On gauge transformations of Poisson structures

Henrique Bursztyn*

Department of Mathematics
University of Toronto
Toronto, Ontario M5S 3G3, Canada

Summary. We discuss various questions in Poisson geometry centered around the notion of gauge transformations associated with 2-forms. The topics in this note include the relationship between gauge transformations and Morita equivalence of Poisson manifolds, gauge transformations of Lie bialgebroids and Poisson groupoids and integration of Dirac structures.

1 Introduction

Connections between Poisson geometry and topological sigma-models have led to the notion of Poisson structures "twisted" by closed 3-forms, see e.g. [23, 33]. Gauge transformations of Poisson structures associated with 2-forms were used in [38] as a tool to study this more general type of Poisson geometry. Roughly speaking, a gauge transformation modifies a given Poisson structure by adding to its leafwise symplectic structure the pullback of a globally defined 2-form. In this paper, we discuss several topics in Poisson geometry revolving around this operation.

The relationship between gauge transformations and Xu's Morita equivalence of Poisson manifolds [44] is suggested by their similar role as Poisson analogues of Morita equivalence of algebras [32, 35], a key notion of equivalence in noncommutative geometry and string theory, see e.g. [15, 37]. On one hand, Xu's purely geometric Morita theory for Poisson manifolds strongly resembles the one for (C^*-)algebras, see e.g. [25]; in fact, at least in some examples, there are quantization procedures concretely relating them [26], though a general correspondence is yet to be found [27]. On the other hand, in the framework of formal deformation quantization [3, 24], two formal Poisson structures are quantized to Morita equivalent algebras only when they are gauge equivalent with respect to an integral 2-form [9, 10, 22]. The connection between Xu's Morita equivalence and gauge transformations was unraveled in [8], and we will recall it in Section 3.

Xu's geometric Morita theory for Poisson manifolds is closely related to the theory of symplectic groupoids [16, 40, 45]. Since symplectic groupoids are objects "integrating" Poisson structures (in a sense analogous to the integration of Lie algebras to Lie groups), clarifying how gauge transformations relate to Xu's Morita equivalence naturally leads one to consider the "integration" of gauge transformations to the level of symplectic groupoids. We will address this issue in two ways: first, in Section 3, we revisit the results of [8]; then, in Section 4, we present a more general approach extending the notion of gauge transformation to Lie bialgebroids and studying its global counterpart on Poisson groupoids.

The most natural framework for the study of gauge transformations, to be recalled in Section 2, is that of *Dirac structures* [17], a common generalization of Poisson structures, closed 2-forms and regular foliations. It turns out that the study of gauge transformations on symplectic groupoids sheds light on an interesting question in Poisson geometry: What are the global objects "integrating" Dirac manifolds? The solution to this problem is given by the *presymplectic groupoids* of [7], which we recall in Section 5 along with the closely related notion of *presymplectic realization*. We also illustrate, following [7, Sec. 7] (see

* `henrique@math.toronto.edu`

also [46]), interesting connections between these objects and certain generalized notions of hamiltonian actions and momentum maps in symplectic geometry [1, 2].

As outlined above, this paper mostly reviews the results in [7, 8]; the only exception is Section 4, which contains a more general approach to [8, Thm. 4.1] leading to interesting objects (multiplicative Dirac structures) generalizing Poisson and presymplectic groupoids [5].

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2 Gauge transformations and Dirac structures

2.1 Gauge transformations of Poisson structures

Let P be a smooth manifold. To fix our notation and terminology, recall that a bivector field $\pi \in \Gamma^\infty(\wedge^2 TP)$ is a **Poisson structure** on P if the bracket $\{f,g\} := \pi(df,dg)$ defines a Lie bracket on $C^\infty(P)$; this condition is equivalent to $[\pi,\pi]_s = 0$, where $[\cdot,\cdot]_s$ is the Schouten bracket. We refer the reader to [12] for definitions and details.

Let (P,π) be a Poisson manifold and consider a *closed* 2-form $B \in \Omega^2(P)$. We will also refer to closed 2-forms as **presymplectic**. Let us consider the addition of the pullback of B to the leafwise symplectic form of π. We call B **π-admissible** if the resulting leafwise 2-form is again symplectic (note that it is always closed, but not necessarily nondegenerate). In this case, this new symplectic foliation on P is associated with a global Poisson structure, which we denote by $\tau_B(\pi)$. The map $\pi \mapsto \tau_B(\pi)$ is a **gauge transformation** of π associated with B.

If B is not π-admissible, then the procedure described in the previous paragraph results in a *presymplectic foliation* of P, which does not correspond to any Poisson structure. We will see, however, that there is a more general geometric structure underlying it.

A convenient way of describing gauge transformations is by means of the bundle maps $\widetilde{\pi} : T^*P \to TP$ and $\widetilde{B} : TP \to T^*P$, defined by $\widetilde{\pi}(\alpha) := \pi(\cdot,\alpha)$ and $\widetilde{B}(X) := B(X,\cdot)$. In these terms, B is π-admissible if and only if

$$\mathrm{Id} + \widetilde{B}\widetilde{\pi} : T^*P \longrightarrow T^*P \tag{1}$$

is invertible. In this case, $\tau_B(\pi)$ is completely determined by the condition that its associated bundle map satisfies

$$\widetilde{\tau_B(\pi)} = \widetilde{\pi}(\mathrm{Id} + \widetilde{B}\widetilde{\pi})^{-1}. \tag{2}$$

In particular, if π is nondegenerate, then so is $\tau_B(\pi)$, and (2) simply says that

$$\widetilde{\tau_B(\pi)}^{-1} = \widetilde{\pi}^{-1} + \widetilde{B}.$$

So any two symplectic structures on a manifold P are gauge equivalent.

2.2 Poisson structures as subbundles of $TP \oplus T^*P$

A key step in unraveling the geometric structure underlying the presymplectic foliations of the previous section is noticing that Poisson structures on P can be completely described as certain subbundles of $E := TP \oplus T^*P$: for each Poisson structure π on P, we set

$$L_\pi := \mathrm{graph}(\widetilde{\pi}) = \{(\widetilde{\pi}(\alpha),\alpha) \mid \alpha \in T^*P\} \subset E. \tag{3}$$

The result of a gauge transformation of π is represented, in these terms, as the subbundle

$$\tau_B(L_\pi) := \{(\widetilde{\pi}(\alpha), \alpha + \widetilde{B}(\widetilde{\pi}(\alpha))) \mid \alpha \in T^*P\} \subset E. \tag{4}$$

As a subbundle, (4) is well defined for any B, though it will be the graph associated with another Poisson structure if and only if $\tau_B(L_\pi) \cap TP = \{0\}$ at all points of P, which is equivalent to B being π-admissible. In this case,

$$\tau_B(L_\pi) = L_{\tau_B(\pi)} = \mathrm{graph}(\widetilde{\tau_B(\pi)}).$$

A natural question now is finding an intrinsic characterization of the subbundles of E of the forms (3) and (4). In order to detect them, we need to consider the following extra structure on the "ambient" bundle E:

1) The symmetric pairing $\langle \cdot, \cdot \rangle : \Gamma^\infty(E) \times \Gamma^\infty(E) \to C^\infty(P)$,

$$\langle (X, \alpha), (Y, \beta) \rangle := \beta(X) + \alpha(Y); \tag{5}$$

2) The bracket $[\cdot, \cdot] : \Gamma^\infty(E) \times \Gamma^\infty(E) \to \Gamma^\infty(E)$,

$$[(X, \alpha), (Y, \beta)] := ([X, Y], \mathcal{L}_X \beta - i_Y d\alpha). \tag{6}$$

The bracket in 2) is the standard **Courant bracket** on E [17], written in its non-skew-symmetric version of [28, 38]. The bundle E, together with $\langle \cdot, \cdot \rangle$ and $[\cdot, \cdot]$, is an example of a *Courant algebroid* [28]. The next result follows from [17].

Proposition 1. *A subbundle $L \subset TP \oplus T^*P$ is of the form $L_\pi = \mathrm{graph}(\widetilde{\pi})$ for a bivector field π if and only if*

i) $\mathrm{rank}(L) = \dim(P)$*;*
ii) $TP \cap L = \{0\}$ *at all points of* P*;*
iii) L *is isotropic with respect to* $\langle \cdot, \cdot \rangle$*, i.e.,* $\langle (X, \alpha), (Y, \beta) \rangle = 0$ *whenever* $(X, \alpha), (Y, \beta) \in L$*;*

In this case, π is a Poisson structure if and only if

iv) $\Gamma^\infty(L)$ *is closed under the Courant bracket* $[\cdot, \cdot]$.

Conditions *i)* and *ii)* say that L is the graph of a bundle map $T^*P \to TP$, which is always associated with a (2,0)-tensor $\pi \in \Gamma^\infty(TP \otimes TP)$; condition *iii)* is then equivalent to π being skew symmetric, i.e., a bivector field. The fact that *iv)* amounts to the integrability condition $[\pi, \pi]_s = 0$ is shown in [17].

