Mat 363: Problem set # 5

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Due Wednesday, April 6

1) Consider a surface patch $\sigma : U \to \mathbb{R}^3$, and let γ be a smooth curve in $S = \sigma(U)$, $\gamma(t) = \sigma(u_1(t), u_2(t))$ (we write (u_1, u_2) for the coordinates in U). Let Γ^k_{ij} denote the associated Christoffel symbols.

a) Let Y be a smooth vector field along γ (i.e., $Y(t) = T_{\gamma(t)}S$ for all t), $Y(t) = \alpha_1(t)\sigma_{u_1} + \alpha_2(t)\sigma_{u_2}$. Prove that the covariant derivative of Y (recall the definition in Problem set #4) is given in coordinates by

$$\frac{DY}{dt} = \sum_{k=1}^{2}(\alpha_k' + \sum_{i,j=1}^{2} \alpha_j u_i' \Gamma^k_{ij})\sigma_{u_k}.$$

b) Prove that γ is a geodesic if and only if it satisfies the following equations:

$$u_k'' + \sum_{i,j=1}^{2} \Gamma^k_{ij} u_i' u_j' = 0, \qquad k = 1, 2.$$

(These equations are equivalent to the geodesic equations that we saw in class in terms of the first fundamental form.)

2) Use the local Gauss-Bonnet theorem to show that if a surface patch has gaussian curvature $K \leq 0$, then it cannot have simple closed geodesics. Why doesn't this apply to the parallels of a circular cylinder? Explain. (Recall that a cylinder is cointained in a single surface patch defined on an annulus.)

3) Let S be the ellipsoid defined by $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, $a, b, c > 0$, let K denote its gaussian curvature. Show that $\iint_S K dA = 4\pi$. Would this result be the same if $S = \{(x, y, z) \,|\, x^2 + y^2 + z^4 = 1\}$? Explain.

4) Let $\sigma : U \to \mathbb{R}^3$ be a surface patch. Suppose that ξ, ξ' are two nowhere vanishing vector fields on $S = \sigma(U)$, and let θ be the angle between them. Let $\gamma(s)$ be a unit-speed, positively-oriented closed simple curve in S with period T.

a) Prove that, along γ, $\theta'(s) = -f'/\sqrt{1 - f^2}$, where $f = \cos(\theta)$.

b) Use Green's theorem to prove that $\int_0^T \theta'(s)ds = 0$, and use this result to show that the definition of the index of a stationary point of a vector field is independent of the "reference vector".

c) Consider in $\mathbb{R}^2$ the vector field $V(x, y) = (\lambda_1 x, \lambda_2 y)$, where $\lambda_1, \lambda_2 \neq 0$. Prove that the index of $p = (0, 0)$ is 1 if λ_1 and λ_2 have the same sign, and -1 otherwise.

d) Can the index of a vector field at a stationary point be 0? If so, give an example.

Bonus problem: Let γ be a smooth simple closed curve on S^2, and let V be a vector field on S^2 never tangent to γ. Prove that each of the two regions determined by γ must contain at least one zero of V, and show (a picture will do it!) that, nonetheless, one can find a vector field on S^2 with only one zero.