsymbol{U} : \boldsymbol{\mathcal{E}}' \longrightarrow \boldsymbol{\mathcal{E}}$ is the desired equivalence. ☐

We will show a result analogous to Proposition 4.11 for hermitian deformations. For a unital *-algebra $\mathcal{A}$, let $\mathsf{Def}^*(\mathcal{A})$ be the set of equivalence classes of hermitian deformations of $\mathcal{A}$ (Definition B.6).

Recall that two idempotents $P, Q \in M_n(\mathcal{A})$ are called *equivalent* if there exist $U, V \in M_n(\mathcal{A})$ with $P = UV$ and $Q = VU$; it is not hard to check that P and Q are equivalent if and only if $P\mathcal{A}^n \cong Q\mathcal{A}^n$, as right $\mathcal{A}$-modules. The reader is referred to [42, Thm. 26] for the proof of the lemma below.

Lemma 4.14. *Let $\mathcal{A}$ be a commutative, unital *-algebra, and suppose elements of the form $1 + L^*L$ are invertible in $M_n(\mathcal{A})$ for all $L \in M_n(\mathcal{A})$. Then any idempotent $P \in M_n(\mathcal{A})$ is equivalent to a projection.*

Proposition 4.15. *Let $\mathcal{A}$ be a unital *-algebra. Suppose elements of the form $1 + L^*L$ are invertible in $M_n(\mathcal{A})$, and let $P_0 \in$ be a full projection. Then there is a bijection Φ :*

$\mathsf{Def}^*(\mathcal{A}) \longrightarrow \mathsf{Def}^*(\mathcal{B})$, *where* $\mathcal{B} = P_0 M_n(\mathcal{A}) P_0$*; hermitian deformations related by* Φ *are Morita* **-equivalent.*

Proof. It follows from Corollary 4.7 that if $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$ is a hermitian deformation of $\mathcal{A}$, then the corresponding deformation of $\mathcal{B}$, constructed in Proposition 4.11, can be chosen to be hermitian (since we can deform P_0 into a projection). Thus Φ from Proposition 4.11 restricts to a map $\Phi : \mathsf{Def}^*(\mathcal{A}) \longrightarrow \mathsf{Def}^*(\mathcal{B})$. To check that this is a bijection, note that the idempotent Q_0, used to define $\hat{\Phi}$ (the inverse of Φ in Proposition 4.11), can be chosen to be a projection due to Lemma 4.14). The map $\hat{\Phi}$ maps hermitian deformations of $\mathcal{B}$ into hermitian deformations of $Q_0 M_m(\mathcal{B}) Q_0$, which is commutative, for it is isomorphic to $\mathcal{A}$ as an associative algebra. But $Q_0 M_m(\mathcal{B}) Q_0$ and $\mathcal{A}$ are Morita *-equivalent, and thus *-isomorphic (Proposition 3.52). Hence $\hat{\Phi}$ also restricts to a map $\hat{\Phi} : \mathsf{Def}^*(\mathcal{B}) \longrightarrow \mathsf{Def}^*(\mathcal{A})$, and the bijection is established. By the same reasoning as in Proposition 4.11, we conclude that hermitian deformations related by Φ are Morita *-equivalent. □

4.2.3 Poisson fibred algebra structures

Let $\mathcal{A}$ be a unital commutative algebra and $P_0 \in M_n(\mathcal{A})$ be a full idempotent. Let $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$ be a formal deformation of $\mathcal{A}$. We keep the notation

$$A_1 \star A_2 = \sum_{r=0}^{\infty} C_r(A_1, A_2)\lambda^r, \quad A_1, A_2 \in \mathcal{A}.$$

Since $\mathcal{A}$ is commutative, it inherits a Poisson algebra structure from $\star$ given by

$$\{A_1, A_2\} := C_1(A_1, A_2) - C_1(A_2, A_1).$$

We saw in Corollary 4.7 how one can define a formal deformation of $\mathcal{B} = P_0 M_n(\mathcal{A}) P_0$ by choosing an idempotent $\boldsymbol{P} \in M_n(\boldsymbol{\mathcal{A}})$ deforming P_0 and considering the isomorphism of $k[[\lambda]]$-modules $I : \mathcal{B}[[\lambda]] \longrightarrow \boldsymbol{P} \star M_n(\boldsymbol{\mathcal{A}}) \star \boldsymbol{P}$, $L_0 \mapsto \boldsymbol{P} \star L_0 \star \boldsymbol{P}$ (see (4.2)). Let us write

$$L_0 \star' S_0 := I^{-1}(I(L_0) \star I(S_0)) = \sum_{r=0}^{\infty} B_r(L_0, S_0)\lambda^r, \quad L_0, S_0 \in \mathcal{B}.$$

For $M, N \in M_n(\mathcal{A})$, let (notice the abuse of notation [2])

$$\{M, N\} := C_1(M, N) - C_1(N, M), \tag{4.6}$$

[2] The bracket in $M_n(\mathcal{A})$ defined in (4.6) induces the bracket $\{\cdot,\cdot\}$ on $\mathcal{A}$ through the identification of $\mathcal{A}$ with the center of $M_n(\mathcal{A})$ in the natural way; we denote both brackets by $\{\cdot,\cdot\}$ by abuse of notation.

where $C_1(M,N) \in M_n(\mathcal{A})$ is defined by $C_1(M,N)_{i,j} = \sum_{r=1}^{n} C_1(M_{i,r}, N_{r,j})$. We will compute the expression for the bracket in $P_0 M_n(\mathcal{A}) P_0 \subseteq M_n(\mathcal{A})$ given by

$$\{L,S\}' := B_1(L,S) - B_1(S,L), \quad L,S \in \mathcal{B}, \tag{4.7}$$

in terms of $\{\cdot,\cdot\}$.

Proposition 4.16. $\{L_0,S_0\}' = P_0\{L_0,S_0\}P_0, \quad L_0,S_0 \in P_0 M_n(\mathcal{A}) P_0$.

Proof. Let us write $I = \sum_{r=0}^{\infty} I_r \lambda^r$ (see (4.3)), where

$$I_r : P_0 M_n(\mathcal{A}) P_0 \longrightarrow M_n(\mathcal{A}).$$

A simple computation shows that $I_0(L_0) = L_0$ and

$$I_1(L_0) = C_1(P_0,L_0) + P_0 C_1(L_0,P_0) + L_0 P_1 + P_1 L_0. \tag{4.8}$$

The equations $\boldsymbol{P} \star I(L_0) = I(L_0)$ and $I(L_0) \star \boldsymbol{P} = I(L_0)$ imply that

$$P_0 C_1(P_0,L_0) + P_0 P_1 L_0 = 0, \quad \text{and} \quad C_1(L_0,P_0)P_0 + L_0 P_1 P_0 = 0. \tag{4.9}$$

It follows from (4.8) and (4.9) that $P_0 I_1(L_0) P_0 = 0$, for all $L_0 \in P_0 M_n(\mathcal{A}) P_0$. Note that if $L_0 + \lambda L_1 + \ldots \in P_0 M_n(\mathcal{A}) P_0[[\lambda]]$ and $I(L_0 + \lambda L_1 + \ldots) = M_0 + \lambda M_1 + \ldots$, then $I_1(L_0) + I_0(L_1) = I_1(L_0) + L_1 = M_1$, and hence

$$P_0(I_1(L_0) + L_1)P_0 = L_1 = P_0 M_1 P_0.$$

But $I(L_0 \star' S_0) = I(L_0) \star I(S_0) = L_0 S_0 + \lambda(C_1(L_0,S_0) + L_0 I_1(S_0) + I_1(L_0) S_0) + \ldots$. Thus

$$B_1(L_0,S_0) = P_0(C_1(L_0,S_0) + L_0 I_1(S_0) + I_1(L_0)S_0)P_0 = P_0 C_1(L_0,S_0) P_0,$$

by (4.9), and the result follows. □

Let $\mathcal{Z}$ denote the center of $\mathcal{B} = P_0 M_n(\mathcal{A}) P_0$. The triple $(\mathcal{B}, \mathcal{Z}, \{\cdot,\cdot\}')$ is a Poisson fibred algebra (see Appendix A), and, as such, the restriction of $\{\cdot,\cdot\}'$ to $\mathcal{Z} \times \mathcal{Z}$ provides $\mathcal{Z}$ with the structure of a Poisson algebra. We can identify $\mathcal{Z}$ and $\mathcal{A}$ through the isomorphism $\Psi : \mathcal{A} \longrightarrow \mathcal{Z}$, $A \mapsto P_0 A P_0 = A P_0$.

Theorem 4.17. $\Psi : (\mathcal{A}, \{\cdot,\cdot\}) \longrightarrow (\mathcal{Z}, \{\cdot,\cdot\}')$ *is an isomorphism of Poisson algebras.*

Proof. The bracket $\{\cdot,\cdot\}' : \mathcal{Z} \times \mathcal{B} \longrightarrow \mathcal{B}$ satisfies

$$\{A, B_1 B_2\}' = \{A, B_1\}' B_2 + B_1 \{A, B_2\}'$$
$$\{A_1 A_2, B\}' = A_1 \{A_2, B\}' + A_2 \{A_1, B\}'.$$

As a result, $\{\cdot, P_0\}' = P_0\{\cdot, P_0\}P_0 = 0$ and $\{P_0, \cdot\}' = P_0\{P_0, \cdot\}P_0 = 0$. It is easy to check that the following Leibniz rule holds for $\{\cdot,\cdot\}$ in $M_n(\mathcal{A})$:

$$\{AM, N\} = A\{M, N\} + M\{A, N\}, \ M, N \in M_n(\mathcal{A}), \ \ A \in \mathcal{A} \cong \text{center}(M_n(\mathcal{A})).$$

Combining these identities, we get

$$\{\Psi(A_1), \Psi(A_2)\}' = \{A_1 P_0, A_2 P_0\}' = \{A_1, A_2\} P_0 = \Psi(\{A_1, A_2\}),$$

for $A_1, A_2 \in \mathcal{A}$. □

4.3 Deformation quantization of hermitian vector bundles

In this section we will consider $\mathcal{A} = C^\infty(M)$, the algebra of complex-valued smooth functions on a manifold M; if $E \to M$ is a smooth m-dimensional complex vector bundle over M, $m \geq 1$, we let $\mathcal{E} = \Gamma^\infty(E)$, considered as a right $\mathcal{A}$-module, and $\mathcal{B} = \mathsf{End}_{\mathcal{A}}(\mathcal{E}) = \Gamma^\infty(\mathsf{End}(E))$. The algebra $\mathcal{A}$ has a natural involution by complex conjugation. If E has a hermitian metric, then there is a corresponding positive definite $\mathcal{A}$-valued inner product on $\mathcal{E}$, and this defines a *-involution on $\mathcal{B}$.

Recall that a star product $\star$ on M, given by $f \star g = \sum_{r=0}^\infty C_r(f,g)\lambda^r$, is called *differential* (resp. *local*, resp. *Vey type*) if each C_r is differential (resp. local, resp. differential of order r) in each entry – see Appendix B.

Definition 4.18. *Let $\boldsymbol{\mathcal{A}} = (C^\infty(M)[[\lambda]], \star)$ be a deformation of $\mathcal{A}$. A* deformation quantization *of E with respect to $\star$ is a deformation of $\mathcal{E} = \Gamma^\infty(E)$ in the sense of Definition 4.9. If E is equipped with a hermitian fiber metric h_0 and $\boldsymbol{\mathcal{A}}$ is a Hermitian deformation of $\mathcal{A}$, then a* hermitian deformation quantization *of (E, h_0) is a deformation of $(\mathcal{E}, h_0)$ as in Definition 4.12. A deformation is called* differential *(resp. local, resp. Vey type) if the corresponding R_r and h_r (as in (4.4), (4.5)) are differential (resp. local, resp. differential of order r) in each argument.*

Recall that any two Hermitian metrics on a complex vector bundle are equivalent (see [43, Ch. I,Thm. 8.8]). Hence we can identify $\mathcal{E}$ with $P_0\mathcal{A}^n$, for some $n \geq 1$ and some projection $P_0 \in M_n(\mathcal{A})$, equipped with its canonical $\mathcal{A}$-valued inner product h_0.

Theorem 4.19. *Let E be a complex (hermitian) vector bundle over a Poisson manifold M and let $\boldsymbol{\mathcal{A}} = (C^\infty(M)[[\lambda]], \star)$ be a (hermitian) deformation quantization of $\mathcal{A}$. Then there exists a (hermitian) deformation quantization of E corresponding to $\boldsymbol{\mathcal{A}}$, which is unique up to equivalence. Moreover, if $\star$ is differential (resp. local, resp. of Vey type), the deformation of E can be chosen to be differential (resp. local, resp. differential of order r).*

Proof. Existence and uniqueness of (Hermitian) deformations follow from Propositions 4.10, 4.13 and the observation before the theorem.

