Mat 363: Problem set # 4

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Due Wednesday, March 23

1) Let S be the pseudosphere with parametrization $\sigma(u,v) = (f(u)\cos(v), f(u)\sin(v), g(u))$, where $f(u) = e^u$ and $g(u) = \sqrt{1-e^{2u}} - \cosh^{-1}(e^{-u})$, for $-\infty < u < 0$. Calculate the principal curvatures of S and show that all its points are hyperbolic.

2) Let S be a smooth oriented surface with unit normal vector field $\mathbf{N}$. We call a regular curve $C \subset S$ a *line of curvature* if for each $p \in C$ the tangent line of C is a principal direction at p.

a) Check that C is a line of curvature if and only if, for any parametrization $\gamma(t)$ of C, we have $\mathbf{N}'(t) = \lambda(t)\gamma'(t)$, where $\mathbf{N}(t) = \mathbf{N}(\gamma(t))$ and λ is some smooth function of t. (In this case, $-\lambda(t)$ is the principal curvature along $\gamma'(t)$).

b) Prove that if S is a surface of revolution, then meridians and parallels are lines of curvature.

c) Show that a (nonrectilinear) geodesic is a line of curvature if and only if it is a plane curve, and give an example of a line of curvature which is a plane curve but not a geodesic.

3) Let S be a smooth oriented surface, fix $p \in S$ and $v \in T_pS$ ($v \neq 0$). Let $\kappa_n(\theta)$ be the normal curvature at p in the direction of a vector $w \in T_pS$ making an angle θ with v. Prove that the mean curvature of S at p is given by

$$H(p) = \frac{1}{\pi}\int_0^\pi \kappa_n(\theta)d\theta.$$

4) Compute the Gaussian curvature $K(u,v)$ for the Moebius strip with parametrization

$$\sigma(u,v) = ((1 + v\sin(u/2))\cos(u), (1 + v\sin(u/2))\sin(u), v\cos(u/2)).$$

5) We start with definitions: Let $\gamma(t)$ be a curve on a smooth surface S, and let Y be a smooth vector field along γ, i.e., $Y(t) \in T_{\gamma(t)}S$ for all t. The *covariant derivative* of Y is the vector field along γ defined by

$$\frac{DY}{dt}(t) := \Pi_{\gamma(t)}Y'(t),$$

where $\Pi_p : \mathbb{R}^3 \to T_pS$ is the orthogonal projection along the direction normal to S. The vector field Y is called *parallel* if $DY/dt = 0$ for all t. (Check that this gives the expected answer if S is a plane.) Let now $\gamma(t)$ be a curve in S, let $p = \gamma(t_0) \in S$ and fix $w_0 \in T_pS$. Using the theorem of existence and uniqueness of solutions of ODEs, one can show that there is a unique parallel vector field X along γ with $X(t_0) = w_0$. If $q = \gamma(t_1)$, then $w_1 := X(t_1)$ is called the *parallel transport* of w_0 along γ to q. The questions start now:

a) Check that a curve $\gamma(t)$ in S is a geodesic iff $\gamma'(t)$ is parallel along γ.

b) Let v and w be parallel fields along a curve γ in a smooth surface S. Prove that $\langle v(t), w(t)\rangle$ is constant (so parallel transport preserves length and angles).

c) Consider the unit sphere S^2 with outward orientation $\mathbf{N} : S^2 \to \mathbb{R}^3$. For each fixed $\phi \in [0, 2\pi)$, consider the curve $\gamma_\phi(t) = (\cos(\phi)\sin(t), \sin(\phi)\sin(t), \cos(t))$, $0 \le t \le \pi$, joining the north pole $p = (0, 0, 1)$ to the south pole $q = (0, 0, -1)$. Fix $w_0 = (1, 0, 0) \in T_pS^2$. Prove that the vector field $X_\phi(t) = (\cos(\phi))\gamma'_\phi(t) - (\sin(\phi))\mathbf{N} \times \gamma'_\phi(t)$ along γ_ϕ is parallel and use it to find the parallel transport of w_0 along γ_ϕ to q for $\phi = 0, \pi/4, \pi/2$. (Draw a picture.)

Bonus problem: Problem **5**), part c), illustrates parallel transport along geodesics on the sphere. Let us now consider a parallel C in the unit sphere S^2 of fixed colatitude θ (i.e, C is the trace of $(\cos(t)\sin(\theta), \sin(t)\sin(\theta), \cos(\theta))$, $0 \le t \le 2\pi$). Let w_0 be a unit vector tangent to C at some point p. Let $w_1 \in T_pS^2$ be the parallel transport of w_0 along C back to p after we go around C once. Prove that the (oriented) angle between w_0 and w_1 is $2\pi(1 - \cos\theta)$.
(Hint: You can try to do it directly or use the following argument: consider the cone which is tangent to the sphere at C; show that the parallel transport of w_0 along C is the same if you consider C as a curve in the sphere or in the cone; now to do parallel transport in the cone, recall that the cone can be "unwrapped" *isometrically* onto a sector of the plane (see the previous problem set), and parallel transport in the plane is just the usual notion.)