MATH 110: Homework #6 - Solutions

Sec 6.3

(19) $y = cx + d$

In this case

$$A = \begin{pmatrix} 3.5 & 1 \\ 4.0 & 1 \\ 4.5 & 1 \\ 5.0 & 1 \end{pmatrix} \quad x = \begin{pmatrix} c \\ d \end{pmatrix} \quad y = \begin{pmatrix} 1.0 \\ 2.2 \\ 2.8 \\ 4.3 \end{pmatrix}$$

$$A^* = \begin{pmatrix} 3.5 & 4.0 & 4.5 & 5.0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$A^*A = \begin{pmatrix} 73.5 & 17 \\ 17 & 4 \end{pmatrix}, \quad A^*y = \begin{pmatrix} 46.4 \\ 10.3 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} 73.5 & 17 \\ 17 & 4 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 46.4 \\ 10.3 \end{pmatrix} \Rightarrow c \approx 2.1$$

Sec. 6.4

2 (a) $T^* = T$, T is selfadjoint

(b) $T^*T = TT^*$, T is normal but <u>not</u> selfadjoint

$\left(T^*(a,b) = (2a+b, -ia+2b)\right)$

5 $T: V \to V$, V complex inner product space

Define

$$T_1 = \frac{1}{2}(T+T^*)$$

$$T_2 = \frac{1}{2i}(T-T^*)$$

(a) $T_1 + iT_2 = \frac{1}{2}(T+T^*) + \frac{1}{2}(T-T^*) = T$

$$T_1^* = \frac{1}{2}(T+T^*)^* = \frac{1}{2}(T^*+T) = T_1$$

$$T_2^* = -\frac{1}{2i}(T-T^*)^* = -\frac{1}{2i}(T^*-T) = \frac{1}{2i}(T-T^*) = T_2.$$

(b) If $T = U_1 + iU_2$, U_1, U_2 self-adjoint, then

$$T_1 = \frac{1}{2}(T+T^*) = \frac{1}{2}(U_1 + iU_2 + U_1^* - iU_2^*) \quad \text{but } U_1 = U_1^*,\ U_2 = U_2^*$$

$$= \frac{1}{2}(2U_1) = U_1$$

$$T_2 = \frac{1}{2i}(T-T^*) = \frac{1}{2i}(U_1 + iU_2 - U_1 + iU_2) = U_2.$$

(c) T is normal $\Leftrightarrow$ $T^*T = TT^*$ $\Leftrightarrow$ $(T_1 - iT_2)(T_1+iT_2) = (T_1+iT_2)(T_1-iT_2)$

$\Leftrightarrow$ $-iT_2T_1 + iT_1T_2 = iT_2T_1 - iT_1T_2$ $\Leftrightarrow$

$2(T_1T_2 - T_2T_1) = 0 \Leftrightarrow T_1T_2 = T_2T_1.$

(6) V inner product space,

$T: V \to V$ Linear operator.

Let W be T-invariant subspace $(T(W) \subseteq W)$.

(a) $T = T^* \Leftrightarrow \langle Tx, y\rangle = \langle x, Ty\rangle \quad \forall x, y \in V \qquad (1)$

Consider $T_W : W \to W$

for $w_1 w_2 \in W$,

$$\langle T_W(w_1), w_2\rangle = \langle w_1, T_W^* w_2\rangle$$

But since $T w_2 \in W$, we have by (1) $\langle T_W(w_1), w_2\rangle = \langle w_1, T_W w_2\rangle$

$$= \langle w_1, T_W^* w_2\rangle \quad \forall w_1 \in W$$

$$\Rightarrow T_W w_2 = T_W^* w_2 \quad \forall w_2$$

$$\Rightarrow (T_W)^* = T_W .$$

(b) Let $w \in W^\perp$.

To check that $T^* w \in W^\perp$, note that, for $v \in W$

$$\langle T^* w, v\rangle = \langle w, Tv\rangle \qquad \text{But } Tv \in W \Rightarrow \langle w, Tv\rangle = 0$$

$$\Rightarrow \langle T^* w, v\rangle = 0 \quad \forall v \in W \qquad \text{so } T^* w \in W^\perp \quad //$$

(c) Let $w_1 w_2 \in W$, W T-invariant and T^*-invariant

then $T_W^* w_2$ is the unique vector in W satisfying

$$\langle T_W w_1, w_2\rangle = \langle w_1, T_W^* w_2\rangle$$

But $\langle T w_1, w_2\rangle = \langle w_1, T^* w_2\rangle$ and $T^* w_2 \in W$ for W is T^*-inva

So $T_W^* w_2 = T^* w_2 \quad \forall w_2 \in W$

$$\Rightarrow (T_W)^* = (T^*)_W \quad //$$

(d) W is T and T^*-invariant

T normal.

Note that $T_W (T_W)^* = T_W (T^*)_W = (TT^*)_W = (T^*T)_W = (T_W)^* T_W^*$

So T_W is normal.

Sec 6.5 :

(3) Let $U_1, U_2 : V \to V$ be two unitary (orthogonal) operators.

then

$$(U_1 U_2)^* U_1 U_2 = U_2^* \underbrace{U_1^* U_1}_{Id} U_2 = U_2^* U_2 = Id$$

So $U_1 \circ U_2$ is unitary (orthogonal).

(12) Let A be $n \times n$ real symmetric or complex normal matrix.

Then A is diagonalizable, and therefore similar to a diagonal matrix:

$$A = Q^{-1} D Q \quad \text{where} \quad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Hence $\det A = \det D = \prod_{i=1}^{n} \lambda_i$

(16) Suppose $A = \begin{pmatrix} a_{11} & \cdots & & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & 0 & \ddots & \\ 0 & 0 & \cdots 0 & a_{nn} \end{pmatrix}$ is unitary. Then its columns form an orthonormal basis of $\mathbb{C}^n$.

Let A_i be the i-th column of A

We must show that $A_i = (0, \ldots, 0, a_i, 0 \ldots 0)$

Clearly A_1 is of this form.

Suppose now $A_1, \ldots, A_k$ are of this form, $1 < k < n$.

Consider $A_{k+1} = (a_{1k+1}, \ldots, a_{k+1\,k+1}, 0, \ldots, 0)$

$j = 1 \ldots k$ $\langle A_{k+1}, A_j \rangle = \langle (a_{1k+1}, \ldots, a_{k+1k+1}, 0, \ldots 0), (0, \ldots, 0, a_j, 0 \ldots 0) \rangle$

$$= a_{jk+1} \bar{a}_j = 0$$

since $\|A_j\| = 1$, $|a_j| = 1$ so $a_j \neq 0$

Hence $a_{jk+1} = 0$ $j = 1 \ldots k$

So $A_{k+1} = (0, \ldots, 0, a_{k+1k+1}, 0 \ldots 0)$

$\blacksquare$