Math 110: Final Exam

August 16, 2001

Name:

Instructions: This is a closed book exam, and you will have 2 hours. Please write your name on every page you use. **Calculators are not allowed!** Make sure to show your work clearly. The exam is worth **80 points**.
Good luck!

Problem 1 (14 pts): Determine whether the following statements are **true** or **false**, giving a short (one sentence) justification for your answers. Make sure to write your answers clearly, otherwise it will not be graded.

In the following, V will denote a finite dimensional vector space.

a) The vector space $M_{2\times 2}(\mathbb{C})$ (over $\mathbb{R}$) is isomorphic to $\mathbb{R}^8$.

Answer:

b) Let W_1 and W_2 be subspaces of V. If $V = W_1 + W_2$, then $\dim(V) = \dim(W_1) + \dim(W_2)$.

Answer:

c) If $A \in M_{n\times n}(\mathbb{C})$, then $\det(cA) = c\det(A)$, $c \in \mathbb{C}$.

Answer:

d) If A, B are self-adjoint $n \times n$ matrices, then AB is also self-adjoint.

Answer:

e) For any inner product on V, $V^\perp = \{0\}$.

Answer:

f) If $T : V \longrightarrow V$ is a normal operator, then so is $T - \mu \mathrm{Id}$, where μ is a scalar.

Answer:

g) Let $T : V \longrightarrow V$ be a linear operator. If T is diagonalizable, then V has an orthonormal basis consisting of eigenvectors of T.

Answer:

Problem 2 (16 pts): Provide examples of the following, justifying your answer.

a) A linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that $\mathbb{R}^3 = N(T) \oplus R(T)$

b) A linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $N(T) = R(T)$.

c) Four non-similar matrices with characteristic polynomial $p(t) = (1-t)^2(2-t)^2$.

d) A matrix $A \in M_{n\times n}$ with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$ and corresponding eigenvectors $v_1 = (1,0)$ and $v_2 = (1,2)$.

Problem 3 (10 pts):
Let $\beta = \{1, 2+x\}$ and $\gamma = \{x, 1+x\}$ be bases for $P_1(\mathbb{R})$. Let $T : P_1(\mathbb{R}) \longrightarrow P_1(\mathbb{R})$ be a linear transformation such that $[T]_\beta^\gamma = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$. Find $T(1)$, $T(x)$ and $T(p(x))$ for an arbitrary $p(x) = a + bx$.

Problem 4 (8 pts):

a) Prove that if $A, B \in M_{n\times n}(\mathbb{C})$ are similar matrices, then $\operatorname{tr}(A) = \operatorname{tr}(B)$.

b) Let $A \in M_{n\times n}(\mathbb{C})$, and suppose $\lambda_1, \ldots, \lambda_k$ are its distincts eigenvalues, with multiplicities $m_1, \ldots, m_k$. Prove that $\operatorname{tr}(A) = m_1\lambda_1 + \ldots + m_k\lambda_k$.

   Hint: Use part a)

Problem 5 (7 pts): Suppose V is a finite dimensional vector space, and let $T : V \longrightarrow V$ be a linear operator. Suppose $T^p = 0$, and assume that there exists $x \in V$ satisfying $T^{p-1}x \neq 0$. Prove that the subspace $W = \operatorname{span}(\{x, Tx, \ldots, T^{p-1}x\})$ has dimension p.
Hint: Show that $\{x, Tx, \ldots, T^{p-1}x\}$ is a basis for W.

Problem 6 (7 pts): Let V be a finite dimensional inner product space. Let $T : V \longrightarrow V$ be a linear operator. Prove that $N(T) = R(T^*)^\perp$.

Problem 7 (10 pts):

Find the Jordan form and Jordan basis of $A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix}$ (check your answer!).

Problem 8 (8 pts): Let

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Find an orthogonal matrix P and a diagonal matrix D such that $P^tAP = D$.
Find the spectral decomposition of A.