MULTISYMPLECTIC GEOMETRY AND LIE GROUPOIDS

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In memory of Jerry Marsden

ABSTRACT. We study higher-degree generalizations of symplectic groupoids, referred to as *multisymplectic groupoids*. Recalling that Poisson structures may be viewed as infinitesimal counterparts of symplectic groupoids, we describe "higher" versions of Poisson structures by identifying the infinitesimal counterparts of multisymplectic groupoids. Some basic examples and features are discussed.

Contents

1. Introduction

Multisymplectic structures are higher-degree analogs of symplectic forms which arise in the geometric formulation of classical field theory much in the same way that symplectic structures emerge in the hamiltonian description of classical mechanics, see [17, 21, 26] and references therein. This symplectic approach to field theory was explored in a number of Marsden's publications, which treated (as it was typical in Marsden's work) theoretical as well as applied aspects of the subject, see e.g. [18, 19, 29, 30]. Multisymplectic geometry (as in [8, 9]) also arises in other settings, such as the study of homotopical structures [35], categorified symplectic geometry [2], and geometries defined by closed forms [28].

Poisson structures are generalizations of symplectic structures which are central to geometric mechanics[1] and permeate Marsden's work. A natural problem in multisymplectic geometry is the identification of "higher" analogs of Poisson structures bearing a relation to multisymplectic forms that extends the way Poisson geometry generalizes symplectic geometry. In this note we discuss one possible approach to tackle this issue.

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[1]E.g., in the description of the interplay between hamiltonian dynamics and symmetries [31], and in the transition from classical to quantum mechanics [7].

Our viewpoint relies on the relationship between Poisson geometry and objects known as *symplectic groupoids* [11, 37]. This relationship is part of a generalized Lie theory in which Poisson structures arise as infinitesimal, or linearized, counterparts of symplectic groupoids, in a way analogous to how Lie algebras correspond to Lie groups. In order to find higher analogs of Poisson structures the route we take is to first consider higher-degree versions of symplectic groupoids, referred to as *multisymplectic groupoids*, and then to identify the geometric objects arising as their infinitesimal counterparts. Recalling that symplectic groupoids are Lie groupoids equipped with a symplectic structure that is compatible with the groupoid multiplication, in the sense that the symplectic form is *multiplicative* (see (2.4) below), multisymplectic groupoids are defined analogously, as Lie groupoids endowed with a multiplicative multisymplectic structure. Our identification of the infinitesimal objects corresponding to multisymplectic groupoids builds on the infinitesimal description of general multiplicative differential forms obtained in [1, 4].

For a manifold M, our "higher-degree" analogs of Poisson structures can be conveniently expressed (in the spirit of Dirac geometry [12]) in terms of subbundles

$$(1.1) \qquad\qquad L \subset TM \oplus \wedge^k T^*M$$

satisfying suitable properties, including an involutivity condition with respect to the "higher" Courant-Dorfman bracket on the space of sections of $TM \oplus \wedge^k T^*M$ (see e.g. [22, Sec. 2]). Related geometric objects have been recently considered in the study of higher analogs of Dirac structures in [38] (see also [36]). But, as it turns out, the higher Poisson structures that arise from multisymplectic groupoids are not particular cases of the higher Dirac structures of [38] (for example, comparing with [38, Def. 3.1], the higher Poisson structures (1.1) considered here are not necessarily lagrangian subbundles, though always isotropic). An alternative characterization of these objects, more in the spirit of the bivector-field description of Poisson structures, is presented in Prop. 5.2.

Another perspective on higher Poisson structures relies on the view of Poisson structures as Lie brackets on the space of smooth functions of a manifold. A natural issue in this context is finding an appropriate extension of the Poisson bracket defined by a symplectic form (see (2.3)) to multisymplectic manifolds. This problem involves notorious difficulties and much work has been done on it, see e.g. [16, 25, 35]. The approach to higher Poisson structures in this note follows a different path and does not address any of the issues involved in the algebraic study of higher Lie-type brackets.

The paper is structured as follows. We review Poisson structures and their connection with symplectic groupoids in Section 2. In Section 3 we recall the basics of multisymplectic forms. The main results are presented in Section 4, in which we introduce multisymplectic groupoids and identify their infinitesimal counterparts. In Section 5 we give different descriptions of these objects and explain some of their properties, while examples are discussed in Section 6.

As one should expect, higher Poisson structures naturally arise in connection with symmetries in multisymplectic geometry. This aspect of the subject is not treated here, though we hope to explore it, as well as its relations with field theory, in future work. Parallel ideas to those in this note can be also carried out in the context of polysymplectic geometry, see [32].

2. Poisson structures and symplectic groupoids

We start by recalling a few different viewpoints to Poisson structures.

A *Poisson structure* on a smooth manifold M is Lie bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$ which is compatible with the pointwise product of functions via the Leibniz rule:

$$(2.1) \qquad \{f,gh\} = \{f,g\}h + \{f,h\}g, \quad f,g,h \in C^\infty(M).$$

The Leibniz condition (2.1) implies that $\{\cdot,\cdot\}$ is necessarily defined by a bivector field $\pi \in \Gamma(\wedge^2 TM)$ via

$$\pi(df,dg) = \{f,g\}, \quad f,g \in C^\infty(M).$$

This leads to the alternative description of Poisson structures on M as bivector fields $\pi \in \Gamma(\wedge^2 TM)$ satisfying $[\pi,\pi] = 0$, where $[\cdot,\cdot]$ is the Schouten-Nijenhuis bracket on multivector fields. (The vanishing of $[\pi,\pi]$ accounts for the Jacobi identity of $\{\cdot,\cdot\}$.) We denote Poisson manifolds by either (M,π) or $(M,\{\cdot,\cdot\})$.