Suppose now $\star$ is differential (resp. local, resp. of Vey type). Choose a deformation $\boldsymbol{P}$ of P_0, and let us consider the $\boldsymbol{\mathcal{A}}$-action on $\mathcal{E}[[\lambda]]$ induced by I as in (4.2). Note that if we write $I = \sum_{r=0}^{\infty} I_r\lambda^r$, it follows from (4.3) that each $I_r : \mathcal{E} \longrightarrow M_n(\mathcal{A})$ is differential (resp. local, resp. of Vey type) of order r. Moreover, $I^{-1} = \sum_{r=0}^{\infty} J_r\lambda^r$ has the same property. From $x \bullet A = I^{-1}(\boldsymbol{P} \star x \star A)$ and $\boldsymbol{h}(x, y) = \langle \boldsymbol{P} \star x, \boldsymbol{P} \star y\rangle$, it follows directly that R_r and h_r have the same desired properties. □

Let us recall the notation introduced in Section 2.1: $\mathsf{Def}_{\mathrm{diff}}(\mathcal{A})$ (resp. $\mathsf{Def}_{\mathrm{diff}}(\mathcal{B})$) denotes the set of equivalence classes of differential deformations of $\mathcal{A} = C^\infty(M)$ (resp. $\mathcal{B} = \Gamma^\infty(\mathsf{End}(E))$); similarly, $\mathsf{Def}^*_{\mathrm{diff}}(\mathcal{A})$ (resp. $\mathsf{Def}^*_{\mathrm{diff}}(\mathcal{B})$) denotes the set of equivalence classes of differential hermitian deformations of $\mathcal{A}$ (resp. $\mathcal{B}$).

Using the identification $\Gamma^\infty(E) = P_0\mathcal{A}^n$, where $P_0 \in M_n(\mathcal{A})$ is an idempotent (projection, in the hermitian case), we can write $\mathcal{B} = P_0M_n(\mathcal{A})P_0$. Let $\boldsymbol{P} \in M_n(\boldsymbol{\mathcal{A}})$ be an idempotent (projection) deforming P_0. We have seen that (Corollary 4.7) the algebra $\boldsymbol{P} \star M_n(\boldsymbol{\mathcal{A}}) \star \boldsymbol{P}$ defines a (Hermitian) deformation of $\mathcal{B}$ via I. By the explicit expression of I (see (4.3)), it follows that if $\star$ is differential (or local/of Vey type), then so is the deformation of $\mathcal{B}$ defined by I.[3] Hence the maps defined in Propositions 4.11,4.15 naturally restrict to

$$\Phi : \mathsf{Def}^{(*)}_{\mathrm{diff}}(\mathcal{A}) \longrightarrow \mathsf{Def}^{(*)}_{\mathrm{diff}}(\mathcal{B}),$$

the same holding for its inverse. (Note that elements of the form $1 + L^*L$ are invertible in $M_n(\mathcal{A})$.) As a result, we have[4]

[3] It would be interesting to compare this construction of star products on $\mathcal{B}$ with Fedosov's construction [28, Sect. 7] in the symplectic case.

[4] See [48] for the case of trivial bundles.

Theorem 4.20. *The map* $\Phi : \mathsf{Def}^{(*)}_{\mathit{diff}}(C^\infty(M)) \longrightarrow \mathsf{Def}^{(*)}_{\mathit{diff}}(\Gamma^\infty(\mathsf{End}(E)))$ *is a bijection.*

For $E \to M$ a hermitian vector bundle over M, we saw in Example 3.72 that $C^\infty(M)$ and $\Gamma^\infty(\mathsf{End}(E))$ are strongly Morita equivalent. The next result shows that this picture can be quantized.

Corollary 4.21. *Let* $\star$ *be a hermitian star product on* M*. Then there exists a unique (up to isomorphism) hermitian deformation* $\star'$ *of* $\Gamma^\infty(\mathsf{End}(E))$ *so that* $(C^\infty(M)[[\lambda]], \star)$ *and* $(\Gamma^\infty(\mathsf{End}(E))[[\lambda]], \star)$ *are strongly Morita equivalent.*

Proof. Just pick $\star'$ so that $[\star'] = \Phi([\star])$; it follows from the very construction of Φ, Lemma 4.5 and Theorem 3.71 that the corresponding deformed algebras are strongly Morita equivalent. □

As a result, deformed algebras related by Φ have equivalent representation theory on pre-Hilbert spaces over $\mathbb{C}[[\lambda]]$ (Corollary 3.59).

Representations of star-product algebras on formal (pre-)Hilbert spaces were investigated in [8, 11, 70]; We believe Corollary 4.21 can be used to extend the results in these papers to the context of hamiltonians taking values on endomorphism bundles [27, 26].

Chapter 5

Morita equivalence of star products

5.1 An action of the Picard group

5.1.1 The algebraic picture

Let $\mathcal{A}$ be a commutative, unital k-algebra. We will see that there is a canonical action of $\mathsf{Pic}_{\mathcal{A}}(\mathcal{A})$ on $\mathsf{Def}(\mathcal{A})$, whose orbits characterize Morita equivalent formal deformations.

Let $P_0 \in M_n(\mathcal{A})$ be a full projection. Let $\mathcal{E}_{\mathcal{A}} = P_0\mathcal{A}^n$ and $\mathcal{B} = P_0 M_n(\mathcal{A})P_0 = \mathsf{End}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}})$. By Proposition 4.11, there is a canonical bijection $\Phi_{\mathcal{E}} : \mathsf{Def}(\mathcal{A}) \longrightarrow \mathsf{Def}(\mathcal{B})$. Recall from Section 2.2.2 that a bimodule ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}}$ is an equivalence bimodule if and only if there exists a full projection $P_0 \in M_n(\mathcal{A})$ so that $\mathcal{E}_{\mathcal{A}} \cong P_0\mathcal{A}^n$, and $P_0 M_n(\mathcal{A})P_0$ is commutative, in which case the natural embedding $\Psi : \mathcal{A} \longrightarrow P_0 M_n(\mathcal{A})P_0$, $A \mapsto P_0 A P_0$ of $\mathcal{A}$ into the center of $P_0 M_n(\mathcal{A})P_0$ is an isomorphism. It is not hard to check that each $(\mathcal{A}, \mathcal{A})$-equivalence bimodule $\mathcal{E}$ defines an automorphism of the set $\mathsf{Def}(\mathcal{A})$,

$$\Phi_{\mathcal{E}} : \mathsf{Def}(\mathcal{A}) \longrightarrow \mathsf{Def}(\mathcal{A}). \tag{5.1}$$

We observe that the map $\Phi_{\mathcal{E}}$ only depends on the isomorphism class of $\mathcal{E}$. We will abuse notation and simply write $\mathcal{E}$ to denote its isomorphism class in $\mathsf{Pic}(\mathcal{A})$ or $\mathsf{Pic}_{\mathcal{A}}(\mathcal{A})$ (if $Ax = xA$ for all $x \in \mathcal{E}$, $A \in \mathcal{A}$).

Let $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star), \boldsymbol{\mathcal{A}'} = (\mathcal{A}[[\lambda]], \star')$ be formal deformations of $\mathcal{A}$.

Lemma 5.1. *$\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{A}'}$ are Morita equivalent if and only if there exists $\mathcal{E} \in \mathsf{Pic}_{\mathcal{A}}(\mathcal{A})$ and $\psi \in \mathsf{Aut}(\mathcal{A})$ with $\Phi_{\mathcal{E}}([\star]) = [\psi^*(\star')]$.*

Proof. If $\Phi_{\mathcal{E}}([\star]) = [\psi^*(\star')]$, then $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{A}}'$ are Morita equivalent by Propositions B.3 and 4.11. Conversely, if $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{A}}'$ are Morita equivalent, then there exists a full idempotent $\boldsymbol{P} \in M_n(\boldsymbol{\mathcal{A}})$ so that $\boldsymbol{\mathcal{A}}' \cong \boldsymbol{P} \star M_n(\boldsymbol{\mathcal{A}}) \star \boldsymbol{P}$. By Lemma 4.4, $\boldsymbol{P} = P_0 + O(\lambda)$ with P_0 full. We know that (see (4.2)) $\boldsymbol{P} \star M_n(\boldsymbol{\mathcal{A}}) \star \boldsymbol{P}$ is isomorphic to $P_0 M_n(\mathcal{A}) P_0[[\lambda]]$ as a $k[[\lambda]]$-module, and since it is also isomorphic to $\mathcal{A}[[\lambda]]$, we must have $P_0 M_n(\mathcal{A}) P_0 \cong \mathcal{A}$. As in Remark 2.25, $\mathcal{E} = P_0 \mathcal{A}^n$ is an $(\mathcal{A}, \mathcal{A})$-equivalence bimodule satisfying $xA = Ax$, for all $x \in \mathcal{E}$ and $A \in \mathcal{A}$, and $\boldsymbol{\mathcal{A}}'$ is isomorphic to the deformations in the class $\Phi_{\mathcal{E}}([\star])$. The result then follows from Proposition B.3. □

We recall that the unit element in $\mathsf{Pic}_{\mathcal{A}}(\mathcal{A})$ is given by (the isomorphism class of) ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$.

Lemma 5.2. *If* ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}} \cong {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$, *then* $\Phi_{\mathcal{E}} = \mathsf{id}$.

Proof. Let $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$ be a formal deformation of $\mathcal{A}$. Then $\boldsymbol{\mathcal{A}}$ itself, regarded as a right module over $\boldsymbol{\mathcal{A}}$, provides a deformation of $\mathcal{E}_{\mathcal{A}}$. Since $\mathsf{End}_{\boldsymbol{\mathcal{A}}}(\boldsymbol{\mathcal{A}}) = \boldsymbol{\mathcal{A}}$, it follows that $\Phi_{\mathcal{E}}([\star]) = [\star]$. □

Lemma 5.3. *Let* $\mathcal{E}$, $\mathcal{E}' \in \mathsf{Pic}_{\mathcal{A}}(\mathcal{A})$, *and* $\mathcal{E}'' = \mathcal{E}' \otimes_{\mathcal{A}} \mathcal{E}$. *Then* $\Phi_{\mathcal{E}''} = \Phi_{\mathcal{E}'} \circ \Phi_{\mathcal{E}}$.

Proof. Let $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$, $\boldsymbol{\mathcal{A}}' = (\mathcal{A}[[\lambda]], \star')$, and $\boldsymbol{\mathcal{A}}'' = (\mathcal{A}[[\lambda]], \star'')$ be formal deformations of $\mathcal{A}$ so that $[\star'] = \Phi_{\mathcal{E}}([\star])$ and $[\star''] = \Phi_{\mathcal{E}'}([\star'])$. Let $\boldsymbol{\mathcal{E}}$ be a deformation of $\mathcal{E}$ corresponding to $\star$, and $\boldsymbol{\mathcal{E}}'$ be a deformation corresponding to $\star'$. We know that $\boldsymbol{\mathcal{A}}' \cong \mathsf{End}_{\boldsymbol{\mathcal{A}}}(\boldsymbol{\mathcal{E}})$ and $\boldsymbol{\mathcal{A}}'' \cong \mathsf{End}_{\boldsymbol{\mathcal{A}}'}(\boldsymbol{\mathcal{E}}')$. As discussed in Section 2.2, $\boldsymbol{\mathcal{E}}' \otimes_{\boldsymbol{\mathcal{A}}'} \boldsymbol{\mathcal{E}}$ is an $(\boldsymbol{\mathcal{A}}'', \boldsymbol{\mathcal{A}}')$-equivalence bimodule, so $\boldsymbol{\mathcal{A}}'' \cong \mathsf{End}_{\boldsymbol{\mathcal{A}}}(\boldsymbol{\mathcal{E}}' \otimes_{\boldsymbol{\mathcal{A}}'} \boldsymbol{\mathcal{E}})$. Since $\boldsymbol{\mathcal{E}}' \otimes_{\boldsymbol{\mathcal{A}}'} \boldsymbol{\mathcal{E}}$ is a f.g.p.m. over $\boldsymbol{\mathcal{A}}$, it follows (see (4.2)) that it is of the form $V[[\lambda]]$, where $V \cong \boldsymbol{\mathcal{E}}' \otimes_{\boldsymbol{\mathcal{A}}'} \boldsymbol{\mathcal{E}} / (\lambda \boldsymbol{\mathcal{E}}' \otimes_{\boldsymbol{\mathcal{A}}'} \boldsymbol{\mathcal{E}})$ as a k-module. But

$$\boldsymbol{\mathcal{E}}' \otimes_{\boldsymbol{\mathcal{A}}'} \boldsymbol{\mathcal{E}} / (\lambda \boldsymbol{\mathcal{E}}' \otimes_{\boldsymbol{\mathcal{A}}'} \boldsymbol{\mathcal{E}}) \cong \mathcal{E}' \otimes_{\mathcal{A}} \mathcal{E}.$$

Hence $\boldsymbol{\mathcal{E}}' \otimes_{\boldsymbol{\mathcal{A}}'} \boldsymbol{\mathcal{E}}$ is a deformation of $\mathcal{E}' \otimes_{\mathcal{A}} \mathcal{E}$, and the conclusion follows. □

The following lemma is a simple corollary of Theorem 4.17.

