Poisson vector bundles, contravariant connections and deformations

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In this note we discuss the semiclassical geometry associated to deformation quantization of vector bundles over Poisson manifolds. We compare the objects obtained as "first-order approximations" to deformed vector bundles with the existent notion of Poisson vector bundles, showing that they agree up to a certain flatness condition. We show how these ideas can be used in the study of Morita equivalence of star products on Poisson manifolds.

§1. Introduction

Poisson modules over Poisson algebras were first considered in connection with the study of quantum groups[23)], and have been recently used to define Poisson K-theory and new invariants of Poisson manifolds[17)].

Poisson geometry is closely related to noncommutative algebras[9)]. In fact, Poisson structures on a manifold M arise as "first-order approximations" to noncommutative algebras defined by deforming[15)] the commutative algebra $C^\infty(M)$, a process known as deformation quantization[2)]. This motivates the idea that the "Poisson category" should occupy an intermediate place between ordinary differential geometry and noncommutative geometry[10)].

A classical result by Serre and Swan[28)] asserts that vector bundles over a manifold M correspond to finitely generated projective modules over $C^\infty(M)$. Analogously, we consider "quantum vector bundles" to be (finitely generated projective) modules over star-product algebras $(C^\infty(M)[[\lambda]], \star)$. It turns out[6)] that these objects always arise from classical vector bundles $E \to M$ by means of deformation quantization of the module structure of $\Gamma^\infty(E)$ over $C^\infty(M)$ with respect to $\star$.

Just as deformations of associative algebra structures give rise to Poisson structures in their semiclassical limit, "first-order approximations" to deformed vector bundles define a geometric structure on the corresponding classical vector bundles: contravariant connections[14), 29)]. This was shown for line bundles in[4)], and, in this note, we will extend this discussion to higher dimensional vector bundles. Unlike the case of deformations of algebras, this "semiclassical" structure on vector bundles is not canonically defined. As we will see, for a star product $\star$ on M, any contravariant connection on a vector bundle $E \to M$ can be obtained as the semiclassical limit of a deformation of E with respect to $\star$.

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As an application of the previous considerations, we recall[4] how line bundles with contravariant connections arise as "first-order approximations" to Morita equivalent star products on Poisson manifolds. The notion of Morita equivalence[1, 21], besides its importance in many areas of mathematics, has been recently shown to be related to physical duality[25, 26], playing an important role in applications of noncommutative geometry to string theory[11].

As shown in[4], one can phrase the problem of classifying Morita equivalent star products on a Poisson manifold (M, π) in terms of a canonical action Φ of the Picard group $\mathrm{Pic}(M) \cong H^2(M, \mathbb{Z})$ on the moduli space of equivalence classes of star products on M. The action Φ is defined by deformation quantization of line bundles over M. We will recall how the curvature of the contravariant connections arising in the semiclassical limit of line bundle deformations can be used to describe the semiclassical limit of Φ in terms of Kontsevich's classification[20] of star products.

The paper is organized as follows. In Section 2 we recall the notions of Poisson vector bundles and contravariant connections. In Section 3 we define star products and deformation quantization of vector bundles. The semiclassical geometry of deformed vector bundles is discussed in Section 4, where we present the main results of this note: we show how contravariant connections arise in the semiclassical limit of vector bundle deformations and compare this semiclassical object with the notion of Poisson vector bundle. We consider deformed line bundles in Section 5, where we describe how their semiclassical geometry provides information about the characterization of Morita equivalent star products on Poisson manifolds.

§2. Poisson vector bundles and contravariant connections

Let $\mathcal{A}$ be a commutative and unital algebra over a field k of characteristic zero.

Definition 2.1 *A* Poisson bracket *on $\mathcal{A}$ is a Lie algebra bracket $\{\cdot, \cdot\}$ satisfying the Leibniz rule*

$$\{A_1, A_2 A_3\} = \{A_1, A_2\}A_3 + A_2\{A_1, A_3\}, \; A_1, A_2 \in \mathcal{A}.$$

The pair $(\mathcal{A}, \{\cdot, \cdot\})$ is called a Poisson algebra*.*

Let $\mathcal{E}$ be a vector space over k. One can define modules in the Poisson category through Poisson extensions of Poisson algebras:

Definition 2.2 *A* Poisson module structure *on $\mathcal{E}$ is the structure of a Poisson algebra on $\mathcal{E} \oplus \mathcal{A}$, extending the bracket on $\mathcal{A}$ and so that $\mathcal{E}.\mathcal{E} = \{\mathcal{E}, \mathcal{E}\} = 0$.*

A simple computation shows that a Poisson module structure on $\mathcal{E}$ is equivalent to a module structure on $\mathcal{E}$ over $\mathcal{A}$ together with a bracket $\{\, , \,\} : \mathcal{E} \times \mathcal{A} \longrightarrow \mathcal{E}$ satisfying

$$\{s, \{A_1, A_2\}\} = \{\{s, A_1\}, A_2\} - \{\{s, A_2\}, A_1\}, \tag{1}$$

$$\{sA_1, A_2\} = \{s, A_2\}A_1 + s\{A_1, A_2\}, \tag{2}$$

$$\{s, A_1 A_2\} = \{s, A_1\}A_2 + \{s, A_2\}A_1, \tag{3}$$

for all $A_1, A_2 \in \mathcal{A}$ and $s \in \mathcal{E}$. We call $(\mathcal{E}, \{\, , \,\})$ a *Poisson module* over $\mathcal{A}$.