Symplectic manifolds are naturally equipped with Poisson structures. Given a symplectic manifold (M,ω), and denoting by X_f the hamiltonian vector field associated with $f \in C^\infty(M)$ via

$$(2.2) \qquad i_{X_f}\omega = df,$$

the Poisson bracket on M is given by

$$(2.3) \qquad \{f,g\} = \omega(X_g, X_f).$$

A more recent perspective on Poisson structures, which is the guiding principle of this note, relies on another type of connection between Poisson structures and symplectic manifolds. It is based on the fact that Poisson geometry fits into a generalized Lie theory, naturally expressed in terms of Lie algebroids and groupoids, see e.g. [11]. In this context, Poisson manifolds are seen as infinitesimal counterparts of global objects called *symplectic groupoids* [37], analogously to how Lie algebras are regarded as infinitesimal versions of Lie groups. We will briefly recall the main aspects of the theory.

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid (the reader can find definitions and further details in [7]). We use the following notation for its structure maps: $\mathsf{s}, \mathsf{t} : \mathcal{G} \to M$ for the source, target maps, $m : \mathcal{G}_\mathsf{s}\times_\mathsf{t}\mathcal{G} \to \mathcal{G}$ for the multiplication map[2], $\epsilon : M \hookrightarrow \mathcal{G}$ for the unit map, and $\mathsf{inv} : \mathcal{G} \to \mathcal{G}$ for the groupoid inversion. We will often identify M with its image under ϵ (the submanifold of $\mathcal{G}$ of identity arrows).

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[2]Here the fibred product $\mathcal{G}_\mathsf{s}\times_\mathsf{t}\mathcal{G} = \{(g,h) \in \mathcal{G}\times\mathcal{G} \,|\, \mathsf{s}(g) = \mathsf{t}(h)\}$ represents the space of composable arrows.

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A differential form $\omega \in \Omega^r(\mathcal{G})$ is called *multiplicative* if it satisfies

(2.4) $$m^*\omega = \mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega,$$

where $\mathrm{pr}_i : \mathcal{G}_{\mathsf{s}}\times_{\mathsf{t}}\mathcal{G} \to \mathcal{G}$, $i = 1, 2$, is the natural projection onto the *i*-th factor[3]. A *symplectic groupoid* is a Lie groupoid $\mathcal{G} \rightrightarrows M$ equipped with a multiplicative symplectic form $\omega \in \Omega^2(\mathcal{G})$. In this case, condition (2.4) is equivalent to the graph of the multiplication map m being a lagrangian submanifold of $\mathcal{G} \times \mathcal{G} \times \overline{\mathcal{G}}$, where $\overline{\mathcal{G}}$ is equipped with the opposite symplectic form $-\omega$. Symplectic groupoids first arose in symplectic geometry in the context of quantization (see e.g. [3, Sec. 8.3]) but turn out to provide a convenient setting for the study of symmetries and reduction [34].

In order to explain how symplectic groupoids are related to Poisson structures, recall that a *Lie algebroid* is a vector bundle $A \to M$ equipped with a bundle map $\rho : A \to TM$, called the *anchor*, and a Lie bracket $[\cdot, \cdot]$ on $\Gamma(A)$ such that

$$[u, fv] = f[u, v] + (\mathcal{L}_{\rho(u)}f)v,$$

for $u, v \in \Gamma(A)$, $f \in C^\infty(M)$. Lie algebroids are infinitesimal versions of Lie groupoids: for a Lie groupoid $\mathcal{G} \rightrightarrows M$, its associated Lie algebroid is defined by $A = \ker(d\mathsf{s})|_M$, with anchor map $d\mathsf{t}|_A : A \to TM$ and Lie bracket on $\Gamma(A)$ induced by the Lie bracket of right-invariant vector fields on $\mathcal{G}$. Much of the usual theory relating Lie algebras and Lie groups carries over to Lie algebroids and groupoids, a notorious exception being Lie's third theorem, i.e., not every Lie algebroid arises as the Lie algebroid of a Lie groupoid (see [13] for a thorough discussion of this issue).

The first indication of a connection between Poisson geometry and Lie algebroids/groupoids is the fact that, if (M, π) is a Poisson manifold, then its cotangent bundle $T^*M \to M$ inherits a Lie algebroid structure, with anchor map given by

(2.5) $$\pi^\sharp : T^*M \to TM, \quad \pi^\sharp(\alpha) = i_\alpha \pi,$$

and Lie bracket on $\Gamma(T^*M) = \Omega^1(M)$ given by

(2.6) $$[\alpha, \beta] = \mathcal{L}_{\pi^\sharp(\alpha)}\beta - \mathcal{L}_{\pi^\sharp(\beta)}\alpha - d(\pi(\alpha, \beta)).$$

The precise relation between Poisson structures and symplectic groupoids is as follows. First, given a symplectic groupoid $(\mathcal{G} \rightrightarrows M, \omega)$, its space of units M inherits a natural Poisson structure π, uniquely determined by the fact that the target map $\mathsf{t} : \mathcal{G} \to M$ is a Poisson map (while $\mathsf{s} : \mathcal{G} \to M$ is anti-Poisson); moreover, denoting by A the Lie algebroid of $\mathcal{G}$, there is a canonical identification between A and the Lie algebroid structure on T^*M induced by π, explicitly given by

$$\mu : A \xrightarrow{\sim} T^*M, \quad \mu(u) = i_u\omega|_{TM}.$$

Here we view TM as a subbundle of $T\mathcal{G}|_M$ via $\epsilon : M \hookrightarrow \mathcal{G}$, so that we can write

(2.7) $$T\mathcal{G}|_M = TM \oplus A.$$

In other words, the Lie groupoid $\mathcal{G}$ integrates the Lie algebroid T^*M defined by π.