Lemma 5.4. *Let* $\star = \sum_{r=0}^{\infty} C_r \lambda^r$ *and* $\star' = \sum_{r=0}^{\infty} C'_r \lambda^r$ *be formal deformations of* $\mathcal{A}$ *such that* $[\star'] = \Phi_{\mathcal{E}}([\star])$, *for some* $\mathcal{E} \in \mathsf{Pic}_{\mathcal{A}}(\mathcal{A})$. *Then* $\star$ *and* $\star'$ *correspond to the same Poisson bracket, i.e.*

$$C_1(A_1, A_2) - C_1(A_2, A_1) = C'_1(A_1, A_2) - C'_1(A_2, A_1), \quad \textit{for all } A_1, A_2 \in \mathcal{A}.$$

As a result, we can state the following

Theorem 5.5. *Let $\mathcal{A}$ be a commutative, unital k-algebra. Then $\Phi : \mathsf{Pic}_{\mathcal{A}}(\mathcal{A}) \to \mathsf{Aut}(\mathsf{Def}(\mathcal{A}))$, $\mathcal{E} \mapsto \Phi_{\mathcal{E}}$, defines an action of $\mathsf{Pic}_{\mathcal{A}}(\mathcal{A})$ on the set $\mathsf{Def}(\mathcal{A})$, preserving Poisson brackets. Moreover, two formal deformations of $\mathcal{A}$, $\star$ and $\star'$, are Morita equivalent if and only if there exists $\psi \in \mathsf{Aut}(\mathcal{A})$ such that $[\star]$ and $[\psi^*(\star')]$ lie in the same orbit of Φ.*

5.1.2 Twisting star products by line bundles

Henceforth $\mathcal{A} = C^\infty(M)$, and we will restrict ourselves to differential deformations of $\mathcal{A}$. In this case, $\mathsf{Pic}_{\mathcal{A}}(\mathcal{A})$ can be naturally identified with $\mathsf{Pic}(M)$, the set of isomorphism classes of complex line bundles over M, with group operation given by fiberwise tensor product: for a line bundle $L \to M$, set $\mathcal{E} = \Gamma^\infty(L) \in \mathsf{Pic}_{\mathcal{A}}(\mathcal{A})$.

The following result is a consequence of Theorems 4.20 and 5.5.

Theorem 5.6. *Let (M, π) be a Poisson manifold. There is a canonical action*

$$\Phi : \mathsf{Pic}(M) \times \mathsf{Def}_{\mathit{diff}}(M, \pi) \longrightarrow \mathsf{Def}_{\mathit{diff}}(M, \pi),$$

and two star products $\star$ and $\star'$ on (M, π) are Morita equivalent if and only if there exists a Poisson diffeomorphism $\psi : M \longrightarrow M$ such that $[\star]$ and $[\psi^*(\star')]$ lie in the same orbit of Φ.

Let Π be the set of Poisson structures on M. It is clear that $\mathsf{Def}_{\mathrm{diff}}(M)$ can be decomposed into a disjoint union

$$\mathsf{Def}_{\mathrm{diff}}(M) = \bigcup_{\pi \in \Pi} \mathsf{Def}_{\mathrm{diff}}(M, \pi). \tag{5.2}$$

We will discuss in Section 5.3 the action Φ restricted to the invariant subsets $\mathsf{Def}_{\mathrm{diff}}(M, \pi)$.

It was shown in [5] that two unital C^*-algebras are strongly Morita equivalent if and only if they are Morita equivalent in the classical sense, as unital rings. A similar result holds for hermitian star-product algebras:

Proposition 5.7. *The *-algebras $\boldsymbol{\mathcal{A}} = (C^\infty(M)[[\lambda]], \star)$ and $\boldsymbol{\mathcal{A}}' = (C^\infty(M)[[\lambda]], \star')$, where $\star, \star'$ are hermitian star products on M, are strongly Morita equivalent if and only if they are Morita equivalent as unital rings.*

Proof. It was shown in Proposition 3.51 that strong Morita equivalence implies Morita equivalence. Let us prove the converse. If $\star$ and $\star'$ are Morita equivalent, then $\boldsymbol{\mathcal{A}}' \cong \mathsf{End}_{\boldsymbol{\mathcal{A}}}(\boldsymbol{P} \star \boldsymbol{\mathcal{A}}^n)$, for some n and idempotent $\boldsymbol{P} \in M_n(\boldsymbol{\mathcal{A}})$. Since $\mathsf{id} + \boldsymbol{M}^* \star \boldsymbol{M}$ is invertible in

$M_n(\mathcal{A})$ for all $\boldsymbol{M} \in M_n(\mathcal{A})$, it follows from Lemma 4.14 that $\boldsymbol{P}$ is equivalent to a projection. So we may assume that $\boldsymbol{P}^* = \boldsymbol{P}$. Hence $\boldsymbol{P} \star M_n(\mathcal{A}) \star \boldsymbol{P} \cong \mathsf{End}_{\mathcal{A}}(\boldsymbol{P} \star \mathcal{A}^n)$ has a natural involution and induces a hermitian star product $\star''$ on M, strongly Morita equivalent to $\star$, for $\boldsymbol{P}$ is strongly full. Since $\star'$ and $\star''$ are equivalent hermitian star products, the algebras $\mathcal{A}'$ and $\mathcal{A}''$ are *-isomorphic (Proposition 2.11), and hence $\mathcal{A}'$ and $\mathcal{A}$ are strongly Morita equivalent. □

Therefore we will focus on classical Morita equivalence of star products in the remainder of this chapter.

We saw that Φ maps equivalence classes containing hermitian star products to equivalence classes containing hermitian star products; it immediately follows that two hermitian star products are strongly Morita equivalent if and only if they lie in the same orbit of Φ. The goal of the next sections is to understand this action and the moduli space $\mathsf{Def}_{\mathrm{diff}}(M,\pi)/\mathsf{Pic}(M)$.

5.2 Semiclassical geometry of quantum line bundles

Let (M,π) be a Poisson manifold and $\star = \sum_{r=0}^{\infty} C_r \lambda^r$ be a star product on M satisfying

$$\frac{1}{\mathrm{i}}(C_1(f,g) - C_1(g,f)) = \pi(df, dg), \;\; f, g \in C^\infty(M).$$

Let $L \to M$ be a complex line bundle over M, and let $\mathcal{E} = \Gamma^\infty(L)$. Let us fix a deformation quantization of L with respect to $\star$, $\boldsymbol{\mathcal{E}} = (\mathcal{E}[[\lambda]], \bullet)$, and pick $\star' \in \Phi_{\mathcal{E}}([\star])$. Since $\mathcal{A}' = (C^\infty(M)[[\lambda]], \star') \cong \mathsf{End}_{\mathcal{A}}(\boldsymbol{\mathcal{E}})$, there is a natural left action of $\mathcal{A}'$ on $\boldsymbol{\mathcal{E}}$ that can be written

$$f \bullet' s = f \cdot s + \sum_{r=1}^{\infty} R'_r(f,s)\lambda^r,$$

for bilinear maps $R'_r : C^\infty(M) \times \mathcal{E} \longrightarrow \mathcal{E}$. It is clear that $\bullet'$, $\bullet$ make $\mathcal{E}[[\lambda]]$ into an $(\mathcal{A}', \mathcal{A})$-bimodule.

Definition 5.8. *Let $L \to M$ be a complex line bundle over a Poisson manifold M. Fix $\star = \sum_r C_r \lambda^r$ on M, and $\star' = \sum_r C'_r \lambda^r \in \Phi_{\mathcal{E}}([\star])$. The triple $(\mathcal{L}[[\lambda]], \bullet, \bullet')$ is called a bimodule quantization of L corresponding to $\star$, $\star'$.*

The following equations relate $\star$, $\star'$, $\bullet$ and $\bullet'$.

$$(f \star' g) \bullet s = f \bullet' (g \bullet' s), \tag{5.3}$$

$$s \bullet (f \star g) = (s \bullet f) \bullet g, \tag{5.4}$$

$$(f \bullet' s) \bullet g = f \bullet' (s \bullet g). \tag{5.5}$$

Let $R : \mathcal{E} \times C^\infty(M) \longrightarrow \mathcal{E}$ be defined by

$$R(s, f) := \frac{1}{\mathrm{i}}(R_1(s, f) - R'_1(f, s)). \tag{5.6}$$

Since $[\star]$ and $\Phi_{\mathcal{E}}([\star])$ correspond to the Poisson bracket $\{\cdot, \cdot\}$ on M, we may assume $C_1 = C'_1$ in Definition 5.8 (see Lemma 5.4).

Proposition 5.9. *R is a contravariant connection on L.*

Proof. We must show that R satisfies (A.8), (A.9) in Appendix A. Note that (5.4) yields, in first order,

$$R'_1(fg, s) + C'_1(f, g)s = R'_1(f, g.s) + fR'_1(g, s). \tag{5.7}$$

Similarly, (5.5) yields

$$R_1(s, fg) + sC_1(f, g) = R_1(sf, g) + R_1(s, g)f. \tag{5.8}$$

We finally note that (5.5) implies that

$$R_1(fs, g) + R'_1(f, s)g = R'_1(f, sg) + fR_1(s, g). \tag{5.9}$$

The difference of equations (5.7) and (5.8) yields

$$\mathrm{i}R(fg, s) = R_1(sf, g) + R_1(s, f)g - R'_1(f, gs) - fR'_1(g, s).$$

But, by (5.9), $R_1(sf, g) = R'_1(f, sg) + fR_1(s, g) - R'_1(f, s)g$. This implies that

$$R(s, fg) = fR(s, g) + gR(s, f),$$

proving that (A.8) is satisfied. Now, switching f and g in (5.7), and subtracting it from (5.8) (assuming $C_1 = C'_1$), we get

$$R(sf, g) = \{f, g\}s + R(s, fg) - gR(s, f) = \{f, g\}s + fR(s, g),$$

proving (A.9). □

We observe that given $\star$ on M, the contravariant connection R on L depends on the choice of $\star'$, $\bullet$ and $\bullet'$. As an example, let us compute it in a concrete situation.

Example 5.10. *Fix n, and let $t(\mathbb{C}^n) = M \times \mathbb{C}^n \to M$ be a trivial bundle. Let $P_0 \in M_n(C^\infty(M))$ be a rank-one idempotent so that $L = P_0 t(\mathbb{C}^n)$ is a line bundle over M. For a fixed star product $\star$ on M, we pick an idempotent $\boldsymbol{P} = P_0 + O(\lambda) \in M_n(\boldsymbol{\mathcal{A}})$. Using I in (4.2) to establish $\mathbb{C}[[\lambda]]$-module isomorphisms $P_0\mathcal{A}^n[[\lambda]] \longrightarrow \boldsymbol{P}\star\boldsymbol{\mathcal{A}}^n$, and $P_0 M_n(\mathcal{A})P_0[[\lambda]] \longrightarrow \boldsymbol{P}\star M_n(\boldsymbol{\mathcal{A}})\star\boldsymbol{P}$, an explicit computation (in the spirit of Proposition 4.16) shows that $R_1(s,f) = P_0C_1(s,f)$ and $R_1'(f,s) = P_0C_1(f,s)$, where $C_1(f,s)_i = C_1(s_i,f)$ and $C_1(s,f)_i = C_1(f,s_i)$, $i = 1\ldots n$. Thus*

$$R(s,f) = P_0\{s,f\} = \nabla_{X_f}s,$$

where $\nabla = P_0 d$ is the adapted connection on L, and X_f the hamiltonian vector field of f.

Corollary 5.11. *Let D be a contravariant connection on L induced by a covariant connection ∇ ($D_{df}s = \nabla_{X_f}s$). Fix $\star$ on M. Then we can choose a bimodule quantization of L so that $R = D$.*

Proof. It follows from [57, Thm. 1.1] that any covariant connection on L can be realized as the pull-back of an adapted connection $P_0 d$, for some embedding $L \hookrightarrow t(\mathbb{C}^n)$ of L into a trivial bundle. The result follows from the computation in Example 5.10. □

5.3 The semiclassical limit of Morita equivalent star products

5.3.1 Semiclassical curvature

Let (M,π) be a Poisson manifold, and suppose $\star = \sum_{r=0}^{\infty} C_r\lambda^r$ and $\star' = \sum_{r=0}^{\infty} C_r'\lambda^r$ are star products on M, with $C_1 = C_1' = \frac{\mathrm{i}}{2}\{\cdot,\cdot\}$. We can associate a Poisson cohomology class to the pair $[\star], [\star']$, measuring the obstruction for these star products being equivalent modulo λ^3 [6, Prop. 3.1].

Lemma 5.12. *Suppose $\star$ and $\star'$ are star products with $C_1 = C_1' = \frac{\mathrm{i}}{2}\{\cdot,\cdot\}$. The map*

$$(df, dg) \mapsto (C_2 - C_2')(f,g) - (C_2 - C_2')(g,f)$$

defines a d_π-closed bivector field $\tau \in \Gamma^\infty(\bigwedge^2 TM)$. Moreover, the class $[\tau]_\pi \in H^2_\pi(M)$ depends only on the equivalence classes of $\star$ and $\star'$.

Proof. The fact that τ is a closed bivector field was proven in [6, Prop. 3.1].