Remark 1. Since the symmetric pairing $\langle \cdot, \cdot \rangle$ has signature zero, conditions *i)* and *ii)* of Proposition 1 are equivalent to L being *maximally isotropic with respect to* $\langle \cdot, \cdot \rangle$.

A direct computation shows that subbundles of the form (4) satisfy all the conditions in Proposition 1 except for *ii)*, which holds if and only if B is π-admissible. This leads us to the geometric object we are after.

2.3 Dirac structures and Lie algebroids

A **Dirac structure** on a smooth manifold P is a subbundle of $E = TP \oplus T^*P$ satisfying conditions *i)*, *iii)* and *iv)* of Prop. 1; in other words, a Dirac structure is a subbundle L of E which is maximally isotropic with respect to $\langle \cdot, \cdot \rangle$ and for which $\Gamma^\infty(L)$ is closed under the Courant bracket $[\cdot, \cdot]$. The set of Dirac structures on P is denoted by $\mathrm{Dir}(P)$.

We can naturally generalize gauge transformations from Poisson to Dirac structures [38]: for $L \in \mathrm{Dir}(P)$ and $B \in \Omega^2(P)$ closed, we set

$$\tau_B(L) := \{(X, \alpha + \widetilde{B}(X)) \mid (X, \alpha) \in L\}.$$

In this extended sense, **gauge transformations** define an action of the abelian group of closed 2-form on $\mathrm{Dir}(P)$.

Of course, Poisson structures are Dirac structures intersecting TP trivially. Let us illustrate Dirac structures with two other examples.

Example 1. (Closed 2-forms)
Maximally isotropic subbundles L of $(E, \langle\cdot,\cdot\rangle)$ satisfying

$$L \cap T^*P = \{0\} \tag{7}$$

at all points of P are precisely the graphs associated with 2-forms $\omega \in \Omega^2(P)$; in this case, condition *iv)* of Prop. 1 translates into $d\omega = 0$. So closed 2-forms correspond to Dirac structures intersecting T^*P trivially.

Example 2. (Regular foliations)
Let F be a subbundle of TP, and let F° denote the subbundle of T^*P annihilating F. Then $L = F \oplus F^\circ$ is clearly isotropic with respect to $\langle\cdot,\cdot\rangle$, and $\mathrm{rank}(L) = \dim(P)$. In this example, a direct computation shows that the "Courant-bracket condition", *iv)* of Prop. 1, becomes the usual integrability condition of Frobenius: if $X, Y \in \Gamma^\infty(F)$, then $[X, Y] \in \Gamma^\infty(F)$, where $[\cdot,\cdot]$ is the Lie bracket of vector fields. So regular foliations are examples of Dirac structures.

Analogously to Poisson manifolds, Dirac manifolds are closely related to the world of Lie algebroids, see e.g. [12].

A **Lie algebroid** over P is a vector bundle $A \to P$ together with a Lie algebra bracket $[\cdot,\cdot]$ on $\Gamma^\infty(A)$ and a bundle map $a : A \to TP$, called the **anchor**, satisfying the Leibniz identity

$$[\xi, f\eta] = f[\xi, \eta] + \mathcal{L}_{a(\xi)}\eta, \quad \xi, \eta \in \Gamma^\infty(A).$$

As a result, a induces a Lie algebra homomorphism $\Gamma^\infty(A) \to \mathcal{X}(P)$. Whenever there is no risk of confusion, we will write $\mathcal{L}_{a(\xi)}$ simply as $\mathcal{L}_\xi$.

If P is a point, we recover the definition of a Lie algebra; another example of a Lie algebroid is $A = TP$, with $a = \mathrm{Id}$. In general, the image $a(A) \subseteq TP$ defines a generalized integrable distribution, determining a foliation of P; the leaves of this foliation are called **orbits** of A.

A Poisson structure π on P always induces a Lie algebroid structure on T^*P, see e.g. [12, Sec. 17.3]: the Lie bracket on $\Omega^1(P)$ is

$$[\alpha, \beta] := \mathcal{L}_{\widetilde{\pi}(\beta)}\alpha - \mathcal{L}_{\widetilde{\pi}(\alpha)}\beta - d(\pi(\alpha, \beta)), \tag{8}$$

and the anchor is $-\widetilde{\pi}$; the orbits of T^*P on P are precisely the symplectic leaves of π.

This picture generalizes to Dirac structures as follows. Let

$$\rho : TP \oplus T^*P \to TP \quad \text{and} \quad \rho^* : TP \oplus T^*P \to T^*P$$

be the natural projections. Then the restriction of the Courant bracket to a Dirac subbundle L defines a Lie algebra bracket on $\Gamma^\infty(L)$, which together with the map $\rho|_L : L \to TP$ makes L into a Lie algebroid over P. When $L = \mathrm{graph}(\widetilde{\pi})$ for a Poisson structure π, the map $\rho^*|_L : L \to T^*P$ establishes an isomorphism of Lie algebroids, where T^*P has the Lie algebroid structure described in the previous paragraph.

Just as the orbits of T^*P carry a symplectic structure when P is a Poisson manifold, the orbits of a general Dirac structure L define a *presymplectic foliation* of P: the leafwise 2-form θ associated with L is defined, at each $x \in P$, by

$$\theta_x(X, Y) = \alpha(Y), \tag{9}$$

where $X, Y \in \rho(L)_x \subset T_xP$ and α is any element in T^*_xP satisfying $(X, \alpha) \in L_x$. One can check that (9) is well defined at each point of $\rho(L)$ and determines a smooth closed leafwise 2-form. Clearly this presymplectic foliation is symplectic if and only if the underlying Dirac structure comes from a Poisson structure.

Remark 2. A direct computation shows that, for any closed 2-form B, the map $\tau_B : E \to E$, $(X, \alpha) \mapsto (X, \alpha + \widetilde{B}(X))$, preserves the Courant bracket. It immediately follows that gauge-equivalent Dirac structures correspond to isomorphic Lie algebroids: if $L \subset E$ is a Dirac subbundle, then $\tau_B : L \to \tau_B(L)$ is an isomorphism.

2.4 Dirac maps

In order to define Dirac maps, we will reformulate the notion of Poisson maps in terms of subbundles of $TP \oplus T^*P$.

Recall that if (P_1, π_1), (P_2, π_2) are Poisson manifolds, then a map $f : P_1 \to P_2$ is a **Poisson map** if $f^* : C^\infty(P_2) \to C^\infty(P_1)$ preserves Poisson brackets. In terms of the bundle maps $\widetilde{\pi_1}$ and $\widetilde{\pi_2}$, this condition is equivalent to

$$(\widetilde{\pi_2})_{f(x)} = T_x f \circ (\widetilde{\pi_1})_x \circ (T_x f)^*, \quad \text{for } x \in P_1, \tag{10}$$

which can be alternatively written as

$$L_{\pi_2} = \{(Tf \circ \widetilde{\pi_1} \circ (Tf)^*(\alpha), \alpha) \mid \alpha \in T^*P_2\}. \tag{11}$$

Since $X = \widetilde{\pi_1}((Tf)^*\alpha)$ if and only if $(X, (Tf)^*\alpha) \in L_{\pi_1}$, we can rewrite (11) just in terms of the subbundles L_{π_1} and L_{π_2}:

$$L_{\pi_2} = \{(Tf(X), \alpha) \mid X \in TP_1,\ \alpha \in T^*P_2,\ (X, (Tf)^*(\alpha)) \in L_{\pi_1}\}.$$

This condition is now easily generalized to Dirac structures: if (P_1, L_1) and (P_2, L_2) are Dirac manifolds, then a map $f : P_1 \to P_2$ is a **Dirac map** if and only if

$$L_2 = \{(Tf(X), \alpha) \mid X \in TP_1,\ \alpha \in T^*P_2,\ (X, (Tf)^*\alpha) \in L_1\}.$$

The reader is referred to [8] for more on Dirac maps.

Remark 3. In the terminology of [8], the maps just defined are **forward Dirac maps**, since they generalize pushforward of bivector fields. There is an analogous notion of **backward Dirac map** generalizing pullback of 2-forms [8], which will not play a role in this paper.

2.5 Twisted Courant brackets

The way of incorporating a closed "background" 3-form $\phi \in \Omega^3(P)$ [38] into the formalism of Dirac structures is by modifying the standard Courant bracket on E to

$$[(X, \alpha), (Y, \beta)]_\phi := ([X, Y], \mathcal{L}_X \beta - i_Y d\alpha + \phi(X, Y, \cdot)). \tag{12}$$

The bundle E together with $\langle \cdot, \cdot \rangle$ and $[\cdot, \cdot]_\phi$ still satisfies the axioms of a Courant algebroid [38], and we refer to $[\cdot, \cdot]_\phi$ as the **ϕ-twisted Courant bracket** on E.

