Henrique Bursztyn
MATH 110

Homework 1 - Solutions

SUMMER SESSION

MATH 110

Section 1.2 :

p19
$ 0.32

10) Check properties of vector spaces...

13) No. For instance, if $(a_1,a_2)+(b_1,b_2)=(a_1,a_2)$, then

$$(b_1,b_2)=(0,1)$$

So $(0,1)$ would have to be the "zero element" in this space (VS3)

But note that $(0,0)+(a_1,a_2)\neq(0,1)$ for all (a_1,a_2).

So VS4) is not satisfied.

19) In any vector space, $v+v=2v$ $(v\in V)$, since $v+v=1.v+1.v=(1+1).v=2v$.

But $(a_1,a_2)+(a_1,a_2)\neq 2(a_1,a_2)$.

So this is NOT a vector space.

Section 1.3

10) You must check that : 1) $0\in W_1$ 2) $w_1, w_2\in W_1 \Rightarrow w_1+w_2\in W_1$

3) $w\in W_1$ $c\in F$ $\Rightarrow$ $cw\in W_1$.

For the second part, note for instance that $0\notin W_2$.

20) If $w_1\in W$, $a_1\in F$, then $a_1w_1\in W_1$ ((3) in (10)); if $a_1,a_2\in F$, $w_1,w_2\in W$, then $a_1w_1, a_2w_2\in W \Rightarrow a_1w_1+a_2w_2\in W$ ((2) in (10)). Similarly, if

$a_1w_1+\dots+a_{n-1}w_{n-1}\in W$, $a_n\in F$, $w_n\in W$, then $a_1w_1+\dots+a_nw_n\in F$ and the result follows by induction

(23) (a) $W_1+W_2=\{z\in V,\ z=x+y \text{ with } x\in W_1, y\in W_2\}$

Note that: • $0\in W_1+W_2$ since $0=0+0$, $0\in W_1$, $0\in W_2$

• If $z_1=x_1+y_1$, $z_2=x_2+y_2\in W_1+W_2$, $x_1,x_2\in W_1$, $y_1,y_2\in W_2$

then $z_1+z_2=\underbrace{(x_1+x_2)}_{\in W_1}+\underbrace{(y_1+y_2)}_{\in W_2} \Rightarrow z_1+z_2\in W_1+W_2$

• If $z\in W_1+W_2$, $c\in F$, $z=x+y$ $x\in W_1$, $y\in W_2$

then $cz=cx+cy$, $cx\in W_1$, $cy\in W_2\rightarrow$

$cz\in W_1+W_2$.

So W_1+W_2 is a subspace.

If $x\in W_1$, then $x=x+0$, $0\in W_2\Rightarrow x\in W_1+W_2$.

So $W_1\subseteq W_1+W_2$.

Same for W_2.

(b) If $W_3\subseteq V$ is subspace, $W_3\supseteq W_1$, $W_3\supseteq W_2$,

then for $z\in W_1+W_2$, $z=x+y$, $x\in W_1$, $y\in W_2$,

we have that $x\in W_3$ (since $W_1\subseteq W_3$) and $y\in W_3$.

W_3 subspace $\Rightarrow x+y=z\in W_3$.

So $W_1+W_2\subseteq W_3$ ∎

It's easy to check that W_1, W_2 are indeed subspaces (but you should do it!) of $V = F^n$.

If $(a_1, \ldots, a_n) \in F^n$, then $(a_1 \ldots a_n) = \underset{\substack{\cap \\ W_1}}{(a_1, \ldots, a_{n-1}, 0)} + \underset{\substack{\cap \\ W_2}}{(0, \ldots, 0, a_n)}$

- So $(a_1 \ldots a_n) \in W_1 + W_2$. Thus $V = W_1 + W_2$ (this shows that $V \subseteq W_1 + W_2$ but it's clear that $W_1 + W_2 \subseteq V$

- If $(a_1, \ldots, a_n) \in W_1 \cap W_2$, then

  $a_1 = \ldots = a_{n-1} = 0$ for $(a_1 \ldots a_n) \in W_1$

  and $a_n = 0$ for $(a_1 \ldots a_n) \in W_2$

  So $(a_1 \ldots a_n) = 0 \Rightarrow W_1 \cap W_2 = \{0\}$

So $\underline{V = W_1 \oplus W_2}$.

