Homework #4 - Solutions

Section 3.2 :

(5) (a) $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = A$

Since the 2 columns of A are linearly independent, $\operatorname{rank} A = 2$.

Thus A is invertible.

$$A^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

(c) $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 3 & -1 \end{pmatrix}$

After Gaussian elimination, we find that its reduced row echelon form is

$$B = \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

so $\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} \right\}$ is a basis for $R(A)$ and

$\operatorname{rank} A = 2$.

So A is not invertible.

(6) (a) $T : P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$, $T(f) = f'' + 2f' - f$

Let $\beta = \{1, x, x^2\}$ basis for $P_2(\mathbb{R})$.

We then have $[T]_\beta = \begin{pmatrix} -1 & 2 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & -1 \end{pmatrix}$

Since $[T]_\beta$ is invertible, <u>so is T</u>.

To find T^{-1}, recall that $[T^{-1}]_\beta = [T]_\beta^{-1}$

and

$$[T]_\beta^{-1} = \begin{pmatrix} -1 & -2 & -10 \\ 0 & -1 & -4 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\begin{tikzcd} P_2(\mathbb{R}) \arrow[r, "T^{-1}"] \arrow[d] & P_2(\mathbb{R}) \arrow[d] \\ \mathbb{R}^3 \arrow[r, "{[T^{-1}]_\beta}"'] & \mathbb{R}^3 \end{tikzcd}$$

It follows that

$$T^{-1}(1) = -1$$

$$T^{-1}(x) = -2 - x$$

$$T^{-1}(x^2) = -10 - 4x - x^2$$

and the <u>unique</u> linear transformations satisfying these conditions is

$$T^{-1}(f) = -f - 2f' - 5f''$$

[the solution $T^{-1}(a + bx + cx^2) = -a + (-2-x)b + c(-10-4x-x^2)$ is also correct.]

(b) $T : P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$, $\beta = \{1, x, x^2\}$

$$T(f)(x) = (x+1) f'(x)$$

$$[T]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

$[T]_\beta$ is <u>not</u> an invertible matrix, so

T is <u>not invertible</u>.

Section 3.4 :

(2) (a) The augmented matrix corresponding to the system is:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 2 & 1 & 1 \\ 3 & 5 & -2 & -1 \end{array}\right)$$

the reduced row echelon form is:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

So $x_1 = 4$, $x_2 = -3$, $x_3 = -1$

(f) the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 2 & 4 & -1 & 6 & 5 \\ 0 & 1 & 0 & 2 & 3 \end{array}\right)$$

the reduced row echelon form is:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

x_1 x_2 x_3

Hence, setting $x_4 = t$ (parameter), we have

$x_3 = 1$,
$x_2 = 1 - 2t$,
$x_1 = 1 + t$

so solutions have the form $(1+t, 1-2t, 1, t)$, $t \in \mathbb{R}$

the set of solutions is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} , t \in \mathbb{R} \right\}$

(6) Reduced row echelon form is

$$\begin{pmatrix} 1 & -3 & 0 & 4 & 0 & 5 \\ 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} | & | & | & | & | & | \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ | & | & | & | & | & | \end{pmatrix}$$

$$a_1 = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 3 \end{pmatrix} \quad a_3 = \begin{pmatrix} -1 \\ 1 \\ 2 \\ -4 \end{pmatrix} \quad a_6 = \begin{pmatrix} 3 \\ -9 \\ 2 \\ 5 \end{pmatrix}$$