Conversely, given a Poisson manifold (M, π) and assuming that its associated Lie algebroid is integrable (i.e., can be realized as the Lie algebroid of a Lie groupoid[4]),

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[3]For a function $f \in \Omega^0(\mathcal{G}) = C^\infty(\mathcal{G})$, condition (2.4) becomes $f(gh) = f(g) + f(h)$, i.e., it says that f is a groupoid morphism into $\mathbb{R}$ (viewed as an abelian group).

[4]See e.g. [37] for a nonintegrable example and [14] for a discussion of obstructions to integrability.

then its s-simply-connected integration $\mathcal{G} \rightrightarrows M$ inherits a symplectic groupoid structure. (As shown in [10], one can obtain $\mathcal{G}$ by means of an infinite-dimensional Marsden-Weinstein reduction.)

The upshot of this discussion is that *Poisson manifolds are the infinitesimal versions of symplectic groupoids.*

Some of the prototypical examples of symplectic groupoids are traditional phase spaces in mechanics. For example, any cotangent bundle T^*Q, equipped with its canonical symplectic form, is a symplectic groupoid over Q with respect to the groupoid structure given by fibrewise addition of covectors; in this case, source and target maps coincide, both being the bundle projection $T^*Q \to Q$, and the corresponding Poisson structure on Q is trivial: $\pi = 0$. A more interesting example is given by the cotangent bundle of a Lie group G. In this case, besides the symplectic groupoid structure over G that we just described, T^*G is also a symplectic groupoid over $\mathfrak{g}^*$, where $\mathfrak{g}$ denotes the Lie algebra of G. The groupoid structure

$$T^*G \rightrightarrows \mathfrak{g}^*$$

is induced by the co-adjoint action of G on $\mathfrak{g}^*$ (see e.g. [34]); source and target maps are given by the momentum maps for the cotangent lifts of the actions of G on itself by left and right translations, while the corresponding Poisson structure on $\mathfrak{g}^*$ is just its natural Lie-Poisson structure. The fact that the target map is a Poisson map may be viewed as the *Lie-Poisson reduction theorem* (see e.g. [31, Sec. 13.1]), another one of Marsden's favorite topics. The correspondence between Poisson structures and symplectic groupoids extends much of the theory relating $\mathfrak{g}^*$ and T^*G to more general settings.

3. Multisymplectic structures

A *multisymplectic structure* [8, 9] on a manifold M is a differential form $\omega \in \Omega^{k+1}(M)$ which is closed and nondegenerate, in the sense that $i_X \omega = 0$ implies that $X = 0$, for $X \in \Gamma(TM)$. Equivalently, the nondegeneracy condition says that the bundle map

$$(3.1) \qquad \omega^\sharp : TM \to \wedge^k T^*M, \quad X \mapsto i_X \omega,$$

is injective. As in [2, 35], we refer to a multisymplectic form of degree $k + 1$ as a *k-plectic* form. Hence a 1-plectic form ω is a usual symplectic structure, in which case the map (3.1) is necessarily surjective; note that the wedge powers ω^r, $r = 2, \ldots, \dim(M)$, are natural examples of higher degree multisymplectic forms. For completeness, we briefly recall some other examples, see e.g. [9].

For a manifold Q, the total space of the exterior bundle $\wedge^k T^*Q$ carries a canonical *k*-plectic form ω_{can}, generalizing the canonical symplectic structure on T^*Q. Indeed, there is a "tautological" k-form θ on $\wedge^k T^*Q$ given by

$$\theta_\xi(X_1, \ldots, X_k) = \xi(dp(X_1), \ldots, dp(X_k)),$$

where $p : \wedge^k T^*Q \to Q$ is the natural bundle projection, $\xi \in \wedge^k T^*Q$, and X_i, $i = 1, \ldots, k$, are tangent vectors to $\wedge^k T^*Q$ at ξ. Then

$$(3.2) \qquad \omega_{can} = d\theta$$

is a k-plectic form on $\wedge^k T^*Q$. These k-plectic manifolds are closely related to the multi-phase spaces in field theory (see e.g. [18, 21] and references therein).

Other examples of k-plectic manifolds include $(k+1)$-dimensional orientable manifolds equipped with volume forms. An important class of 2-plectic manifolds is given by compact, semi-simple Lie groups G, equipped with the Cartan 3-form $H \in \Omega^3(G)$, i.e., the bi-invariant 3-form uniquely defined by the condition $H(u, v, w) = \langle u, [v, w] \rangle$, where u, v, $w \in \mathfrak{g}$ and $\langle \cdot, \cdot \rangle$ is the Killing form (see e.g. [2, 9]). Hyper-Kähler manifolds are examples of 3-plectic manifolds: if ω_1, ω_2, ω_3 are the three Kähler forms on a hyper-Kähler manifold M, then the form $\omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3 \in \Omega^4(M)$ is 3-plectic [9, 28].

In physical applications (such as quantization), an important issue concerns the identification of an appropriate analog of the Poisson bracket (2.3) on a k-plectic manifold (M, ω); there is an extensive literature on this problem, see [8, 16, 25, 35]. As a starting point, one usually considers forms $\alpha \in \Omega^{k-1}(M)$ for which there exists a (necessarily unique) vector field X_α such that $i_{X_\alpha} \omega = d\alpha$; such forms are called *hamiltonian*. Then, on the space of hamiltonian $(k-1)$-forms, one defines the bracket

$$(3.3) \qquad \{\alpha, \beta\} = i_{X_\alpha} i_{X_\beta} \omega,$$

which is a direct generalization of the Poisson bracket (2.3) when $k = 1$. This skew-symmetric bracket turns out to be well defined on the space of hamiltonian $(k-1)$-forms, but the Jacobi identity usually fails (see e.g. [8, 35]):

$$(3.4) \qquad \{\alpha, \{\beta, \gamma\}\} + \{\gamma, \{\alpha, \beta\}\} + \{\beta, \{\gamma, \alpha\}\} = -d i_{X_\alpha} i_{X_\beta} i_{X_\gamma} \omega.$$