Suppose $\hat{\star}$ is a star product equivalent to $\star$:

$$f \hat{\star} g = \sum_{r=0}^{\infty} \widehat{C}_r \lambda^r = \boldsymbol{T}^{-1}(\boldsymbol{T}(f) \star \boldsymbol{T}(g)),$$

where $\boldsymbol{T} = \mathsf{id} + \sum_{r=1}^{\infty} T_r \lambda^r$ is an equivalence transformation. Assume $\widehat{C}_1 = C_1 = \frac{\mathrm{i}}{2}\{\cdot,\cdot\}$. We must show that, if $\widehat{\tau}$ is the closed bivector given by the skew symmetric part of $(\widehat{C}_2 - C_2')$, then $[\widehat{\tau}]_\pi = [\tau]_\pi$. The condition $\widehat{C}_1 = C_1$ implies that $T_1 \in \mathsf{Der}(C^\infty(M))$. Hence $T_1 = \mathcal{L}_X$, for some vector field $X \in \chi(M)$. A simple computation just using the definitions shows that

$$\widehat{\tau} = \tau - d_\pi X,$$

and the result follows. □

If M is a symplectic manifold, then the bivector τ defines a closed 2-form $\widetilde{\tau}$ by

$$\widetilde{\tau}(X_f, X_g) = \tau(df, dg), \tag{5.10}$$

where X_f and X_g are the hamiltonian vector fields of f and g, respectively. The de Rham class $[\widetilde{\tau}]$ is the one corresponding to $[\tau]_\pi$ under the natural isomorphism between de Rham and Poisson cohomologies (see (A.7)). We can state

Lemma 5.13. *Let M be a symplectic manifold, and let $\star, \star'$ be star products with $C_1 = C_1' = \frac{\mathrm{i}}{2}\{\cdot,\cdot\}$. Then $[\widetilde{\tau}] \in H^2_{dR}(M)$ depends only on $[\star], [\star']$.*

Suppose now $\star$ and $\star'$ satisfy $\star' \in \Phi_{\mathcal{E}}([\star])$, where $\mathcal{E} = \Gamma^\infty(L)$ for a line bundle $L \to M$. Let $(\mathcal{E}[[\lambda]], \bullet, \bullet')$ be a bimodule quantization of L corresponding to $\star, \star'$. Let $R = \frac{1}{\mathrm{i}}(R_1 - R_1')$ be the contravariant connection on L defined by $\bullet, \bullet'$, and let Θ_R denote its curvature.

Theorem 5.14. *$\tau(f,g)s = \Theta_R(df, dg)s$, for all $f, g \in C^\infty(M)$, $s \in \mathcal{E}$.*

Proof. From (5.4) we get, in second order,

$$R_2'(fg, s) + R_1'(C_1'(f,g), s) + C_2'(f,g)s = R_2'(f, gs) + R_1'(f, R_1'(g,s)) + fR_2'(g,s). \tag{5.11}$$

Similarly, from (5.5) we get

$$R_2(s, fg) + R_1(s, C_1(f,g)) + sC_2(f,g) = R_2(sf, g) + R_1(R_1(s,f), g) + R_2(s,f)g. \tag{5.12}$$

Finally, from (5.5) we have

$$R_2(fs,g)+R_1(R'_1(f,s),g)+R'_2(f,s)g=R'_2(f,sg)+R'_1(f,R_1(s,g))+fR_2(s,g). \quad (5.13)$$

Since we assume that $C_1=C'_1$, subtracting (5.11) from (5.12) yields

$$\begin{aligned} \mathrm{i}R(s,C_1(f,g))-\mathrm{i}R(\mathrm{i}R(s,f),g)+\mathrm{i}R(\mathrm{i}R(s,g),f) & \\ +(C_2-C'_2)(f,g)s &= R'_1(g,R_1(s,f))-R'_1(R'_1(g,s),f) \\ &\quad +R_2(sf,g)-R'_2(f,gs) \\ &\quad +R_2(s,f)g+\mathrm{i}R(R_1(s,g),f)) \\ &\quad +\mathrm{i}R(R'_1(s,f),g)-fR'_2(g,s) \\ &\quad +R'_2(fg,s)-R_2(s,fg). \end{aligned}$$

Using (5.13), we then get

$$\begin{aligned} \mathrm{i}R(s,C_1(f,g))-\mathrm{i}R(\mathrm{i}R(s,f),g)+\mathrm{i}R(\mathrm{i}R(s,g),f) & \\ +(C_2-C'_2)(f,g)s &= R(R'_1(f,s),g)+R(R_1(s,g),f) \\ &\quad +R'_2(fg,s)-R_2(s,fg) \\ &\quad +R_2(sf,g)+R_2(gs,f) \\ &\quad -R'_2(f,gs)-R'_2(g,sf). \end{aligned}$$

Taking the skew-symmetric part of this equation, and recalling that $\mathrm{i}\{f,g\}=C_1(f,g)-C_1(g,f)$, we finally have

$$\tau(f,g)s=R(s,\{f,g\})-R(R(s,f),g)+R(R(s,g),f).$$

□

Consider the natural map

$$i: H^2(M,\mathbb{Z}) \longrightarrow H^2_{dR}(M). \quad (5.14)$$

We denote $H^2_{dR}(M,\mathbb{Z}):=i(H^2(M,\mathbb{Z}))$.

Corollary 5.15. *Let $\star$ and $\star'$ be Morita equivalent star products on M, with $\star'\in\Phi_{\mathcal{E}}([\star])$, $\mathcal{E}=\Gamma^\infty(L)$. Then $\frac{\mathrm{i}}{2\pi}[\tau]_\pi=c_1^\pi(L)\in H^2_\pi(M,\mathbb{Z})$, where $c_1^\pi(L)=\pi^*c_1(L)$ is the Poisson-Chern class of L. In particular, if M is symplectic, $\frac{\mathrm{i}}{2\pi}[\tilde\tau]=c_1(L)\in H^2_{dR}(M,\mathbb{Z})$.*

In the next two subsections, we will interpret these results in terms of the characteristic classes of star products.

5.3.2 The symplectic case

Let (M, ω) be a symplectic manifold. Recall from Section 2.1.2 that there is a bijection

$$\mathrm{c} : \mathsf{Def}_{\mathrm{diff}}(M, \omega) \longrightarrow [\omega] \oplus \lambda H^2_{dR}(M)[[\lambda]]. \tag{5.15}$$

The class $\mathrm{c}(\star)$ is the characteristic class of $\star$. Let $\star, \star'$ be star products on M, and let $\widetilde{\tau}$ be the closed 2-form introduced in (5.10). The following result was proven in [37, Prop. 6.2].

Lemma 5.16. $[\widetilde{\tau}] = \frac{1}{\mathrm{i}\lambda}(c(\star) - c(\star'))$ *mod* λ.

We want to study the action $\Phi : \mathsf{Pic}(M) \times \mathsf{Def}_{\mathrm{diff}}(M, \omega) \longrightarrow \mathsf{Def}_{\mathrm{diff}}(M, \omega)$. Let $S : H^2_{dR}(M)[[\lambda]] \longrightarrow H^2_{dR}(M)$ be the semiclassical limit map, $S(\sum_{r=0}^{\infty}[\omega_r]\lambda^r) = [\omega_1]$. With the identification (5.15), we consider $S : \mathsf{Def}_{\mathrm{diff}}(M, \omega) \longrightarrow H^2_{dR}(M)$. Let $L \to M$ be a complex line bundle, and let $\mathcal{E} = \Gamma^\infty(L)$.

Theorem 5.17. *The following diagram commutes:*

$$\begin{array}{ccc} \mathsf{Def}_{\mathit{diff}}(M,\omega) & \xrightarrow{\Phi_{\mathcal{E}}} & \mathsf{Def}_{\mathit{diff}}(M,\omega) \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle S} \\ H^2_{dR}(M) & \xrightarrow{\widehat{\Phi}_{\mathcal{E}}} & H^2_{dR}(M), \end{array}$$

where $\widehat{\Phi}_{\mathcal{E}}([\alpha]) = [\alpha] - 2\pi c_1(L)$.

Proof. The proof follows directly from Lemma 5.16 and Corollary 5.15. □

Recall that $\mathsf{Pic}(M) \cong H^2(M, \mathbb{Z})$ and the kernel of i (see (5.14)) is given by the torsion elements in $H^2(M, \mathbb{Z})$. We then have the following

Corollary 5.18. *Let (M, ω) be a symplectic manifold, and suppose $H^2(M, \mathbb{Z})$ is free. Then the action $\Phi : H^2(M, \mathbb{Z}) \times \mathsf{Def}_{\mathit{diff}}(M, \omega) \longrightarrow \mathsf{Def}_{\mathit{diff}}(M, \omega)$ is faithful.*

Recall that for star products $\star, \star'$ on M, their relative Deligne class is defined by $t(\star, \star') = \mathrm{c}(\star) - \mathrm{c}(\star') \in \lambda H^2_{dR}(M)[[\lambda]]$. We write $t(\star, \star') = \lambda t_0(\star, \star') + O(\lambda^2)$. We have the following immediate consequence of Theorem 5.17 phrased in terms of relative classes.

Corollary 5.19. *If $\star, \star'$ are Morita equivalent star products on a symplectic manifold (M, ω), then there exists a symplectomorphism $\psi : M \longrightarrow M$ such that $\frac{\mathrm{i}}{2\pi} t_0(\star, \psi^*(\star')) \in H^2_{dR}(M, \mathbb{Z})$. Conversely, for any star product $\star$ on M and $[\alpha] \in H^2_{dR}(M, \mathbb{Z})$, there is a star product $\star'$ Morita equivalent to $\star$ such that $t(\star, \star') = \frac{2\pi}{\mathrm{i}}[\alpha]\lambda + O(\lambda^2)$.*

5.3.3 The Poisson case

Let us recall some results mentioned in Section 2.1. For an arbitrary Poisson manifold (M, π), Kontsevich constructed in [44] a bijection

$$c : \mathsf{Def}_{\mathrm{diff}}(M, \pi) \longrightarrow \{\pi_\lambda = \pi + \lambda\pi_1 + \ldots \in \chi^2(M)[[\lambda]], [\pi_\lambda, \pi_\lambda] = 0\}/F, \tag{5.16}$$

where F is the group $\{\exp(\sum_{r=1}^{\infty} D_r \lambda^r),\ D_r \in \mathsf{Der}(C^\infty(M))\}$, acting on formal Poisson structures by $\boldsymbol{T}(\pi_\lambda) = \pi'_\lambda$ if and only if $\pi'_\lambda(df, dg) = \boldsymbol{T}^{-1}\pi_\lambda(d(\boldsymbol{T}(f)), d(\boldsymbol{T}(g)))$, for $\boldsymbol{T} \in F$. As recalled in Section B.4, this correspondence is a result of a more general fact [44]: there exists an L_∞-quasi-isomorphism $\mathcal{U}$ from the graded Lie algebra of multivectors fields on M (with zero differential and Schouten bracket), $\mathfrak{g}_1$, into the graded Lie algebra of multidifferential operators on M (with Hochschild differential and Gerstenhaber bracket), $\mathfrak{g}_2$. Given such an $\mathcal{U}$, for every formal Poisson structure π_λ we can define a star product $\star_{\pi_\lambda}$ by

$$f \star_{\pi_\lambda} g := fg + \sum_{r=1}^{\infty} \frac{1}{r!} \mathcal{U}_r(\underbrace{\pi_\lambda \wedge \ldots \wedge \pi_\lambda}_{r})(f \otimes g)\lambda^r, \tag{5.17}$$

where $\mathcal{U}_r : \bigwedge^r \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ are the Taylor coefficients of $\mathcal{U}$. Moreover, Kontsevich showed that one can choose $\mathcal{U}_1 : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ to be the natural embedding of multivector fields into multidifferential operators.

If $\pi_\lambda = \pi + \lambda\pi_1 + \ldots$ is a formal Poisson structure on M, the integrability equation $[\pi_\lambda, \pi_\lambda] = 0$ immediately implies that $d_\pi \pi_1 = 0$.

Lemma 5.20. *If $\pi_\lambda = \pi + \lambda\pi_1 + \ldots$ and $\pi'_\lambda = \pi + \lambda\pi'_1 + \ldots$ are equivalent formal Poisson structures, then $[\pi_1]_\pi = [\pi'_1]_\pi$.*

Proof. Let $\boldsymbol{T} = \exp(\sum_{r=1}^{\infty} D_r\lambda^r) \in F$. A simple computation shows that if $\boldsymbol{T}(\pi_\lambda) = \pi'_\lambda$, then

$$\pi'_1 = \pi_1 - d_\pi X_1,$$

where $X_1 \in \chi(M)$ is defined by $\mathcal{L}_{X_1} = D_1$. Thus $[\pi_1]_\pi = [\pi'_1]_\pi$. □

With the identification given in (5.16), we define the semiclassical limit map $S : \mathsf{Def}_{\text{diff}}(M,\pi) \longrightarrow H^2_\pi(M)$ by

$$S([\pi_\lambda]) = [\pi_1]_\pi,$$

where $[\pi_\lambda]$ is the equivalence class of the formal Poisson structure $\pi_\lambda = \pi + \lambda\pi_1 + \ldots$.