Just as before, we define **ϕ-twisted Dirac structures** as maximally isotropic subbundles of E with respect to $\langle \cdot, \cdot \rangle$ whose sections are closed under the $[\cdot, \cdot]_\phi$. Then **ϕ-twisted Poisson structures** are bivectors $\pi \in \mathcal{X}^2(P)$ whose graphs are Dirac structures; a direct computation shows that this is equivalent to π satisfying

$$[\pi, \pi]_s = 2(\wedge^3 \widetilde{\pi})(\phi).$$

Similarly, a 2-form ω gives rise to a ϕ-twisted Dirac structure if and only if

$$d\omega + \phi = 0,$$

in which case it is called a **ϕ-twisted presymplectic form**.

A ϕ-twisted Dirac structure induces a singular foliation on P with ϕ-twisted presymplectic leaves.

Example 3. (Cartan-Dirac structures on Lie groups [38, Ex. 4.2])

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Suppose $(\cdot, \cdot)_\mathfrak{g}$ is a bi-invariant metric on G, which we use to identify TG and T^*G. On $TG \oplus TG$, we consider the maximally isotropic subbundle

$$L = \{(v_r - v_l, \frac{1}{2}(v_l + v_r)),\ v \in \mathfrak{g}\},$$

where v_l (resp. v_r) is obtained from $v \in \mathfrak{g}$ by left (resp. right) translation. A direct computation shows that L is a ϕ-twisted Dirac structure, where ϕ is the associated bi-invariant Cartan 3-form, defined on Lie algebra elements by

$$\phi(u, v, w) = \frac{1}{2}(u, [v, w])_{\mathfrak{g}}.$$

We call L the **Cartan-Dirac structure** on G associated with $(\cdot, \cdot)_{\mathfrak{g}}$ [7]. In this example, the presymplectic leaves of L coincide with the conjugacy classes of G.

3 Gauge transformations, symplectic groupoids and Morita equivalence

Following [8], as a step to clarify the relationship between gauge transformations, symplectic groupoids and Xu's Morita equivalence, we discuss gauge transformations on dual pairs.

3.1 Gauge transformations of dual pairs

Let (S, ω_S) be a symplectic manifold, and let (P_1, π_1) and (P_2, π_2) be Poisson manifolds. A **dual pair** [39] is a pair of Poisson maps

$$\begin{tikzcd}[column sep=small] & S \arrow[dl, "J_1"'] \arrow[dr, "J_2"] & \\ P_1 & & P_2 \end{tikzcd}$$

for which the J_1- and J_2-fibers are the symplectic orthogonal of one another. A dual pair is called **full** if J_1 and J_2 are surjective submersions and **complete** if the maps J_1 and J_2 are complete. (Recall that a Poisson map $J : Q \to P$ is **complete** if whenever $f \in C^\infty(P)$ has compact support, the hamiltonian vector field $X_{J^*f} \in \mathcal{X}(Q)$ is complete.)

The following result, proven in [8], describes the effect of gauge transformations on dual pairs.

Theorem 1. *Let $(P_1, \pi_1) \stackrel{J_1}{\leftarrow} (S, \omega_S) \stackrel{J_2}{\rightarrow} (P_2, \pi_2)$ be a [full] dual pair, and let B_i be a closed 2-form on P_i, $i = 1, 2$. Let $\widehat{\omega}_S = \omega_S + J_1^*B_1 + J_2^*B_2$. Then $\widehat{\omega}_S$ is a symplectic form if and only if B_i is π_i-admissible, $i = 1, 2$, in which case*

$$\begin{tikzcd}[column sep=small] & (S, \widehat{\omega}_S) \arrow[dl, "J_1"'] \arrow[dr, "J_2"] & \\ (P_1, \tau_{B_1}(\pi_1)) & & (P_2, \tau_{B_2}(\pi_2)) \end{tikzcd} \qquad (1)$$

is again a [full] dual pair.

In general, $\tau_{B_1}(L_{\pi_1})$ and $\tau_{B_2}(L_{\pi_2})$ are just Dirac structures, and $\widehat{\omega}_S$ is a presymplectic form. In this more general setting, the maps J_1 and J_2 are Dirac maps, and the diagram (1) is an example of a **pre-dual pair** in the sense of [8].

Before we move on, let us recall some necessary facts about Lie and symplectic groupoids.

3.2 Lie and symplectic groupoids

Lie groupoids are the global counterparts of Lie algebroids. In order to fix our notation, let us recall that a Lie groupoid over a manifold P consists of another manifold $\mathcal{G}$ together with surjective submersions $t, s : \mathcal{G} \to P$, called **target** and **source**, a partially defined **multiplication** $m : \mathcal{G}^{(2)} \to \mathcal{G}$, where $\mathcal{G}^{(2)} = \{(g, h) \in \mathcal{G} \times \mathcal{G},\ s(g) = t(h)\}$ is the set of **composable pairs**, a **unit section** $\varepsilon : P \to \mathcal{G}$, and an **inversion** $i : \mathcal{G} \to \mathcal{G}$, all related by the appropriate axioms[2] (see e.g. [12, Sec. 13]). To simplify the notation, we will often identify an element x in P with its image $\varepsilon(x) \in \mathcal{G}$.

[2] A groupoid is as a small category in which all morphisms are isomorphisms; in this sense, P corresponds to the set of objects and $\mathcal{G}$ to the arrows, and the compatibility axioms between the structure maps follow.

Note that, if P is a point, a Lie groupoid becomes just a Lie group.

A Lie groupoid $\mathcal{G}$ over P has an associated Lie algebroid $A(\mathcal{G}) = \mathrm{Ker}(Ts)|_P \to P$, with anchor

$$a = Tt|_P : A(\mathcal{G}) \to TP,$$

and bracket induced by the Lie bracket of vector fields on $\mathcal{G}$.

An **integration** of a Lie algebroid $A \to P$ is a Lie groupoid $\mathcal{G}$ together with an identification $A(\mathcal{G}) \cong A$. Unlike Lie algebras, Lie algebroids are not always integrable, see [19] and references therein. However, if A is integrable, then there exists a canonical Lie groupoid $\mathcal{G}(A)$ with simply-connected s-fibers[3], unique up to isomorphism, called the *Weinstein groupoid* in [19].

Example 4. (Pair and fundamental groupoids)

Let P be a manifold. Then $\mathcal{G} = P \times P$ acquires a groupoid structure if we set $s(x,y) = y$, $t(x,y) = x$ and $m((x,y)(y,z)) = (x,z)$. This is called the **pair groupoid**. In this case, $A(\mathcal{G}) = TP$. In general, the s-simply-connected integration of TP is $\Pi(P)$, the **fundamental groupoid** of P, which naturally covers $P \times P$.

Example 5. (Tangent and cotangent groupoids)

Let $\mathcal{G}$ be a Lie groupoid over P. Then $T\mathcal{G}$ is a Lie groupoid over TP with source (resp. target) $Ts : T\mathcal{G} \to TP$ (resp. $Tt : T\mathcal{G} \to TP$); the multiplication $\bullet$ is defined by

$$U \bullet V = Tm(U,V),$$

where m is the multiplication on $\mathcal{G}$, see [29]. Note that $(T\mathcal{G})^{(2)} = T(\mathcal{G}^{(2)})$.

The cotangent bundle $T^*\mathcal{G}$ is a Lie groupoid over $A(\mathcal{G})^*$, see [16, 29]: The source and target maps are defined by

$$\widetilde{s}(\mu_g)(\xi) = \mu_g(Tl_g(\xi - Tt(\xi))) \text{ and } \widetilde{t}(\mu_g)(\eta) = \mu_g(Tr_g(\eta)),$$

where $\mu_g \in T^*_g\mathcal{G}$, $\xi \in A(\mathcal{G})_{s(g)}$ and $\eta \in A(\mathcal{G})_{t(g)}$; l_g (resp. r_g) is the left (resp. right) multiplication by $g \in \mathcal{G}$. The composition $\circ$ on $T^*\mathcal{G}$ is defined by

$$(\mu_g \circ \nu_h)(U_g \bullet V_h) = \mu_g(U_g) + \nu_h(V_h), \text{ for } (V_g, U_h) \in T_{(g,h)}\mathcal{G}^{(2)}. \tag{2}$$

A 2-form ω on a Lie groupoid $\mathcal{G}$ is called **multiplicative** if the graph of m is an *isotropic* submanifold of $(\mathcal{G},\omega) \times (\mathcal{G},\omega) \times (\mathcal{G},-\omega)$; this is equivalent to the condition

$$m^*\omega = \mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega, \tag{3}$$

where $\mathrm{pr}_i : \mathcal{G}^{(2)} \to \mathcal{G}$, $i = 1,2$, are the natural projections.

If π is a Poisson structure on P inducing an integrable Lie algebroid structure on T^*P, then the associated groupoid $\mathcal{G}(P) := \mathcal{G}(T^*P)$ carries a canonical multiplicative symplectic structure ω [13, 20]. A groupoid together with a multiplicative symplectic form is called a **symplectic groupoid** [16, 40].

Remark 4. The groupoids $\mathcal{G}(P)$ associated with Poisson manifolds coincide with reduced phase-spaces of Poisson sigma-models, in such a way that the canonical symplectic forms arise through (infinite-dimensional) symplectic reduction [13].