Much work has been done to deal with this "defect" on the jacobiator of (3.3), either by forcing its elimination or by somehow making sense of it. One approach relies on noticing that closed $(k-1)$-forms are automatically hamiltonian, so one can consider the quotient space of hamiltonian forms modulo closed forms (see e.g. [8]); the bracket (3.3) descends to this quotient and, since the right-hand side of (3.4) is exact, the quotient inherits a genuine Lie-algebra structure[5]. By using multivector fields, one can also consider hamiltonian forms of other degrees and show that these Lie algebras fit into larger graded Lie algebras. A more recent approach, see [2, 35], shows that, without taking quotients (so as to force the vanishing of the jacobiator), the bracket (3.3) on hamiltonian forms can be naturally understood in terms of structures from homotopy theory; namely, this bracket is part of a Lie k-algebra (a special type of L_∞-algebra). A missing ingredient in these generalizations of the Poisson bracket (2.3) is a corresponding analog of the Leibniz rule (2.1). For a discussion in this direction, see e.g. [23, 25].

Just as symplectic manifolds are particular cases of Poisson manifolds, one could wonder about the analog of Poisson manifolds in multisymplectic geometry. As recalled in Section 2, the Leibniz rule is central for the general definition of a Poisson structure. So, as indicated by the previous discussion on Poisson brackets on k-plectic manifolds, it is not evident how to define such analogs in terms of algebraic/Lie-type structures on spaces of forms. A different, more geometric, perspective to this problem will be discussed next.

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[5]In the case of exact k-plectic manifolds, a different way to eliminate the jacobiator defect is presented in [16], based on a modification of the bracket (3.3) using the k-plectic potential.

4. Multisymplectic groupoids and their infinitesimal versions

We start with a straightforward generalization of symplectic groupoids to multi-symplectic geometry: A *multisymplectic groupoid* is a Lie groupoid equipped with a multisymplectic form that is multiplicative, in the sense of (2.4). We will also use the terminology *k-plectic groupoid* when the multisymplectic form has degree $k+1$.

Recalling that Poisson structures arise as infinitesimal versions of symplectic groupoids, as briefly explained in Section 2, we will now identify the infinitesimal objects corresponding to multisymplectic groupoids.

Let $\mathcal{G} \rightrightarrows M$ be an s-simply-connected Lie groupoid, let $A \to M$ be its Lie algebroid, with anchor map $\rho : A \to TM$. The following result is established in [1, 4]: there is a 1-1 correspondence between closed, multiplicative forms $\omega \in \Omega^{k+1}(\mathcal{G})$ and vector-bundle maps $\mu : A \to \wedge^k T^*M$ (covering the identity map on M) satisfying:

$$(4.1) \qquad i_{\rho(u)}\mu(v) = -i_{\rho(v)}\mu(u),$$
$$(4.2) \qquad \mu([u,v]) = \mathcal{L}_{\rho(u)}\mu(v) - i_{\rho(v)}d(\mu(u)),$$

for $u, v \in \Gamma(A)$. Such maps μ are called (closed) *IM $(k+1)$-forms* (where IM stands for *infinitesimally multiplicative*). Using (2.7), one can write the explicit relation between ω and μ as

$$(4.3) \qquad i_{X_k}\ldots i_{X_1}\mu_x(u) = \omega_x(u, X_1, \ldots, X_k),$$

for $u \in A|_x$ and $X_i \in TM|_x$, $x \in M$.

We now discuss a slight refinement of this result taking into account the non-degeneracy condition of multisymplectic forms. We will need a few properties of multiplicative forms on Lie groupoids, all of which follow from (2.4). If ω is a multiplicative form on $\mathcal{G}$, then the following holds:

$$(4.4) \qquad \epsilon^*\omega = 0, \qquad \mathsf{inv}^*\omega = -\omega,$$

and

$$(4.5) \qquad i_{u^r}\omega = \mathsf{t}^*\mu(u), \quad \forall u \in \Gamma(A),$$

where u^r is the vector field on $\mathcal{G}$ determined by $u \in \Gamma(A)$ via right translations; see [6, Sec. 3] for the proofs of these identities (the proofs there work in any degree, though the statements refer to 2-forms). Using the second equation in (4.4) and (4.5), we also obtain

$$(4.6) \qquad i_{\overline{u}^l}\omega = -\mathsf{s}^*\mu(u),$$

where $\overline{u}^l = \mathsf{inv}_*(u^r)$ (note that this vector field coincides with the one defined by left translations of $\overline{u} = d\mathsf{inv}(u) \in \Gamma(\ker(d\mathsf{t})|_M)$).

Proposition 4.1. *A closed, multiplicative form $\omega^{k+1}(\mathcal{G})$ is nondegenerate if and only if its corresponding IM form $\mu : A \to \wedge^k T^*M$ satisfies*
  (1) $\ker\mu = \{0\}$,
  (2) $(\mathrm{Im}(\mu))^\circ = \{X \in TM \,|\, i_X\mu(u) = 0 \ \forall u \in A\} = \{0\}$.

Proof. Assume that ω is nondegenerate, and let us verify that (1) and (2) hold. If $u \in \ker\mu$, then (by (4.5)) $i_u\omega = \mathsf{t}^*\mu(u) = 0$, so $u = 0$ and (1) follows. Let now $X \in (\mathrm{Im}(\mu))^\circ|_x$, $x \in M$. Then $i_u i_X\omega = -i_X\mathsf{t}^*\mu(u) = \mathsf{t}^* i_X\mu(u) = 0$ for all $u \in A|_x$. We claim that this implies that $i_X\omega = 0$, so that $X = 0$ by nondegeneracy, and hence (2) holds. To see that, it suffices to check that $i_{Z_k}\ldots i_{Z_1}i_X\omega = 0$ for arbitrary

$Z_i \in T\mathcal{G}|_x$, $i = 1, \ldots, k$. Using (2.7), we write $Z_i = X_i + u_i$, for $X_i \in TM|_x$ and $u_i \in A|_x$. Expanding out $i_{Z_k} \ldots i_{Z_1} i_X \omega$ using multilinearity, we see that the term $i_{X_k} \ldots i_{X_1} i_X \omega$ vanishes by the first condition in (4.4), and all the other terms vanish as a consequence of the fact that $i_u i_X \omega = 0 \; \forall u \in A$.

