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MATH 110

Homework # 2

SUMMER SESSION

MATH 110

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Section 2.1

(2) $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$

- T is linear:

  - $T((a_1,a_2,a_3) + (b_1,b_2,b_3)) = T((a_1+b_1, a_2+b_2, a_3+b_3))$
  $$= ((a_1+b_1)-(a_2+b_2), 2(a_3+b_3))$$
  $$= (a_1-a_2, 2a_3) + (b_1-b_2, 2b_3)$$
  $$= T(a_1,a_2,a_3) + T(b_1,b_2,b_3)$$
  - $T(c(a_1,a_2,a_3)) = T(ca_1, ca_2, ca_3) = (ca_1 - ca_2, 2ca_3)$
  $$= c(a_1-a_2, 2a_3)$$
  $$= cT(a_1,a_2,a_3)$$

  So T is linear.

- N(T)

By definition, $(a_1, a_2, a_3) \in N(T) \iff T(a_1,a_2,a_3) = 0$

But $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$, so

$T(a_1,a_2,a_3) = 0 \iff a_1 = a_2, \ a_3 = 0$

So

$$N(T) = \{ (a_1,a_2,a_3) \in \mathbb{R}^3 \ / \ a_1 = a_2, a_3 = 0 \}$$
$$= \{ (a,a,0), a \in \mathbb{R} \}$$
$$= \{ a.(1,1,0), a \in \mathbb{R} \}$$

Thus $\{(1,1,0)\}$ is a basis for N(T).

- Nullity of T = 1

• R(T)

$$R(T) = \{ (a_1 - a_2, 2a_3) \ , \ a_1, a_2, a_3 \in \mathbb{R} \}$$

Note that $(a_1 - a_2, 2a_3) = a_1(1,0) + a_2(-1,0) + a_3(0,2)$

So $R(T)$ is generated by $\{(1,0)\ (-1,0)\ (0,2)\}$

and a basis for $R(T)$ is given by $\{(1,0)\ (0,2)\}$

• Rank $(T) = 2$

Check: $\text{nullity}(T) + \text{rank}(T) = 1 + 2 = 3 = \dim \mathbb{R}^3$ ✓

• Since $N(T) \neq \{0\}$, T is not one-to-one

$R(T) = \mathbb{R}^2 \Rightarrow T$ is onto

$\boxtimes$

(3) $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$

• T is linear. Computation just as in (2).

• N(T) $\quad (a_1, a_2) \in N(T) \iff (a_1 + a_2, 0, 2a_1 - a_2) = 0$

$$\iff \begin{cases} a_1 + a_2 = 0 \\ 2a_1 - a_2 = 0 \end{cases} \Rightarrow a_1 = 0, \ a_2 = 0$$

So $N(T) = \{0\}$ Basis for $N(T) = \phi$ (empty set)

• nullity of $T = 0$

• R(T) $\quad R(T) = \{ (a_1 + a_2, 0, 2a_1 - a_2) \}$

Note that $(a_1 + a_2, 0, 2a_1 - a_2) = a_1(1,0,2) + a_2(1,0,-1)$

So $\{(1,0,2), (1,0,-1)\}$ generate $R(T)$ and are l.i $\Rightarrow$ this set is a basis.

- rank $T = 2$

- nullity T + rank $T = 0 + 2 = 2 = \dim \mathbb{R}^2$

- Since $N(T) = \{0\}$, <u>T is one-to-one</u>

  $\dim R(T) = 2 < \dim \mathbb{R}^3$, T is <u>not</u> onto.

(4) $T : M_{2\times 3}(F) \longrightarrow M_{2\times 2}(F)$

$$T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}$$

- <u>T is linear</u> : check as in (2)

- <u>$N(T)$</u> : $T(A) = 0 \iff \begin{cases} 2a_{11} - a_{12} = 0 \\ a_{13} + 2a_{12} = 0 \end{cases} \iff \begin{array}{l} a_{11} = \frac{a_{12}}{2} \\ a_{13} = -2a_{12} \end{array}$

So $A \in N(T)$ has the form $A = \begin{pmatrix} \frac{a}{2} & a & -2a \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$, $a, a_{21}, a_{22}, a_{23} \in F$

We can write $A = a\begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} + a_{21}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + a_{22}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} +$

$+ a_{23}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Since $\left\{ \begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$ generates $N(T)$

and is l.i, it is a <u>basis</u>.

