Homework #7 - Solutions

Section 6.5

(2) (a) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $\quad p(t) = (1-t)^2 - 4 = t^2 - 2t - 3$

Eigenvalues: $\lambda_1 = -1 \quad \lambda_2 = 3$

Eigenvectors:

λ_1: $A + I = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$, $N(A+I) = \{(x,y) \,/\, x = -y\} = E_{\lambda_1}$

o.n. basis: $\{(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})\}$

λ_2: $A - 3I = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$, $E_{\lambda_2} = \{(x,y) \,/\, x = y\}$

o.n. basis: $\{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\}$

If $P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$, then

$$P^* A P = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \,//$$

(b) $A = \begin{pmatrix} 2 & 3-3i \\ 3+3i & 5 \end{pmatrix}$ $\quad p(t) = t^2 - 7t - 8$

Eigenvalues: $\lambda_1 = -1, \quad \lambda_2 = 8$

Eigenvectors: λ_1: $A + I = \begin{pmatrix} 3 & 3-3i \\ 3+3i & 5 \end{pmatrix}$, o.n. for $E_{\lambda_1} = \{\frac{(i-1,\ 1)}{\sqrt{3}}\}$

λ_2: $A - 8I = \begin{pmatrix} -6 & 3-3i \\ 3+3i & -3 \end{pmatrix}$, o.n. for $E_{\lambda_2} = \{\frac{(1-i,\ 2)}{\sqrt{6}}\}$

$P = \begin{pmatrix} \frac{i-1}{\sqrt{3}} & \frac{1-i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{pmatrix}$ and $P^* A P = \begin{pmatrix} -1 & 0 \\ 0 & 8 \end{pmatrix}$ //

(11)

Let $v_1 = \left(\frac{1}{3}\ \frac{2}{3}\ \frac{2}{3}\right)$, $\|v_1\| = 1$

one can complete $\{v_1\}$ to a basis $\{v_1, v_2, v_3\}$ (for example: $v_2 = (0\ 1\ 0)$, $v_3 = (0\ 0\ 1)$)

and apply G-S to get o.n. basis.

Ex : $$U = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{-2}{\sqrt{45}} & \frac{-4}{\sqrt{45}} & \frac{5}{\sqrt{45}} \end{pmatrix}$$

(21)

(a) $x^2 + 4xy + y^2 = \langle Av, v \rangle$ $\quad v = \begin{pmatrix} x \\ y \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

Eigenvalues: $-1, 3$

eigenvectors: $v_1 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ $\quad v_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\quad P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

New coord.:

$v' = P^* v$ on

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

and $x^2 + 4xy + y^2 = \langle APv', Pv' \rangle = \langle P^*APv', v' \rangle$

$$= -(x')^2 + 3(y')^2.$$

(b) $2x^2 + 2xy + y^2 = \langle Av, v \rangle$ $\quad v = \begin{pmatrix} x \\ y \end{pmatrix}$

$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad$ Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 3$

Eigenvectors : $v_1 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

$v_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$P=\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

New Coordinates:

$$v' = P^* v, \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} 2x^2 + 2xy + 2y^2 &= \langle Av, v\rangle \\ &= \langle APv', Pv'\rangle \\ &= \langle P^*APv', v'\rangle \\ &= \underline{(x')^2 + 3(y')^2} . \end{aligned}$$

Section 6.6:

(2) $V = \mathbb{R}^2$

$W = \text{span}\{(1,2)\} \subseteq V$. $\quad \beta = \{e_1, e_2\}$ canonical basis

Let $u = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ unit vector, o.n. basis for W.

Then for $v = (x,y) \in V$,

$$\begin{aligned} T(v) &= \langle v, u\rangle u \\ &= \left(\frac{x}{\sqrt{5}} + \frac{2y}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \\ &= \left(\frac{x+2y}{5}, \frac{2x+4y}{5}\right) = T(x,y) \end{aligned}$$

for $V = \mathbb{R}^3$,

$W = \text{Span}\{(1,0,1)\}$, we have $u = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$,

$v = (x, y, z)$,

$$T(v) = \langle v, u\rangle u = \left(\frac{x}{\sqrt{2}} + \frac{z}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = \left(\frac{x+z}{2}, 0, \frac{x+z}{2}\right)$$

//

④ Let $W \subseteq V$ f.d. subspace

$T: V \to V$ be orthogonal projection on W.