Let (M, π) be a Poisson manifold, where $\pi \in \chi^2(M)$ is the Poisson tensor. Let $C^\infty(M)$ be the algebra of smooth complex-valued functions on M, and let

$$\{f, g\} := \pi(df, dg), \quad f, g \in C^\infty(M) \tag{4}$$

be the Poisson bracket. The Poisson tensor π defines a bundle map

$$\tilde{\pi} : \Omega^1(M) \longrightarrow \chi(M), \ \ \alpha \mapsto \pi(\cdot, \alpha), \tag{5}$$

and the vector field $X_f := \tilde{\pi}(df)$ is called the hamiltonian vector field of f. We use $\tilde{\pi}$ to define a Lie algebra bracket on $\Omega^1(M)$:

$$[\alpha, \beta] := -\mathcal{L}_{\tilde{\pi}(\alpha)}\beta + \mathcal{L}_{\tilde{\pi}(\beta)}\alpha - \mathrm{d}(\pi(\alpha, \beta)), \tag{6}$$

so that $-\tilde{\pi}$ is a Lie algebra homomorphism making T^*M into a Lie algebroid [9].

Let $E \to M$ be a complex m-dimensional vector bundle, and let $\Gamma^\infty(E)$ be the space of smooth sections of E, regarded as a right finitely generated projective module over $C^\infty(M)$.

Definition 2.3 *A* Poisson vector bundle *structure on $E \to M$ is a Poisson module structure on $\Gamma^\infty(E)$.*

Example 2.4 *Suppose ∇ is a flat connection on E. Then $\{s, f\} := \nabla_{X_f} s$ makes E into a Poisson vector bundle over M.*

In order to formulate the notion of a Poisson vector bundle in terms of connections, we need the following definition [14, 29].

Definition 2.5 *A* contravariant connection *on a vector bundle $E \to M$ is a $\mathbb{C}$-bilinear map $D : \Gamma^\infty(E) \times \Omega^1(M) \longrightarrow \Gamma^\infty(E)$ satisfying*

$$D_{f\alpha} s = f D_\alpha s, \tag{7}$$

$$D_\alpha(fs) = f D_\alpha s + \alpha(X_f) s, \tag{8}$$

for $\alpha \in \Omega^1(M)$, $f \in C^\infty(M)$, and $s \in \Gamma^\infty(E)$.

Example 2.6 *Any ordinary connection ∇ on $E \to M$ defines a contravariant connection by $D_{df} s := \nabla_{X_f} s$, for $s \in \Gamma^\infty(E)$, $f \in C^\infty(M)$. If π is nondegenerate (symplectic), any contravariant connection on E arises in this way.*

Definition 2.7 *The* curvature *of a contravariant connection D is the map $\Theta_D : \Omega^1(M) \times \Omega^1(M) \longrightarrow \mathrm{End}(\Gamma^\infty(E))$ given by*

$$\Theta_D(\alpha, \beta)s = D_\alpha D_\beta s - D_\beta D_\alpha s + D_{[\alpha, \beta]} s, \tag{9}$$

where $[\ ,\ ]$ is the bracket (6).

Let $(\Gamma^\infty(E), \{\ ,\ \})$ be a Poisson vector bundle. Note that (3) implies that $\{s, f\}$ at a point $x \in M$ depends only on $df(x)$. The formula

$$D_{df} s := \{s, f\}$$

defines a contravariant connection on E, and (1) is equivalent to D being flat. As a result, we have

Proposition 2.8 *A Poisson vector bundle structure on $E \to M$ is equivalent to a flat contravariant connection on E.*

We will see in Section 4 that this flatness condition is not satisfied by the infinitesimal part of formal deformations of vector bundles in general.

§3. Deformation quantization

In this section we recall the definitions of formal deformations of algebras and modules.

3.1. *Star products*

Let $\mathcal{A}$ be a k-algebra. We recall the definition of a formal deformation of $\mathcal{A}$ [15]:

Definition 3.1 *A* formal deformation *of $\mathcal{A}$ is an associative $k[[\lambda]]$-bilinear multiplication $\star$ on $\mathcal{A}[[\lambda]]$ of the form*

$$A \star A' = \sum_{r=0}^{\infty} C_r(A, A')\lambda^r, \qquad A, A' \in \mathcal{A}, \tag{10}$$

where the maps $C_r : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ are k-bilinear, and C_0 is the original product on $\mathcal{A}$. (We extend $\star$ to $\mathcal{A}[[\lambda]]$ using λ-linearity.)

A formal deformation of $\mathcal{A}$ will be denoted by $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$; two formal deformations of $\mathcal{A}$, $\boldsymbol{\mathcal{A}}_1 = (\mathcal{A}[[\lambda]], \star_1)$ and $\boldsymbol{\mathcal{A}}_2 = (\mathcal{A}[[\lambda]], \star_2)$, are *equivalent* if there exist k-linear maps $T_r : \mathcal{A} \longrightarrow \mathcal{A}$, $r \geq 1$, so that $T = \mathsf{id} + \sum_{r=1}^{\infty} T_r \lambda^r : \boldsymbol{\mathcal{A}}_1 \longrightarrow \boldsymbol{\mathcal{A}}_2$ satisfies

$$A \star_1 A' = T^{-1}(T(A) \star_2 T(A')), \qquad \forall A, A' \in \mathcal{A}[[\lambda]]. \tag{11}$$

Such a T is called an *equivalence transformation*, and we denote the equivalence class of a deformation $\star$ by $[\star]$. If $\mathcal{A}$ is unital, then so is any formal deformation $\boldsymbol{\mathcal{A}}$; moreover, any formal deformation of $\mathcal{A}$ is equivalent to one for which the unit is the same as the one for $\mathcal{A}$ [16].

