MATH 110 - Homework # 5 - Solutions

Section 6.1 :

(2) $V = \mathbb{C}^3$ with standard inner product

$x = (2, 1+i, i)$, $y = (2-i, 2, 1+2i)$

· $\langle x, y \rangle = 2.(2+i) + (1+i).2 + i(1-2i)$

$= 8 + 5i$

· $\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{7}$

· $\|y\| = \sqrt{\langle y, y\rangle} = \sqrt{14}$

· $\|x+y\| = \sqrt{\langle x+y, x+y\rangle} = \sqrt{37}$

So $|\langle x, y\rangle| = \sqrt{89} \leq \|x\|\|y\| = \sqrt{98}$

and

$\|x+y\| = \sqrt{37} \leq \|x\| + \|y\| = \sqrt{7} + \sqrt{14}$

(8)

(a) $\langle (a,b), (c,d)\rangle = ac - bd$ is not an inner product

since $\langle (1,1), (1,1)\rangle = 0$ and $(1,1) \neq 0$. (Property (4) is violated)

(b) $\langle A, B\rangle = tr(A+B)$ $A, B \in M_{2\times 2}(\mathbb{R})$

If $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $\langle A, A\rangle = 0$ and $A \neq 0$

(9) Let V be a finite dimensional vector space and $\beta = \{x_1 \ldots x_n\}$ be a basis for V.

Suppose $\langle x_i, y \rangle = 0 \quad i = 1 \ldots n$

Because β is a basis, we can write $y = a_1 x_1 + \ldots + a_n x_n$, $a_i \in F$.

then

$$\langle y, y \rangle = \langle a_1 x_1 + \ldots + a_n x_n, y \rangle$$
$$= a_1 \underbrace{\langle x_1, y \rangle}_{0} + \ldots + a_n \underbrace{\langle x_n, y \rangle}_{0} = 0$$

So $y = 0$ ∎

(17) Let $T : V \to V$ be linear operator, V inner product space.

Suppose $\|T(x)\| = \|x\|$.

In order to show that T is one-to-one, it suffices to show that $N(T) = \{0\}$ (ie, $Tx = 0 \Rightarrow x = 0$)

But if $Tx = 0$, then $\|Tx\| = 0 = \|x\|$.

But $\|x\| = \sqrt{\langle x, x \rangle} = 0 \Rightarrow \langle x, x \rangle = 0 \Rightarrow x = 0$ ∎

Section 6.2

(2) (a) $V=\mathbb{R}^3$, $S=\{\overbrace{(1,0,1)}^{w_1}, \overbrace{(0,1,1)}^{w_2}, \overbrace{(1,3,3)}^{w_3}\}$

Gram-Schmidt: $v_1 = w_1$

$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = (0,1,1) - \frac{1}{2}(1,0,1) = (-\tfrac{1}{2}, 1, \tfrac{1}{2})$$

$$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle w_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2$$

$$= (1,3,3) - \frac{4}{2}(1,0,1) - \frac{4}{3/2}(-\tfrac{1}{2}, 1, \tfrac{1}{2})$$

$$= (\tfrac{1}{3}, \tfrac{1}{3}, -\tfrac{1}{3})$$

Normalizing, we get orthonormal basis: $\left\{ \frac{1}{\sqrt{2}}(1,0,1), \sqrt{\frac{2}{3}}(-\frac{1}{2}, 1, \frac{1}{2}), \sqrt{3}(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}) \right\}$

Fourier coefficients of $x=(1,1,2)$: $a_1 = \frac{3}{\sqrt{2}}$, $a_2 = \frac{3}{2}\sqrt{\frac{2}{3}}$, $a_3 = 0$

(c) $V = P_2(\mathbb{R})$, $S = \{\overset{w_1}{1}, \overset{w_2}{x}, \overset{w_3}{x^2}\}$ $\quad \langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$