- <u>nullity T</u> $= 4$

- <u>$R(T)$</u> : $R(T) = \left\{ \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix} : a_{11}, a_{12}, a_{13} \in \mathbb{R} \right\}$

But $\begin{pmatrix} 2a_{11}-a_{12} & a_{13}+2a_{12} \\ 0 & 0 \end{pmatrix} = a_{11}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + a_{12}\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} + a_{13}\begin{pmatrix} 0 & \\ 0 & c \end{pmatrix}$

So $\left\{ \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}}_{v_2}, \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{v_3} \right\}$ generates $R(T)$, but it's <u>not</u> l.i.

note that $v_2 = -\frac{1}{2}v_1 + 2v_3$

Note that $\left\{ \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ is l.i. and still generates $R(T)$, so it's a basis

- Rank $(T) = 2$

- nullity T + rank T = $4 + 2 = 6 = \dim M_{3\times 2}(F)$ ✓

- Since $N(T) \neq \{0\}$, T is <u>not</u> one-to-one

  $\text{rank}(T) = 2 < \dim M_{2\times 2}(F) \Rightarrow T$ is <u>not</u> onto ∎

9 (a) $T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(a_1, a_2) = (1, a_2)$

Recall that any <u>linear</u> transformation must satisfy $T(0)=0$

Since $T(0,0) = (1,0) \neq (0,0)$, T is <u>not</u> linear.

(b) Note that $T((0,1)+(0,2))$

$= T(0,3) = (0,9)$

But $T(0,1) = (0,1)$, $T(0,2) = (0,4)$ $\therefore$ $T((0,1)+(0,2)) \neq T(0,1) + T$

and T is <u>NOT</u> linear.

(10) $T: \mathbb{R}^2 \to \mathbb{R}^2$ is linear

$T(1,0) = (1,4)$

$T(1,1) = (2,5)$

To compute $T(2,3)$, note that

$(2,3) = -(1,0) + 3(1,1)$

so $T(2,3) = T(-(1,0) + 3(1,1))$

$= -T(1,0) + 3T(1,1)$

$= -(1,4) + 3(2,5)$

$= (6,15) - (1,4) = (5,11)$

Since $(1,4)$ and $(2,5)$ are l.i., it follows that $\operatorname{rank}(T) = 2$.

By the dimension thm, $\operatorname{nullity}(T) = 0$ and therefore <u>T is 1-1</u>. ∎

<u>Section 2.2</u> :

(2) $T: \mathbb{R}^n \to \mathbb{R}^m$ $\quad \beta = \{e_1, \dots e_n\}$

$\gamma = \{e_1 \dots e_m\}$

(a) $T: \mathbb{R}^2 \to \mathbb{R}^3$

$T(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$

$T(1,0) = (2,3,1) = 2e_1 + 3e_2 + e_3$

$T(0,1) = (-1,4,0) = -1e_1 + 4e_2 + 0e_3$

$$[T]_\beta^\gamma = \begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}$$

(c) $T: \mathbb{R}^3 \to \mathbb{R}$, $T(a_1, a_2, a_3) = 2a_1 + a_2 - 3a_3$

$[T]_\beta^\gamma = (2, 1, -3)$

(3) $T : \mathbb{R}^2 \to \mathbb{R}^3$

$T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$

Let $\beta = \{e_1, e_2\}$,

$\gamma = \{(1,1,0), (0,1,1), (2,2,3)\}$

Note that: $T(e_1) = T(1,0) = (1,1,2) = a(1,1,0) + b(0,1,1) + c(2,2,3)$

solving it for a,b,c, we get $a = -\frac{1}{3}, b = 0, c = \frac{2}{3}$

so $T(e_1) = -\frac{1}{3}(1,1,0) + 0(0,1,1) + \frac{2}{3}(2,2,3)$

similarly, $T(e_2) = (-1, 0, 1) = -1(1,1,0) + (0,1,1) + 0(2,2,3)$

so $$[T]_\beta^\gamma = \begin{pmatrix} -1/3 & -1 \\ 0 & 1 \\ 2/3 & 0 \end{pmatrix}$$

