Clay Mathematics Proceedings

Classifying Morita equivalent star products

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Abstract. These notes are based on lectures given at CIMPA's school *Topics in noncommutative geometry*, held in Buenos Aires in 2010. The main goal is to expound the classification of deformation quantization algebras up to Morita equivalence.

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1. Introduction

Deformation quantization [**1**] (see e.g. [**15**] for a survey) is a quantization scheme in which algebras of quantum observables are obtained as formal deformations of classical observable algebras. For a smooth manifold M, let $C^\infty(M)$ denote the algebra of complex-valued smooth functions on M, and let $C^\infty(M)[[\hbar]]$ be the space of formal power series in a parameter $\hbar$ with coefficients in $C^\infty(M)$; deformation quantization concerns the study of associative products $\star$ on $C^\infty(M)[[\hbar]]$, known as *star products*, deforming the pointwise product on $C^\infty(M)$,

$$f \star g = fg + O(\hbar),$$

in the sense of Gerstenhaber [**13**]. The noncommutativity of a star product $\star$ is controlled, in first order, by a Poisson structure $\{\cdot,\cdot\}$ on M, in the sense that

$$f \star g - g \star f = \mathrm{i}\hbar\{f, g\} + O(\hbar^2).$$

Two fundamental issues in deformation quantization are the existence and isomorphism classification of star products on a given Poisson manifold, and the most general results in these directions follow from Kontsevich's formality theorem [**18**]. In these notes we treat another kind of classification problem in deformation quantization, namely that of describing when two star products define *Morita equivalent* algebras. This study started in [**2, 5**] (see also [**17**]), and here we will mostly review the results obtained in [**3**] (where detailed proofs can be found), though from a less technical perspective.

Morita equivalence [**21**] is an equivalence relation for algebras, which is based on comparing their categories of representations. This type of equivalence is weaker than the usual notion of algebra isomorphism, but strong enough to capture essential algebraic properties. The notion of Morita equivalence plays a central role in noncommutative geometry and has also proven relevant at the interface of noncommutative geometry and physics, see e.g. [**17, 20, 27**]. Although there are more analytical versions of deformation quantization and Morita equivalence used in noncommutative geometry (especially in the context of C^*-algebras, see e.g. [**24, 25**] and [**8**, Chp. II, App. A]), our focus in these notes is on deformation quantization and Morita equivalence in the purely algebraic setting.

The classification of Morita equivalent star products on a manifold M [**3**] builds on Kontsevich's classification result [**18**], which establishes a bijective correspondence between the moduli space of star products on M, denoted by $\mathrm{Def}(M)$, and the set $\mathrm{FPois}(M)$ of equivalence classes of formal families of Poisson structures on M,

$$\mathcal{K}_* : \mathrm{FPois}(M) \xrightarrow{\sim} \mathrm{Def}(M). \tag{1.1}$$

Morita equivalence of star products on M defines an equivalence relation on $\mathrm{Def}(M)$, and these notes explain how one recognizes Morita equivalent star products in terms of their classes in $\mathrm{FPois}(M)$, through Kontsevich's correspondence (1.1). We divide the discussion in two steps: first, we identify a canonical group action on $\mathrm{Def}(M)$ whose orbit relation coincides with Morita equivalence of star products (Thm. 5.2); second, we find the expression for the corresponding action on $\mathrm{FPois}(M)$, making the quantization map (1.1) equivariant (Thm. 7.1).

This paper is structured much in the same way as the lectures presented at the school. In Section 2, we briefly discuss how deformation quantization arises from the quantization problem is physics; Section 3 reviews the basics on star products and the main results on deformation quantization; Morita equivalence is recalled in Section 4, while Section 5 presents a description of Morita equivalence for star products as orbits of a suitable group action. Section 6 discusses the B-field action on (formal) Poisson structures, and Section 7 presents the main results on the classification of Morita equivalent star products.

Notation and conventions: For a smooth manifold M, $C^\infty(M)$ denotes its algebra of smooth complex-valued functions. Vector bundles $E \to M$ are taken to be complex, unless stated otherwise. $\mathcal{X}^\bullet(M)$ denotes the graded algebra of (complex) multivector fields on M, $\Omega^\bullet(M)$ is the graded algebra of (complex) differential forms, while $\Omega^p_{cl}(M)$ denotes the space of closed p-forms on M. We use the notation $H^\bullet_{dR}(M)$ for de Rham cohomology. For any vector space V over $k = \mathbb{R}$ or $\mathbb{C}$, $V[[\hbar]]$ denotes the space of formal power series with coefficients on V on a formal parameter $\hbar$, naturally seen as a module over $k[[\hbar]]$.