Let $\star$ and $\star'$ be star products on (M,π), with $\mathrm{c}(\star) = [\pi + \lambda\pi_1 + \ldots]$ and $\mathrm{c}(\star') = [\pi + \lambda\pi_1' + \ldots]$. Let τ be as in Lemma 5.12.

Lemma 5.21. $[\tau]_\pi = \mathrm{i}([\pi_1]_\pi - [\pi_1']_\pi)$.

Proof. Since in our convention $C_1^{\text{skew}} = \frac{\mathrm{i}}{2}\{\cdot,\cdot\}$, we use Kontsevich's explicit construction for the formal Poisson structure $\frac{\mathrm{i}}{2}\pi_\lambda$. The expression of Kontsevich's star products in terms of the maps $\mathcal{U}_r$ is

$$\begin{aligned} f \star_{\pi_\lambda} g &= fg + \lambda\mathcal{U}_1(\frac{\mathrm{i}}{2}\pi_\lambda)(f\otimes g) + \frac{\lambda^2}{2}\mathcal{U}_2(\frac{\mathrm{i}}{2}\pi_\lambda \wedge \frac{\mathrm{i}}{2}\pi_\lambda)(f\otimes g) + \ldots \\ &= fg + \frac{\lambda\mathrm{i}}{2}\pi(df,dg) + \lambda^2(\frac{\mathrm{i}}{2}\pi_1(df,dg) - \frac{1}{8}\mathcal{U}_2(\pi\wedge\pi)(f\otimes g)) + \ldots \end{aligned}$$

Since $\star$ is equivalent to $\star_{\pi_\lambda}$, and $\star'$ is equivalent to $\star_{\pi_\lambda'}$, by Lemma 5.21 it suffices to compute τ for $\star_{\pi_\lambda}$ and $\star_{\pi_\lambda'}$. It is clear from the expression just above that $\tau = \mathrm{i}(\pi_1 - \pi_1')$. □

Let $L \to M$ be a complex line bundle, and $\mathcal{E} = \Gamma^\infty(L)$. The following result follows from Lemma 5.21 and Theorem 5.14.

Theorem 5.22. *The following diagram commutes:*

$$\begin{array}{ccc} \mathsf{Def}_{\mathit{diff}}(M,\pi) & \xrightarrow{\Phi_{\mathcal{E}}} & \mathsf{Def}_{\mathit{diff}}(M,\pi) \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle S} \\ H^2_\pi(M) & \xrightarrow{\widehat{\Phi}_{\mathcal{E}}} & H^2_\pi(M), \end{array}$$

where $\widehat{\Phi}_{\mathcal{E}}([\alpha]) = [\alpha] + 2\pi c_1^\pi(L)$.

Hence, for a star product $\star$ on (M,π), each element in $H^2_\pi(M,\mathbb{Z}) = \pi^* H^2_{dR}(M,\mathbb{Z})$ corresponds to a different equivalence class of star products Morita equivalent to $\star$. Theorem 5.22 shows that the semiclassical limit of Φ is trivial when π induces the trivial map in cohomology; the case $\pi = 0$ will be discussed in the next section.

A bivector field π_1 on a Poisson manifold (M,π) is called an *infinitesimal deformation* of π if $d_\pi\pi_1 = 0$. It is generally difficult to determine whether an infinitesimal deformation can be extended to a formal Poisson structure $\pi_\lambda = \pi + \lambda\pi_1 + \ldots$ (in other words, which infinitesimal deformations π_1 are such that $[\pi_1]_\pi$ lies in the image of S).

Corollary 5.23. *Suppose π_1 is an infinitesimal deformation that extends to a formal Poisson structure π_λ. Then the same holds for $\pi_1 + \alpha$ if $\frac{1}{2\pi}[\alpha]_\pi \in H^2_\pi(M, \mathbb{Z})$.*

5.3.4 Deformations of the zero Poisson structure

As mentioned in the previous section, Theorem 5.22 does not provide much information about the orbits of star products corresponding to the null Poisson structure. We will show that the picture, in this case, is analogous to Sections 5.3.2, 5.3.3, but in higher orders of λ.

Let (M, π) be a Poisson manifold, with $\pi = 0$. For simplicity, we will identify equivalence classes of star products on M with their characteristic classes as in (5.16). In order to understand the action of Φ on $\mathsf{Def}_{\text{diff}}(M, \pi)$, consider the disjoint union

$$\mathsf{Def}_{\text{diff}}(M, \pi) = \bigcup_{m \geq 1} \mathsf{Def}^m_{\text{diff}}(M, \pi) \cup [0], \tag{5.18}$$

where $[0]$ denotes the equivalence class of the trivial formal Poisson structure on M, and $\mathsf{Def}^m_{\text{diff}}(M, \pi)$ is the set of equivalence classes of formal Poisson structures of the form $\pi^m_\lambda = \lambda^m(\pi_m + \lambda\pi_{m+1} + O(\lambda^2))$, $\pi_m \neq 0$, $m \geq 1$. Note that we can decompose each $\mathsf{Def}^m_{\text{diff}}(M, \pi)$ into a disjoint union of sets $\mathsf{Def}^m_{\text{diff}}(M, \pi_m)$, given by equivalence classes of formal Poisson structures of the form $\lambda^m(\pi_m + O(\lambda))$ for a *fixed* Poisson structure $\pi_m \neq 0$. We can always choose a star product $\star = \sum_{r=0}^\infty C_r \lambda^r$ corresponding to a class in $\mathsf{Def}^m_{\text{diff}}(M, \pi_m)$ with $C_1 = C_2 = \ldots = C_m = 0$. It is easy to check that all the results in the previous sections of Chapter 5 hold for such star products, with a shift in order by λ^m. For instance, the same arguments as in Theorem 5.6 show that $\mathsf{Def}^m_{\text{diff}}(M, \pi_m)$ is invariant under Φ.

Corollary 5.24. *The trivial class $[0]$ is a fixed point for Φ.*

Let $S_m : \mathsf{Def}^m_{\text{diff}}(M, \pi_m) \longrightarrow H^2_{\pi_m}(M)$ be defined by $S_m(\lambda^m(\pi_m + \lambda\pi_{m+1} + O(\lambda^2))) = [\pi_{m+1}]_{\pi_m}$. Let $L \to M$ be a line bundle and $\mathcal{E} = \Gamma^\infty(L)$. Just as in Theorem 5.22, one can show

Theorem 5.25. *The following diagram commutes:*

$$\begin{array}{ccc}
\mathsf{Def}^m_{\mathit{diff}}(M, \pi_m) & \xrightarrow{\Phi_{\mathcal{E}}} & \mathsf{Def}^m_{\mathit{diff}}(M, \pi_m) \\
{\scriptstyle S_m}\downarrow & & \downarrow{\scriptstyle S_m} \\
H^2_{\pi_m}(M) & \xrightarrow{\widehat{\Phi}_{\mathcal{E}}} & H^2_{\pi_m}(M),
\end{array}$$

where $\widehat{\Phi}_{\mathcal{E}}([\alpha]) = [\alpha] + 2\pi(\pi^*_m c_1(L))$.

Hence, for a star product in $\mathsf{Def}^m_{\mathrm{diff}}(M,\pi_m)$, each element in $H^2_{\pi_m}(M,\mathbb{Z})$ corresponds to an equivalence class of star products Morita equivalent to it.

5.3.5 Final remarks

The discussion presented in Sections 5.3.2, 5.3.3, 5.3.4 gives an explicit description of the semiclassical limit of the action $\Phi : \mathsf{Def}_{\mathrm{diff}}(M) \times \mathsf{Pic}(M) \longrightarrow \mathsf{Def}_{\mathrm{diff}}(M)$ restricted to $\mathsf{Def}_{\mathrm{diff}}(M,\pi)$, for a Poisson structure π on M. As a result, for a fixed star product $\star$ with $[\star] \in \mathsf{Def}_{\mathrm{diff}}(M,\pi)$, we saw that different elements in the image of the group $\mathsf{Pic}(M) \cong H^2(M,\mathbb{Z})$ under the map $\pi^* \circ i$,[1]

$$H^2(M,\mathbb{Z}) \xrightarrow{\ i\ } H^2_{dR}(M) \xrightarrow{\ \pi^*\ } H^2_\pi(M),$$

correspond to different points of the Φ-orbit of $[\star]$ (and hence to different classes of star products Morita equivalent to $\star$). A natural question that arises is whether elements in the kernel of $\pi^* \circ i$ can act nontrivially on $\mathsf{Def}_{\mathrm{diff}}(M,\pi)$ under Φ; in particular, whether torsion elements in $\mathsf{Pic}(M)$ can act in a nontrivial way.

For star products on symplectic manifolds, the answer seems to be no; in fact, a different approach to this problem using Čech cohomology arguments (as in [22, 37]) shows that the semiclassical action gives the full picture in the symplectic case. These ideas have been carried out with Stefan Waldmann. In the case of arbitrary Poisson manifolds, the higher order terms of the action Φ are still unknown. This will be the subject of future work.

[1] For the zero Poisson structure, we should consider $\mathsf{Def}^m_{\mathrm{diff}}(M,\pi_m)$.

Appendix A

Poisson geometry

Details can be found in [17, 49, 68].

A.1 Poisson manifolds

Definition A.1. *A* Poisson manifold *is a smooth manifold M together with a Lie algebra bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$ satisfying the Leibniz rule:*

$$\{f, gh\} = \{f, g\}h + f\{g, h\}, \quad \textit{for all } f, g, h \in C^\infty(M). \tag{A.1}$$

We call $\{\cdot,\cdot\}$ a Poisson bracket *on M.*

Important examples of Poisson manifolds symplectic manifolds and dual of Lie algebras.

For a fixed $f \in C^\infty(M)$, by (A.1) $\{\cdot, f\}$ is a derivation on $C^\infty(M)$. Hence there is a unique vector field $X_f \in \chi(M)$ so that

$$\mathcal{L}_{X_f} = \{\cdot, f\}.$$

We call X_f the *hamiltonian vector field* of f.

The Leibniz rule (A.1) implies that there exists a bivector field $\pi \in \Gamma^\infty(\bigwedge^2 TM) = \chi^2(M)$ so that

$$\{f, g\} = \pi(df, dg). \tag{A.2}$$

The bivector field π is called the *Poisson tensor*. If π is nondegenerate, then it induces a symplectic form on M. For an arbitrary $\pi \in \chi^2(M)$, the bracket defined by (A.2) does not necessarily satisfy the Jacobi identity. In order to formulate the condition on

π corresponding to the Jacobi identity of $\{\cdot,\cdot\}$, we need the Schouten bracket (see [68] for details).

For vector fields $X, Y \in \chi(M)$, let $[X,Y] = \mathcal{L}_X Y$ denote their Lie bracket. We can extend this operation to multivector fields by the formula

$$[X_1 \wedge \ldots \wedge X_p, Y] = \sum_{i=1}^{p} (-1)^{i+1} X_1 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge X_p \wedge [X_i, Y], \tag{A.3}$$

where $X_i, Y \in \chi(M)$, and $\hat{X}_i$ means X_i is missing.

Theorem A.2. *There is a unique bilinear operation* $[\cdot,\cdot] : \chi^p(M) \times \chi^q(M) \longrightarrow \chi^{p+q-1}(M)$ *such that*

1. $[\cdot,\cdot]$ *is natural with respect to the restriction to open sets.*
2. $[f,g] = 0$*,* $[X,f] = \mathcal{L}_X f$ *and* $[X,Y] = \mathcal{L}_X Y$*, for* $f, g \in C^\infty(M)$ *and* $X, Y \in \chi(M)$*.*
3. *(A.3) holds.*

In addition, the bracket $[\cdot,\cdot]$ *satisfies*

4. $[X,Y] = -(-1)^{pq}[Y,X]$*, for* $X \in \chi^{p+1}(M), Y \in \chi^{q+1}(M)$*.*
5. $[X, Y \wedge Z] = [X,Y] \wedge Z + (-1)^{(q+1)(p+1)} Y \wedge [X,Z],\ X \in \chi^{p+1}(M), Y \in \chi^{q+1}(M)$*.*
6. $[X,[Y,Z]] + (-1)^{r(p+q)}[Z,[X,Y]] + (-1)^{p(q+r)}[Y,[Z,X]] = 0$*, for* $X \in \chi^{p+1}(M), Y \in \chi^{q+1}(M), Z \in \chi^{r+1}(M)$*.*

If we define the degree of elements in $\chi^{p+1}(M)$ to be p, then $\chi^\bullet(M)$ is a *differential graded Lie algebra* with respect to the Schouten bracket and with differential $d = 0$ (see Appendix B).

One can show [68] that a bivector $\pi \in \chi^2(M)$ defines a Poisson structure if and only if $[\pi,\pi] = 0$.

A bivector field π defines a bundle morphism

$$\tilde{\pi} : T^*M \longrightarrow TM, \quad \alpha \mapsto \pi(\cdot,\alpha). \tag{A.4}$$

When π is Poisson, the image of $\tilde{\pi}$ defines a singular integrable distribution on M, whose leaves carry a natural symplectic structure. This singular foliation is called the *symplectic foliation* of M.