$v_1 = 1$

$v_2 = x - 1/2$

$$v_3 = x^2 - \frac{\langle x^2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle x^2, v_2\rangle}{\langle v_2, v_2\rangle} v_2 = x^2 - \frac{1}{3} - \frac{(\frac{1}{4}-\frac{1}{6})}{(\frac{1}{3}-\frac{1}{4})}(x - \tfrac{1}{2})$$

$$= x^2 - x + \frac{1}{6}$$

orthonormal basis:

$$\left\{ 1, \sqrt{12}(x - \tfrac{1}{2}), \sqrt{\tfrac{360}{211}}(x^2 - x + \tfrac{1}{6}) \right\}$$

(4) $S = \{\overbrace{(1,0,i)}^{v_1}, \overbrace{(1,2,1)}^{v_2}\} \subseteq \mathbb{C}^3$ $\qquad \langle x, y\rangle = x_1\overline{y_1} + x_2\overline{y_2} + x_3\overline{y_3}$

$$S^\perp = \{ v \in \mathbb{C}^3 \,/\, \langle v, v_1\rangle = 0, \langle v, v_2\rangle = 0 \}$$

for $v = (a,b,c)$, we have $v \in S^\perp \Leftrightarrow$ $\begin{aligned} a\cdot 1 + b\cdot 0 + c(-i) &= 0 \\ a + 2b + c &= 0 \end{aligned}$

Solving the system we find:

$$a = ic, \quad b = -\frac{(i+1)c}{2}$$

$$\therefore \quad S^{\perp} = \{ (a,b,c) \in \mathbb{C}^3 \,/\, a = ic,\ b = -\tfrac{(i+1)}{2}c \}$$

$$= \{ c\,(i, -\tfrac{(i+1)}{2}, 1),\ c \in \mathbb{C} \}$$

$\blacksquare$

(7) Suppose $\{w_1 \dots w_n\}$ is orthogonal, $w_i \neq 0$.

Let $\{v_1 \dots v_n\}$ be the vectors obtained by Gram-Schmidt.

We will show that $v_i = w_i$ by induction.

Proceeding by G-S, we set $v_1 = w_1$.

Now suppose $v_1 = w_1, \dots, v_{k-1} = w_{k-1}$

we must show that $v_k = w_k$.

By G-S,

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\langle v_j, v_j \rangle} v_j$$

But since $v_1 = w_1, \dots, v_{k-1} = w_{k-1}$

and $\{w_1 \dots w_{k-1}\}$ is orthogonal, $\langle w_k, v_j \rangle = 0 \ \forall j = 1 \dots k-1$.

So $v_k = w_k$

$\blacksquare$

(19) $V = C[-1,1]$, $\langle f, g \rangle = \int_{-1}^{1} f(t) g(t)\, dt$

Let $W = P_2(\mathbb{R}) \subseteq C[-1,1]$ subspace, $\beta = \{1, x, x^2\}$.

(a) Applying G-S to β, we get:

$$v_1 = 1$$

$$v_2 = x - \int_{-1}^{1} t\, dt = x$$

$$v_3 = x^2 - \frac{1}{2}\int_{-1}^{1} t^2\, dt - \cancel{\frac{\left(\int_{-1}^{1} t^3\, dt\right)}{\left(\int_{-1}^{1} t^2\, dt\right)}}^{\,0}\, x$$

$$= x^2 - \frac{1}{3}$$

$$\therefore \{1, x, x^2 - \tfrac{1}{3}\}$$

(b) Just normalize the vectors

$$\gamma = \left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\,x, \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) \right\}$$

(c) The closest second-degree polynomial to $h(t) = e^t$ is the orthogonal projection of $h(t)$ on $W = P_2(\mathbb{R})$.

The orthogonal projection of $h(t)$ on W is given by

$$u = \left\langle h(t), \frac{1}{\sqrt{2}} \right\rangle \frac{1}{\sqrt{2}} + \left\langle h(t), \sqrt{\frac{3}{2}}\,x \right\rangle \sqrt{\frac{3}{2}}\,x$$

$$+ \left\langle h(t), \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) \right\rangle \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) =$$

$$= \frac{1}{2}\langle h(t), 1 \rangle + \frac{3}{2}\langle h(t), x \rangle\, x + \frac{45}{8}\left\langle h(t), x^2 - \frac{1}{3} \right\rangle \left(x^2 - \frac{1}{3}\right)$$

From now on it's just calculus...