Conversely, suppose that (1) and (2) hold, and let $X \in T_g\mathcal{G}$ be such that $i_X \omega = 0$. Then
$$i_{u^r} i_X \omega = 0 = -i_X(\mathsf{t}^* \mu(u))$$
for all $u \in \Gamma(A)$, which means that $d\mathsf{t}(X) \in (\mathrm{Im}(\mu))^\circ$, so $d\mathsf{t}(X) = 0$ by (2). Hence X is tangent to the t-fiber at g, and we can find $v \in \Gamma(A)$ so that $\mathsf{inv}_*(v^r)|_g = \overline{v}^l|_g = X$. By (4.6), at the point g we have
$$i_X \omega = i_{\overline{v}^l} \omega = -\mathsf{s}^* \mu(v),$$
so $i_X \omega = 0$ implies that $\mu(v) = 0$, hence $v = 0$ by (1), and $X = \overline{v}^l|_g = 0$. $\qquad\square$

It follows that the infinitesimal counterpart of a k-plectic groupoid is a closed IM $(k+1)$-form $\mu : A \to \wedge^k T^*M$ additionally satisfying conditions (1) and (2) of Prop. 4.1. A natural terminology for the resulting object is *IM k-plectic form*. In this paper, we will alternatively refer to them as *higher Poisson structures of degree k*, or simply *k-Poisson structures* (being aware that this may clash with the terminology for different objects in the literature). Before giving different characterizations of k-Poisson structures and examples, we briefly explain how 1-Poisson structures are the same as ordinary Poisson structures.

4.1. **The case $k = 1$.** For a bundle map $\mu : A \to T^*M$, note that condition (1) in Prop. 4.1 says that μ is injective, while (2) says that μ is surjective. It follows that a 1-Poisson structure is a bundle map $\mu : A \to T^*M$ satisfying (4.1), (4.2) (i.e., a closed IM 2-form), and that is an isomorphism.

Note that given a Poisson structure π on M, if we consider the associated Lie algebroid $A = T^*M$, see (2.5) and (2.6), it is clear that

(4.7) $$\mu = \mathrm{id} : A \to T^*M$$

is a 1-Poisson structure. It turns out that any 1-Poisson structure is equivalent[6] to one of this type. To justify this claim, it will be convenient to view Poisson structures from the broader perspective of Dirac geometry [12].

Let us consider the bundle $\mathbb{T}M := TM \oplus T^*M \to M$ equipped with the non-degenerate, symmetric fibrewise bilinear pairing $\langle \cdot, \cdot \rangle$ given at each $x \in M$ by

(4.8) $$\langle (X, \alpha), (Y, \beta) \rangle := \beta(X) + \alpha(Y),$$

for $X, Y \in T_x M$, $\alpha, \beta \in T_x^* M$, and with the Courant-Dorfman bracket $[\![\cdot, \cdot]\!] : \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$,

(4.9) $$[\![(X, \alpha), (Y, \beta)]\!] := ([X, Y], \mathcal{L}_X \beta - i_Y d\alpha).$$

Poisson structures on M are equivalent to subbundles $L \subset \mathbb{T}M$ satisfying

(d1) $L = L^\perp$, i.e., L is *lagrangian* with respect to $\langle \cdot, \cdot \rangle$,
(d2) $L \cap TM = \{0\}$,
(d3) $[\![\Gamma(L), \Gamma(L)]\!] \subseteq \Gamma(L)$.

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[6]We say that two IM $(k+1)$-forms $\mu_1 : A_1 \to \wedge^k T^*M$ and $\mu_2 : A_2 \to \wedge^k T^*M$ are *equivalent* if there is a Lie-algebroid isomorphism $\phi : A_1 \to A_2$ such that $\mu_2 \circ \phi = \mu_1$; these are infinitesimal versions of isomorphism of Lie groupoids preserving multiplicative forms.

Condition (d1) is equivalent to L being isotropic, i.e., $L \subseteq L^\perp$, and the dimension condition $\text{rank}(L) = \dim(M)$. Using the exact sequence

$$L \cap TM \to L \to T^*M$$

induced by the natural projection $\text{pr}_2 : \mathbb{T}M \to T^*M$, we see that (d2) is equivalent to saying that L projects isomorphically onto T^*M. It follows that conditions (d1) and (d2) can be alternatively written as

(d1') $L \subseteq L^\perp$,
(d2') $\text{pr}_2|_L : L \to T^*M$ is an isomorphism.

Given a subbundle $L \subset \mathbb{T}M$, conditions (d1') and (d2') are equivalent to L being the graph of a skew-adjoint bundle map $T^*M \to TM$; such maps are always of the form $\alpha \mapsto i_\alpha \pi$, where π is a bivector field. The involutivity condition (d3) amounts to $[\pi, \pi] = 0$.