For a formal deformation (10) of $\mathcal{A}$, a simple computation using associativity of $\star$ shows that

$$\{A_1, A_2\} := C_1(A_1, A_2) - C_1(A_2, A_1) = \frac{1}{\lambda}(A_1 \star A_2 - A_2 \star A_1) \bmod \lambda \tag{12}$$

is a Poisson bracket on $\mathcal{A}$, and if two formal deformations are equivalent, then they determine the same Poisson bracket through (12).

Let $\mathcal{A} = C^\infty(M)$. We recall the definition of a star product [2] on M.

Definition 3.2 *A formal deformation $\star = \sum_{r=0}^{\infty} C_r \lambda^r$ of $\mathcal{A}$ is called a* star product *if each C_r is a bidifferential operator.*

The set of equivalence classes of star products on M is denoted by $\mathrm{Def}(M)$, and, if π is a Poisson structure on M, we let

$$\mathrm{Def}(M, \pi) := \{[\star] \in \mathrm{Def}(M) \mid f \star g - g \star f = \lambda \pi(df, dg) \bmod \lambda^2\}. \tag{13}$$

3.2. *Deformation quantization of vector bundles*

This section recalls some results on deformations of vector bundles over Poisson manifolds [6]. The reader is referred to [31] for physical applications.

Let $E \to M$ be a complex m-dimensional vector bundle, and let $\star$ be a star product on M. Motivated by Serre-Swan's theorem, we consider the following definition.

Definition 3.3 *A* deformation quantization *of $E \to M$ with respect to $\star$ is a $\mathbb{C}[[\lambda]]$-bilinear map $\bullet : \Gamma^\infty(E)[[\lambda]] \times C^\infty(M)[[\lambda]] \longrightarrow \Gamma^\infty(E)[[\lambda]]$ satisfying $s \bullet (f \star g) = (s \bullet f) \bullet g$ and such that*

$$s \bullet f = \sum_{r=0}^{\infty} \lambda^r R_r(s, f),$$

where each $R_r : \Gamma^\infty(E) \times C^\infty(M) \longrightarrow \Gamma^\infty(E)$ is bidifferential and $R_0(f, g) = sf$ (multiplication of sections by functions).

Thus $(\Gamma^\infty(E), \bullet)$ is a right module over $\boldsymbol{\mathcal{A}} = (C^\infty(M)[[\lambda]], \star)$ deforming the right $C^\infty(M)$-module $\Gamma^\infty(E)$.

Two vector bundle deformations $\bullet, \bullet'$ are *equivalent* if there exist differential operators $T_r : \Gamma^\infty(E) \longrightarrow \Gamma^\infty(E)$ so that $T = \mathsf{id} + \sum_{r=1}^{\infty} \lambda^r T_r$ satisfies

$$T(s \bullet' f) = T(s) \bullet f, \;\; s \in \Gamma^\infty(E), \; f \in C^\infty(M).$$

The following result was proven in [6].

Proposition 3.4 *Let $E \to M$ be a complex m-dimensional vector bundle, and let $\star$ be a star product on M. Then there exists a deformation $\bullet$ of E with respect to $\star$, which is unique up to equivalence.*

One can check that the right module $(\Gamma^\infty(E)[[\lambda]], \bullet)$ is finitely generated and projective over $(C^\infty(M)[[\lambda]], \star)$, and any finitely generated projective module over $(C^\infty(M)[[\lambda]], \star)$ arises as a deformation quantization of a classical vector bundle. This motivates the interpretation of these deformed modules as quantum vector bundles in the framework of deformation quantization. Deformed vector bundles can be also described locally through deformed trivialization maps and transition matrices [5].

Let $\mathcal{E} = \Gamma^\infty(E)$, regarded as a right module over $C^\infty(M)$. Let $\bullet$ be a deformation quantization of $\mathcal{E}$ with respect to a star product $\star$, and consider the right $\boldsymbol{\mathcal{A}}$-module $\boldsymbol{\mathcal{E}} = (\mathcal{E}[[\lambda]], \bullet)$, where $\boldsymbol{\mathcal{A}} = (C^\infty(M)[[\lambda]], \star)$. It is simple to check that $\mathrm{End}(\mathcal{E}) \cong \Gamma^\infty(\mathrm{End}(E))$, where $\mathrm{End}(E)$ is the bundle of endomorphisms of E. In the deformed picture, we have $\mathrm{End}(\boldsymbol{\mathcal{E}}) \cong \Gamma^\infty(\mathrm{End}(E))[[\lambda]]$ as $\mathbb{C}[[\lambda]]$-modules, and any explicit identification

$$\Gamma^\infty(\mathrm{End}(E))[[\lambda]] \stackrel{\sim}{\longrightarrow} \mathrm{End}(\boldsymbol{\mathcal{E}}) \tag{14}$$

induces a formal associative deformation $\star'$ of the algebra $\Gamma^\infty(\mathrm{End}(E))$; as observed in [6], this identification can be chosen so that $\star'$ is given by bidifferential cochains.