The bundle map $\tilde{\pi}$ defines a map on sections $\tilde{\pi} : \Omega^1(M) \longrightarrow \chi(M)$ in such a way that $\tilde{\pi}(df) = X_f$. We can use $\tilde{\pi}$ to define a Lie algebra bracket on $\Omega^1(M)$:

$$[\alpha, \beta] = -\mathcal{L}_{\tilde{\pi}(\alpha)}\beta + \mathcal{L}_{\tilde{\pi}(\beta)}\alpha - d(\pi(\alpha, \beta)), \quad \alpha, \beta \in \Omega^1(M). \tag{A.5}$$

The map $-\tilde{\pi} : \Omega^1(M) \longrightarrow \chi(M)$ is a Lie algebra homomorphism, and this makes T^*M into a *Lie algebroid* (see [17, Chp. 16]).

A.2 Poisson cohomology

Let (M, π) be a Poisson manifold. The Poisson tensor $\pi \in \chi^2(M)$ can be used to define a differential

$$d_\pi : \chi^k(M) \longrightarrow \chi^{k+1}(M), \quad d_\pi = [\pi, \cdot], \tag{A.6}$$

where $[\cdot, \cdot]$ is the Schouten bracket ($d_\pi^2 = 0$ follows from $[\pi, \pi] = 0$ and the graded Jacobi identity) .

Definition A.3. *The cohomology groups of the complex $(\chi^\bullet, d_\pi)$ are called the Poisson cohomology groups of M and denoted $H_\pi^k(M)$.*

The map $\tilde{\pi}$ in (A.4) induces a map $\pi^* : \Omega^{\bullet (M)} \longrightarrow \chi^\bullet(M)$ intertwining differentials, and therefore gives rise to a morphism

$$\pi^* : H_{dR}^k(M) \longrightarrow H_\pi^k(M), \tag{A.7}$$

which is an isomorphism when π is symplectic. We define integral (resp. real) Poisson cohomology as the image of integral (resp. real) de Rham cohomology classes on M under π^*, i.e., $H_\pi^k(M, \mathbb{Z}) = \pi^* H_{dR}^k(M, \mathbb{Z})$ (resp. $H_\pi^k(M, \mathbb{R}) = \pi^* H^k(M, \mathbb{R})$).

A.3 Contravariant connections and Poisson-Chern classes

The key ingredient in defining contravariant connections on vector bundles over Poisson manifolds is to think of T^*M as a “new” tangent bundle to M, using its Lie algebroid structure. Contravariant connections were introduced in [67] in the study of geometric quantization of Poisson manifolds; a thorough treatment of the subject can be found in [30].

Let $E \to M$ be a complex vector bundle over a Poisson manifold (M, π).

Definition A.4. *A* contravariant connection *on E is a $\mathbb{C}$-linear map* $D : \Gamma^\infty(E)\otimes\Omega^1(M) \longrightarrow \Gamma^\infty(E)$ *so that*

i.) $D_{f\alpha}s = fD_\alpha s$

ii.) $D_\alpha(fs) = fD_\alpha s + \alpha(X_f)s$,

for $\alpha \in \Omega^1(M)$, $f \in C^\infty(M)$. *The* curvature *of a contravariant connection D is* $\Theta_D : \Omega^1(M) \otimes \Omega^1(M) \longrightarrow \mathsf{End}(\Gamma^\infty(E))$,

$$\Theta_D(\alpha, \beta)s = D_\alpha D_\beta s - D_\beta D_\alpha s + D_{[\alpha,\beta]}s.$$

It is easy to see that, if ∇ is any connection (in the usual sense) on E, then it induces a contravariant connection by $D_{df} = \nabla_{X_f}$. On symplectic manifolds this is the only way contravariant connections can arise. Thus this notion is mostly important in degenerate situations.

A bilinear map $D' : \Gamma^\infty(E) \times C^\infty(M) \longrightarrow \Gamma^\infty(E)$, satisfying

$$D'(s, f \cdot g) = D'(s, f)g + D'(s, g)f, \tag{A.8}$$

$$D'(s \cdot f, g) = D'(s, g)f + s\{f, g\}, \tag{A.9}$$

provides an equivalent definition of a contravariant connection. The definitions are related by the formula

$$D'(s, f) = D_{df}s.$$

If $E = L \to M$ is a line bundle, then the curvature Θ_D of a contravariant connection defines a bivector field on M, closed with respect to d_π [67]. As in the case of usual connections, its Poisson cohomology is a well-defined class, independent of the connection.

Definition A.5. *let D be a contravariant connection on a line bundle $L \to M$, and let Θ_D be its curvature. We call the class* $\mathrm{c}_1^\pi(L) := \frac{\mathrm{i}}{2\pi}[\Theta_D]_\pi \in H^2_\pi(M)$ *the Poisson-Chern class of L.*

It is clear that $\mathrm{c}_1^\pi(L) = \pi^*(\mathrm{c}_1(L))$.

Appendix B

Deformation theory

B.1 Deformations of associative algebras

Let k be a commutative, unital ring of characteristic zero, and let $\mathcal{A}$ be an associative k-algebra.

Definition B.1. *A* formal deformation *of $\mathcal{A}$ is an associative $k[[\lambda]]$-bilinear multiplication $\star$ on $\mathcal{A}[[\lambda]]$ of the form*

$$A \star A' = \sum_{r=0}^{\infty} C_r(A, A')\lambda^r, \qquad A, A' \in \mathcal{A}, \tag{B.1}$$

where the maps $C_r : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ are bilinear,[1] *and C_0 is the original product on $\mathcal{A}$.*

A formal deformation of $\mathcal{A}$ will be denoted by $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$.

Definition B.2. *Two formal deformations of $\mathcal{A}$, $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$ and $\boldsymbol{\mathcal{A}}' = (\mathcal{A}[[\lambda]], \star')$, are called* equivalent *if there exist k-linear maps $T_r : \mathcal{A} \longrightarrow \mathcal{A}$, $r \geq 1$, so that $\boldsymbol{T} = \mathsf{id} + \sum_{r=1}^{\infty} T_r \lambda^r : \boldsymbol{\mathcal{A}} \longrightarrow \boldsymbol{\mathcal{A}}'$ satisfies*

$$A_1 \star A_2 = \boldsymbol{T}^{-1}(\boldsymbol{T}(A_1) \star' \boldsymbol{T}(A_2)), \qquad \forall A_1, A_2 \in \mathcal{A}[[\lambda]]. \tag{B.2}$$

Such a $\boldsymbol{T}$ is called an equivalence transformation.

It is shown in [33] that if $\mathcal{A}$ is unital, then so is any formal deformation $\boldsymbol{\mathcal{A}}$; moreover, any formal deformation of $\mathcal{A}$ is equivalent to one for which the unit is the same as for $\mathcal{A}$.

[1] We extend $\star$ to $\mathcal{A}[[\lambda]]$ using λ-linearity.

The group of automorphisms of $\mathcal{A}$, $\mathsf{Aut}(\mathcal{A})$, acts on formal deformations by $\star' = \psi^*(\star)$ if and only if $A \star' A' = \psi^{-1}(\psi(A) \star \psi(A'))$, $A, A' \in \mathcal{A}$. Since any $k[[\lambda]]$-algebra isomorphism $S : (\mathcal{A}[[\lambda]], \star) \longrightarrow (\mathcal{A}[[\lambda]], \star')$ is of the form $S = S_0 + \sum_{r=1}^{\infty} \lambda^r S_r$, with k-linear $S_r : \mathcal{A} \longrightarrow \mathcal{A}$, and $S_0 \in \mathsf{Aut}(\mathcal{A})$, a simple computation shows

Proposition B.3. *Let $\star$ and $\star'$ be formal deformations of $\mathcal{A}$. Then they are isomorphic if and only if there exists $\psi \in \mathsf{Aut}(\mathcal{A})$ with $[\psi^*(\star')] = [\star]$.*

Let $\boldsymbol{V}$ be a module over $k[[\lambda]]$. Consider the sequence of submodules $I_n = \lambda^n \boldsymbol{V} \subseteq \boldsymbol{V}$. We define the λ-*adic topology* of $\boldsymbol{V}$ by taking the sequence I_n to be a base for the open neighborhoods of 0, and imposing the condition that addition should be continuous (so that $v + I_n$, with $v \in \boldsymbol{V}$, form a basis for all open sets). In the case of $\boldsymbol{V} = \mathcal{A}[[\lambda]]$, the λ-adic topology is induced by the metric $d(A, B) = 2^{-o(A-B)}$, where $o(\sum_i A_i \lambda^i) = \min\{i \mid A_i \neq 0\}$[2]. One can check that $\mathcal{A}[[\lambda]]$ is λ-adic complete, λ-torsion free, and Hausdorff[3].

Let $\mathcal{A}$ be a *-algebra.

Definition B.4. *A formal deformation $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$ of $\mathcal{A}$ is called* hermitian *if*

$$(A \star A')^* = A'^* \star A^* \text{ for all } A, A' \in \mathcal{A}.$$

We assume that $\lambda^* = \lambda$. The following result can be found in [53, 22].

Lemma B.5. *Let $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{A}}'$ be equivalent hermitian deformations of $\mathcal{A}$. Then there exists an equivalence transformation $\boldsymbol{T}$ satisfying $\boldsymbol{T}(A^*) = \boldsymbol{T}(A)^*$.*

Definition B.6. *We denote the set of equivalence classes of formal deformations of $\mathcal{A}$ by* $\mathsf{Def}(\mathcal{A})$*; the set of equivalence classes of hermitian deformations of $\mathcal{A}$ is denoted by* $\mathsf{Def}^*(\mathcal{A})$.

B.2 Poisson algebras

Let $\mathcal{A}$ be an associative, commutative, unital k-algebra. Motivated by Definition A.1, we have the following

---

[2] And $o(0) = \infty$

[3] If k is a field, it is easy to check that any $k[[\lambda]]$-module $\boldsymbol{V}$ satisfying these 3 properties is (isomorphic to one) of the form $\mathcal{A}[[\lambda]]$. We can define a formal deformation of a k-algebra $\mathcal{A}$ to be a λ-adic complete, λ-torsion free, Hausdorff $k[[\lambda]]$-algebra $\boldsymbol{\mathcal{A}}$ together with a k-algebra isomorphism $\mathcal{A} \longrightarrow \boldsymbol{\mathcal{A}}/\lambda\boldsymbol{\mathcal{A}}$.

Definition B.7. *A* Poisson bracket *on $\mathcal{A}$ is a Lie algebra bracket $\{\cdot,\cdot\}$ satisfying*

$$\{A_1, A_2A_3\} = \{A_1, A_2\}A_3 + A_2\{A_1, A_3\}. \tag{B.3}$$

The pair $(\mathcal{A}, \{\cdot,\cdot\})$ is called a Poisson algebra*.*

Let $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$ be a formal deformation of $\mathcal{A}$, with $A_1 \star A_2 = \sum_{r=0}^{\infty} C_r(A_1, A_2)\lambda^r$, for $A_1, A_2 \in \mathcal{A}$. A simple computation using the associativity of $\star$ shows [17, Sect. 19]

Proposition B.8. *The bracket*

$$\{A_1, A_2\} := C_1(A_1, A_2) - C_1(A_2, A_1)$$

is a Poisson bracket on $\mathcal{A}$. Moreover, equivalent formal deformations correspond to the same Poisson bracket.

Definition B.9. *The set of equivalence classes of formal deformations of $\mathcal{A}$ corresponding to a Poisson structure $\{\cdot,\cdot\}$ is denoted by $\mathsf{Def}(\mathcal{A}, \{\cdot,\cdot\})$.*

Let $\mathcal{A}$ be a unital k-algebra, not necessarily commutative, and let $\mathcal{Z}$ be its center. The following generalization of a Poisson algebra was introduced in [59].

Definition B.10. *A* Poisson fibred algebra structure *on $(\mathcal{A}, \mathcal{Z})$ is a bracket*

$$\{\cdot,\cdot\} : \mathcal{Z} \times \mathcal{A} \longrightarrow \mathcal{A}$$

satisfying the following:

1. *The restriction of $\{\cdot,\cdot\}$ to $\mathcal{Z} \times \mathcal{Z}$ makes $\mathcal{Z}$ into a Poisson algebra.*

2. *The following Leibniz identities hold* [4]*.*

$$\{Z, A_1A_2\} = \{Z, A_1\}A_2 + A_1\{Z, A_2\},$$
$$\{Z_1Z_2, A\} = Z_1\{Z_2, A\} + Z_2\{Z_1, A\}.$$

The notions of *equivalence* of Poisson fibred algebras and *curvature* of Poisson fibred structures were studied in [59], along with the geometric interpretation of these objects in terms of Lie algebroids.

Let $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$, $A_1 \star A_2 = \sum_{r=0}^{\infty} B_r(A_1, A_2)\lambda^r$, be a formal deformation of $\mathcal{A}$. We have a noncommutative analogue of Proposition B.8 [59, Prop. 1.2].