From the reduced row echelon form of A, we know that

$$a_2 = -3a_1 \Rightarrow a_2 = \begin{pmatrix} -3 \\ 6 \\ 3 \\ -9 \end{pmatrix}$$

$$a_4 = 4a_1 + 3a_3 = \begin{pmatrix} 4 \\ -8 \\ -4 \\ 12 \end{pmatrix} + \begin{pmatrix} -3 \\ 3 \\ 6 \\ -12 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 2 \\ 0 \end{pmatrix} \quad \text{and}$$

$$a_6 = 5a_1 + 2a_3 - a_5 \quad \therefore$$

$$a_5 = 5a_1 + 2a_3 - a_6 = \begin{pmatrix} 5 \\ -10 \\ -5 \\ 15 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \\ 4 \\ -8 \end{pmatrix} - \begin{pmatrix} 3 \\ -9 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -3 \\ 2 \end{pmatrix}$$

$$\therefore \quad A = \begin{pmatrix} 1 & -3 & -1 & 1 & 0 & 3 \\ -2 & 6 & 1 & -5 & 1 & -9 \\ -1 & 3 & 2 & 2 & -3 & 2 \\ 3 & -9 & -4 & 0 & 2 & 5 \end{pmatrix}$$

(7) $u_1 = (2, -3, 1)$ $\quad u_2 = (1, 4, -2)$ $\quad u_3 = (-8, 12, -4)$ $\quad u_4 = (1, 37, -17)$ $\quad u_5 = (-3, -5, 8)$

Consider
$$\begin{pmatrix} 2 & 1 & -8 & 1 & -3 \\ -3 & 4 & 12 & 37 & -5 \\ 1 & -2 & -4 & -17 & 8 \end{pmatrix}$$

Its reduced row echelon form is :

$$\begin{pmatrix} 1 & 0 & * & * & 0 \\ 0 & 1 & * & * & 0 \\ 0 & 0 & * & * & 1 \end{pmatrix}$$

$\therefore$ $\{u_1, u_2, u_5\}$ form a basis.

Section 4.2

(13) $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ $\quad \det A = -8$

(21) $A = \begin{pmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{pmatrix}$

$$\det A = 1 \underbrace{\det\begin{pmatrix} 1 & 1 & 2 \\ 4 & -1 & 1 \\ 3 & 0 & 1 \end{pmatrix}}_{4} - 2 \underbrace{\det\begin{pmatrix} -3 & 1 & 2 \\ 0 & 4 & 1 \\ 2 & 3 & 1 \end{pmatrix}}_{-17} - 3 \underbrace{\det\begin{pmatrix} -3 & 1 & 1 \\ 0 & 4 & -1 \\ 2 & 3 & 0 \end{pmatrix}}_{-19}$$

$$= 4 + 34 + 57 = 95$$

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(23) Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & O & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix}$

$$\det A = a_{11} \cdot \det\begin{pmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ & O & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix} + 0 \quad \text{(expansion along first \underline{column})}$$

$$= a_{11} \cdot a_{22} \cdot \det\begin{pmatrix} a_{33} & \cdots & a_{3n} \\ O & \ddots & a_{nn} \end{pmatrix} = a_{11} \cdot a_{22} \cdots a_{nn}$$

(or one can prove it by induction)

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Section 4.3

(9) Suppose that $M \in M_{n\times n}(\mathbb{C})$ is nilpotent.

then $\exists$ positive integer k st. $\underbrace{M \dots M}_{k \text{ times}} = M^k = 0$

So

$$\det(M^k) = \det(0) = 0.$$

But $\det(M^k) = \det(M \dots M) = \det(M) \dots \det(M) = \det(M)^k$.

(since $\det(AB) = \det A \det B$)

$\therefore$

$$\det(M)^k = 0 \Rightarrow \det(M) = 0 \quad \left(\text{since } \det M \in \mathbb{C}\right)$$

(11) Suppose $Q \in M_{n\times n}(\mathbb{R})$ is orthogonal, ie $QQ^T = I_n$

So $\det(QQ^t) = \det(I_n) = 1.$

But $\det(QQ^t) = \det(Q)\det(Q^t) = \det(Q)^2$ (since $\det(Q^t) = \det(Q)$)

$\therefore$

$$\det(Q)^2 = 1 \Rightarrow \det Q = \pm 1$$

Section 5.1

(3)

(b) $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$ $\quad F = \mathbb{R}$

. characteristic polynomial: $p(t) = \det(A - tI) = -t^3 + 6t^2 - 11t + 6$

(i). eigenvalues: $\lambda_1 = 1$
$\lambda_2 = 2$
$\lambda_3 = 3$

(ii) . eigenvectors:

$\underline{\lambda = 1}$: Consider $A - I = \begin{pmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{pmatrix}$

$N(A-I) = \{ t(1,1,-1),\ t \in \mathbb{R} \}$

So the eigenvectors corresponding to 1 are of the form $t(1,1,-1)$, $t \neq 0$.