As discussed in [4], due to Proposition 3.4, this procedure gives rise to a well-defined map

$$\Phi_E : \mathrm{Def}(M) \longrightarrow \mathrm{Def}(\Gamma^\infty(\mathrm{End}(E))), \;\; [\star] \mapsto [\star'], \tag{15}$$

where $\mathrm{Def}(\Gamma^\infty(\mathrm{End}(E)))$ denotes the moduli space of equivalence classes of formal differential deformations of $\Gamma^\infty(\mathrm{End}(E))$. We recall [6]

Proposition 3.5 *Φ_E is a bijection.*

§4. Semiclassical geometry of deformed vector bundles

In this section we extend the discussion in [4] to higher dimensional vector bundles.

Let $\mathcal{A}= C^\infty(M)$, and let $\star = \sum_{r=0}^\infty \lambda^r C_r$ be a star product on M. As observed in (12), the skew-symmetric part of C_1 defines a Poisson bracket $\{\,,\,\}$ on $\mathcal{A}$. Thus the "first-order approximation" to the deformed algebra $\boldsymbol{\mathcal{A}} = (\mathcal{A}[[\lambda]], \star)$ defines a geometric structure on M, namely a Poisson tensor.

If $E \to M$ is a vector bundle, then, by Proposition 3.4, we can deform the module structure of $\Gamma^\infty(E)$ over $C^\infty(M)$ with respect to the star product $\star$, defining a (finitely generated projective) right module $\boldsymbol{\mathcal{E}} = (\Gamma^\infty(E)[[\lambda]], \bullet)$ over $\boldsymbol{\mathcal{A}}$. As in the case of formal deformations of algebras, we consider the "first-order approximation" to the module deformation $\bullet = \sum_{r=0}^\infty \lambda^r R_r$,

$$R_1 : \Gamma^\infty(E) \times C^\infty(M) \longrightarrow \Gamma^\infty(E). \tag{16}$$

We will discuss here a way to "skew symmetrize" R_1 and the geometric meaning of the resulting object.

As observed in the previous section, we can fix a $\mathbb{C}[[\lambda]]$-module isomorphism (14), and define a formal deformation $\star' = \sum_{r=0}^\infty \lambda^r C'_r$ of $\Gamma^\infty(\mathrm{End}(E))$. Since $(\Gamma^\infty(\mathrm{End}(E))[[\lambda]], \star') \cong \mathrm{End}(\boldsymbol{\mathcal{E}})$, there is a left module structure on $\Gamma^\infty(E)[[\lambda]]$ over $(\Gamma^\infty(\mathrm{End}(E))[[\lambda]], \star')$,

$$\bullet' : \Gamma^\infty(\mathrm{End}(E))[[\lambda]] \times \Gamma^\infty(E)[[\lambda]] \longrightarrow \Gamma^\infty(E)[[\lambda]], \tag{17}$$

$\bullet' = \sum_{r=0}^\infty \lambda^r R'_r$, where $R_r : \Gamma^\infty(\mathrm{End}(E)) \times \Gamma^\infty(E) \longrightarrow \Gamma^\infty(E)$ are $\mathbb{C}$-bilinear (and can be chosen to be bidifferential operators), and $R'_0(L,s)(x) = L(x)s(x)$, for $x \in M$. Note that the deformed module structures $\bullet'$ and $\bullet$ make $\Gamma^\infty(E)[[\lambda]]$ into a bimodule over $(\Gamma^\infty(\mathrm{End}(E))[[\lambda]], \star')$ and $(C^\infty(M)[[\lambda]], \star)$. This implies the compatibility equations

$$(L \star' U) \bullet' s = L \bullet' (U \bullet' s), \tag{18}$$
$$s \bullet (f \star g) = (s \bullet f) \bullet g, \tag{19}$$
$$(L \bullet' s) \bullet f = L \bullet' (s \bullet f), \tag{20}$$

for $f, g \in C^\infty(M)$, $L, U \in \Gamma^\infty(\mathrm{End}(E))$, $s \in \Gamma^\infty(E)$.

Let $\mathcal{Z}$ denote the center of $\Gamma^\infty(\mathrm{End}(E))$. We note that

$$\{L, U\}' = C'_1(L,U) - C'_1(U,L), \quad L, U \in \Gamma^\infty(\mathrm{End}(E)),$$

defines a Poisson bracket when restricted to $\mathcal{Z}$ [23)], in such a way that the natural algebra isomorphism

$$i : C^\infty(M) \longrightarrow \mathcal{Z}, \tag{21}$$

preserves Poisson brackets [4)]. Hence, with this identification, we can always assume that $C'_1|_{\mathcal{Z}} = C_1$ [*)]. Moreover, we can consider

$$R'_1|_{\mathcal{Z}} : C^\infty(M) \times \Gamma^\infty(E) \longrightarrow \Gamma^\infty(E),$$

[*)] Although $C'_1|_{\mathcal{Z}}$ is a Poisson bracket, it seems a hard question as to when $\star'$ defines a deformation of $\mathcal{Z}$.

and define a "skew symmetric" version of R_1:

$$R = R_1 - R_1'|_{\mathcal{Z}} : \Gamma^\infty(E) \times C^\infty(M) \longrightarrow \Gamma^\infty(E), \tag{22}$$

$R(s,f) = R_1(s,f) - R_1'(f,s)$, for $f \in C^\infty(M) \cong \mathcal{Z}$, $s \in \Gamma^\infty(E)$.