On the other hand, if $(\mathcal{G},\omega)$ is a symplectic groupoid over P, then the compatibility (3) alone implies several interesting properties. In particular,

i) s- and t-fibers are symplectically orthogonal to one another;
ii) $\varepsilon : P \to \mathcal{G}$ is a lagrangian embedding, and i is an anti-symplectomorphism;
iii) P inherits a Poisson structure π, uniquely determined by the condition that $t : \mathcal{G} \to P$ (resp. $s : \mathcal{G} \to P$) is a Poisson map (resp. anti-Poisson map).

[3] Simply connected will always mean connected with trivial fundamental group.

iv) The symplectic form ω induces a map

$$\mathrm{Ker}(Ts)|_P \xrightarrow{\sim} T^*P, \quad \xi \mapsto i_\xi\omega|_{TP} \tag{4}$$

which establishes an isomorphism of Lie algebroids

$$A(\mathcal{G}) \cong T^*P. \tag{5}$$

In particular, $\dim(\mathcal{G}) = 2\dim(P)$.

An **integration** of a Poisson manifold (P, π) is a symplectic groupoid $\mathcal{G}$ over P for which the induced Poisson structure on P coincides with π. The isomorphism (5) also makes $\mathcal{G}$ into an integration of T^*P in the sense of Lie algebroids.

Example 6. (Symplectic manifolds)
A symplectic manifold (M, ω_M) is always integrable. A symplectic groupoid over it is the pair groupoid $M \times M$ equipped with the symplectic structure $\omega = \omega_M \times (-\omega_M)$. In this case, $\mathcal{G}(M) = \Pi(M)$ equipped with the symplectic form obtained by pulling back ω by the cover map $\Pi(M) \to M \times M$.

Example 7. (Dual of Lie algebras)
Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{g}^*$, with the Lie-Poisson structure, is always integrable. Indeed, if G is a Lie group integrating $\mathfrak{g}$, then a symplectic groupoid over $\mathfrak{g}^*$ is the semi-direct product [12] $G \ltimes \mathfrak{g}^*$, with respect to the coadjoint action. The symplectic structure on this groupoid is induced by the canonical symplectic form on T^*G via the identification $T^*G \cong G \times \mathfrak{g}^*$. Note that this is a particular case of the cotangent groupoid of Example 5 when P is a point.

Various other examples of symplectic groupoids can be found in [16, 20]; obstructions to integrability are discussed in detail in [20].

3.3 Gauge transformations and Morita equivalence of Poisson manifolds

We call two Poisson manifolds (P_1, π_1) and (P_2, π_2) **gauge equivalent up to Poisson diffeomorphism** if there exists a Poisson diffeomorphism $f : (P_1, \pi_1) \to (P_2, \tau_B(\pi_2))$ for some closed 2-form B on P_2. The following properties are immediate:

1) Two symplectic manifolds are gauge equivalent up to Poisson diffeomorphism if and only if they are symplectomorphic;
2) Poisson manifolds which are gauge equivalent up to Poisson diffeomorphism have isomorphic foliations (though generally different leafwise symplectic structures);
3) Gauge equivalence up to Poisson diffeomorphism preserves the Lie algebroids associated with the Poisson structures; as a result, it preserves all Poisson cohomology groups [12, 38].

Following [8], we now recall how gauge equivalence relates to Xu's Morita equivalence.
Two Poisson manifolds (P_1, π_1), (P_2, π_2) are called **Morita equivalent** [44] if there exists a complete, full dual pair

$$\begin{tikzcd} & (S, \omega_S) \arrow[dl, "J_2"'] \arrow[dr, "J_1"] & \\ (P_1, \pi_1) & & (P_2, -\pi_2) \end{tikzcd} \tag{6}$$

with J_i-simply-connected fibers, i=1,2; we call this diagram a **Morita bimodule**. Let us describe some properties of Morita equivalence, in analogy with gauge equivalence:

1) For symplectic manifolds, the fundamental group is a complete Morita invariant [44]; in particular, any simply-connected symplectic manifold is Morita equivalent to a point.
2) Morita equivalent Poisson manifolds have isomorphic spaces of symplectic leaves, see e.g. [8, 20, 45].
3) Morita equivalence preserves first Poisson cohomology groups [18, 21].

Remark 5. i) A symplectic groupoid $\mathcal{G}$ over P is a self-Morita equivalence of P. In fact, one can prove that a Poisson manifold is Morita equivalent to itself if and only if it can be integrated to a symplectic groupoid [42].

ii) The regularity conditions in the definition of a Morita bimodule allow them to be "composed" through a certain "tensor product" operation [44, 45]. This makes Morita equivalence a transitive relation and endows the self-Morita equivalences of a Poisson manifold with an interesting group structure [11].

So, despite its name, Morita equivalence only becomes an equivalence relation when restricted to the class of integrable Poisson manifolds.

The comparison of properties 1), 2) and 3) for gauge and Morita equivalences suggests that they should be related and that the former should be stronger than the latter. We now show that this is in fact the case.

Let (P, π) be an integrable Poisson manifold with s-simply-connected symplectic groupoid $(\mathcal{G}, \omega)$. Let B be a π-admissible closed 2-form on P. Theorem 1 implies that

$$\begin{tikzcd} & (\mathcal{G}, \omega - s^*B) \arrow[dl, "t"'] \arrow[dr, "s"] & \\ (P, \pi) & & (P, -\tau_B(\pi)) \end{tikzcd}$$

is a full dual pair, which is in fact complete [8, Thm. 5.1]. Since s- and t-fibers are simply connected by assumption, it follows that

Theorem 2. *The Poisson manifolds (P, π) and $(P, \tau_B(\pi))$ are Morita equivalent. As a consequence, gauge equivalence up to Poisson diffeomorphism implies Morita equivalence.*

Let us make some remarks on the converse.

Remark 6. i) A comparison between properties 1) for gauge and Morita equivalences immediately implies that the converse of Theorem 2 does not hold: just consider two nonisomorphic symplectic manifolds with the same fundamental group. As shown in [8, Ex. 5.2], by studying certain symplectic fibrations, one can also find examples of Morita equivalent Poisson structures on the *same* manifold which are not gauge equivalent up to Poisson diffeomorphism.

ii) There is, however, a generic class of Poisson structures on 2-dimensional surfaces for which gauge and Morita equivalences coincide. Let Σ be a compact connected oriented surface. We call a Poisson structure π on Σ **topologically stable** if the zero set of π consists of n smooth disjoint, closed curves on Σ, for some $n \geq 0$, where π vanishes at most linearly [34]. Building on [8, 34], it is proven in [11] that two surfaces equipped with topologically stable Poisson structures are Morita equivalent if and only if they are gauge equivalent up to Poisson diffeomorphism. Furthermore, one can use this fact to construct a complete set of gauge/Morita invariants for these structures, which can all be assembled into a labelled graph [11].

3.4 Gauge transformations of symplectic groupoids

Let $(\mathcal{G}, \omega)$ be a symplectic groupoid associated with the Poisson manifold (P, π). By Remark 2, the Lie algebroids associated to Poisson structures gauge equivalent to π are all isomorphic. Therefore $\mathcal{G}$ integrates all of them as a Lie groupoid. Since gauge-equivalent Poisson structures are generally not isomorphic, gauge transformations must affect ω. We will now recall how.

Since $(P, \pi) \xleftarrow{t} (\mathcal{G}, \omega) \xrightarrow{s} (P, -\pi)$ is a full dual pair, Theorem 1 implies that, if B is a closed π-admissible 2-form on P, then

$$\begin{tikzcd} & (\mathcal{G}, \tau_B(\omega)) \arrow[dl, "t"'] \arrow[dr, "s"] & \\ (P, \tau_B(\pi)) & & (P, -\tau_B(\pi)) \end{tikzcd}$$

is also a full dual pair, where

$$\tau_B(\omega) := \omega + t^*B - s^*B. \tag{7}$$

This gives an indication of the next result.

Theorem 3. *$(\mathcal{G}, \tau_B(\omega))$ is a symplectic groupoid integrating $(P, \tau_B(\pi))$.*

A direct computation, see [8, Thm. 4.1], shows that $\tau_B(\omega)$ is multiplicative, so that $(\mathcal{G}, \tau_B(\omega))$ is a symplectic groupoid; it integrates $\tau_B(\pi)$ since $t : (\mathcal{G}, \tau_B(\omega)) \to (P, \tau_B(\pi))$ is a Poisson map.

We will prove a more general result in the next section.

4 Gauge transformations of Lie bialgebroids and Poisson groupoids

4.1 Lie bialgebroids and Poisson groupoids

A common generalization of symplectic groupoids and Poisson-Lie groups was introduced by Weinstein in [41]: a bivector field $\pi_{\mathcal{G}}$ on a Lie groupoid $\mathcal{G}$ is called **multiplicative** if the graph of the multiplication m is a *coisotropic*[4] submanifold of $(\mathcal{G}, \pi_{\mathcal{G}}) \times (\mathcal{G}, \pi_{\mathcal{G}}) \times (\mathcal{G}, -\pi_{\mathcal{G}})$; a **Poisson groupoid** is a Lie groupoid together with a multiplicative Poisson structure.