[4] These identities imply that $\{Z, 1\} = \{1, A\} = 0$, for all $Z \in \mathcal{Z},\ A \in \mathcal{A}$.

Proposition B.11. *The bracket*

$$\{A_1, A_2\} := B_1(A_1, A_2) - B_1(A_2, A_1)$$

defines a Poisson fibred algebra structure on $(\mathcal{A}, \mathcal{Z})$.

B.3 Hochschild cohomology

Let $\mathcal{A}$ be a k-module. Let $\mathcal{C}^\bullet(\mathcal{A}) = \oplus_p \mathcal{C}^p(\mathcal{A})$, where $\mathcal{C}^p(\mathcal{A}) = \mathsf{Hom}(\mathcal{A}^{\otimes^p}, \mathcal{A})$. For $C \in C^p(\mathcal{A})$, $C' \in C^q(\mathcal{A})$, we define the *Gerstenhaber product* [5]

$$C \diamond C'(A_1, \ldots, A_{p+q-1}) := \sum_{i=0}^{p-1} (-1)^{i(q-1)} C(A_1, \ldots, A_i, C'(A_{i+1}, \ldots, A_{i+q}), A_{i+q+1}, \ldots, A_{p+q-1}).$$

Definition B.12. *The bracket* $[\cdot,\cdot]_G : \mathcal{C}^p(\mathcal{A}) \times \mathcal{C}^q(\mathcal{A}) \longrightarrow \mathcal{C}^{p+q-1}(\mathcal{A})$ *defined by*

$$[C, C']_G = C \diamond C' - (-1)^{(p-1)(q-1)} C' \diamond C$$

is called the Gerstenhaber bracket *on* $\mathcal{C}^\bullet(\mathcal{A})$.

We note that $\mu \in C^2(\mathcal{A})$ provides $\mathcal{A}$ with an associative algebra structure if and only if $[\mu, \mu]_{\mathrm{G}} = 0$.

Suppose $\mathcal{A}$ is an associative, unital k-algebra, with algebra structure given by $\mu \in \mathcal{C}^2(\mathcal{A})$. We define a differential $d_{\mathrm{H}} : \mathcal{C}^p(\mathcal{A}) \longrightarrow \mathcal{C}^{p+1}(\mathcal{A})$ by $d_{\mathrm{H}}(C) := (-1)^{p+1}[\mu, C]_{\mathrm{G}}$.

Definition B.13. *We call* d_H *the* Hochschild differential*, and the complex* $(\mathcal{C}^\bullet(\mathcal{A}), d_H)$ *the* Hochschild complex *of* $\mathcal{A}$*. The associated cohomology is called the* Hochshild cohomology *of* $\mathcal{A}$*, and denoted* $H^\bullet(\mathcal{A})$*.*

If we assign degree p to elements in $\mathcal{C}^{p+1}(\mathcal{A})$, then $\mathcal{C}^\bullet(\mathcal{A})$, equipped with d_{H} and $[\cdot,\cdot]_{\mathrm{G}}$, becomes a differential graded Lie algebra.

A sequence of 2-cochains $C_r \in \mathcal{C}^2(\mathcal{A})$ defines a formal deformation of $\mathcal{A}$ if and only if $C_0 = \mu$ and $[\sum_{r=0}^\infty C_r \lambda^r, \sum_{r=0}^\infty C_r \lambda^r]_{\mathrm{G}} = 0$[6], which is equivalent to $C = \sum_{r=1}^\infty C_r \lambda^r$ satisfying the *Maurer-Cartan equation*

$$d_{\mathrm{H}} C - \frac{1}{2}[C, C]_{\mathrm{G}} = 0.$$

[5] This product is not associative.

[6] We are considering the λ-linear extension of $[\cdot,\cdot]_{\mathrm{G}}$ to $\mathcal{C}^\bullet(\mathcal{A})[[\lambda]]$.

Let $\mathcal{A} = C^\infty(M)$. In this case we can consider special classes of Hochshild cochains:

$$\begin{aligned} \mathcal{C}^p_{\text{diff}}(\mathcal{A}) &:= \{C \in \mathcal{C}^p(\mathcal{A}) \mid C \text{ is differential in each entry}\}, \\ \mathcal{C}^p_{\text{loc}}(\mathcal{A}) &:= \{C \in \mathcal{C}^p(\mathcal{A}) \mid C \text{ is local in each entry}\}, \\ \mathcal{C}^p_{\text{cont}}(\mathcal{A}) &:= \{C \in \mathcal{C}^p(\mathcal{A}) \mid C \text{ is continuous}\}, \end{aligned}$$

where an operator $T : C^\infty(M) \longrightarrow C^\infty(M)$ is *local* if $\operatorname{supp}(T(f)) \subseteq \operatorname{supp}(f)$, and continuity is with respect to the usual Fréchet topology of $C^\infty(M)$.

Formal deformations of $\mathcal{A} = C^\infty(M)$ with cochains in $\mathcal{C}^p_{\text{diff}}(\mathcal{A})$ (resp. $\mathcal{C}^p_{\text{loc}}(\mathcal{A})$, $\mathcal{C}^p_{\text{cont}}(\mathcal{A})$) are called *differential (resp. local, continuous)* star products on M. One can show that these special types of cochains are stable under Gerstenhaber bracket and Hochshild differential. Thus it makes sense to define the corresponding Hochshild cohomology groups: $H^\bullet_{\text{diff}}(\mathcal{A})$, $H^\bullet_{\text{loc}}(\mathcal{A})$, and $H^\bullet_{\text{cont}}(\mathcal{A})$. It turns out that $H^p_{\text{diff}}(\mathcal{A}) \cong H^p_{\text{loc}}(\mathcal{A}) \cong H^p_{\text{cont}}(\mathcal{A}) \cong \Gamma^\infty(\bigwedge^p TM)$ [41, 69, 35, 44].

It was shown in [69] that any differential p-cocycle C can be written as $d_{\text{H}}F + B$, where F is a $p-1$ differential cochain and B is a skew symmetric differential p-cocycle. Analogous results for local and continuous cochains were proven in [35, 16, 56].

Theorem B.14. *Any differential (resp. local, resp. continuous) Hochshild p-cocycle C on $C^\infty(M)$ can be written as*

$$C(f_1, \dots, f_p) = d_H F(f_1, \dots, f_p) + B(df_1, \dots, df_p),$$

where F is a differential (resp. local, resp. continuous) $p-1$-cochain and $B \in \Gamma^\infty(\bigwedge^p TM)$.

A simple consequence of Theorem B.14 is

Corollary B.15. *Any differential (resp. local, continous) star product on a Poisson manifold $(M, \{\cdot,\cdot\})$ is equivalent to one with $C_1 = \frac{1}{2}\{\cdot,\cdot\}$.*

Another consequence of Theorem B.14 is [35, Prop. 2.3]

Proposition B.16. *Any local (resp. continous) star product on M is equivalent to a differential one.*

Thus we have natural bijections between the sets $\mathsf{Def}_{\text{diff}}(M)$, $\mathsf{Def}_{\text{loc}}(M)$ and $\mathsf{Def}_{\text{cont}}(M)$.

B.4 Kontsevich's formality theorem

Let us recall that a triple $(\mathfrak{g}, [\cdot,\cdot], d)$ is a *differential graded Lie algebra* if

i.) $\mathfrak{g} = \oplus_n \mathfrak{g}^n$ is $\mathbb{Z}$-graded,

ii.) $[\cdot,\cdot] : \mathfrak{g}^p \times \mathfrak{g}^q \longrightarrow \mathfrak{g}^{p+q}$ satisfies

$$[g_1, g_2] = -(-1)^{pq}[g_2, g_1],$$
$$[g_1, [g_2, g_3]] + (-1)^{r(p+q)}[g_3, [g_1, g_2]] + (-1)^{p(q+r)}[g_2, [g_3, g_1]] = 0,$$

iii.) $d : \mathfrak{g}^p \longrightarrow \mathfrak{g}^{p+1}$ satisfies $d^2 = 0$ and

$$d[g_1, g_2] = [dg_1, g_2] + (-1)^p[g_1, dg_2],$$

for $g_1 \in \mathfrak{g}^p, g_2 \in \mathfrak{g}^q$ and $g_3 \in \mathfrak{g}^r$.

Let $\mathfrak{g}[1]$ be the differential graded Lie algebra $\mathfrak{g}$ with degree shifted by one. Let $\widetilde{\mathcal{C}}(\mathfrak{g}) := S(\mathfrak{g}[1]) = \oplus_{n\geq 1} S^n(\mathfrak{g}[1])$ be the cofree cocommutative graded coalgebra without counit cogenerated by $\mathfrak{g}[1]$. The differential graded Lie algebra structure of $\mathfrak{g}$ induces a degree 1 coderivation Q on $\widetilde{\mathcal{C}}(\mathfrak{g})$ satisfying $[Q, Q] = 0$[7].

Definition B.17. *An* L_∞-quasi-isomorphism *between two differential graded Lie algebras* $\mathfrak{g}_1$, $\mathfrak{g}_2$ *is a morphism of coalgebras*

$$\mathcal{U} : \widetilde{\mathcal{C}}(\mathfrak{g}_1) \longrightarrow \widetilde{\mathcal{C}}(\mathfrak{g}_2),$$

of degree zero, satisfying $\mathcal{U} \circ Q_1 = Q_2 \circ \mathcal{U}$, *and such that its restriction to* $\mathfrak{g}_1$ *is a quasi-isomorphism of the complexes* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$.

Such a map $\mathcal{U}$ is uniquely determined by its *Taylor coefficients*, which are linear maps of graded vector spaces

$$\mathcal{U}_n : \wedge^n \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2[1],$$

defined by composing $\mathcal{U}$ on the right with the canonical projection $\widetilde{\mathcal{C}}(\mathfrak{g}_1) \longrightarrow \mathfrak{g}_1$. [8]

Let $\mathfrak{g}_1$ be the differential Lie algebra of multivector fields on a manifold M, with Schouten bracket and zero differential, and let $\mathfrak{g}_2$ be the differential graded Lie algebra of

[7] A graded vector space V with a derivation Q of degree 1 on $\widetilde{\mathcal{C}}(V)$ satisfying $[Q, Q] = 0$ is called an L_∞-algebra.

[8] We are using the natural isomorphism $S^n(\mathfrak{g}_1[1]) \cong \bigwedge^n \mathfrak{g}_1$.

multidifferential operators, with Gerstenhaber bracket and Hochschild differential. Recall that the gradings are shifted by one, i.e., a $p+1$-vector field (resp. $p+1$-differential operator) is a homogeneous element in $\mathfrak{g}_1$ (resp. $\mathfrak{g}_2$) of degree p. There is a natural embedding $\mathcal{U}_1^{(0)} : \Gamma^\infty(\bigwedge^\bullet TM) \longrightarrow \mathcal{C}^\bullet_{\mathrm{diff}}(C^\infty(M))$, by regarding vector fields as differential operators. This map is a quasi-isomorphism of complexes [44, Sec. 4.6.1.1], but does not preserve the Lie algebra structures. Kontsevich proved the following theorem [44, Sec. 4.6.2].

Theorem B.18 (Kontsevich's Formality theorem). *There exists an L_∞-quasi-isomorphism $\mathcal{U}$ from $\mathfrak{g}_1$ to $\mathfrak{g}_2$ with $\mathcal{U}_1 = \mathcal{U}_1^{(0)}$.*

Let $\boldsymbol{m} = \lambda\mathbb{C}[[\lambda]]$. Let $\lambda\pi_\lambda = \lambda(\pi_0 + \lambda\pi_1 + \ldots)$ be a solution to the Maurer-Cartan equation on $\mathfrak{g}_1^1 \otimes \boldsymbol{m}$,

$$d(\lambda\pi_\lambda) - \frac{1}{2}[\lambda\pi_\lambda, \lambda\pi_\lambda] = 0.$$

Since $d = 0$, we have $[\pi_\lambda, \pi_\lambda] = 0$, and π_λ is a formal Poisson structure on M. As observed by Kontsevich, $\mathcal{U}$ carries π_λ into an element $\tilde{C} = \sum_{k\geq 1} \frac{\lambda^k}{k!}\mathcal{U}_k(\pi_\lambda \wedge \ldots \wedge \pi_\lambda) \in \mathfrak{g}_2^1 \otimes \boldsymbol{m}$, also satisfying the Maurer-Cartan equation. Let μ denote the multiplication on $C^\infty(M)$. It is clear that

$$d_{\mathrm{H}}(\tilde{C}) - \frac{1}{2}[\tilde{C}, \tilde{C}]_{\mathrm{G}} = 0 \iff [\mu + \tilde{C}, \mu + \tilde{C}]_{\mathrm{G}} = 0.$$

Therefore $\star = \mu + \tilde{C} \in \mathcal{C}^2_{\mathrm{diff}}(C^\infty(M))$ defines a differential star product on M corresponding to the Poisson structure π_0. The fact that $\mathcal{U}$ is an L_∞-quasi-isomorphism implies the equivalence of the so-called *deformation functors* (moduli spaces of solutions of Maurer-Cartan equations), and the following correspondence holds [44].