$\underline{\lambda = 2}$: $A - 2I = \begin{pmatrix} -2 & -2 & -3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{pmatrix}$

$N(A - 2I) = \{ t(1,-1,0),\ t \in \mathbb{R} \}$

Eigenvectors corresponding to 2 are $t(1,-1,0)$, $t \neq 0$.

$\underline{\lambda = 3}$:

$A - 3I = \begin{pmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{pmatrix}$

$N(A-3I) = \{ t(1,0,-1),\ t \in \mathbb{R} \}$

Eigenvectors are of the form $t(1,0,-1)$, $t \neq 0$.

(iii) $\beta = \{ (1,1,-1), (1,-1,0), (1,0,-1) \}$, $[A]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

(iv) $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$, $Q^{-1}AQ = D$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

(c)

$A = \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix}$ $F = \mathbb{C}$,

$p(t) = t^2 - 1$

(i) eigenvalues

$\lambda_1 = 1$, $\lambda_2 = -1$

(ii) eigenvectors

$\underline{\lambda_1 = 1}$: $A - I = \begin{pmatrix} i-1 & 1 \\ 2 & -i-1 \end{pmatrix}$

$N(A-I) = \{t(1, 1-i), t \in \mathbb{R}\}$

Eigenvectors are of the form $t(1, 1-i)$, $t \neq 0$

$\underline{\lambda_2 = -1}$: $A + I = \begin{pmatrix} i+1 & 1 \\ 2 & -i+1 \end{pmatrix}$

$N(A+I) = \{t(1, -1-i), t \in \mathbb{R}\}$,

Eigenvectors are $t(1, -1-i)$, $t \neq 0$.

(iii) $\beta = \{(1, 1-i), (1, -1-i)\}$, $[A]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(iv) $Q = \begin{pmatrix} 1 & -1-i \\ 1-i & 1 \end{pmatrix}$, $Q^{-1}AQ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(4)

$T: P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$

$$T(f(x)) = f(x) + x f'(x)$$

Let $\beta = \{1, x, x^2\}$.

then $[T]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. So $p(t) = (1-t)(2-t)(3-t)$

the eigenvalues of T are $1, 2, 3$.

the desired basis is just $\beta = \{1, x, x^2\}$.

(8) (a) Note that T is invertible $\underline{\text{iff}}$ $\det(T) \neq 0$.

In general, λ is eigenvalue of T iff $\det(T - \lambda Id) = 0$.

So 0 is not an eigenvalue iff $\det(T) \neq 0$.

Hence 0 is not an eigenvalue iff T is invertible.

(b) Suppose λ is eigenvalue of T.

Then $\exists\, v \neq 0$ with $Tv = \lambda v$.

If T is invertible, then $T^{-1}(Tv) = T^{-1}(\lambda v) = \lambda T^{-1} v$

$$\therefore \quad v = \lambda T^{-1} v$$

$\underline{\underline{\text{Since } \lambda \neq 0}}$ (for T is invertible, see (a)), we can write

$$T^{-1} v = \frac{1}{\lambda} v$$

which means that $\frac{1}{\lambda}$ is eigenvalue of T^{-1}.

The converse follows by $\underline{\text{symmetry}}$ of the statement.

$\underline{\text{Remark}}$: Note that we actually showed that if v is eigenvector of T corresponding to λ, then v is also an eigenvector of T^{-1} corresponding to $1/\lambda$.