As in the case of symplectic groupoids, several properties follow directly from the definition:

i) $\{s^*C^\infty(P), t^*C^\infty(P)\} = 0$ in $C^\infty(\mathcal{G})$;
ii) $\varepsilon : P \to \mathcal{G}$ is a coisotropic embedding, and i is an anti-Poisson map;
iii) There is a unique Poisson structure π on P for which which t is a Poisson map (and s is anti-Poisson).

Example 8. (Pair groupoid)
If (P, π) is any Poisson manifold, then the pair groupoid $P \times P$ endowed with the Poisson structure $\pi \times (-\pi)$ is a Poisson groupoid.

It follows from this example that any Poisson structure can be realized as the base of a Poisson groupoid (even if it is not integrable) and there can be many Poisson groupoids inducing it.

The first hint about the nature of the infinitesimal counterpart of a Poisson groupoid $\mathcal{G}$ is the observation [41] that, since P sits in $\mathcal{G}$ as a coisotropic submanifold, its conormal bundle, which can be identified with $A(\mathcal{G})^*$, has itself an induced Lie algebroid structure. So a Poisson groupoid is associated with a pair of Lie algebroids $(A(\mathcal{G}), A(\mathcal{G})^*)$; the compatibility between them was explained in [29].

Suppose A is a Lie algebroid and that its dual A^* also carries a Lie algebroid structure. Then (A, A^*) is a **Lie bialgebroid** if

$$d_*[\xi, \eta] = \mathcal{L}_\xi d_* \eta - \mathcal{L}_\eta d_* \xi \tag{1}$$

for all $\xi, \eta \in \Gamma^\infty(A)$. Here $d_* : \Gamma^\infty(\wedge^k A) \to \Gamma^\infty(\wedge^{k+1} A)$ is the differential associated with the Lie algebroid structure of A^*, and $[\cdot, \cdot]$ is the Schouten-type bracket of A [12, 29],

$$[\cdot, \cdot] : \Gamma^\infty(\wedge^k A) \times \Gamma^\infty(\wedge^m A) \to \Gamma^\infty(\wedge^{k+m-1} A),$$

extending the Lie bracket on $\Gamma^\infty(A)$. The Lie derivatives $\mathcal{L}_\xi$ and $\mathcal{L}_\eta$ in (1) mean $[\xi, \cdot]$ and $[\eta, \cdot]$, respectively.

The pair $(A(\mathcal{G}), A(\mathcal{G})^*)$ on a Poisson groupoid $\mathcal{G}$ is a Lie bialgebroid [29]; conversely, if (A, A^*) is a Lie bialgebroid and A is integrable, then there is a unique Poisson structure on $\mathcal{G}(A)$ making it into a Poisson groupoid with Lie bialgebroid (A, A^*) [30].

Note that a Lie bialgebroid over a base point is a Lie bialgebra. In this case, the associated Poisson groupoids are Poisson-Lie groups.

[4] If (P, π) is a Poisson manifold, then a submanifold $N \subset P$ is coisotropic if, at each point, $\alpha, \beta \in TN^\circ$ implies that $\pi(\alpha, \beta) = 0$.

Example 9. (Poisson structures and symplectic groupoids)

If the Poisson structure on a Poisson groupoid $\mathcal{G}$ is nondegenerate, then the associated symplectic form makes it into a symplectic groupoid. The Lie bialgebroid in this case is the pair $(A(\mathcal{G}), A(\mathcal{G})^*) = (T^*P, TP)$, where T^*P has the Lie algebroid structure coming from π (the induced Poisson structure on P), and TP has the canonical Lie algebroid structure. Conversely, if π is a Poisson structure on P, then (T^*P, TP) is a Lie bialgebroid, and a Poisson groupoid associated with it is a symplectic groupoid integrating P.

If (A, A^*) is a Lie bialgebroid with anchors $a : A \to TP$ and $a_* : A^* \to TP$ then

$$a \circ a_*^* : T^*P \to TP \tag{2}$$

defines a Poisson structure on P [29]; here $a_*^* : T^*P \to A$ is the dual of a_*. We will refer to (2) as the Poisson structure **induced** by (A, A^*). If $\mathcal{G}$ is an associated Poisson groupoid, then this Poisson structure is the unique one making t into a Poisson map (see property *iii*) after the definition of Poisson groupoids).

4.2 Gauge transformations

Let (P, π) be a Poisson manifold, and let $(A, A^*) = (T^*P, TP)$ be the Lie bialgebroid associated with π, as in Example 9; since $A = T^*P$ and $A^* = TP$, we denote the associated Lie bracket on $\Omega^1(P)$ by $[\cdot,\cdot]$ and the usual Lie bracket of vector fields by $[\cdot,\cdot]_*$.

Suppose B is a closed π-admissible 2-form on P, and let

$$\varphi_B = (\mathrm{Id} + \widetilde{B}\widetilde{\pi}) : T^*P \to T^*P, \quad \text{and} \quad \psi_B = \varphi_B^* = (\mathrm{Id} + \widetilde{\pi}\widetilde{B}) : TP \to TP.$$

Then the Lie algebroid structure on T^*P induced by $\tau_B(\pi)$, denoted by $(T^*P)_B$, has anchor $a_B = \widetilde{\pi}\varphi_B^{-1}$ and bracket

$$[\alpha, \beta]^B := \varphi_B[\varphi_B^{-1}(\alpha), \varphi_B^{-1}(\beta)].$$

Although $(T^*P)_B$ is isomorphic to T^*P, the Lie bialgebroids (T^*P, TP) and $((T^*P)_B, TP)$ are *not* isomorphic in general. Note that the Lie bialgebroid $((T^*P)_B, TP)$ is isomorphic to $(T^*P, (TP)_B)$, where $(TP)_B$ has anchor $(a_*)_B = \psi_B^{-1}$ and bracket

$$[X, Y]_*^B := \psi_B[\psi_B^{-1}(X), \psi_B^{-1}(Y)].$$

We now generalize this operation for arbitrary Lie bialgebroids (details to appear in [5]).

Let (A, A^*) be a Lie bialgebroid, with anchors $a : A \to TP$, $a_* : A^* \to TP$, and brackets $[\cdot,\cdot]$, $[\cdot,\cdot]_*$, respectively. Let π be the induced Poisson structure on P.

Let $B \in \Omega^2(P)$ be a π-admissible 2-form. One can check that π-admissibility is equivalent to the invertibility of the map

$$\psi_B := (\mathrm{Id} + a^*\widetilde{B}a_*) : A^* \to A^*. \tag{3}$$

Here $a^* : T^*P \to A^*$ is the dual of a. Then a **gauge transformation** of (A, A^*) associated with B is the pair of Lie algebroids

$$\tau_B(A, A^*) := (A, (A^*)_B),$$

where the anchor and bracket of $(A^*)_B$ are given by

$$(a_*)_B = a_*(\psi_B)^{-1}, \quad [u, v]_*^B := \psi_B[\psi_B^{-1}(u), \psi_B^{-1}(v)]_*. \tag{4}$$

Using that B is closed, one can check that the pair $\tau_B(A, A^*) = (A, (A^*)_B)$ is a Lie bialgebroid, but this will also follow from Theorem 4 below. The bundle map associated with the Poisson structure induced by $\tau_B(A, A^*)$ on P is

$$a(\mathrm{Id} + a_*^*\widetilde{B}a)^{-1}a_*^* : T^*P \to TP, \tag{5}$$

and a simple computation shows that this map coincides with $\widetilde{\pi}(\mathrm{Id} + \widetilde{B}\widetilde{\pi})^{-1}$. Hence the Poisson structure induced by $\tau_B(A, A^*)$ is just $\tau_B(\pi)$.

Since a gauge transformation of a Lie bialgebroid (A, A^*) only affects A^*, its effect on the associated Poisson groupoids is only a change of the Poisson structure. This picture is clarified by the next result.

Theorem 4. *Let $(\mathcal{G}, \pi_{\mathcal{G}})$ be a Poisson groupoid over P, with Lie bialgebroid (A, A^*), and induced Poisson structure π on P. Let $B \in \Omega^2(P)$ be a closed 2-form, and let $B_{\mathcal{G}} := t^*B - s^*B \in \Omega^2(\mathcal{G})$. Then B is π-admissible if and only if $B_{\mathcal{G}}$ is $\pi_{\mathcal{G}}$-admissible, in which case $(\mathcal{G}, \tau_{B_{\mathcal{G}}}(\pi_{\mathcal{G}}))$ is a Poisson groupoid with Lie bialgebroid $\tau_B(A, A^*)$.*

The first assertion follows from an extension of [8, Lem. 2.12] saying that Tt induces an isomorphism between the kernels of the leafwise presymplectic forms associated with the Dirac structures $\tau_{B_{\mathcal{G}}}(L_{\pi_{\mathcal{G}}})$ and $\tau_B(L_\pi)$.

