Mat 363: Problem set # 2

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Due Monday, Feb. 7 in class

1) We saw in class that if 0 is a regular value of a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$, then $f^{-1}(0) \subseteq \mathbb{R}^3$ is a smooth surface. Find an example of a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$ for which 0 is *not* a regular value but $f^{-1}(0)$ is still a smooth surface.

2) Let $a \neq 0$, and consider the space curves $p(t) = (0, 0, t)$ and $q(t) = (a, t, 0)$, $t \in \mathbb{R}$. Show that the straight line through $p(t)$ and $q(t)$ describes a subset of $\mathbb{R}^3$ given by $y(x - a) + zx = 0$. Is this a smooth surface?

3) Let R, r be positive real numbers with $r < R$. Consider the 2-torus T^2 defined by rotating the circle in the xz-plane with center $(R, 0, 0)$ and radius r around the z-axis.

a) Show geometrically that a point (x, y, z) is in T^2 if and only if it satisfies

$$z^2 + (R - \sqrt{x^2 + y^2})^2 = r^2.$$

b) Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$,

$$f(x, y, z) = (x^2 + y^2 + z^2 + R^2 - r^2)^2 - 4R^2(x^2 + y^2).$$

Conclude from part a) that $T^2 = f^{-1}(0)$.

c) Prove that T^2 is a smooth surface by showing that 0 is a regular value for the function f in part b).

4) Consider the unit sphere $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. One way to define an atlas for S^2 is to use the *stereographic projection* $g : \mathbb{R}^3 \backslash \{(0, 0, 1)\} \to \mathbb{R}^2$, where g sends a point $p \in \mathbb{R}^3 \backslash \{(0, 0, 1)\}$ to the point of intersection of the xy-plane with the straight line through p and $(0, 0, 1)$. Let π be the restriction of g to $S^2 \backslash \{(0, 0, 1)\}$.

a) Prove that $\pi(x, y, z) = \frac{(x,y)}{1-z}$

b) Prove that π is invertible and find an expression for the surface patch $\sigma = \pi^{-1} : \mathbb{R}^2 \to S^2 \backslash \{(0, 0, 1)\}$.

Bonus problem: Prove that $S \subseteq \mathbb{R}^3$ is a smooth surface if and only if each point $p \in S$ has an open neighborhood V so that $S \cap V$ is the graph of a function of the form $z = h(x, y)$ or $y = g(x, z)$ or $x = f(y, z)$.