30) $(\Rightarrow)$ If $V = W_1 \oplus W_2$, suppose we can write $z = x_1 + y_1$, $z = x_2 + y_2$

$x_1, x_2 \in W_1$, $y_1, y_2 \in W_2$.

Then $0 = (x_1 - x_2) + (y_1 - y_2) \Rightarrow (x_1 - x_2) = -(y_1 - y_2)$

Since $x_1 - x_2 \in W_1$, $y_1 - y_2 \in W_2$, it follows that $(x_1 - x_2) \in W_1 \cap W_2$
$(y_1 - y_2) \in W_1 \cap W_2$

Since $W_1 \cap W_2 = \{0\}$, $x_1 - x_2 = 0 \Rightarrow x_1 = x_2$
$y_1 - y_2 = 0 \Rightarrow y_1 = y_2$ //

So the decomposition of z is unique.

$(\Leftarrow)$ Suppose $z \in W_1 \cap W_2$.

We can write $z = x + y$, $x \in W_1$, $y \in W_2$

Since $z \in W_1$, $x \in W_1$, $y = z - x \in W_1$. Hence $z = \tilde{x} + 0$, $\tilde{x} = (x+y) \in W$
$0 \in W_2$

By uniqueness of decomposition, $y = 0$.

Similarly, $x = 0$ and $z = 0$. So $W_1 \cap W_2 = \{0\}$ //

Section 1.4 :

(3) (a) the system is $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} \Rightarrow \begin{matrix} a_1 = 4 \\ a_2 = -3 \end{matrix}$

(b) the system is $\begin{pmatrix} -3 & 2 \\ 2 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \rightarrow \begin{matrix} a_1 = 5 \\ a_2 = 8 \end{matrix}$

(c) the system $\begin{pmatrix} 1 & -2 \\ -2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ has NO solution - so the first vector is not a combination of the others.

(4) Similar to (3):

(a) $a_1 = 3$
$a_2 = -2$

(b) No solution

(c) $a_1 = 4$
$a_2 = -3$

(8) $a M_1 + b M_2 + c M_3 = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ is symmetric so $\text{span}\{M_1, M_2, M_3\} \subseteq$ symm. 2x2 matrices

On the other hand, any symmetric 2x2 matrix

$\begin{pmatrix} a & c \\ c & b \end{pmatrix}$ can be written $aM_1 + bM_2 + cM_3$, so
symm. 2x2 matrices $\subseteq \text{span}\{M_1, M_2, M_3\}$

So $\text{span}\{M_1\ M_2\ M_3\} = \{2\times 2 \text{ symm. matrices}\}$.

(12) If $v \in \text{span}(S_1 \cup S_2)$, then $v = a_1 u_1 + \dots + a_n u_n$, $u_i \in S_1$ or $u_i \in S_2$.

Let $x = \sum a_{j_k} u_{j_k}$, $u_{j_k} \in S_1$, $y = \sum a_{i_k} u_{i_k}$, $u_{i_k} \in S_2$.

Then $v = x + y$, $x \in \text{span}(S_1)$, $y \in \text{span}(S_2)$

So $\text{span}(S_1 \cup S_2) \subseteq \text{span}\, S_1 + \text{span}\, S_2$

To show the reverse inclusion, note that if $v \in \operatorname{Span} S_1 + \operatorname{span} S_2$,

then $v = x + y$, $x \in \operatorname{Span} S_1$, $y \in \operatorname{Span} S_2$

So $x = a_1 v_1 + \cdots + a_n v_n$, $v_i \in S_1$

$y = b_1 u_1 + \cdots + b_k u_k$, $u_j \in S_2$

So $v = a_1 v_1 + \cdots + a_n v_n + b_1 u_1 + \cdots + b_k u_k$

$= \sum c_j w_j$ where $w_j \in S_1$ (if it's v_j)

or $w_j \in S_2$ (if it's a u_j)

Section 1.5

5) It's simple to check that $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ spans 2×2 diag matrices $\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$ and is linearly independent.

14) $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ \bigcirc & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix}$, $a_{ii} \neq 0$.