Before we prove the second assertion, let us recall that a bivector $\pi_{\mathcal{G}}$ on a Lie groupoid $\mathcal{G}$ is multiplicative if and only if $\widetilde{\pi_{\mathcal{G}}} : T^*\mathcal{G} \to T\mathcal{G}$ is a morphism of Lie groupoids [30], where $T^*\mathcal{G}$ and $T\mathcal{G}$ are the groupoids of Example 5; in this case, the induced map of identity sections $A(\mathcal{G})^* \to TP$ is the anchor of the Lie algebroid structure on $A(\mathcal{G})^*$.

Lemma 1. *Let $B \in \Omega^2(P)$ be closed, and let $B_{\mathcal{G}} = t^*B - s^*B$. Then $\widetilde{B_{\mathcal{G}}} : T\mathcal{G} \to T^*\mathcal{G}$ is a groupoid morphism (which is equivalent to $B_{\mathcal{G}}$ being multiplicative).*

Proof. A direct computation using the definitions in Example 5 shows that $\widetilde{B_{\mathcal{G}}}$ commutes with source and target maps.

Now suppose $U_g, V_h \in T\mathcal{G}^{(2)}$. Evaluating $\widetilde{B_{\mathcal{G}}}(U_g) \circ \widetilde{B_{\mathcal{G}}}(V_h)$ at $U'_g \bullet V'_h$ yields

$$\widetilde{B_{\mathcal{G}}}(U_g)(U'_g) + \widetilde{B_{\mathcal{G}}}(V_h)(V'_h) = t^*B_{\mathcal{G}}(U_g, U'_g) - s^*B_{\mathcal{G}}(U_g, U'_g) + t^*B_{\mathcal{G}}(V_h, V'_h) - s^*B_{\mathcal{G}}(V_h, V'_h).$$

But $U_g, V_h \in T\mathcal{G}^{(2)}$ means that $Ts(U_g) = Tt(V_h)$, so we obtain

$$\widetilde{B_{\mathcal{G}}}(U_g) \circ \widetilde{B_{\mathcal{G}}}(V_h)(U'_g \bullet V'_h) = t^*B_{\mathcal{G}}(U_g, U'_g) - s^*B_{\mathcal{G}}(V_h, V'_h). \tag{6}$$

On the other hand,

$$\begin{aligned}\widetilde{B_{\mathcal{G}}}(U_g \bullet V_h)(U'_g \bullet V'_h) = t^*B_{\mathcal{G}}(Tm(U_g, V_h), Tm(U'_g, V'_h)) - \\ s^*B_{\mathcal{G}}(Tm(U_g, V_h), Tm(U'_g, V'_h)).\end{aligned} \tag{7}$$

But $TtTm(U, V) = Tt(U)$ and $TsTm(U, V) = Ts(V)$ on any groupoid. Plugging these identities into (7), we get (6) and, as a result, $\widetilde{B_{\mathcal{G}}}(U_g \bullet V_h) = \widetilde{B_{\mathcal{G}}}(U_g) \circ \widetilde{B_{\mathcal{G}}}(V_h)$.

We can now complete the proof of Theorem 4. Since $\widetilde{\pi_{\mathcal{G}}} : T^*\mathcal{G} \to T\mathcal{G}$ and $\widetilde{B_{\mathcal{G}}} : T\mathcal{G} \to T^*\mathcal{G}$ are groupoid morphisms, so is the composition $\widetilde{\pi_{\mathcal{G}}}(\mathrm{Id} + \widetilde{B_{\mathcal{G}}}\widetilde{\pi_{\mathcal{G}}})^{-1}$. Hence $\tau_{B_{\mathcal{G}}}(\pi_{\mathcal{G}})$ is a multiplicative Poisson structure.

A simple computation shows that

$$\widetilde{\tau_{B_{\mathcal{G}}}(\pi_{\mathcal{G}})}|_{A^*} = a_*(\mathrm{Id} + a^*\widetilde{B}a_*)^{-1},$$

which coincides with $(a_*)_B$; finally, a longer but direct computation shows that the bracket induced by $\tau_{B_{\mathcal{G}}}(\pi_{\mathcal{G}})$ on $\Gamma^\infty(A^*)$ is $[\cdot, \cdot]^B_*$.

Note that Theorem 3 now follows as a corollary.

Remark 7. (Dirac groupoids)

What is the object resulting from a gauge transformation on a Poisson groupoid if B is *not* admissible? In general, $\tau_{B_{\mathcal{G}}}(L_{\pi_{\mathcal{G}}})$ is just a Dirac structure on $\mathcal{G}$, which can be shown to be *multiplicative* in the sense that it is a subgroupoid of the product groupoid $T\mathcal{G} \times T^*\mathcal{G}$;[5] furthermore, the Dirac structure $\tau_B(L_\pi)$ on P is *induced* from $\tau_{B_{\mathcal{G}}}(L_{\pi_{\mathcal{G}}})$ in the sense that

$$t : (\mathcal{G}, \tau_{B_{\mathcal{G}}}(L_{\pi_{\mathcal{G}}})) \to (P, \tau_B(L_\pi))$$

is a Dirac map. As we will discuss in the next section, this last property does not follow from multiplicativity in general; it requires a certain "nondegeneracy condition" on the Dirac structure. These observations

[5] Note that bivector fields and 2-forms on $\mathcal{G}$ are multiplicative in the usual sense if and only if their graphs are subgroupoids of $T\mathcal{G} \times T^*\mathcal{G}$.

motivate the investigation of Lie groupoids carrying multiplicative Dirac structures. The particular case of multiplicative presymplectic forms was studied thoroughly in [7] (see the next section); the general case will be treated elsewhere [5].

The infinitesimal versions of multiplicative Dirac structures on $\mathcal{G}$ encompass Lie bialgebroids and Dirac structures on P. The associated integration problem should be handled by means of an extension of the methods in [30], where the correspondence between Lie algebroid and groupoid morphisms should now be phrased in terms of subalgebroids and subgroupoids of the associated product groupoids. Finally, one can also consider "background" closed 3-forms; for example, if B is not closed, then $\tau_{B_\mathcal{G}}(\pi_\mathcal{G})$ becomes twisted with respect to $s^*\phi - t^*\phi$, $\phi = dB$, and its infinitesimal counterpart $\tau_B(A, A^*)$ is closely related to the objects in [36], indicating a generalization of [14] to twisted Poisson groupoids.

5 Integration of Dirac structures

5.1 Back to gauge equivalence of symplectic groupoids

Let us discuss the objects obtained by gauge transformation of symplectic groupoids in more detail. Let $(\mathcal{G}, \omega)$ be a symplectic groupoid, $B \in \Omega^2(P)$ a closed 2-form, and let us consider the Dirac structure $\tau_B(L_\pi)$ and the presymplectic form $\tau_B(\omega) = \omega + t^*B - s^*B$. As pointed out in Remark 7, $\tau_B(L_\pi)$ and $\tau_B(\omega)$ satisfy the following properties:

i) $\tau_B(\omega)$ is multiplicative;
ii) $t : (\mathcal{G}, \tau_B(\omega)) \to (P, \tau_B(L_\pi))$ is a Dirac map (while s is anti-Dirac);
iii) $A(\mathcal{G}) \cong \tau_B(L_\pi)$.

In order to extract the properties defining the global objects integrating Dirac manifolds from this example, we explore property *iii*) a bit further, as it reveals a certain compatibility between the kernel of $\tau_B(\omega)$ and the groupoid structure.

The identification $A(\mathcal{G}) \cong \tau_B(L_\pi)$ can be divided into two steps: first, $A(\mathcal{G}) \cong T^*P \cong L_\pi$; second, $L_\pi \cong \tau_B(L_\pi)$. For the first part, the isomorphism is given by

$$A(\mathcal{G}) \longrightarrow L_\pi, \quad \xi \mapsto (\widetilde{\pi}(i_\xi\omega|_{TP}), i_\xi\omega|_{TP}).$$

Claim. $\widetilde{\pi}(i_\xi\omega|_{TP}) = Tt(\xi)$.

To prove the claim, note that since $t : \mathcal{G} \to P$ is a Poisson map, $Tt(\xi) = \widetilde{\pi}(\alpha)$ if and only if $Tt^*\alpha = i_\xi\omega$. Using the multiplicativity (3) of ω, one shows that $(Tt)^*(i_\xi\omega|_{TP}) = i_\xi\omega$, which implies the result (see [7, Sec. 3]).

Composing this map with a gauge transformation we get

$$A(\mathcal{G}) \xrightarrow{\sim} \tau_B(L_\pi), \quad \xi \mapsto (Tt(\xi), i_\xi\tau_B(\omega)|_{TP}), \tag{1}$$

which is the desired isomorphism of Lie algebroids.