Let $\mu : A \to T^*M$ be a 1-Poisson structure, and let us consider the bundle map

$$(4.10) \qquad (\rho, \mu) : A \to \mathbb{T}M,$$

where $\rho : A \to TM$ is the anchor. Since μ is an isomorphism, the map (4.10) is injective, and its image is a subbundle $L \subset \mathbb{T}M$ satisfying (d2'). Note that condition (4.1) for μ amounts to condition (d1') for L, while (4.2) becomes (d3). It follows that L represents a Poisson structure on M, explicitly given by

$$\pi(\alpha, \beta) = i_{\rho(\mu^{-1}(\alpha))}\beta, \qquad \alpha, \beta \in T^*M.$$

It is clear from (4.2) that $\mu : A \to T^*M$ is an isomorphism of Lie algebroids, where T^*M has the Lie-algebroid structure induced by π (as in (2.5) and (2.6)), showing the equivalence between μ and the 1-Poisson structure (4.7) associated with π.

As we see next, one has a similar interpretation of general k-Poisson structures in terms of higher Courant-Dorfman brackets (as in [22, Sec. 2]), leading to objects closely related to those studied in [38].

5. Descriptions of k-Poisson structures

Let us consider the vector bundle

$$\mathbb{T}M^{(k)} := TM \oplus \wedge^k T^*M;$$

we denote by $\text{pr}_1 : \mathbb{T}M^{(k)} \to TM$ and $\text{pr}_2 : \mathbb{T}M^{(k)} \to \wedge^k T^*M$ the natural projections. The same expressions as in (4.8) and (4.9) lead to a symmetric $\wedge^{k-1}T^*M$-valued pairing $\langle \cdot, \cdot \rangle$ on the fibres of $\mathbb{T}M^{(k)}$ and a bracket $[\![\cdot, \cdot]\!]$ on $\Gamma(\mathbb{T}M^{(k)})$, that we will keep referring to as the Courant-Dorfman bracket.

Given a subbundle $L \subset \mathbb{T}M^{(k)}$, we keep denoting by $L^\perp$ its orthogonal relative to $\langle \cdot, \cdot \rangle$; note that, for $k > 1$, it may happen that $L^\perp$ does not have constant rank (see Section 6). We will keep calling L *isotropic* if $L \subset L^\perp$, and *involutive* if its space of sections $\Gamma(L)$ is closed under $[\![\cdot, \cdot]\!]$. For a subbundle $D \subseteq \wedge^k T^*M$, we let

$$D^\circ := \{X \in TM \mid i_X \alpha = 0 \, \forall \alpha \in D\}$$

be its annihilator.

Whenever $L \subset \mathbb{T}M^{(k)}$ is an isotropic and involutive subbundle, it inherits a Lie-algebroid structure with anchor map $\text{pr}_1|_L : L \to TM$ and Lie bracket $[\![\cdot, \cdot]\!]|_{\Gamma(L)}$ on $\Gamma(L)$. In particular, it follows that the distribution

$$(5.1) \qquad \text{pr}_1(L) \subseteq TM$$

is integrable and its integral leaves (the "orbits" of the Lie algebroid) define a singular foliation on M, see [15, Sec. 8.1]. One may also directly check that

$$(5.2) \qquad \mathrm{pr}_2|_L : L \to \wedge^k T^* M$$

is a closed IM k-form. Since $\ker(\mathrm{pr}_2|_L) = L \cap TM$ and

$$(\mathrm{pr}_2(L))^\circ = L^\perp \cap TM \supseteq L \cap TM,$$

it is clear that (5.2) is a k-Poisson structure if and only if

$$(5.3) \qquad L^\perp \cap TM = \{0\}.$$

By considering the bundle map (5.2), we will think of any isotropic, involutive subbundle $L \subseteq \mathbb{T}M^{(k)}$ satisfying (5.3) as a k-Poisson structure. It turns out that all k-Poisson structures on M are of this type.

Proposition 5.1. *Any k-Poisson structure $\mu : A \to \wedge^k T^* M$ is equivalent to a subbundle $L \subset \mathbb{T}M^{(k)}$ that is isotropic, involutive, and satisfies (5.3).*

Proof. Let $\mu : A \to \wedge^k T^* M$ be a k-Poisson structure. The bundle map $(\rho, \mu) : A \to \mathbb{T}M$ is an isomorphism onto its image (due to condition (1) in Prop. 4.1), which is a subbundle $L \subset \mathbb{T}M^{(k)}$ that is isotropic, involutive, and satisfies (5.3) (as a result of (4.1), (4.2) and condition (2) in Prop. 4.1, respectively). It is clear that $(\rho, \mu) : A \to L$ is an isomorphism of Lie algebroids, which establishes the desired equivalence. $\square$

We conclude that the infinitesimal versions of k-plectic groupoids can be seen as isotropic, involutive subbundles $L \subset \mathbb{T}M^{(k)}$ satisfying (5.3). Note that the condition $L = L^\perp$ (see (d1)) may not hold for $k > 1$ (we will see simple examples in Section 6); in the case $k = 1$, the condition $L^\perp \cap TM = (\mathrm{pr}_2(L))^\circ = \{0\}$ implies that $\mathrm{pr}_2(L) = T^* M$, so that $L = L^\perp$.

Another characterization of k-Poisson structures, closer in spirit to the description of Poisson structures via bivector fields, is as follows.

Proposition 5.2. *There is a one-to-one correspondence between subbundles $L \subset \mathbb{T}M^{(k)}$ as in Prop. 5.1 and pairs (D, λ), where $D \subseteq \wedge^k T^* M$ is a subbundle and $\lambda : D \to TM$ is a bundle map (covering the identity) satisfying the following conditions: (a) $D^\circ = \{0\}$, (b) $i_{\lambda(\alpha)}\beta = -i_{\lambda(\beta)}\alpha$, for $\alpha, \beta \in D$, and (c) the space $\Gamma(D)$ is involutive with respect to the bracket (c.f. (2.6))*

$$(5.4) \qquad [\alpha, \beta]_\lambda := \mathcal{L}_{\lambda(\alpha)}\beta - i_{\lambda(\beta)}d\alpha = \mathcal{L}_{\lambda(\alpha)}\beta - \mathcal{L}_{\lambda(\beta)}\alpha - d(i_{\lambda(\alpha)}\beta),$$

and $\lambda : \Gamma(D) \to \Gamma(TM)$ preserves brackets.

