Mat 363: Problem set # 3

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Due Friday, March 4

1) Let S be the paraboloid $z = x^2 + y^2$, and consider the surface patch $\sigma(r, \varphi) = (r\cos(\varphi), r\sin(\varphi), r^2)$, for $r > 0$ and $0 < \varphi < 2\pi$.

a) Find E, F and G (coefficients of the first fundamental form) associated with σ.

b) Use that $dA = \sqrt{EG - F^2}$ to compute the area of the region of S defined by $z \leq h$ (for a fixed $h > 0$).

2) Let S be the one-sheeted cone (minus the vertex) $z = \sqrt{x^2 + y^2}$, $(x, y) \neq (0, 0)$. Let $U \subset \mathbb{R}^2 = \{(u, v)\}$ be the open set given by $u > 0$ and $0 < v < \pi\sqrt{2}$. Consider the following two maps from U into $\mathbb{R}^3$: $\sigma(u, v) = (\frac{u}{\sqrt{2}}\cos(v\sqrt{2}), \frac{u}{\sqrt{2}}\sin(v\sqrt{2}), \frac{u}{\sqrt{2}})$ and $\widetilde{\sigma}(u, v) = (u\cos(v), u\sin(v), 0)$.

a) Show that σ is a surface patch for S. How much of the cone does it cover? Compute the functions E, F and G in these coordinates.

b) Describe the image of $\widetilde{\sigma}$, and show that $\widetilde{E} = E$, $\widetilde{F} = F$, $\widetilde{G} = G$.

c) Write down an explicit isometry from $\sigma(U) \subset S$ onto an open subset of the xy-plane (and conclude that S is locally isometric to $\mathbb{R}^2$).

3) A diffeomorphism $f : S_1 \to S_2$ is called *area-preserving* if for each region $R \subset S_1$, the areas of R and $f(R)$ are equal. Show that a conformal map is area-preserving if and only if it is an isometry. (In particular, any isometry is area-preserving.)

4) Consider the surface patch of the sphere $\sigma(\theta, \varphi) = (\cos(\varphi)\sin(\theta), \sin(\varphi)\sin(\theta), \cos(\theta))$. Find E, F and G, and then:

a) For $\epsilon > 0$, find the area A of the region defined by $0 \leq \theta \leq \epsilon$, and the length l of the curve defined by $\theta = \epsilon$.

b) Check that l and A do not satisfy the isoparametric inequality, and (using problem **3)**) conclude that the sphere is *not* locally isometric to a plane.

Bonus problem: Let S^2 be the unit sphere. A *loxodrome* is a curve in S^2 that intersects the meridians at a fixed angle. A *loxodromic triangle* is a triangle in S^2 cut out by three loxodromes. Prove that if the triangle does not contain the poles $\pm(0, 0, 1)$ then the sum of its interior angles is π.
(Hint: the change of variables $u = \log(\tan(\frac{1}{2}\theta))$, $0 < \theta < \pi$, and $v = \varphi$ in the patch σ of problem **4)** defines a new surface patch $\widetilde{\sigma}(u, v)$ of S^2 known as the *Mercator's projection* (see book, p.p. 83); show that $\widetilde{\sigma}$ is conformal and that loxodromes correspond to straight lines in the uv-plane under $\widetilde{\sigma}$).