Mat 363: Problem set # 1

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Due Friday, Jan. 21 in class

1) Consider the astroid $\gamma(t) = (\cos^3(t), \sin^3(t))$.

a) Show that $\|\dot{\gamma}(t)\| = \frac{3}{2}|\sin(2t)|$.

b) Show that the arclength function is given by $s(t) = \frac{3}{2}\sin^2(t)$, for $t \in [0, \pi/2]$. Find the total length of the curve when t goes from 0 to 2π.

c) Find the expression for $\tilde{\gamma}(s)$, the arclength reparametrization of $\gamma(t)$, $0 < t < \pi/2$.

2) Find the signed curvature κ_s and the equation of the evolute for the following curves:

a) the catenary $\gamma(t) = (t, \cosh(t))$.

b) the ellipse $\gamma(t) = (a\cos(t), b\sin(t))$.

3) As discussed in class, for a curve $\gamma(s)$ in $\mathbb{R}^n$ parametrized by arclength, its curvature is defined by $\|\gamma''(s)\|$. For an arbitrary regular curve $\gamma : (a, b) \to \mathbb{R}^n$, $n \geq 2$, prove that its curvature can be computed by

$$\kappa(t) = \frac{\sqrt{\|\dot{\gamma}\|^2\|\ddot{\gamma}\|^2 - \langle\dot{\gamma}, \ddot{\gamma}\rangle^2}}{\|\dot{\gamma}\|^3}. \tag{1}$$

Show also directly from (1) that this formula agrees with the one given in class for $n = 3$.

4) Fix $X, Y \in \mathbb{R}^n$, let $\gamma : [a, b] \to \mathbb{R}^n$ be any smooth curve satisfying $\gamma(a) = X, \gamma(b) = Y$.

a) Let $w \in \mathbb{R}^n$ be any constant vector with $\|w\| = 1$. Prove that

$$\langle Y - X, w\rangle = \int_a^b \langle\dot{\gamma}(t), w\rangle dt \leq \int_a^b \|\dot{\gamma}(t)\| dt.$$

(Hint: for the first equality, use the fundamental theorem of calculus).

b) Use a) to prove that $\|Y - X\| \leq \int_a^b \|\dot{\gamma}(t)\| dt$. Conclude that the curve of shortest length joining two points in $\mathbb{R}^n$ is the straight line joining them.

Bonus problem: Let $\gamma : (a, b) \to \mathbb{R}^2$ be a curve parametrized by arclength. Fix $s_0 \in (a, b)$, and suppose that $\kappa(s_0) \neq 0$. Consider $h_1, h_2 > 0$ two small positive real numbers, and let $\Gamma(h_1, h_2)$ be the circle passing through the points $\gamma(s_0)$, $\gamma(s_0 - h_1)$ and $\gamma(s_0 + h_2)$. Prove that the limit of $\Gamma(h_1, h_2)$ when h_1, h_2 go to zero (what's the meaning of this limit and why does it exist?) is the osculating circle at s_0.