Let U be orthog. proj. on $W^{\perp}$.

For $v \in V$, $v = x + y$, $x \in W$, $y \in W^{\perp}$ (in a unique way)

$$T(v) = x$$
$$U(v) = y$$

But $(I-T)(v) = v - x = y = U(v) \quad \forall u$

So $\underline{U = I - T}$ ∎

Section 7.1: ② and ③

ⓐ $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$ $\quad p(t) = (1-t)(3-t)+1$

$$= t^2 - 4t + 4 = (t-2)^2$$

eigenvalues: $\underline{\underline{\lambda = 2}}$.

Note $\dim E_2 = \dim(A-2I) = 1$

So we need $\underline{\underline{1}}$ cycle; ie, we need $x \in \mathbb{R}^2$ st

$$(A-2I)x \neq 0$$

$A - 2I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$ $\quad$ we can pick $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

then $\beta = \{(A-2I)x, x\}$ is Jordan basis $(= \{(-1,-1), (1,0)\})$

$$[A]_\beta = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

ⓒ $T: P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$

$T(f) = 2f - f'$

Let $\gamma = \{1, x, x^2\}$

$$A = [T]_\gamma = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{pmatrix}$$

$p(t) = (2-t)^3$

$\lambda = 2$ is the only eigenvalue.

$\dim E_\lambda = \dim N(A - 2I) = 1$

So we need 1 cycle in $K_\lambda = \mathbb{R}^3$:

we must find $u \in \mathbb{R}^3$ s.t. $\underline{(A-2I)^2 u \neq 0}$

Note: $(A - 2I) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$

$(A-2I)^2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

So we can choose $u = (0, 0, 1)$, which corresponds to $v = x^2$ in $P_2(\mathbb{R})$.

and $\beta = \{ (T-2I)^2 v, (T-2I)v, v \}$ is a Jordan basis.

$= \{ 2, -2x, x^2 \} \subseteq P_2(\mathbb{R})$

$$[T]_\beta = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

7. Let $U: V \to V$ Lin. operator, $\dim V < \infty$.

(a)

Must show that $N(U) \subseteq N(U^2) \subseteq \ldots$

Note that if $x \in N(U^k)$ for $k \geq 1$,

then $U^k x = 0$.

But then $U^{k+1}x = U(U^k x) = U(0) = 0$.

So $N(U^k) \subseteq N(U^{k+1})$ $\underline{\forall k} \geq 1$.

$\Rightarrow N(U) \subseteq N(U^2) \subseteq \ldots$

(b) Note that $R(U^{m+1}) \subseteq R(U^m)$ (since if $x = U^{m+1}y$, then $x = U^m(Uy) \in R(U^m)$)

So $\operatorname{rank}(U^{m+1}) = \operatorname{rank} U^m \Rightarrow \underline{R(U^{m+1}) = R(U^m)}$.

To show that $\operatorname{rank}(U^k) = \operatorname{rank}(U^m)$ $\forall k \geq m$, it suffices to show that $R(U^{k+1}) = R(U^k)$ for all $k \geq m$.

But if $x = U^k y$, then

$X = U^{k-m}U^m y$. But $U^m y \in R(U^m) = R(U^{m+1}) \Rightarrow \exists z$ st

$U^m y = U^{m+1} z$.

So $x = U^{k-m}U^{m+1}z = U^{k+1}z$. So $x \in R(U^{k+1})$ and $R(U^k) = R(U^{k+1})$.

(c)

By (b), $\operatorname{RANK}(U^m) = \operatorname{RANK}(U^{m+1})$

$\Rightarrow \operatorname{rank}(U^k) = \operatorname{rank}(U^m)$ $\forall k \geq m$.

By dim theorem,

$\operatorname{nullity}(U^k) = \operatorname{nullity}(U^m)$ $\forall k \geq m$

Since $N(U^m) \subseteq N(U^k) \Rightarrow N(U^m) = N(U^k)$ $\forall k \geq m$

QED