Proposition 4.1 *Suppose that $\star'$ is such that $C_1'|_{\mathcal{Z}} = C_1$. Then R, defined in (22), is a contravariant connection on E.*

Proof: We must check that R satisfies the Leibniz rules (2) and (3). This follows from the compatibility equations (18), (19),(20) in order λ, as in [4]. □

Since our construction of R involves non-canonical choices, it is natural to ask which contravariant connections on E arise as the semiclassical limit of deformations of E with respect to a star product $\star$.

Suppose that we change $\star'$ by an equivalent formal deformation $\star'' = \sum_{r=0}^\infty \lambda^r C_r''$, with $C_1''|_{\mathcal{Z}} = C_1'|_{\mathcal{Z}}$. Let $T = \mathsf{id} + \sum_{r=1}^\infty \lambda^r T_r$ be an equivalence transformation,

$$L \star'' U = T^{-1}(T(L) \star T(U)), \;\; L, U \in \Gamma^\infty(\mathrm{End}(E)). \tag{23}$$

Defining $L \bullet'' s = T(L) \bullet' s = \sum_{r=0}^\infty \lambda^r R_r''(L,s)$, it follows that

$$R_1''(L,s) = R_1'(L,s) + T_1(L)s,$$

and the contravariant connection $R' = R_1 - R_1''|_{\mathcal{Z}}$ satisfies

$$R(s,f) - R'(s,f) = T_1(f)s. \tag{24}$$

Theorem 4.2 *Let $\star$ be a star product on M and $\bullet$ a deformation of $E \to M$ with respect to $\star$. Let D be a contravariant connection on E. Then we can choose $\star'$, $[\star'] = \Phi_E([\star])$, so that $C_1'|_{\mathcal{Z}} = C_1$ and $R = D$.*

Proof: Note that any two contravariant connections on $E \to M$ must differ by a linear map $X : \Omega^1(M) \longrightarrow \Gamma^\infty(\mathrm{End}(E))$, or, equivalently, by a linear map

$$X_1 : C^\infty(M) \longrightarrow \Gamma^\infty(\mathrm{End}(E))$$

satisfying $X_1(fg) = fX_1(g) + X_1(f)g$ (the definitions are related by $X(df) = X_1(f)$). For any such X_1, we can find $T_1 : \Gamma^\infty(\mathrm{End}(E)) \longrightarrow \Gamma^\infty(\mathrm{End}(E))$ with $T_1|_{\mathcal{Z}} = X_1$. Choose $\star'$, and let $T = \mathsf{id} + \lambda T_1$. Let $\star''$ be defined as in (23). A simple computation shows that the condition $T_1(fg) = fT_1(g) + T_1(f)g$, for $f, g \in \mathcal{Z}$, implies that $C_1''|_{\mathcal{Z}} = C_1'|_{\mathcal{Z}}$. Hence, by (24), the corresponding contravariant connections satisfy

$$R(s,f) - R'(s,f) = T_1(f)s = X_1(f)s.$$

Since X_1 is arbitrary, this concludes the proof. □

Thus the bracket $\{\,,\} = R$ arising as a first-order approximation to a formal deformation of a module over a Poisson algebra fails to satisfy (1) in general, and the extension of $\{\,,\}$ to $\mathcal{E} \oplus \mathcal{A}$ defines just an almost Poisson algebra.

§5. Application: Morita equivalence of star products

The reader is referred to [4] for details on results in this section.

5.1. *Picard groups acting on star products*

Let $L \to M$ be a complex line bundle over a manifold M. In this case, we have an identification

$$C^\infty(M) \xrightarrow{\sim} \Gamma^\infty(\mathrm{End}(L)),$$

and therefore, by Proposition 3.5, each line bundle L gives rise to an automorphism of $\mathrm{Def}(M)$,

$$\Phi_L : \mathrm{Def}(M) \longrightarrow \mathrm{Def}(M). \tag{25}$$

The map Φ_L depends only on the isomorphism class of L in $\mathrm{Pic}(M) \cong H^2(M,\mathbb{Z})$, and, as mentioned in the previous section, star products related by Φ correspond to the same Poisson bracket on M [4].

Proposition 5.1 *Let (M,π) be a Poisson manifold. The map*

$$\Phi : \mathrm{Pic}(M) \times \mathrm{Def}(M,\pi) \longrightarrow \mathrm{Def}(M,\pi), \quad ([L],[\star]) \mapsto [\star'] = \Phi_L([\star])$$

defines an action of $\mathrm{Pic}(M)$ *on* $\mathrm{Def}(M,\pi)$.

This action is related to an important equivalence relation between star products [4]:

Proposition 5.2 *Let $\star$ and $\star'$ be star products on M. The algebras $(C^\infty(M)[[\lambda]],\star)$ and $(C^\infty(M)[[\lambda]],\star')$ are Morita equivalent if and only if there exists a Poisson diffeomorphism $\psi : M \longrightarrow M$ such that $[\star]$ and $[\psi^*(\star')]$ lie in the same orbit of Φ*[*)]

Recall that two unital algebras are Morita equivalent if they have equivalent categories of left modules [21), 1)]. Thus the action Φ relates star products with equivalent representation theories.

