Math 110: Midterm II

July 31, 2001

Name:

Instructions: This is a closed book exam, and you will have 60 minutes. Please write your name on every page you use. **Calculators are not allowed!** Make sure to show your work clearly. The exam is worth **40 points**.
Good luck!

Problem 1 (14 pts): Determine whether the following statements are **true** or **false**, giving a short (one sentence) justification for your answer. Make sure to write your answer clearly, otherwise it will not be graded.

In the following, V will denote a finite dimensional vector space.

a) The vector spaces $M_{4\times 7}(\mathbb{R})$ and $\mathbb{R}^{11}$ are isomorphic.

Answer:

b) Let $T : V \longrightarrow V$ be a linear operator. Let $A = [T]_\beta$, where β is a basis for V. If A has linearly independent columns, then T is invertible.

Answer:

c) Let $T : V \longrightarrow V$ be a linear operator, and let β and β' be bases for V. Suppose $Q = [\mathrm{Id}]_{\beta}^{\beta'}$. Then $[T]_\beta = Q[T]_{\beta'}Q^{-1}$.

Answer:

d) Let $A \in M_{n\times n}(\mathbb{C})$. If $A^7 = 0$, then A is not invertible.

Answer:

e) The matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is diagonalizable.

Answer:

f) Let W_1 and W_2 be subspaces of V. Then $(W_1 + W_2)^\perp = W_1^\perp + W_2^\perp$.

Answer:

g) Let T and U be linear operators on V. Suppose λ is an eigenvalue of T and μ is an eigenvalue of U. Then $\lambda + \mu$ is an eigenvalue of $T + U$.

Answer:

Problem 2 (7 pts): Let $T : P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ be defined by $T(p(x)) = p(x)+(1+x)p'(x)$.

a) Find the eigenvalues of T.

b) Is T diagonalizable? If so, find a basis β for $P_2(\mathbb{R})$ so that $[T]_\beta$ is diagonal.

c) Is T invertible? (You should be able to answer it with no computations, just by using what you found in a).)

Problem 3 (7 pts): Let $V = \mathbb{R}^3$, and let W be the subspace spanned by $v_1 = (1, 1, 0)$ and $v_2 = (1, 2, 1)$. Define $\langle x, y\rangle = x_1y_1 + 2x_2y_2 + 2x_3y_3$, and consider the inner product space $(\mathbb{R}^3, \langle\, ,\, \rangle)$

a) Find an orthonormal basis for $W^\perp$.

b) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation given by $T(x) = u$, where u is the orthogonal projection of x on W. Find the expression of T.

c) Is T diagonalizable? invertible? (You should be able to answer this question with no further computations.)

Problem 4 (7 pts):

a) Let $A \in M_{n\times n}(F)$ with characteristic polynomial $p(t) = (\lambda_1 - t)\ldots(\lambda_n - t)$, where $\lambda_1 \ldots \lambda_n$ are distinct. Prove that $\det(A) = \lambda_1 \ldots \lambda_n$.

b) Find an example of two **non-similar** matrices with the same characteristic polynomial.

Problem 5 (5 pts): Let V be a finite dimensional vector space, and let $T : V \longrightarrow V$ and $U : V \longrightarrow V$ be linear operators. Suppose that $\beta = \{v_1, \ldots, v_n\}$ is a basis for V and that each v_i is an eigenvector for both T and U. Prove that $TU = UT$.