Homework # 3 - Solutions - MATH 110

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Section 2.4 :

(8) Let $A, B \in M_{n\times n}(F)$, and suppose $AB = I_n$.

(a) We must show that A and B are invertible.

Thinking of A and B as linear transformations from $F^n \to F^n$

(in the notation of the book, $[L_A]_\beta = A$, $[L_B]_\beta = B$, β = canonical basis of F^n),

we see that A is invertible $\Leftrightarrow$ A is 1-1 $\Leftrightarrow$ A is onto (same for B), as a consequence of the dim theorem.

Claim: If $AB = I_n$, then B is 1-1.

Pf: Suppose $Bx = 0$.

$$\text{Then } A(Bx) = 0 \Rightarrow \underbrace{(AB)}_{I_n}x = 0 \Rightarrow x = 0. \blacksquare$$

As a consequence, B is invertible.

Claim: A is onto.

Pf: Any $y \in F^n$ can be written as

$$y = Ax, \text{ for } x = By. \blacksquare$$

So A is invertible.

(b) Since A, B are invertible and $AB = I_n$, we have

$$A^{-1}(AB) = A^{-1}I_n \Rightarrow B = A^{-1} \xrightarrow{\text{taking inverses}} B^{-1} = A \blacksquare$$

(13) $T: V \to W$ linear transf. V, W finite dimensional.

Let $\beta = \{v_1 \ldots v_n\}$ be a basis for V.

We must show that T is an isomorphism (= invertible = one-to-one + onto)

iff $T(\beta) = \{T(v_1), \ldots, T(v_n)\}$ is a basis for W.

($\Rightarrow$) Suppose T is an isomorphism.

To show that $\{T(v_1), \ldots, T(v_n)\}$ is a basis, we must show it's li and generating

- l.i: $a_1 T(v_1) + \ldots + a_n T(v_n) = 0$

$\Leftrightarrow T(a_1 v_1 + \ldots + a_n v_n) = 0$ since T is linear

But T is 1-1 $\Rightarrow a_1 v_1 + \ldots + a_n v_n = 0$

But $\{v_1 \ldots v_n\}$ are l.i. $\Rightarrow a_i = 0$.

So $\{T(v_1), \ldots T(v_n)\}$ is l.i.

- generating.

Let $w \in W$. We must show that w can be written as a linear combination of the $T(v_i)$, $i = 1 \ldots n$.

Since T is onto, $w = T(v)$ for some $v \in V$.

But $v = a_1 v_1 + \ldots + a_n v_n$, since $\{v_1 \ldots v_n\}$ is a basis

So $w = T(v) = T(a_1 v_1 + \ldots + a_n v_n) = a_1 T(v_1) + \ldots + a_n T(v_n)$

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($\Leftarrow$) Suppose $T(\beta) = \{T(v_1), \ldots, T(v_n)\}$ is a basis. We'll show T is an isomorphism, ie, 1-1 and onto.

1-1: $T(v) = 0 \Rightarrow$

$T(a_1 v_1 + \dots + a_n v_n) = 0$, $v = a_1 v_1 + \dots + a_n v_n$

$\Rightarrow a_1 T(v_1) + \dots + a_n T(v_n) = 0 \Rightarrow a_i = 0$ for $T(v_i)$ are l.i.

So $v = 0$.

onto: $w \in W$ can be written as

$w = a_1 T(v_1) + \dots + a_n T(v_n)$ for $\{T(v_1) \dots T(v_n)\}$ is a basis,

$\Rightarrow w = T(\underbrace{a_1 v_1 + \dots + a_n v_n}_{v}) = T(v)$

So T is onto.

(14) $\Phi : M_{n\times n}(F) \longrightarrow M_{n\times n}(F)$, $B \in M_{n\times n}(F)$ invertible

$\Phi(A) = B^{-1} A B$

We must show that Φ is an isomorphism, ie, Φ is 1-1 and onto.

1-1: $\Phi(A) = 0 \Leftrightarrow B^{-1} A B = 0 \Leftrightarrow \underbrace{B(B^{-1} A B) B^{-1}}_{= A} = 0$

So $A = 0. \Rightarrow \Phi$ is 1-1.

(this is enough by dimension thm)

But we can check Φ is onto directly:

$C \in M_{n\times n}(F)$, $C = \Phi(B C B^{-1})$,

since $\Phi(B C B^{-1}) = B^{-1}(B C B^{-1}) B = C$

Section 2.5 :

(2) (b) $\beta = \{(-1,3)\ (2,-1)\}$ $\beta' = \{(0,10)\ (5,0)\}$

$$[Id]_{\beta'}^{\beta} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

$(0,10) = 4(-1,3) + 2(2,-1)$

$(5,0) = (-1,3) + 3(2,-1)$

(c) $\beta = \{(2,5)\ (-1,-3)\}$ $\beta' = \{e_1, e_2\}$

$$[Id]_{\beta'}^{\beta} = \begin{bmatrix} 3 & -1 \\ 5 & -2 \end{bmatrix}$$

$(1,0) = 3(2,5) + 5(-1,-3)$

$(0,1) = -1(2,5) - 2(-1,-3)$

(4) $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a+b \\ a-3b \end{pmatrix}$$

Let $\beta = \{e_1, e_2\}$

$\beta' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

Note that $[T]_\beta = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$

$Q = [Id]_{\beta'}^{\beta} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

$Q^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

So $[T]_{\beta'} = Q^{-1}[T]_\beta Q$

$= \begin{pmatrix} 8 & 13 \\ -5 & 9 \end{pmatrix}$

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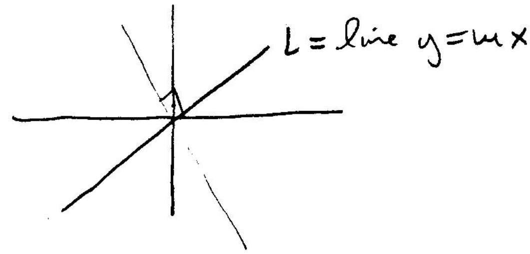
a



Let $\beta' = \{ \underbrace{(1,m)}_{v_1}, \underbrace{(-m,1)}_{v_2} \}$

If T is reflection about L, then

$$Tv_1 = v_1$$
$$Tv_2 = -v_2$$

So $[T]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Let $\beta = \{e_1, e_2\}$

$$Q = [Id]_{\beta'}^{\beta} = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}, \quad Q^{-1} = \frac{1}{1+m^2} \begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix}$$

So
$$[T]_\beta = Q [T]_{\beta'} Q^{-1}$$
$$= \frac{1}{m^2+1} \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix}$$
$$= \frac{1}{m^2+1} \begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$$

So $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1-m^2}{m^2+1} x + \frac{2m}{m^2+1} y \\ \frac{2m}{m^2+1} x + \frac{m^2-1}{m^2+1} y \end{pmatrix}$

b Exactly the same, but now $[T]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$