Proof. Given a k-Poisson structure $L \subset TM \oplus \wedge^k T^* M$, note that $\mathrm{pr}_2|_L : L \to \wedge^k T^* M$ is injective (since $\ker(\mathrm{pr}_2|_L) = L \cap TM \subseteq L^\perp \cap TM = \{0\}$). Setting $D = \mathrm{pr}_2(L)$ and $\lambda = \mathrm{pr}_1 \circ (\mathrm{pr}_2|_L)^{-1}$, we see that $L = \{(\lambda(\alpha), \alpha) \,|\, \alpha \in D\}$. Then (5.3) is equivalent to condition (a), while (b) means that L is isotropic. The involutivity of L is equivalent to condition (c). $\square$

For $k = 1$, as previously remarked, $D = T^* M$ (as a result of (a)), while (b) says that $\lambda = \pi^\sharp$, for a bivector field π. The involutivity condition in (c) is automatically satisfied, and the bracket-preserving property is equivalent to the Poisson condition $[\pi, \pi] = 0$ (see e.g. [5, Lem. 2.3]).

For a k-Poisson structure defined by (D, λ) as in Prop. 5.2, D acquires a Lie algebroid structure with bracket (5.4) and anchor λ, in such a way that $\mathrm{pr}_2|_L : L \to D$ is an isomorphism of Lie algebroids. In terms of (D, λ), the singular foliation on M determined by the k-Poisson structure (see (5.1)) is given by the integral leaves of the distribution $\lambda(D) \subseteq TM$. Moreover, each leaf $\mathcal{O}$ inherits a $(k+1)$-form ω by

$$(5.5) \qquad \omega(Y_0, Y_1, \ldots, Y_k) = i_{Y_k} \ldots i_{Y_1} \alpha,$$

where $Y_i \in \lambda(D)|_{\mathcal{O}} = T\mathcal{O}$, and $\alpha \in D$ is such that $Y_0 = \lambda(\alpha)$; indeed, property (b) in Prop. 5.2 assures that ω is well defined. One may also verify, using (c) in Prop. 5.2, that ω is closed. For $k = 1$, one recovers the symplectic foliation that underlies any Poisson structure and completely determines it. However, for $k > 1$, it is no longer true that the leafwise closed $(k+1)$-forms are nondegenerate, nor that a k-Poisson structure is uniquely determined by them, see Remark 6.5 (c.f. [38, Prop. 3.8]).

The description of k-Poisson structures in Prop. 5.2 also makes the notion of morphism of k-Poisson manifolds more evident: if (D_i, λ_i) is a k-Poisson structure on M_i, $i = 1, 2$, then a map $\phi : M_1 \to M_2$ is a k-*Poisson morphism* if, for all $x \in M_1$, $\phi^*(D_2|_{\phi(x)}) \subseteq D_1|_x$ and $d\phi(\lambda_1(\phi^*\alpha)) = \lambda_2(\alpha)$, for all $\alpha \in D_2|_{\phi(x)}$.

6. Some examples and final remarks

We now give some examples of k-Poisson structures. The first two examples are from [38].

Example 6.1. *Let $\omega \in \Omega^{k+1}(M)$ be a k-plectic form. Then its graph*

$$L = \{(X, i_X\omega),\ X \in TM\} \subset \mathbb{T}M^{(k)}$$

satisfies $L = L^\perp$ and is involutive (as a consequence of ω being closed, see [38, Prop. 3.2]). Also, $L^\perp \cap TM = L \cap TM = \ker(\omega) = \{0\}$ by nondegeneracy. In terms of Prop. 5.2, $D = \mathrm{Im}(\omega^\sharp)$ and $\lambda = (\omega^\sharp)^{-1} : D \to TM$. So, just as any symplectic structure is a Poisson structure, any k-plectic form is a particular type of k-Poisson structure. A k-plectic groupoid integrating this k-Poisson structure is the pair groupoid $M \times M$, with k-plectic structure $p_1^*\omega - p_2^*\omega$ where p_i, $i = 1, 2$, denote the two natural projections from $M \times M$ to M.

Considering a k-plectic groupoid $\mathcal{G} \rightrightarrows M$ with the k-Poisson structure of Example 6.1, one may use (4.5) to check that the target map $\mathtt{t} : \mathcal{G} \to M$ is a k-Poisson morphism, extending the well-known property of symplectic groupoids, see Section 2.

We saw in Section 4.1 that Poisson bivector fields are the same as 1-Poisson structures. Other types of higher Poisson structures are obtained from top-degree multivector fields as follows.

Example 6.2. *Let $\pi \in \Gamma(\wedge^{k+1}TM)$ be a multivector field of top degree, i.e., $k = \dim(M) - 1$. Then its graph*

$$L = \{(i_\alpha\pi, \alpha) \mid \alpha \in \wedge^k T^*M\} \subseteq \mathbb{T}M^{(k)}$$

is isotropic and involutive – and, besides Poisson bivector fields, these are the only examples of non-zero multivector fields whose graphs have these properties, see [38, Prop. 3.4]. Also, since $\mathrm{pr}_2(L) = \wedge^k T^*M$, it is clear that $\mathrm{pr}_2(L)^\circ = L^\perp \cap TM = \{0\}$, so L is a k-Poisson structure. The foliations defined by these k-Poisson structures are usually singular: leaves are either open subsets of M or singular points (where π

vanishes). The restriction of π to each open leaf is nondegenerate, and the induced $(k+1)$-forms ω on these leaves (see (5.5)) are the volume forms dual to π, i.e., they are defined by $i_{(i_\alpha\pi)}\omega = \alpha, \forall \alpha \in \wedge^k T_x^* M$. The groupoids integrating these k-Poisson structures have been mostly studied when $\dim(M) = 2$ (so π is a bivector field), see [20, 33].

