Mat 301: Final Worksheet

1) Prove that $GL(n, \mathbb{R})/SL(n, \mathbb{R}) \approx \mathbb{R}^*$.

2) Let $G = \mathbb{Z}_4 \oplus \mathbb{Z}_4$, and consider the subgroup $K = \{(0,0),(2,0),(0,2),(2,2)\}$. Write G/K as an internal direct product.

3) Is there a homomorphism from $U(5)$ *onto* $U(10)$? If so, how many?

4) Find a noncyclic subgroup of order 4 in $\mathbb{Z}_4 \oplus \mathbb{Z}_{10}$.

5) Write $\mathbb{Z}_{20} \oplus \mathbb{Z}_{30} \oplus \mathbb{Z}_{40} \oplus \mathbb{Z}_{50} \oplus \mathbb{Z}_{100}$ as a product of the form $\mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \ldots \oplus \mathbb{Z}_{m_k}$ where m_i divides m_{i-1}.

6) Are there more abelian groups of order 120 than abelian groups of order 54 (up to isomorphism)? Explain.

7) Prove that if K and H are normal subgrups of G, then $K \cap H$ is a normal subgrup of G.

8) Let K be a normal subgroup of G. If K is cyclic, prove that every subgroup of K is also normal in G. (Is this still true if K is not cyclic?)

9) Consider the permutations $\alpha = (34)(15)(2)(6)$, $\beta = (25)(13)(4)(6)$ in S_6. Show that they are conjugate to one another (i.e., find $\sigma \in S_6$ so that $\alpha = \sigma\beta\sigma^{-1}$. Show that there are no normal subgroups of order 2 in S_6.

10) Prove that $|\text{Inn}(G)| = 1$ if and only if G is abelian.

11) Let G be a finite group, and assume that there is a homomorphism from G onto $\mathbb{Z}_{10}$. Prove that G has normal subgroups of index 2 and 5.

12) Prove that $\mathbb{Z}_n$ has an even number of generators if $n > 2$.

13) Let G be a group of order 5, and suppose that $\phi : \mathbb{Z}_{30} \to G$ is a homomorphism which is onto. What is the kernel of ϕ? Explain.

14) Find a group of infinite order containing exactly three elements of order 2. Can this group be cyclic?

15) Let $G = S_3$, $H = \langle(123)\rangle$ and $K = \langle(12)\rangle$. Check that $G = HK$ and $H \cap K = \{(1)\}$. Is G isomorphic to $H \oplus K$? Explain.

16) Suppose $|G| = 168$ and G has more than six elements of order 7. How many subgroups of order 7 does G have?

17) Prove that $U(16)/\langle 7\rangle$ is cyclic.

18) Show that $\mathbb{Q}/\mathbb{Z}$ (under addition) is a group of infinite order in which all the elements have finite order.

19) Show that a group of order 99 has a normal subgroup of order 11.

20) Show that all proper subgroups of S_3 are cyclic.