From $c_1 \begin{pmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} a_{12} \\ a_{22} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + c_n \begin{pmatrix} a_{1n} \\ \\ \\ \\ a_{nn} \end{pmatrix} = 0 \Rightarrow$

$a_{11} c_1 + \quad + a_{1n} c_n = 0 \quad (1)$

$a_{22} c_2 + \quad + a_{2n} c_n = 0 \quad (2)$

$\vdots$

$a_{n-1\,n-1} c_{n-1} + c_n a_{n-1\,n} = 0 \quad (n-$

$a_{nn} c_n = 0 \quad (n$

Since $a_{nn} \neq 0$ we have $c_n = 0$. (eq (n))

But $c_n = 0 \Rightarrow a_{n-1\,n-1} c_{n-1} = 0 \Rightarrow c_{n-1} = 0$ for $a_{n-1\,n-1} \neq 0$ (eq (n-1))

and so on.

17) $f(t) = e^{rt}$, $g(t) = e^{st}$. Suppose $c_1 f + c_2 g = 0$

This means that $c_1 f(t) + c_2 g(t) = 0 \ \forall t$. Setting $t = 0$, we have

$c_1 + c_2 = 0$, or $c_1 = -c_2 \therefore c_1 e^{rt} - c_1 e^{st} = c_1 (e^{rt} - e^{st}) = 0$

So either $e^{rt} = e^{st} \Rightarrow e^r = e^s \Rightarrow r = s$, which is not the case,

or $c_1 = 0 = c_2$.

Thus f, g are l.i. //

Section 1.6:

(2) Must check wheter the given set of vectors is l.i.

a) $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix} \right\}$ is l.i so form a basis of $\mathbb{R}^3$

b) $\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}$ is l.i, so it's a basis of $\mathbb{R}^3$.

12) Solutions of the system are of the form $t(1,1,1)$, $t \in \mathbb{R}$

So $\{(1,1,1)\}$ is a basis.

14) Matrices with zero trace are of the form

$$\left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \sum_{i=1}^{n} a_{ii} = 0 \right\} \quad \text{or} \quad \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & -\sum_{i=1}^{n-1} a_{ii} \end{pmatrix} \right\}$$

Let $M_{ij} = \begin{pmatrix} 0 & & & \\ 0 \cdots & 1 & \cdots 0 \\ 0 & & & 0 \end{pmatrix}$ (ij)-slot

A basis for the set of zero trace matrices is (check it!)

$$\underbrace{\begin{pmatrix} 1 & & 0 \\ & 0 & \\ 0 & & -1 \end{pmatrix}}_{M_{11} - M_{nn}}, \cdots, \underbrace{\begin{pmatrix} 0 & & 0 \\ & -1 & \\ 0 & & 1 \end{pmatrix}}_{M_{n-1\,n-1} - M_{nn}}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & & & 0 \\ 0 & & & 0 \end{pmatrix}, \cdots, \underbrace{\begin{pmatrix} 0 & & & 0 \\ 0 & \cdots & 0 \, 1 & 0 \end{pmatrix}}_{M_{ij} \;\; i \neq j}$$

$\beta = \{ M_{ii} - M_{nn},\ i = 1 \ldots n-1 \} \cup \{ M_{ij},\ i \neq j\ \ i = 1 \ldots n \}$

Counting basis elements, we get

$\dim \{\text{zero trace Matrices}\} = \underline{\underline{n^2 - 1}}$

29)

W_1, $\dim W_1 = m$ $\qquad m \geq n$

W_2, $\dim W_2 = n$

(a) $W_1 \cap W_2 \subseteq W_2$, subspace. So $\dim(W_1 \cap W_2) \leq \dim W_2 = n$

(recall: if $W \subseteq V$ subspace, $\dim W \leq \dim V$ - why?)

(b) If $\beta_1 = \{u_1 \dots u_m\}$ basis of W_1

$\beta_2 = \{v_1 \dots v_n\}$ basis for W_2

then $\beta_1 \cup \beta_2 = \{u_1 \dots u_m, v_1 \dots v_n\}$ generates $W_1 + W_2$ (not nec. l.i.) why?

Hence $\dim(W_1 + W_2) \leq n + m$

(recall: if $\{x_1 \dots x_k\}$ generates V then $\dim V \leq k$ - why?)

$\boxed{\text{IIII}}$