The fact that the particular k-Poisson structures of Examples 6.1 and 6.2 are infinitesimal versions of k-plectic groupoids was observed in [38, Prop. 3.7].

In the preceding examples, the bundle L always satisfied $L = L^\perp$. For examples where this condition fails, consider subbundles

(6.1) $$L \subseteq \wedge^k T^* M \subset \mathbb{T}M^{(k)}.$$

These are automatically isotropic and involutive. Note that

$$L^\perp = L^\circ \oplus \wedge^k T^* M, \quad \text{and} \quad L^\perp \cap TM = L^\circ.$$

So L is a k-Poisson structure as long as $L^\circ = \{0\}$, and $L \subsetneq L^\perp$ as long as L is properly contained in $\wedge^k T^* M$. A k-plectic groupoid integrating it is L itself, viewed as a vector bundle (with groupoid structure given by fibrewise addition), equipped with the k-plectic form given by the pullback of the canonical multisymplectic form on $\wedge^k T^* M$ (see (3.2)); the fact that this pullback is nondegenerate boils down to the condition $L^\perp \cap TM = L^\circ = \{0\}$.

Example 6.3. *For $L = \wedge^k T^* M$, note that $L^\circ = \{0\}$ (and hence L is a k-Poisson structure on M) if and only if $\dim(M) \geq k$.*

Example 6.4. *Let ξ be a nondegenerate k-form on M, and let $L \subset \wedge^k T^* M$ be the line bundle generated by ξ,*

$$L|_x = \{c\xi_x \mid c \in \mathbb{R}\}, \quad x \in M.$$

Then $L^\circ = \ker(\xi) = 0$, so L is a k-Poisson structure.

Remark 6.5. *Note that all k-Poisson structures of the type (6.1) determine the same foliation, the leaves of which are the points of M.*

A general observation is that one can take direct products of k-Poisson structures: if L_1 and L_2 are k-Poisson structures on M_1 and M_2, respectively, we define their product by

$$L := \{(X + Y, \alpha + \beta) \mid (X, \alpha) \in L_1, (Y, \beta) \in L_2\} \subseteq TM \oplus \wedge^k T^* M,$$

where $M = M_1 \times M_2$ and we simplify the notation by identifying forms on M_i with their pullbacks to M via the projections. One may directly verify that L is a k-Poisson structure on M. Moreover, if $(\mathcal{G}_i \rightrightarrows M_i, \omega_i)$ is a k-symplectic groupoid integrating L_i, $i = 1, 2$, the direct product $\mathcal{G}_1 \times \mathcal{G}_2 \rightrightarrows M_1 \times M_2$ (equipped with the k-plectic form $\omega_1 + \omega_2$) is a k-plectic groupoid that integrates L. The following is a concrete example.

Example 6.6. *Let (M, ω) be a k-plectic manifold, and let N be a manifold with $\dim(N) \geq k$. Then the subbundle*

$$L = \{(X, i_X\omega + \alpha) \mid X \in TM, \alpha \in \wedge^k T^* N\} \subset T(M \times N) \oplus \wedge^k T^*(M \times N)$$

is a k-Poisson structure on $M \times N$ (c.f. [38, Thm. 3.12]), the direct product of the k-plectic form on M with the k-Poisson structure $L = \wedge^k T^* N$ on N (see Example 6.3).

The leaves of L are $M \times \{t\}$, $t \in N$, with induced $(k+1)$-form (as in (5.5)) given by ω.

The next observation illustrates that k-Poisson structures become more rigid than Poisson structures when $k > 1$.

Remark 6.7. *Let M and N be as is Example 6.6, let $f \in C^\infty(N)$, and consider the smooth family $\omega_t = f(t)\omega$, $t \in N$, of k-plectic forms on M. For $k = 1$, this family defines a Poisson structure on $M \times N$, uniquely determined by the fact that its symplectic leaves are $(M \times \{t\}, \omega_t)$. A higher generalization of this Poisson structure is given by the (isotropic) subbundle $L \subset T(M \times N) \oplus \wedge^k T^*(M \times N)$ defined by*

$$L|_{(x,t)} = \{(X, i_X\omega_t + \alpha) \mid X \in T_x M, \alpha \in \wedge^k T_t^* N\}.$$

As it turns out, for $k > 1$, one may verify that such L is involutive if and only if $df = 0$, i.e., f is (locally) constant.

We finally mention another product-type operation for multisymplectic manifolds leading to higher Poisson structures that are not multisymplectic.

Example 6.8. *Let (M_i, ω_i) be a k_i-plectic manifold, $i = 1, 2$. Let $M = M_1 \times M_2$ and $\omega = \omega_1 \wedge \omega_2 \in \Omega^{k_1+k_2+2}(M)$ (we keep the simplified notation of identifying forms on M_i with their pullbacks to M via the projections $M \to M_i$). Then*

$$L = \{(X, i_X\omega) = (X, (i_X\omega_1) \wedge \omega_2) \mid X \in TM_1\} \subset TM \oplus \wedge^{k_1+k_2+1}T^*M$$

can be checked to be a $(k_1 + k_2 + 1)$-Poisson structure. Its leaves are of the form $M_1 \times \{y\}$, for $y \in M_2$, and the induced $(k_1 + k_2 + 2)$-form on each leaf is zero. An integrating k-plectic groupoid is given by the direct product of the pair groupoid $M_1 \times M_1$ (see Example 6.1) and the trivial groupoid over M_2, endowed with the multiplicative $(k_1 + k_2 + 2)$-form given by $(p_1^*\omega_1 - p_2^*\omega_1) \wedge \omega_2$.

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