5.2 Presymplectic groupoids

Let us now consider a Lie groupoid $\mathcal{G}$ equipped with a multiplicative closed 2-form ω. The analogue of the map (1) is

$$A(\mathcal{G}) \to TP \oplus T^*P, \quad \xi \mapsto (Tt(\xi), i_\xi(\omega)|_{TP}). \tag{2}$$

Guided by the discussion in Subsection 5.1, we would like this map to be injective and its image to be a Dirac structure. A direct computation shows that the map (2) is injective if and only if

$$\mathrm{Ker}(\omega_x) \cap \mathrm{Ker}(T_x s) \cap \mathrm{Ker}(T_x t) = \{0\} \tag{3}$$

for all $x \in P$; in this case, the fibers of its image have the same dimension as P (which is necessary if it is to be a Dirac structure) if and only if $\dim(\mathcal{G}) = 2\dim(P)$. One can check that the compatibility between the Lie bracket on $\Gamma^\infty(A(\mathcal{G}))$ and the Courant bracket on $TP\oplus T^*P$ follows from the multiplicativity of ω [7, Prop. 3.5]. So the image of (2) is a Dirac structure on P if and only if (3) holds and $\dim(\mathcal{G}) = 2\dim(P)$. It turns out that this is all we need.

A **presymplectic groupoid** is a Lie groupoid $\mathcal{G}$ with $\dim(\mathcal{G}) = 2\dim(P)$ equipped with a closed multiplicative 2-form ω satisfying (3). In the presence of a closed 3-form $\phi \in \Omega^3(P)$, we require ω to be **relatively ϕ-closed**, i.e.,

$$d\omega = s^*\phi - t^*\phi,$$

and $(\mathcal{G}, \omega, \phi)$ is called a **ϕ-twisted presymplectic groupoid**. It is proved in [7] that [twisted] presymplectic groupoids are the global counterparts of [twisted] Dirac structures.

Theorem 5. *Let $(\mathcal{G}, \omega)$ be a [ϕ-twisted] presymplectic groupoid. Then the image of the map (2) is a [ϕ-twisted] Dirac structure L on P, isomorphic to $A(\mathcal{G})$ as a Lie algebroid via (2) itself; furthermore, $t : (\mathcal{G}, \omega) \to (P, L)$ is a Dirac map (and s is anti-Dirac) and L is the unique Dirac structure on P with this property. (We call $(\mathcal{G}, \omega)$ an* **integration** *of L.)*

Conversely, let L be a [ϕ-twisted] Dirac structure on P whose associated Lie algebroid is integrable, and let $\mathcal{G}(L)$ be its s-simply connected integration. Then there exists a unique 2-form ω making $(\mathcal{G}(L), \omega)$ into a presymplectic groupoid integrating L.

The following examples illustrate the correspondence established in the theorem.

Example 10. (Poisson structures)
A presymplectic groupoid $\mathcal{G}$ over P induces a Poisson structure on P if and only if ω is symplectic. So we obtain the well-known correspondence between Poisson manifolds and symplectic groupoids. In the presence of a closed 3-form, Thm. 5 recovers the correspondence between twisted symplectic groupoids and twisted Poisson structures proved in [14].

Example 11. (Closed 2-forms)
As in the case of symplectic manifolds, presymplectic manifolds can be integrated by their fundamental groupoids with presymplectic form defined just as in Example 6.

Example 12. (Cartan-Dirac structures and the AMM-groupoid)
Let G be a Lie group with bi-invariant metric $(\cdot,\cdot)_{\mathfrak{g}}$. The **AMM groupoid** [4] over G is the action groupoid $\mathcal{G} = G \ltimes G$ with respect to the conjugation action equipped with the 2-form [2]

$$\omega_{(g,x)} = \frac{1}{2}\left((\mathrm{Ad}_x p_g^*\lambda, p_g^*\lambda)_{\mathfrak{g}} + (p_g^*\lambda, p_x^*(\lambda + \bar{\lambda}))_{\mathfrak{g}}\right)$$

where p_g and p_x denote the projections onto the first and second components of $G \times G$, and λ and $\overline{\lambda}$ are the left and right Maurer-Cartan forms. One can check that the AMM-groupoid integrates L, the Cartan-Dirac structure on G associated with $(\cdot,\cdot)_{\mathfrak{g}}$. If G is simply connected, then $(G\ltimes G, \omega)$ is isomorphic to the canonical s-simply connected integration of L; in general, one must pull-back ω to $\widetilde{G} \ltimes G$, where $\widetilde{G}$ is the universal cover of G [7, Thm. 7.6].

See [7, Sec. 8] for a detailed account of integration of Dirac structures associated with regular foliations.

5.3 Presymplectic realizations and hamiltonian actions

A **symplectic realization** of a Poisson manifold (P, π) is a Poisson map from a symplectic manifold (S, ω_S) to P (see e.g. [12]). Let us briefly recall how these objects connect symplectic groupoids to the theory of hamiltonian actions.

If $J : S \to P$ is a symplectic realization, then it induces a map $a_S : T^*P \to TS$ assigning to each $\alpha \in T^*_{J(y)}P$ a vector $Z \in T_yS$ uniquely defined by

$$i_Z\omega_S = J^*\alpha.$$

This map satisfies the compatibility $TJ \circ a_S = \widetilde{\pi}$ and induces a Lie algebra homomorphism $\Omega^1(P) \to \mathcal{X}^1(S)$; in fact, a_S defines an *action* of the Lie algebroid T^*P on S [20]. When J is complete, this action can be integrated to an action of the groupoid $\mathcal{G}(P)$ on S with moment J [20, 31]. We denote this action by $m_S : \mathcal{G}(P) \times_P S \to S$; furthermore, this action is *symplectic* in the sense that

$$m_S^*\omega_S = \mathrm{pr}_{\mathcal{G}}^*\omega + \mathrm{pr}_S^*\omega_S, \tag{4}$$

where $\mathrm{pr}_{\mathcal{G}} : \mathcal{G}(P) \times_P S \to \mathcal{G}$ and $\mathrm{pr}_S : \mathcal{G}(P) \times_P S \to S$ are the natural projections. In this context, symplectic realizations are the infinitesimal versions of symplectic groupoid actions.

Example 13. (Hamiltonian actions)

Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$. A symplectic realization $J : S \to \mathfrak{g}^*$ induces an action of the Lie algebroid $T^*\mathfrak{g}^* = \mathfrak{g} \ltimes \mathfrak{g}^*$ (this semi-direct product is with respect to the coadjoint action), which is equivalent to an action of $\mathfrak{g}$ on S for which J is $\mathfrak{g}$-equivariant [31]. It is easy to check that the $\mathfrak{g}$-action on S is hamiltonian with momentum map J.

If the realization J is complete, then the Lie algebroid action integrates to an action of $\mathcal{G}(\mathfrak{g}^*) = G \ltimes \mathfrak{g}^*$, which is equivalent to a hamiltonian G-action on S for which J is an equivariant momentum map.

Poisson actions can be similarly described as symplectic realizations of duals of Poisson-Lie groups, see [43] and references therein.

The natural way to extend the notion of symplectic realization to the realm of Dirac manifolds is to consider Dirac maps from presymplectic manifolds. So let (P, L) be a Dirac manifold, let (S, ω_S) be a presymplectic manifold and let $J : S \to P$ be a Dirac map. Let us try to define an action of L on S analogous to the one for Poisson structures: if $(X, \alpha) \in L_{J(y)}$, then J being a Dirac map means that there exists a vector $Z \in T_yS$ satisfying

$$T_yJ(Z) = X, \quad \text{and} \quad J^*\alpha = i_Z\omega_S; \tag{5}$$

note, however, that these equations determine Z uniquely if and only if

$$\mathrm{Ker}(TJ) \cap \mathrm{Ker}(\omega_S) = \{0\}. \tag{6}$$

In this case, the map

$$a_S : L \to TS, \quad (X, \alpha) \mapsto Z,$$

defines an action of the Lie algebroid L on S, generalizing the property of symplectic realizations.

Bringing closed 3-forms into the picture, we arrive at the following definition: a **presymplectic realization** of a ϕ-twisted Dirac manifold (P, L) is a Dirac map $J : (S, \omega_S) \to (P, L)$, where ω_S is a $J^*\phi$-twisted presymplectic form (i.e., $d\omega_S + J^*\phi = 0$), such that $\mathrm{Ker}(TJ) \cap \mathrm{Ker}(\omega_S) = \{0\}$.

An immediate consequence of this definition is

Corollary 1. *Any presymplectic realization $J : (S, \omega_S) \to (P, L)$ of a ϕ-twisted Dirac structure is canonically equipped with an infinitesimal action of the Lie algebroid of L.*

Furthermore, if the realization $J : S \to P$ is complete[6] *and L is integrable, then there is an induced smooth action of the presymplectic groupoid $(\mathcal{G}(L), \omega)$ on S satisfying* (4).

Having the notion of presymplectic realizations, we can repeat the discussion in Example 13 replacing $\mathfrak{g}^*$ by other Dirac manifolds.

