Worksheet #5 · Sample problems

① (Kathy, Minsang) Let A, B be $n\times n$ matrices. Prove that

$tr(AB) = tr(BA)$.

What can you say about the trace of two similar matrices.

② (Shasta) Let $T: P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ be defined by $T(f) = f'' + 2f' - f$

Is T invertible? If so, compute T^{-1}.

③ Provide examples to the following statements:

ⓐ (Vicky) Two invertible $n\times n$ matrices A, B, with $\det A = \det B$,

but which are not similar.

ⓑ (Vicky) Two diagonalizable $n\times n$ matrices A, B, with $\det A = \det B$,

which are not similar.

ⓒ (Asa) A normal matrix $A \in M_{n\times n}(\mathbb{R})$ which is not diagonalizable.

ⓓ (Asa) Two normal matrices A, B whose product is not normal.

④ (Nik) Let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$, $T(p(x)) = p(x) + (2+x)p'(x)$.

Show that T is diagonalizable and find a basis where T is diagonal.

⑤ (Wei) Consider the basis of $\mathbb{R}^3$: $\gamma = \{(1,1,1), (1,1,0), (1,0,0)\}$.

Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be s.t. $[T]_\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

What is $[T]_\beta$, where β is the canonical basis of $\mathbb{R}^3$?

(6) Let V be a finite dimensional vector space over $\mathbb{C}$.
(Kane) Let U, T be linear operators on V. Show that, if $TU = UT$, then U and T have a common eigenvector.

(7) Let V be a finite dimensional inner product space.
(David) Let $T: V \to V$ be a linear operator.
Show that if T is self-adjoint and nilpotent (ie, $\exists p$ st. $T^p = 0$) then $T = 0$.

(8) For $A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$, find an orthogonal matrix P
(Bessie) st $P^* A P = D$, where D is diagonal.
What's the spectral decomposition of A?

(9) Let $A = \begin{pmatrix} 1 & -1 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & -5 \end{pmatrix}$.
(Ohesum)
What's its Jordan form J?
Find Q st. $Q^{-1} A Q = J$.