MATH 110: Worksheet #3

Let V be a finite dimensional inner product space.

<u>Definition</u>: A linear operator $T: V \to V$ is called <u>positive definite</u> (positive semidefinite) if T is self-adjoint and $\langle Tx, x \rangle > 0$ ($\langle Tx, x \rangle \geq 0$) $\forall x \in V$, $x \neq 0$.

Let $T: V \to V$ be self-adjoint.

① Prove that T is positive definite (semidefinite) iff all its eigenvalues are positive (nonnegative).

Let $A = [T]_\beta$, β orthonormal basis of V

② Prove that T is positive definite iff $A = B^* B$ for some square matrix B.

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③ Let V be finite dimensional inner product space, and let $\beta = \{v_1, \ldots, v_n\}$ be an orthonormal basis.

Let $U: V \to V$ be a linear operator.

If $\|U(v_i)\| = \|v_i\|$ $\forall v_i \in \beta$, must U be unitary?

④ Let V be a finite dimensional inner product space.

Let $T: V \to V$ be a linear operator.

ⓐ Define the adjoint T^*, and define normal, self-adjoint and unitary (orthogonal) operators.

ⓑ Prove: • If T is unitary, then it's normal.

• If T is self-adjoint, then it's normal.

ⓒ Find an example of: • T self-adjoint but not unitary.

• T unitary but not self-adjoint.

ⓓ Find an example of a diagonalizable T that is not normal.

⑤ Find the spectral decomposition of

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$