- If $\alpha = \{\underset{v_1}{(1,2)}, \underset{v_2}{(2,3)}\}$, then

$T(v_1) = (-1, 1, 4) = -\frac{7}{3}(1,1,0) + 2(0,1,1) + \frac{2}{3}(2,2,3)$

$T(v_2) = (-1, 2, 7) = -\frac{11}{3}(1,1,0) + 3(0,1,1) + \frac{4}{3}(2,2,3)$

so $$[T]_\alpha^\gamma = \begin{pmatrix} -7/3 & -11/3 \\ 2 & 3 \\ 2/3 & 4/3 \end{pmatrix}$$ //

(4) $T : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$, $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + bx^2$

Let $\beta = \left\{ \underset{v_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{v_3}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{v_4}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right\}$, $\gamma = \{\underset{u_1}{1}, \underset{u_2}{x}, \underset{u_3}{x^2}\}$

- $T(v_1) = 1 = u_1 + 0u_2 + 0.u_3$
- $T(v_2) = 1 + x^2 = u_1 + 0\,u_3 + u_3$
- $T(v_3) = 0 = 0u_1 + 0u_2 + 0u_3$
- $T(v_4) = 2x = 0u_1 + 2u_2 + 0u_3$

$$\therefore \quad [T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$ //

Section 2.3 Let $g = 3+x$

(3) $T: P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$, $T(f) = f'g + 2f$

$U: P_2(\mathbb{R}) \longrightarrow \mathbb{R}^3$, $U(a+bx+cx^2) = (a+b, c, a-b)$

$\beta = \{1, x, x^2\}$

$\gamma = \{e_1, e_2, e_3\}$

(a) $[U]_\beta^\gamma$: $U(1) = (1,0,1) = e_1 + 0e_2 + e_3$

$U(x) = (1,0,-1) = e_1 + 0e_2 - e_3$

$U(x^2) = (0,1,0) = 0e_1 + e_2 + 0.e_3$

$$[U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$[T]_\beta$: $T(1) = 2 = 2.1 + 0.x + 0.x^2$

$T(x) = 3 + x + 2x = 3 + 3x = 3.1 + 3.x + 0.x^2$

$T(x^2) = 2x(3+x) + 2x^2 = 6x + 4x^2 = 0.1 + 6.x + 4.x^2$

$$[T]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}$$

$UT: P_2(\mathbb{R}) \longrightarrow \mathbb{R}^3$, $UT(a+bx+cx^2) = U((b+2cx)(3+x) + 2(a+bx+cx^2))$

$= U(3b + (b+6c)x + 2cx^2 + 2a + 2b\,x + 2c\,x^2)$

$= U((3b+2a) + (3b+6c)x + 4c\,x^2)$

$= (6b + 2a + 6c, 4c, 2a - 6c)$

$[UT]_\beta^\gamma$: $(UT)(1) = (2,0,2) = 2e_1 + 0e_2 + 2e_3$

$(UT)(x) = (6,0,0) = 6e_1 + 0e_2 + 0e_3$

$(UT)(x^2) = (6,4,-6) = 6e_1 + 4e_2 - 6e_3$

So

$$[UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}$$

$$= [U]_\beta^\gamma [T]_\beta \quad //$$

(b) $h(x) = 3 - 2x + x^2$

$[h]_\beta = (3, -2, 1)$

$[Uh]_\gamma = [(1, 1, 5)]_\gamma$

$= (1, 1, 5)$

Note that

$$[Uh]_\gamma = \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}}_{[U]_\beta^\gamma} \underbrace{\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}}_{[h]_\beta} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$$

4 (a) $T: M_{2\times 2}(F) \to M_{22}(F)$, $T(A) = A^t$ $\quad \alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$A = \begin{pmatrix} 1 & 4 \\ -1 & 6 \end{pmatrix}$,

By thm 2.14: $[T(A)]_\alpha = [T]_\alpha [A]_\alpha$

$$[T]_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad [A]_\alpha = \begin{pmatrix} 1 \\ 4 \\ -1 \\ 6 \end{pmatrix} \quad \therefore \quad [T(A)]_\alpha = \begin{pmatrix} 1 \\ -1 \\ 4 \\ 6 \end{pmatrix}$$

(b) $T: M_{2\times 2}(F) \to F$, $T(A) = tr(A)$, $\gamma = \{1\}$ basis for F

$$[T]_\alpha^\gamma = (1, 0, 0, 1), \quad [T(A)]_\gamma = [T]_\alpha^\gamma [A]_\alpha = (1, 0, 0, 1) \begin{pmatrix} 1 \\ 4 \\ -1 \\ 6 \end{pmatrix} = \quad //$$