Theorem B.19. *There is a bijection between equivalence classes of formal Poisson structures on a Poisson manifold (M, π) and $\mathsf{Def}_{\mathit{diff}}(M)$.*

Bibliography

[1] Ara, P.: *Morita equivalence for rings with involution.* Algebr. Represent. Theory **2**.3 (1999), 227–247.

[2] Ara, P.: *Morita equivalence and Pedersen ideals.* Proc. Amer. Math. Soc. **129**.4 (2001), 1041–1049 (electronic).

[3] Bass, H.: *Algebraic K-theory.* W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[4] Bayen, F., Flato, M., Frønsdal, C., Lichnerowicz, A., Sternheimer, D.: *Deformation Theory and Quantization.* Ann. Phys. **111** (1978), 61–151.

[5] Beer, W.: *On Morita equivalence of nuclear C^*-algebras.* J. Pure Appl. Algebra **26**.3 (1982), 249–267.

[6] Bertelson, M., Cahen, M., Gutt, S.: *Equivalence of Star Products.* Class. Quantum Grav. **14** (1997), A93–A107.

[7] Bordemann, M., Neumaier, N., Waldmann, S.: *Homogeneous Fedosov Star Products on Cotangent Bundles I: Weyl and Standard Ordering with Differential Operator Representation.* Commun. Math. Phys. **198** (1998), 363–396.

[8] Bordemann, M., Neumaier, N., Waldmann, S.: *Homogeneous Fedosov star products on cotangent bundles II: GNS representations, the WKB expansion, traces, and applications.* J. Geom. Phys. **29** (1999), 199–234.

[9] Bordemann, M., Waldmann, S.: *A Fedosov star product of the Wick type for Kähler manifolds.* Lett. Math. Phys. **41**.3 (1997), 243–253.

[10] Bordemann, M., Waldmann, S.: *Formal GNS Construction and WKB Expansion in Deformation Quantization.* In: Sternheimer, D., Rawnsley, J., Gutt, S. (eds.): *Deformation Theory and Symplectic Geometry*, *Mathematical Physics Studies* no. **20**, 315–319. Kluwer Academic Publisher, Dordrecht, Boston, London, 1997.

[11] Bordemann, M., Waldmann, S.: *Formal GNS Construction and States in Deformation Quantization.* Commun. Math. Phys. **195** (1998), 549–583.

[12] Bursztyn, H., Waldmann, S.: **-Ideals and Formal Morita Equivalence of *-Algebras.* Internat. J. Math. (to appear) . Preprint math.QA/0005227.

[13] Bursztyn, H., Waldmann, S.: *Deformation quantization of Hermitian vector bundles.* Lett. Math. Phys. **53**.4 (2000), 349–365. math.QA/0009170.

[14] Bursztyn, H., Waldmann, S.: *On positive deformations of $*$-algebras.* In: Dito, G., Sternheimer, D. (eds.): *Conférence Moshé Flato 1999, Vol. II (Dijon).* [25], 69–80. math.QA/9910112.

[15] Bursztyn, H., Waldmann, S.: *Algebraic Rieffel induction, formal Morita equivalence, and applications to deformation quantization.* J. Geom. Phys. **37**.4 (2001), 307–364. math.QA/9912182.

[16] Cahen, M., Gutt, S., De Wilde, M.: *Local cohomology of the algebra of C^∞ functions on a connected manifold.* Lett. Math. Phys. **4**.3 (1980), 157–167.

[17] Cannas da Silva, A., Weinstein, A.: *Geometric models for noncommutative algebras.* American Mathematical Society, Providence, RI, 1999.

[18] Cattaneo, A. S., Felder, G., Tomassini, L.: *From local to global deformation quantization of Poisson manifolds.* math.QA/0012228.

[19] Connes, A.: *Noncommutative geometry.* Academic Press, Inc., San Diego, CA, 1994.

[20] Connes, A., Douglas, M. R., Schwarz, A.: *Noncommutative geometry and matrix theory: compactification on tori.* J. High Energy Phys. **1998**.2, Paper 3, 35 pp. (electronic).

[21] Davidson, K. R.: *C^*-algebras by example.* American Mathematical Society, Providence, RI, 1996.

[22] Deligne, P.: *Déformations de l'algèbre des fonctions d'une variété symplectique: comparaison entre Fedosov et De Wilde, Lecomte.* Selecta Math. (N.S.) **1**.4 (1995), 667–697.

[23] DeWilde, M., Lecomte, P. B. A.: *Existence of Star-Products and of Formal Deformations of the Poisson Lie Algebra of Arbitrary Symplectic Manifolds.* Lett. Math. Phys. **7** (1983), 487–496.

[24] Dirac, P. A. M.: *The Principles of Quantum Mechanics.* Oxford, at the Clarendon Press, 1947. 3d ed.

[25] Dito, G., Sternheimer, D. (eds.): *Conférence Moshé Flato 1999. Vol. II*, Dordrecht, 2000. Kluwer Academic Publishers Group. Quantization, deformations, and symmetries.

[26] Emmrich, C., Römer, H.: *Multicomponent Wentzel-Kramers-Brillouin Approximation on arbitrary Symplectic Manifolds: A Star Product Approach.* J. Math. Phys. **39**.7 (1998), 3530–3546.

[27] Emmrich, C., Weinstein, A.: *Geometry of the transport equation in multicomponent WKB approximations.* Comm. Math. Phys. **176**.3 (1996), 701–711.

[28] Fedosov, B. V.: *A Simple Geometrical Construction of Deformation Quantization.* J. Diff. Geom. **40** (1994), 213–238.

[29] Fedosov, B. V.: *Deformation Quantization and Index Theory.* Akademie Verlag, Berlin, 1996.

[30] Fernandes, R. L.: *Connections in Poisson geometry. I. Holonomy and invariants.* J. Differential Geom. **54**.2 (2000), 303–365. math.DG/0001129.

[31] Folland, G. B.: *Harmonic analysis in phase space.* Princeton University Press, Princeton, NJ, 1989.

[32] Gerstenhaber, M.: *On the Deformation of Rings and Algebras.* Ann. Math. **79** (1964), 59–103.

[33] Gerstenhaber, M., Schack, S. D.: *Algebraic Cohomology and Deformation Theory.* In: Hazewinkel, M., Gerstenhaber, M. (eds.): *Deformation Theory of Algebras and Structures and Applications.* [39], 13–264.

[34] Gutt, S.: *An explicit *-product on the cotangent bundle of a Lie group.* Lett. Math. Phys. **7**.3 (1983), 249–258.

[35] Gutt, S.: *On some second Hochschild cohomology spaces for algebras of functions on a manifold.* Lett. Math. Phys. **39**.2 (1997), 157–162.

[36] Gutt, S.: *Variations on deformation quantization.* In: Dito, G., Sternheimer, D. (eds.): *Conférence Moshé Flato 1999, Vol. I (Dijon)*, 217–254. Kluwer Acad. Publ., Dordrecht, 2000. math.DG/0003107.

[37] Gutt, S., Rawnsley, J.: *Equivalence of star products on a symplectic manifold; an introduction to Deligne's Čech cohomology classes.* J. Geom. Phys. **29** (1999), 347–392.

[38] Handelman, D.: *Rings with involution as partially ordered abelian groups.* Rocky Mountain J. Math. **11**.3 (1981), 337–381.

[39] Hazewinkel, M., Gerstenhaber, M. (eds.): *Deformation Theory of Algebras and Structures and Applications*. Kluwer Academic Press, Dordrecht, 1988.

[40] Hirzebruch, F.: *Topological methods in algebraic geometry*. Springer-Verlag, Berlin, 1995. Translated from the German and Appendix One by R. L. E. Schwarzenberger, With a preface to the third English edition by the author and Schwarzenberger, Appendix Two by A. Borel, Reprint of the 1978 edition.

[41] Hochschild, G., Kostant, B., Rosenberg, A.: *Differential forms on regular affine algebras*. Trans. Amer. Math. Soc. **102** (1962), 383–408.

[42] Kaplansky, I.: *Rings of operators*. W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[43] Karoubi, M.: *K-theory*. Springer-Verlag, Berlin, 1978. An introduction, Grundlehren der Mathematischen Wissenschaften, Band 226.

[44] Kontsevich, M.: *Deformation Quantization of Poisson Manifolds, I* . Preprint q-alg/9709040.

[45] Lam, T. Y.: *Lectures on modules and rings*. Springer-Verlag, New York, 1999.

[46] Landsman, N. P.: *Quantized reduction as a tensor product* . Preprint math-ph/0008004.

[47] Landsman, N. P.: *Mathematical Topics between Classical and Quantum Mechanics. Springer Monographs in Mathematics*. Springer-Verlag, Berlin, Heidelberg, New York, 1998.

[48] Lecomte, P., Roger, C.: *Formal deformations of the associative algebra of smooth matrices*. Lett. Math. Phys. **15**.1 (1988), 55–63.

[49] Marsden, J., Ratiu, T.: *Introduction to mechanics and symmetry*. Springer-Verlag, New York, second. edition, 1999. A basic exposition of classical mechanical systems.

[50] Morita, K.: *Duality for modules and its applications to the theory of rings with minimum condition*. Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **6** (1958), 83–142.

[51] Muhly, P. S., Renault, J. N., Williams, D. P.: *Equivalence and isomorphism for groupoid C^*-algebras*. J. Operator Theory **17**.1 (1987), 3–22.

[52] Nest, R., Tsygan, B.: *Algebraic Index Theorem for Families*. Adv. Math. **113** (1995), 151–205.

[53] Neumaier, N.: *Local ν-Euler Derivations and Deligne's Characteristic Class of Fedosov Star Products* . Preprint Freiburg FR-THEP-99/3, math.QA/9905176.

[54] Omori, H., Maeda, Y., Yoshioka, A.: *Weyl Manifolds and Deformation Quantization.* Adv. Math. **85** (1991), 224–255.

[55] Pedersen, G. K.: *C^*-algebras and their automorphism groups.* Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1979.

[56] Pflaum, M.: *On continuous Hochschild homology and cohomology groups.* Lett. Math. Phys. **44**.1 (1998), 43–51.

[57] Porta, H., Recht, L.: *Classification of linear connections.* J. Math. Anal. Appl. **118**.2 (1986), 547–560.

[58] Raeburn, I., Williams, D. P.: *Morita equivalence and continuous-trace C^*-algebras.* American Mathematical Society, Providence, RI, 1998.

[59] Reshetikhin, N., Voronov, A., Weinstein, A.: *Semiquantum geometry.* J. Math. Sci. **82**.1 (1996), 3255–3267. Algebraic geometry, 5.

[60] Rieffel, M. A.: *Induced representations of C^*-algebras.* Advances in Math. **13** (1974), 176–257.

[61] Rieffel, M. A.: *Morita equivalence for C^*-algebras and W^*-algebras.* J. Pure Appl. Algebra **5** (1974), 51–96.

[62] Rieffel, M. A.: *Morita equivalence for operator algebras.* 285–298. Amer. Math. Soc., Providence, R.I., 1982.

[63] Rieffel, M. A.: *Deformation quantization of Heisenberg manifolds.* Comm. Math. Phys. **122**.4 (1989), 531–562.

[64] Rieffel, M. A., Schwarz, A.: *Morita equivalence of multidimensional noncommutative tori.* Internat. J. Math. **10**.2 (1999), 289–299.

[65] Schwarz, A.: *Morita equivalence and duality.* Nuclear Phys. B **534**.3 (1998), 720–738.

[66] Sternheimer, D.: *Deformation quantization: twenty years after.* In: *Particles, fields, and gravitation (Łódź, 1998)*, 107–145. Amer. Inst. Phys., Woodbury, NY, 1998. Preprint math.QA/9809056.

[67] Vaisman, I.: *On the geometric quantization of Poisson manifolds.* J. Math. Phys. **32**.12 (1991), 3339–3345.

[68] Vaisman, I.: *Lectures on the geometry of Poisson manifolds.* Birkhäuser Verlag, Basel, 1994.

[69] Vey, J.: *Déformation du Crochet de Poisson sur une Variété symplectique.* Comm. Math. Helv. **50** (1975), 421–454.

[70] Waldmann, S.: *Locality in GNS Representations of Deformation Quantization.* Commun. Math. Phys. **210** (2000).

[71] Watts, C. E.: *Intrinsic characterizations of some additive functors.* Proc. Amer. Math. Soc. **11** (1960), 5–8.

[72] Weinstein, A.: *Deformation Quantization.* Séminaire Bourbaki 46ème année **789** (1994).

[73] Weinstein, A., Xu, P.: *Hochschild cohomology and characterisic classes for star-products.* In: Khovanskij, A., Varchenko, A., Vassiliev, V. (eds.): *Geometry of differential equations. Dedicated to V. I. Arnold on the occasion of his 60th birthday*, 177–194. American Mathematical Society, Providence, 1998.

[74] Xu, P.: *Morita equivalence of Poisson manifolds.* Comm. Math. Phys. **142**.3 (1991), 493–509.

[75] Xu, P.: *Morita equivalent symplectic groupoids.* In: *Symplectic geometry, groupoids, and integrable systems (Berkeley, CA, 1989)*, 291–311. Springer, New York, 1991.