Math 110: Midterm I

July 12, 2001

Name: SOLUTIONS

Instructions: This is a closed book exam, and you will have 55 minutes. Please write your name on every page you use. Calculators are not allowed! Make sure to show your work clearly. The exam is worth **40 points**.
Good luck!

Problem 1 (12 pts): Determine whether the following statements are **true** or **false** (write your answer clearly, otherwise it will not be graded). If the statement is **false**, provide a counter example.

2 a) Let V be a vector space, and let W_1, W_2, U be subspaces of V. If

$$W_1 \oplus U = W_2 \oplus U$$

then $W_1 = W_2$.

Answer: FALSE. Let $V = \mathbb{R}^2$, $W_1 =$ y-axis, $W_2 =$ Line $y=x$, $U =$ x-axis
$W_1 \neq W_2$ but $W_1 \oplus U = W_2 \oplus U = \mathbb{R}^2$.

2 b) If $S = \{v_1, \ldots, v_n\} \subseteq V$ generates the vector space V, then any other set generating V has at least n elements.

Answer: FALSE. $S = \{(1,0)(0,1)(1,1)\}$ generates $\mathbb{R}^2$, S has 3 elements
But $\{(1,0)(0,1)\}$ generates with 2 elements.

2 c) If β is a basis of a vector space V, then any subset of β is linearly independent.

Answer: TRUE

2 d) The **real** vector spaces $P_3(\mathbb{R})$, $M_{2\times 2}(\mathbb{R})$ and $\mathbb{C}^2$ have the same dimension.

Answer: TRUE (all have dimension 4)

2 e) If V is a finite dimensional vector space and $T : V \longrightarrow V$ is a linear tranformation, then T is injective if and only if T is surjective.

Answer: TRUE (this is a consequence of the dim thm)

2 f) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear transformation. Then $\mathbb{R}^2 = N(T) \oplus R(T)$.

Answer: FALSE.

$T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (0,x)$ is such that

$N(T) =$ y-axis, $R(T) =$ y-axis (so $N(T) \oplus R(T) =$ y-axis)

Problem 2 (8 pts): Consider the linear tranformation $T : P_3(\mathbb{R}) \longrightarrow \mathbb{R}^2$ given by

$$T(p) = (p''(0), p'(0)).$$

Find bases for $N(T)$ and $R(T)$, compute the nullity and rank of T, and determine whether T is one-to-one or onto.

If $p(x) = a + bx + cx^2 + dx^3$, then $p'(x) = b + 2cx + 3dx^2$, $p'(0) = b$

$p''(x) = 2c + 6dx$, $p''(0) = 2c$

So $T(p(x)) = (2c, b)$.

Then $T(p(x)) = 0 \Leftrightarrow b = 0, c = 0$, and

$N(T) = \{ p(x) / a + dx^3, \ a, d \in \mathbb{R} \}$

(2) • basis for $N(T) = \{1, x^3\}$

$R(T) = \mathbb{R}^2$ (this follows directly from or by the dim theorem — $\dim N(T) = 2$, $\dim P_3(\mathbb{R}) = 4$ $\Rightarrow \dim R(T) = 2$)

(2) • basis for $R(T) = \{ (1,0) \ (0,1) \}$ (or any other...)

(1) • nullity of $T = 2$

(1) • rank $T = 2$

(1) • T is NOT 1-1 (for $N(T) \neq \{0\}$)

(1) • T is ONTO .

Problem 3 (8 pts): Let $\beta = \{(1,0),(1,1)\}$ and $\gamma = \{(0,1),(-1,1)\}$ be bases of $\mathbb{R}^2$, and let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear tranformation such that

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

What is $T(1,0)$? What is $T(0,1)$? What is $T(x,y)$ in general?

For $v = (1,0)$, $v = 1.(1,0) + 0.(1,1)$.

note that $[v]_\beta = (1,0)$, so

$[Tv]_\gamma = [T]_\beta^\gamma [v_\beta] = (0,1)$

Hence $Tv = 0.(0,1) + 1.(-1,1) = \boxed{(-1,1) = T(1,0)}$ 3

For $v = (0,1)$, $v = -(1,0) + (1,1)$

$[v]_\beta = (-1,1)$

So $[Tv]_\gamma = [T]_\beta^\gamma [v]_\beta = (1,-1)$

Hence

$Tv = (0,1) - (-1,1)$

$= \boxed{(1,0) = T(0,1)}$ 3

- $T(x,y) = xT(1,0) + yT(0,1)$

$= x(-1,1) + y(1,0) = (y-x, x)$ 2

7

Problem 4 (7 pts): Let V_1 and V_2 be finite dimensional vector spaces, and suppose $T : V_1 \longrightarrow V_2$ is a one-to-one linear transformation. Let $W_1 \subseteq V_1$ be a subspace.

3 pts a) Prove that $T(W_1) \subseteq V_2$ is a subspace of V_2.

4 pts b) Prove that $\dim(T(W_1)) = \dim(W_1)$.