Section 5.2

2 (d) $A = \begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix}$, $p(t) = (3-t)(t^2-2t-3)$

$= -(3-t)^2(t+1)$

So the eigenvalues are $\lambda_1 = 3$, $\lambda_2 = -1$

$\lambda_1 = 3$ has multiplicity 2

$\lambda_2 = -1$ has multiplicity 1

So A will be diagonalizable iff $\dim E_3 = 2$, $E_3 = N(A-3I)$.

$$A - 3I = \begin{pmatrix} 4 & -4 & 0 \\ 8 & -8 & 0 \\ 6 & -6 & 0 \end{pmatrix}$$

Hence $\dim N(A-3I) = 2$ and A is diagonalizable.

$N(A-3I) = \{ (t, t, s),\ t, s \in \mathbb{R} \}$

$= \{ t(1,1,0) + s(0,0,1),\ t, s \in \mathbb{R} \}$

basis for $E_3 = \{ (1,1,0), (0,0,1) \}$

$A + I = \begin{pmatrix} 8 & -4 & 0 \\ 8 & -4 & 0 \\ 6 & -6 & 2 \end{pmatrix}$, $N(A+I) = \{ t(1,2,3),\ t \in \mathbb{R} \}$

basis for $E_{-1} = \{ (1,2,3) \}$.

So if $Q = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$, then $Q^{-1}AQ = D$,

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

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③

ⓐ $T: P_3(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$, $T(f) = f' + f''$

Fix basis $\beta = \{1, x, x^2, x^3\}$

$$[T]_\beta = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 6 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \underline{p(t) = t^4}$$

the only eigenvalue is $\underline{\lambda = 0}$ with multiplicity 4.

$N(T - 0I) = N(T)$.

Clearly $\dim N(T) < 4$ (otherwise T would be the zero transformation).

So T is <u>not</u> diagonalizable.

ⓑ $T: P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$

$T(ax^2 + bx + c) = cx^2 + bx + a$

Fix $\beta = \{1, x, x^2\}$,

$$A = [T]_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad p(t) = -t(1-t)^2$$

Eigenvalues: $\lambda_1 = 0$, $\lambda_2 = 1$ (multiplicity 2)

- $N(A) = \{t(-1, 0, 1),\ t \in \mathbb{R}\}$

basis for $E_0 = \{(-1, 0, 1)\}$

- $N(A-I) = \{(0,t,s),\ t,s\in\mathbb{R}\}$ $\qquad A-I = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$= \{t(0,1,0) + s(0,0,1),\ t,s\in\mathbb{R}\}$

basis for $E_1 = \{(0,1,0), (0,0,1)\}$.

So $\dim E_1 = 2 =$ multiplicity of $\lambda=1$ $\Rightarrow$ A is diagonalizable

$\Rightarrow T$ is diagonalizable.

A is diagonal in the basis $\{(-1,0,1), (0,1,0), (0,0,1)\}$

So if $\beta' = \{-1+x^2,\ x,\ x^2\}$,

$$[T]_{\beta'} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ is diagonal}$$

(12) Let $T: V \to V$ be invertible, $\dim V = n < \infty$.

(a) Let λ be eigenvalue of T. We know by Ex. 8, sect. 5.1, that $\frac{1}{\lambda}$ is eigenvalue of T^{-1}. We must show that $E_\lambda = \tilde{E}_{1/\lambda}$,

$$E_\lambda = \{v\in V \,/\, Tv = \lambda v\},\quad \tilde{E}_{1/\lambda} = \{v\in V \,/\, T^{-1}v = \tfrac{1}{\lambda}v\}.$$

Note that if $v\in E_\lambda$, then $Tv = \lambda v \Rightarrow T^{-1}(Tv) = \lambda T^{-1}(v) \Rightarrow$

$$v = \lambda T^{-1}v \Rightarrow T^{-1}v = \tfrac{1}{\lambda}v$$

So $E_\lambda \subseteq \tilde{E}_{1/\lambda}$

We show that $\tilde{E}_{1/\lambda} \subseteq E_\lambda$ analogously. Then $E_\lambda = \tilde{E}_{1/\lambda}$.

(b) Suppose T is diagonalizable. Then there exists a basis $\{v_1, \dots v_n\}$ of V consisting of eigenvectors of T. But since each v_i is also an eigenvector of T^{-1}, T^{-1} is also diagonalizable.