Section 6.3

(3)

(a) $V=\mathbb{R}^2$, $T(a,b) = (2a+b, a-3b)$, $x=(3,5)$

recall: $\langle Tu, x\rangle = \langle u, T^*x\rangle$,

$T^*x = \overline{g(e_1)}e_1 + \overline{g(e_2)}e_2$, where $g(u) = \langle Tu, x\rangle$

$\therefore$ $g(e_1) = \langle Te_1, x\rangle = \langle (2,1), (3,5)\rangle = 11$

$g(e_2) = \langle Te_2, x\rangle = \langle (1,-3), (3,5)\rangle = -12$

$\therefore$ $T^*x = \underline{(11,-12)}$

(b) $V = \mathbb{C}^2$, $T(z_1, z_2) = (2z_1 + iz_2, (1-i)z_1)$, $x = (3-i, 1+2i)$

$g(u) = \langle Tu, x\rangle$

$g(e_1) = \langle (2, (1-i)), (3-i, 1+2i)\rangle = 2(3+i) + (1-i)(1-2i)$

$= 6+2i+1-3i-2$

$= 5-i$

$g(e_2) = \langle (i, 0), (3-i, 1+2i)\rangle$

$= i(3+i) = -1+3i$

$T^*x = \overline{g(e_1)}e_1 + \overline{g(e_2)}e_2 = (5+i, -1-3i)$ //

(8) $T: V \to V$ linear operator, V f. dimensional.

T is invertible, hence $N(T) = \{0\}$ (ie, $Tu=0 \Rightarrow u=0$)

To show that T^* is invertible, it suffices to show that $R(T^*) = V$ (T^* is onto)

But $R(T^*) = V$ iff $R(T^*)^\perp = \{0\}$.

therefore, to show that T^* is invertible it suffices to show that $R(T^*)^\perp = \{0\}$.

Let then $v \in R(T^*)^\perp$. This means that

$$\langle v, T^*u \rangle = 0 \quad \forall u \in V.$$

But $\langle v, T^*u \rangle = \langle Tv, u \rangle$, so

$$\langle Tv, u \rangle = 0 \quad \forall u \in V$$

$$\Rightarrow Tv = 0 \Rightarrow v = 0 \quad \text{since } N(T) = \{0\}.$$

So $R(T^*)^\perp = \{0\}$ and T^* is invertible.

To see that $(T^*)^{-1} = (T^{-1})^*$, note that, for all $u, v \in V$,

$$\langle (T^{-1})^* T^* u, v \rangle = \langle T^* u, T^{-1} v \rangle$$
$$= \langle u, T T^{-1}(v) \rangle = \langle u, v \rangle.$$

Since this holds for all $v \in V$, we must have

$$(T^{-1})^* T^* u = u \quad \forall u \in V \Rightarrow (T^{-1})^* T^* = Id, \text{ or}$$
$$(T^{-1})^* = (T^*)^{-1} \quad \blacksquare$$

(14) V inner product space. Let $y, z \in V$.

Define $T: V \to V$ by $T(x) = \langle x, y \rangle z \quad \forall x \in V$.

Clearly $T(x_1 + c x_2) = \langle x_1 + c x_2, y \rangle z = \langle x_1, y \rangle z + c \langle x_2, y \rangle z$
$= T(x_1) + c T(x_2)$, so T is Linear.

$T^* v$ must satisfy $\langle Tx, v \rangle = \langle x, T^* v \rangle$.

But $\langle Tx, v \rangle = \langle \langle x, y \rangle z, v \rangle = \langle x, y \rangle \langle z, v \rangle$
$= \langle x, \overline{\langle z, v \rangle} y \rangle = \langle x, \langle v, z \rangle y \rangle$.

So $\boxed{T^* v = \langle v, z \rangle y}$ $\blacksquare$