Math 110: Midterm I

July 12, 2001

Name:

Instructions: This is a closed book exam, and you will have 55 minutes. Please write your name on every page you use. Calculators are not allowed! Make sure to show your work clearly. The exam is worth **40 points**.
Good luck!

Problem 1 (12 pts): Determine whether the following statements are **true** or **false** (write your answer clearly, otherwise it will not be graded). If the statement is **false**, provide a counter example.

a) Let V be a vector space, and let W_1, W_2, U be subspaces of V. If

$$W_1 \oplus U = W_2 \oplus U$$

then $W_1 = W_2$.

Answer:

b) If $S = \{v_1, \ldots, v_n\} \subseteq V$ generates the vector space V, then any other set generating V has at least n elements.

Answer:

c) If β is a basis of a vector space V, then any subset of β is linearly independent.

Answer:

d) The **real** vector spaces $P_3(\mathbb{R})$, $M_{2\times 2}(\mathbb{R})$ and $\mathbb{C}^2$ have the same dimension.

Answer:

e) If V is a finite dimensional vector space and $T : V \longrightarrow V$ is a linear transformation, then T is injective if and only if T is surjective.

Answer:

f) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear transformation. Then $\mathbb{R}^2 = N(T) \oplus R(T)$.

Answer:

Problem 2 (8 pts): Consider the linear tranformation $T : P_3(\mathbb{R}) \longrightarrow \mathbb{R}^2$ given by

$$T(p) = (p''(0), p'(0)).$$

Find bases for $N(T)$ and $R(T)$, compute the nullity and rank of T, and determine whether T is one-to-one or onto.

Problem 3 (8 pts): Let $\beta = \{(1,0),(1,1)\}$ and $\gamma = \{(0,1),(-1,1)\}$ be bases of $\mathbb{R}^2$, and let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear tranformation such that

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

What is $T(1,0)$? What is $T(0,1)$? What is $T(x,y)$ in general?

Problem 4 (8 pts): Let V_1 and V_2 be finite dimensional vector spaces, and suppose $T : V_1 \longrightarrow V_2$ is a one-to-one linear transformation. Let $W_1 \subseteq V_1$ be a subspace.

a) Prove that $T(W_1) \subseteq V_2$ is a subspace of V_2.

b) Prove that $\dim(T(W_1)) = \dim(W_1)$.

Problem 5 (4 pts): Let V be a finite dimensional vector space, and let $W_1, W_2 \subseteq V$ be subspaces. Prove that

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

(This result implies, in particular, that $\dim(W_1 \oplus W_2) = \dim(W_1) + \dim(W_2)$).

Hint: Start with a basis for $W_1 \cap W_2$. Then extend it to bases of W_1 and W_2, and find a basis for $W_1 + W_2$.