(a)

1 • Since $0 \in W_1$ and $T(0)=0$, $0 \in T(W_1)$

3 • Suppose $y_1, y_2 \in T(W_1)$. Then $y_1 = T(x_1)$, $y_2 = T(x_2)$, $x_1 \in W_1$, $x_2 \in W_2$.

Thus $$y_1 + y_2 = T(x_1) + T(x_2) = T(x_1 + x_2).$$

Since $x_1 + x_2 \in W_1$ (for W_1 is a subspace),

$$y_1 + y_2 \in T(W_1).$$

• For $y \in T(W_1)$, $y = T(x)$, $x \in W_1$, c scalar, $cy = cT(x) = T(cx)$

But $cx \in W_1$, so $cy \in T(W_1)$

Therefore $T(W_1) \subseteq V_2$ is a subspace.

(b)

Let $\{w_1, \dots, w_k\}$ be a basis of W_1 (so $\dim W_1 = k$)

We claim that $\{Tw_1, \dots, Tw_k\} \subseteq T(W_1)$ is a basis of $T(W_1)$ (and this implies $\dim T(W_1) = \dim W_1$)

• $\{Tw_1, \dots, Tw_k\}$ generates:

If $y \in T(W_1)$, then $y = T(w)$, $w \in W_1$.

But $w = a_1 w_1 + \dots + a_k w_k$, so $y = T(a_1 w_1 + \dots + a_k w_k) = a_1 T(w_1) + \dots + a_k T(w_k)$

• $\{Tw_1, \dots, Tw_k\}$ is l.i.:

$$a_1 Tw_1 + \dots + a_k Tw_k = 0 \Rightarrow T(a_1 w_1 + \dots + a_k w_k) = 0$$

Since T is one-to-one, $N(T) = \{0\}$, so $a_1 w_1 + \dots + a_k w_k = 0$.

But $w_1, \dots, w_k$ are l.i., hence $a_1 = a_2 = \dots = a_k = 0$.

5 [redacted] pts
[redacted]

Problem 5 ([redacted] pts): Let V be a finite dimensional vector space, and let $W_1, W_2 \subseteq V$ be subspaces. Prove that

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

(This result implies, in particular, that $\dim(W_1 \oplus W_2) = \dim(W_1) + \dim(W_2)$).
Hint: Find a basis of $W_1 + W_2$ starting with a basis of $W_1 \cap W_2$.

Let $\{u_1, \dots, u_k\}$ be a basis of $W_1 \cap W_2$, $(\dim W_1 \cap W_2 = k)$

Since $W_1 \cap W_2 \subseteq W_1$, we can extend $\{u_1 \dots u_k\}$ to a basis $\{u_1, \dots, u_k, v_1, \dots, v_m\}$ of W_1 $(\dim W_1 = k+m)$

Since $W_1 \cap W_2 \subseteq W_2$, we can extend $\{u_1 \dots u_k\}$ to a basis $\{u_1 \dots u_k, w_1, \dots, w_n\}$ of W_2. $(\dim W_2 = k+n)$

<u>claim</u>: $\beta = \{u_1, \dots, u_k, v_1, \dots, v_m, w_1, \dots, w_n\}$ is a basis of $W_1 + W_2$

$\Big($hence $\dim(W_1+W_2) = k+m+n$
$= \dim W_1 + \dim W_2 - k$
$= \dim W_1 + \dim W_2 - \dim W_1 \cap W_2 \Big)$

• β generates:

If $z \in W_1 + W_2$, $z = x + y$, $x \in W_1$, $y \in W_2$.

So $x = a_1 u_1 + \dots + a_k u_k + b_1 v_1 + \dots + b_m v_m$, $y = c_1 u_1 + \dots + c_k u_k + d_1 w_1 + \dots + d_n w_n$ $\quad a_i, b_i, c_i, d_i$ scalars

$\therefore \quad z = x + y = (a_1 + c_1) u_1 + \dots + (a_k + c_k) u_k + b_1 v_1 + \dots + b_m v_m + d_1 w_1 + \dots + d_n w_n$

so β generates

• β is l.i: Suppose

$$a_1 u_1 + \dots + a_k u_k + b_1 v_1 + \dots + b_m v_m + c_1 w_1 + \dots + c_n w_n = 0 \qquad (1)$$

we must show that $a_i = b_i = c_i = 0$.

we can write $\underbrace{\sum a_i u_i + \sum b_j v_j}_{\in W_1} = \underbrace{-\sum c_r w_r}_{\in W_2}$. So $\sum c_r w_r \in W_1 \cap W_2$, and hence can be written as $\sum d_r u_r$

So we have $\sum a_i u_i + \sum b_j v_j = \sum d_r u_r \Rightarrow \sum (a_i - d_i) u_i + \sum b_j v_j = 0$

but $\{u_1 \dots u_k v_1 \dots v_m\}$ is l.i. $\Rightarrow b_j = 0$, $j = 1 \dots m$

Now eq. (1) is just $a_1 u_1 + \dots + a_k u_k + c_1 w_1 + \dots + c_n w_n = 0 \Rightarrow a_i = 0, c_j = 0$

since $u_1, \dots u_k, w_1, \dots w_n$ are l.i.

[redacted]