Example 14. (*Quasi-hamiltonian actions [2]*)

Let G be a simply connected Lie group equipped with the Cartan-Dirac structure L associated with a bi-invariant metric $(\cdot, \cdot)_{\mathfrak{g}}$. Suppose $J : S \to G$ is a presymplectic realization of G; here S is equipped with a $J^*\phi$-twisted presymplectic form ω_S, where ϕ is the Cartan 3-form. In this case, we can identify $L \cong \mathfrak{g} \ltimes G$, where the semi-direct product is with respect to the action by conjugation. As in Example 13, an action of L on S is equivalent to an action $a_S : \mathfrak{g} \to TS$ preserving ω_S for which J is $\mathfrak{g}$-equivariant. In terms of a_S, the conditions that J is a Dirac map and (6) are expressed equivalently by [7]

[6] In the sense that the vector field associated with $l \in \Gamma^\infty(L)$ is complete whenever l has compact support.

1. $i_{a_S(v)}(\omega_S) = \frac{1}{2}J^*(\lambda + \bar{\lambda}, v)_{\mathfrak{g}}$, where λ is the left invariant Maurer-Cartan 1-form;
2. $\mathrm{Ker}(\omega_S)_y = \{a_S(v)_y : v \in \mathrm{Ker}(\mathrm{Ad}_{J(y)} + 1)\}$, at each $y \in S$.

These properties define the infinitesimal form of quasi-hamiltonian spaces [2].

As in Examples 13, if J is complete, then we have an action of the AMM-groupoid $\mathcal{G} = G \ltimes G$ on S; this action is equivalent to a G-action on S for which J is equivariant. This makes S into a quasi-hamiltonian G-space with group-valued moment map J [2].

Replacing symplectic realizations by Poisson maps from arbitrary Poisson manifolds in Example 13 yields hamiltonian actions on Poisson manifolds; analogously, a Dirac manifold together with a Dirac map (satisfying a suitable nondegeneracy condition like (6)) into a Lie group equipped with a Cartan-Dirac structure is equivalent to a hamiltonian quasi-Poisson manifold [1]. This topic will be treated in more detail in a separate paper [6].

References

1. Alekseev, A., Kosmann-Schwarzbach, Y., Meinrenken, E.: *Quasi-Poisson manifolds.* Canad. J. Math. **54** (2002), 3–29.
2. Alekseev, A., Malkin, A., Meinrenken, E.: *Lie group valued moment maps.* J. Differential Geom. **48** (1998), 445–495.
3. Bayen, F., Flato, M., Frønsdal, C., Lichnerowicz, A., Sternheimer, D.: *Deformation Theory and Quantization.* Ann. Phys. **111** (1978), 61–151.
4. Behrend. K., Xu, P., Zhang, B.: *Equivariant gerbes over compact simple Lie groups.* C. R. Acad. Sci. Paris **333** (2003), 251–256.
5. Bursztyn, H.: *Multiplicative Dirac structures.* In preparation.
6. Bursztyn, H., Crainic, M.: *Dirac structures, moment maps and quasi-Poisson manifolds.* Preprint.
7. Bursztyn, H., Crainic, M., Weinstein, A., Zhu, C.: *Integration of twisted Dirac brackets*, Duke Math. J., to appear. Math.DG/0303180.
8. Bursztyn, H., Radko, O.: *Gauge equivalence of Dirac structures and symplectic groupoids.* Ann. Inst. Fourier (Grenoble) **53** (2003), 309–337.
9. Bursztyn, H., Waldmann, S.: *The characteristic classes of Morita equivalent star products on symplectic manifolds.* Comm. Math. Phys. **228** (2002), 103–121.
10. Bursztyn, H., Waldmann, S.: In preparation.
11. Bursztyn, H., Weinstein: *Picard groups in Poisson geometry*, Moscow Math. J., to appear. Math.SG/0304048.
12. Cannas da Silva, A., Weinstein, A.: *Geometric models for noncommutative algebras.* American Mathematical Society, Providence, RI, 1999.
13. Cattaneo, A., Felder, G.: *Poisson sigma models and symplectic groupoids.* In: *Quantization of singular symplectic quotients*, Progr. Math. 198, 61–93, Birkhäuser, Basel, 2001.
14. Cattaneo, A., Xu, P.: *Integration of twisted Poisson structures.* J. Geom. Phys., to appear. Math.SG/0302268.
15. Connes, A., Douglas, M. R., Schwarz, A.: *Noncommutative geometry and matrix theory: compactification on tori.* J. High Energy Phys. **1998**.2, Paper 3, 35 pp. (electronic).
16. Coste, A., Dazord, P., Weinstein, A.: *Groupoïdes symplectiques.* In: *Publications du Département de Mathématiques. Nouvelle Série. A, Vol. 2*, i–ii, 1–62. Univ. Claude-Bernard, Lyon, 1987.
17. Courant, T.: *Dirac manifolds.* Trans. Amer. Math. Soc. **319** (1990), 631–661.
18. Crainic, M.: *Differentiable and algebroid cohomology, van Est isomorphisms, and characteristic classes.* Math.DG/0006064, to appear in Comment. Math. Helv.
19. Crainic, M., Fernandes, R.: *Integrability of Lie brackets.* Ann. of Math. (2) **157** (2003), 575–620.
20. Crainic, M., Fernandes, R.: *Integrability of Poisson brackets.* Math.DG/0210152.
21. Ginzburg, V. L., Lu, J.-H.: *Poisson cohomology of Morita-equivalent Poisson manifolds.* Internat. Math. Res. Notices **10** (1992), 199-205.
22. Jurco, B., Schupp, P., Wess, J.: *Noncommutative line bundle and Morita equivalence.* Lett. Math. Phys. **61** (2002), 171–186.
23. Klimcik, C., Strobl, T.: *WZW-Poisson manifolds.* J. Geom. Phys. **4** (2002), 341-344.
24. Kontsevich, M.: *Deformation Quantization of Poisson Manifolds, I.* Preprint q-alg/9709040.

25. LANDSMAN, N. P.: *Mathematical Topics between Classical and Quantum Mechanics. Springer Monographs in Mathematics.* Springer-Verlag, Berlin, Heidelberg, New York, 1998.
26. LANDSMAN, N. P.: *Operator algebras and Poisson manifolds associated to groupoids.* Comm. Math. Phys. **222** (2001), 97–116.
27. LANDSMAN, N. P.: *Quantization as a functor*, math-ph/0107023.
28. LIU, Z.-J., WEINSTEIN, A., XU , P.: *Manin triples for Lie bialgebroids.* J. Diff. Geom. **45** (1997), 547–574.
29. MACKENZIE, K., XU , P.: *Lie bialgebroids and Poisson groupoids.* Duke Math. J. **73** (1994), 415–452.
30. MACKENZIE, K., XU , P.: *Integration of Lie bialgebroids.* Topology **39** (2000), 445–467.
31. MIKAMI, K., WEINSTEIN, A.: *Moments and reduction for symplectic groupoid actions.* Publ. RIMS, Kyoto Univ. **24** (1988), 121–140.
32. MORITA, K.: *Duality for modules and its applications to the theory of rings with minimum condition.* Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **6** (1958), 83–142.
33. PARK, J.-S.: *Topological open p-branes.* In: *Symplectic geometry and Mirror symmetry (Seoul, 2000)*, 311-384, World Sci. Publishing, Rover Edge, NJ, 2001.
34. RADKO, O.: *A classification of topologically stable Poisson structures on a compact oriented surface.* J. Symp. Geometry. **1** (2002), 523-542.
35. RIEFFEL, M. A.,: *Morita equivalence for C^*-algebras and W^*-algebras,* J. Pure Appl. Algebra **5** (1974), 51–96.
36. ROYTENBERG, D.: *Quasi-Lie bialgebroids and twisted Poisson manifolds.* Lett. Math. Phys. **61** (2002), 123-137.
37. SCHWARZ, A.: *Morita equivalence and duality.* Nuclear Phys. B **534**.3 (1998), 720–738.
38. ŠEVERA, P., WEINSTEIN, A.: *Poisson geometry with a 3-form background.* Prog. Theor. Phys. Suppl. **144** (2001), 145–154.
39. WEINSTEIN, A.: *The local structure of Poisson manifolds.* J. Diff. Geom. **18** (1983), 523–557.
40. WEINSTEIN, A.: *Symplectic groupoids and Poisson manifolds.* Bull. Amer. Math. Soc. (N.S.) **16** (1987), 101–104.
41. WEINSTEIN, A.: *Coisotropic calculus and Poisson groupoids.* J. Math. Soc. Japan **40** (1988), 705–727.
42. WEINSTEIN, A.: *Affine Poisson structures.* Internat. J. Math. **1** (1990), 343–360.
43. WEINSTEIN, A.: *The geometry of momentum.* Math.SG/0208108
44. XU, P.: *Morita equivalence of Poisson manifolds.* Comm. Math. Phys. **142** (1991), 493–509.
45. XU, P.: *Morita equivalent symplectic groupoids.* In: *Symplectic geometry, groupoids, and integrable systems (Berkeley, CA, 1989)*, 291–311. Springer, New York, 1991.
46. XU, P.: *Morita equivalence and momentum maps.* Math.SG/0307319.