5.2. *Poisson cohomology and Poisson-Chern classes*

Let us recall a few facts about Poisson cohomology and characteristic classes of line bundles over Poisson manifolds.

If M is manifold and $\pi \in \chi^2(M)$ is a Poisson tensor, then we can define a differential

$$d_\pi : \chi^k(M) \longrightarrow \chi^{k+1}(M), \quad d_\pi = [\pi, \cdot],$$

where $[\,,\,]$ is the Schouten-Nijenhuis bracket [30]. The cohomology groups of the complex $(\chi^\bullet(M), d_\pi)$ are the *Poisson cohomology* groups of (M,π), and they are denoted by $H^k_\pi(M)$.

The map $\tilde{\pi}$ in (5) induces a map $\pi^* : \Omega^\bullet(M) \longrightarrow \chi^\bullet(M)$ intertwining differentials, and hence gives rise to a homomorphism in cohomology

$$\pi^* : H^k_{dR}(M) \longrightarrow H^k_\pi(M),$$

which is an isomorphism when π is symplectic. We define integral (resp. real) Poisson cohomology as the image of integral (resp. real) de Rham cohomology on M under π^*.

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*) $\star'' = \psi^*(\star')$ is defined by $f \star'' g = (\psi^*)^{-1}(\psi^*(f) \star' \psi^*(g))$, $\psi^*(f) = f \circ \psi$.

Let $L \to M$ be a complex line bundle over the Poisson manifold (M, π), and let D be a contravariant connection on L. The curvature of D, Θ_D, defines a d_π-closed bivector on M, and its Poisson cohomology class is independent of the choice of contravariant connection [29]. We call the class $c_1^\pi(L) = \frac{i}{2\pi}[\Theta_D]_\pi \in H^2_\pi(M)$ the *Poisson-Chern class* of L. A simple computation shows that $c_1^\pi(L) = \pi^*(c_1(L))$, where $c_1(L)$ is the Chern class of L.

5.3. *First-order approximation to Morita equivalent star products*

We will now show how the semiclassical geometry of line bundle deformations over a Poisson manifold (M, π_0) can be used to describe the semiclassical limit of the action Φ in terms of Kontsevich's parametrization of equivalence classes of star products.

For a Poisson manifold (M, π_0), Kontsevich showed [20] that there is a bijection between equivalence classes of star products and equivalence classes of formal Poisson structures on M. We denote this correspondence by

$$\mathrm{c} : \mathrm{Def}(M, \pi_0) \longrightarrow \{\pi_\lambda = \pi_0 + \lambda\pi_1 + \ldots \in \chi^2(M)[[\lambda]], [\pi_\lambda, \pi_\lambda] = 0\}/F, \qquad (26)$$

where F is the group $\{\exp(\sum_{r=1}^\infty D_r\lambda^r), D_r \in \mathrm{Der}(C^\infty(M))\}$, acting on formal Poisson structures by

$$T(\pi_\lambda) = \pi'_\lambda \text{ if and only if } \pi'_\lambda(df, dg) = T^{-1}\pi_\lambda(d(T(f)), d(T(g))),$$

for $T \in F$. We denote the equivalence class of π_λ by $[\pi_\lambda]$.

This correspondence is a result of a more general fact [20]: there exists an L_∞-quasi-isomorphism $\mathcal{U}$ from the graded Lie algebra of multivectors fields on M (with zero differential and Schouten bracket), $\mathfrak{g}_1$, into the graded Lie algebra of multidifferential operators on M (with Hochschild differential and Gerstenhaber bracket), $\mathfrak{g}_2$. Given such a $\mathcal{U}$, for every formal Poisson structure π_λ we can define a star product $\star_{\pi_\lambda}$ by

$$f \star_{\pi_\lambda} g := fg + \sum_{r=1}^\infty \frac{\lambda^r}{r!}\mathcal{U}_r(\underbrace{\pi_\lambda \wedge \ldots \wedge \pi_\lambda}_{r})(f \otimes g), \qquad (27)$$

where $\mathcal{U}_r : \bigwedge^r \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ are the Taylor coefficients of $\mathcal{U}$. Moreover, Kontsevich showed that one can choose $\mathcal{U}_1 : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ to be the natural embedding of multivector fields into multidifferential operators.

Let $\star = \sum_{r=0}^\infty \lambda^r C_r$ and $\star' = \sum_{r=0}^\infty \lambda^r C'_r$ be arbitrary star products on (M, π_0), with $C_1 = C'_1 = \frac{1}{2}\{\,,\,\}$. The map

$$(df, dg) \mapsto (C_2 - C'_2)(f, g) - (C_2 - C'_2)(g, f)$$

defines a d_π-closed bivector field $\tau \in \chi^2(M)$ [3] whose class $[\tau]_\pi \in H^2_\pi(M)$ depends only on the equivalence classes $[\star]$, $[\star']$ [4]. (The cohomology class of τ measures the obstruction for $\star$ and $\star'$ being equivalent modulo λ^3.)

If $\pi_\lambda = \pi_0 + \lambda\pi_1 + \ldots$ is a formal Poisson structure on M, the integrability equation $[\pi_\lambda, \pi_\lambda] = 0$ immediately implies that $d_{\pi_0}\pi_1 = 0$. We observe

Lemma 5.3 *If $\pi_\lambda = \pi_0 + \lambda\pi_1 + \ldots$ and $\pi'_\lambda = \pi_0 + \lambda\pi'_1 + \ldots$ are equivalent formal Poisson structures, then $[\pi_1]_\pi = [\pi'_1]_\pi$.*

Suppose $\star$ and $\star'$ are star products with $\mathrm{c}(\star) = [\pi_0 + \lambda\pi_1 + \ldots]$ and $\mathrm{c}(\star') = [\pi_0 + \lambda\pi'_1 + \ldots]$. A simple computation using (27) shows [4]

Lemma 5.4 $[\tau]_\pi = [\pi_1]_\pi - [\pi'_1]_\pi$.

Let $L \to M$ be a complex line bundle. Let $\star = \sum_{r=0}^\infty \lambda^r C_r$ be a star product on M, and choose $\star' = \sum_{r=0}^\infty \lambda^r C'_r$ so that $[\star'] = \Phi_L([\star])$ and $C_1 = C'_1 = \frac{1}{2}\{\,,\,\}$. Let $\bullet = \sum_{r=0}^\infty \lambda^r R_r$ be a deformation quantization of L with respect to $\star$, and $\bullet' = \sum_{r=0}^\infty \lambda^r R'_r$ be a deformed left module structure on $\Gamma^\infty(L)$ over $(C^\infty(M)[[\lambda]], \star')$. A long but simple computation using (18), (19), (20) in order λ^2 shows [4]

Lemma 5.5 *The bivector τ corresponding to $\star$ and $\star'$ satisfies $\tau = \Theta_R$, where Θ_R is the curvature of the contravariant connection $R = R_1 - R'_1$.*

For a formal Poisson structure $\pi_\lambda = \pi_0 + \lambda\pi_1 + \ldots$, with the identification given in (26), we define the semiclassical limit map

$$S : \mathrm{Def}(M, \pi_0) \longrightarrow H^2_\pi(M), \quad S([\pi_\lambda]) = [\pi_1]_\pi.$$

The following result follows from Lemmas 5.4 and 5.5.

Theorem 5.6 *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Def}(M,\pi_0) & \xrightarrow{\Phi_L} & \mathrm{Def}(M,\pi_0) \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle S} \\ H^2_\pi(M) & \xrightarrow{\widehat{\Phi}_L} & H^2_\pi(M), \end{array}$$

where $\widehat{\Phi}_L([\alpha]) = [\alpha] - \frac{2\pi}{\mathrm{i}} c_1^\pi(L) = [\alpha] - \frac{2\pi}{\mathrm{i}} \pi_0^* c_1(L)$.

This result shows that, in the semiclassical limit, the action of Φ "twists" star products by Poisson-Chern classes. As a consequence, for a star product $\star$ on (M, π_0), each element in $H^2_\pi(M,\mathbb{Z}) = \pi_0^* H^2_{dR}(M,\mathbb{Z})$ corresponds to a different equivalence class of star products Morita equivalent to $\star$.

A full description of Φ can be given when π_0 is symplectic [5]: If (M, ω) is a symplectic manifold, the set of equivalence classes of star products on M is described in terms of the second de Rham cohomology of M [3, 12, 13, 22, 32] through a bijection

$$\mathrm{c} : \mathrm{Def}(M, \omega) \longrightarrow ([\omega]/\mathrm{i}\lambda) + H^2_{dR}(M)[[\lambda]]. \tag{28}$$

The class $\mathrm{c}(\star)$ is called the (Fedosov-Deligne's) *characteristic class* of $\star$. In this case, we have

$$\Phi_L([\omega_\lambda]) = [\omega_\lambda] + 2\pi \mathrm{i} c_1(L), \tag{29}$$

where $[\omega_\lambda] = ([\omega]/\mathrm{i}\lambda) + \sum_{r=1}^\infty \lambda^r [\omega_r]$. The approach taken in [5] to prove this result is based on the Čech-cohomological description of relative classes developed in [18] and on a local description of deformed vector bundles (see [19] for related ideas).

Thus two star products on a symplectic manifold are Φ-related if and only if their relative class is 2πi-integral. As discussed in [5], this integrality condition is related to Dirac's quantization condition for magnetic charges. This result also provides a characterization of strong Morita equivalent [8), 7)] hermitian star products, and can be used to produce induced *-representations of star products as in the theory of C^*-algebras [24)].

5.4. *Final remarks*

Let (M, ω) be a symplectic manifold. We recall that there is a correspondence between $([\omega]/\mathrm{i}\lambda) + H^2_{dR}(M)[[\lambda]]$ and equivalence classes of formal Poisson structures deforming the bracket given by ω (by "inversion" of formal symplectic forms). With this identification, (29) shows that Φ acts on formal Poisson brackets by "gauge transformations" in the sense of [27)]. A similar picture seems to hold for general Poisson structures under the identification (26). It would be interesting to investigate whether or when gauge equivalent Poisson structures have Morita equivalent symplectic groupoids [34)], as a way to link the notions of Morita equivalence for star products and Poisson structures [33)].

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References

1) Bass, H.: *Algebraic K-theory.* W. A. Benjamin, Inc., New York-Amsterdam, 1968.
2) Bayen, F., Flato, M., Frønsdal, C., Lichnerowicz, A., Sternheimer, D.: *Deformation Theory and Quantization.* Ann. Phys. **111** (1978), 61–151.
3) Bertelson, M., Cahen, M., Gutt, S.: *Equivalence of Star Products.* Class. Quantum Grav. **14** (1997), A93–A107.
4) Bursztyn, H.: *Semiclassical geometry of quantum line bundles and Morita equivalence of star products* . Preprint math.QA/0105001.
5) Bursztyn, H., Waldmann, S.: *The characteristic classes of Morita equivalent star products on symplectic manifolds* . Preprint math.QA/0106178.
6) Bursztyn, H., Waldmann, S.: *Deformation quantization of Hermitian vector bundles.* Lett. Math. Phys. **53**.4 (2000), 349–365. math.QA/0009170.
7) Bursztyn, H., Waldmann, S.: **-Ideals and Formal Morita Equivalence of *-Algebras.* Internat. J. Math **12**.5 (2001), 555–577. math.QA/0005227.
8) Bursztyn, H., Waldmann, S.: *Algebraic Rieffel induction, formal Morita equivalence, and applications to deformation quantization.* J. Geom. Phys. **37**.4 (2001), 307–364. math.QA/9912182.
9) Cannas da Silva, A., Weinstein, A.: *Geometric models for noncommutative algebras.* American Mathematical Society, Providence, RI, 1999.
10) Connes, A.: *Noncommutative geometry.* Academic Press, Inc., San Diego, CA, 1994.
11) Connes, A., Douglas, M. R., Schwarz, A.: *Noncommutative geometry and matrix theory: compactification on tori.* J. High Energy Phys. **1998**.2, Paper 3, 35 pp. (electronic).
12) Deligne, P.: *Déformations de l'algèbre des fonctions d'une variété symplectique: comparaison entre Fedosov et De Wilde, Lecomte.* Selecta Math. (N.S.) **1**.4 (1995), 667–697.
13) Fedosov, B. V.: *A Simple Geometrical Construction of Deformation Quantization.* J. Diff. Geom. **40** (1994), 213–238.
14) Fernandes, R. L.: *Connections in Poisson geometry. I. Holonomy and invariants.* J. Differential Geom. **54**.2 (2000), 303–365. math.DG/0001129.
15) Gerstenhaber, M.: *On the Deformation of Rings and Algebras.* Ann. Math. **79** (1964),

59–103.
16) Gerstenhaber, M., Schack, S. D.: *Algebraic Cohomology and Deformation Theory.* In: Hazewinkel, M., Gerstenhaber, M. (eds.): *Deformation Theory of Algebras and Structures and Applications*, 13–264. Kluwer Academic Press, Dordrecht, 1988.
17) Ginzburg, V. L.: *Grothendieck Groups of Poisson Vector Bundles* . math.DG/0009124.
18) Gutt, S., Rawnsley, J.: *Equivalence of star products on a symplectic manifold; an introduction to Deligne's Čech cohomology classes.* J. Geom. Phys. **29** (1999), 347–392.
19) Jurco, B., Schupp, P., Wess, J.: *Noncommutative line bundle and Morita equivalence* . Preprint hep-th/0106110.
20) Kontsevich, M.: *Deformation Quantization of Poisson Manifolds, I* . Preprint q-alg/9709040.
21) Morita, K.: *Duality for modules and its applications to the theory of rings with minimum condition.* Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **6** (1958), 83–142.
22) Nest, R., Tsygan, B.: *Algebraic Index Theorem for Families.* Adv. Math. **113** (1995), 151–205.
23) Reshetikhin, N., Voronov, A., Weinstein, A.: *Semiquantum geometry.* J. Math. Sci. **82**.1 (1996), 3255–3267. Algebraic geometry, 5.
24) Rieffel, M. A.: *Induced representations of C^*-algebras.* Advances in Math. **13** (1974), 176–257.
25) Rieffel, M. A., Schwarz, A.: *Morita equivalence of multidimensional noncommutative tori.* Internat. J. Math. **10**.2 (1999), 289–299.
26) Schwarz, A.: *Morita equivalence and duality.* Nuclear Phys. B **534**.3 (1998), 720–738.
27) Ševera, P., Weinstein, A.: *Poisson geometry with a 3-form background.* To appear in the Proceedings of the Workshop on Noncommutative Geometry and String Theory, Keio University (2001). math.SG/0107133.
28) Swan, R. G.: *Vector bundles and projective modules.* Trans. Amer. Math. Soc. **105** (1962), 264–277.
29) Vaisman, I.: *On the geometric quantization of Poisson manifolds.* J. Math. Phys. **32**.12 (1991), 3339–3345.
30) Vaisman, I.: *Lectures on the geometry of Poisson manifolds.* Birkhäuser Verlag, Basel, 1994.
31) Waldmann, S.: *Deformation of Hermitian vector bundles and non-commutative field theory.* Proceedings of the Workshop on Noncommutative Geometry and String Theory, Keio University (2001).
32) Weinstein, A., Xu, P.: *Hochschild cohomology and characterisic classes for star-products.* In: Khovanskij, A., Varchenko, A., Vassiliev, V. (eds.): *Geometry of differential equations. Dedicated to V. I. Arnold on the occasion of his 60th birthday*, 177–194. American Mathematical Society, Providence, 1998.
33) Xu, P.: *Morita equivalence of Poisson manifolds.* Comm. Math. Phys. **142**.3 (1991), 493–509.
34) Xu, P.: *Morita equivalent symplectic groupoids.* In: *Symplectic geometry, groupoids, and integrable systems (Berkeley, CA, 1989)*, 291–311. Springer, New York, 1991.