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2. A word on quantization

Quantization is usually understood as a map assigning quantum observables to classical ones. In general, classical observables are represented by smooth functions on a symplectic or Poisson manifold (the classical "phase space"), whereas quantum observables are given by (possibly unbounded) operators acting on some (pre-)Hilbert space. A "quantization map" is expected to satisfy further compatibility properties (see e.g. [**20**] for a discussion), roughly saying that the algebraic features of the space of classical observables (e.g. pointwise multiplication and Poisson bracket of functions) should be obtained from those of quantum observables (e.g. operator products and commutators) in an appropriate limit "$\hbar \to 0$". As we will see in Section 3, deformation quantization offers a purely algebraic formulation of quantization. In order to motivate it, we now briefly recall the simplest quantization procedure in physics, known as *canonical quantization*.

Let us consider the classical phase space $\mathbb{R}^{2n} = T^*\mathbb{R}^n$, equipped with global coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$, and the canonical Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial q^j}\frac{\partial g}{\partial p_j} - \frac{\partial g}{\partial q^j}\frac{\partial f}{\partial p_j}, \quad f, g \in C^\infty(\mathbb{R}^{2n}), \tag{2.1}$$

so that the brackets of canonical coordinates are

$$\{q^k, p_\ell\} = \delta^k_\ell,$$

for $k, \ell = 1, \dots, n$. Quantum mechanics tells us that the corresponding Hilbert space in this case is $\mathrm{L}^2(\mathbb{R}^n)$, the space of wave functions on the configuration space $\mathbb{R}^n = \{(q^1, \dots, q^n)\}$. To simplify matters when dealing with unbounded operators, we will instead consider the subspace $C_0^\infty(\mathbb{R}^n)$ of compactly supported functions on $\mathbb{R}^n$. In canonical quantization, the classical observable $q^k \in C^\infty(T^*\mathbb{R}^n)$ is taken to the multiplication operator $Q^k : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)$, $\psi \mapsto Q^k(\psi)$, where

$$Q^k(\psi)(q) := q^k \psi(q), \quad \text{for } q = (q^1, \dots, q^n) \in \mathbb{R}^n, \tag{2.2}$$

while the classical observable $p_\ell \in C^\infty(T^*\mathbb{R}^n)$ is mapped to the differentiation operator $P_\ell : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)$ given by

$$\psi \overset{P_\ell}{\mapsto} -\mathrm{i}\hbar \frac{\partial \psi}{\partial q^\ell}. \tag{2.3}$$

Here $\hbar$ is Planck's constant. The requirements $q^k \mapsto Q^k$, $p_\ell \mapsto P_\ell$, together with the condition that the constant function 1 is taken to the identity operator $\mathrm{Id} : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)$, constitute the core of canonical quantization.

A natural issue is whether one can extend the canonical quantization procedure to assign operators to more general functions on $T^*\mathbb{R}^n$, including higher order monomials of q^k and p_ℓ. Since on the classical side $q^k p_\ell = p_\ell q^k$, but on the quantum side we have the canonical commutation relations

$$[Q^k, P_\ell] = Q^k P_\ell - P_\ell Q^k = \mathrm{i}\hbar\delta^k_\ell, \tag{2.4}$$

any such extension relies on the choice of an *ordering prescription*, for which one has some freedom. As a concrete example, we consider the *standard ordering*, defined by writing, for a given monomial on q^k and p_ℓ, all momentum variables p_ℓ to the right, and then replacing q^k by Q^k and p_ℓ by P_ℓ; explicitly, this means that $q^{k_1}\cdots q^{k_r}p_{\ell_1}\cdots p_{\ell_s}$ is quantized by the operator $Q^{k_1}\cdots Q^{k_r}P_{\ell_1}\cdots P_{\ell_s}$. If f is a polynomial in q^k and p_ℓ, $k,\ell = 1,\dots,n$, we can explicitly write this *standard-ordered quantization map* as

$$f \;\mapsto\; \sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{\hbar}{\mathrm{i}}\right)^r \frac{\partial^r f}{\partial p_{k_1}\cdots\partial p_{k_r}}\Big|_{p=0}\frac{\partial^r}{\partial q^{k_1}\cdots\partial q^{k_r}}. \tag{2.5}$$

One may verify that formula (2.5) in fact defines a linear bijection

$$\varrho_{\mathrm{Std}}\colon \mathrm{Pol}(T^*\mathbb{R}^n)\longrightarrow \mathrm{DiffOp}(\mathbb{R}^n) \tag{2.6}$$

between the space $\mathrm{Pol}(T^*\mathbb{R}^n)$ of smooth function on $T^*\mathbb{R}^n$ that are polynomial in the momentum variables $p_1,\dots,p_n$, and the space $\mathrm{DiffOp}(\mathbb{R}^n)$ of differential operators with smooth coefficients on $\mathbb{R}^n$. In order to compare the pointwise product and Poisson bracket of classical observables with the operator product and commutator of quantum observables, one may use the bijection (2.6) to pullback the operator product to $\mathrm{Pol}(T^*\mathbb{R}^n)$,

$$f\star_{\mathrm{Std}} g := \varrho_{\mathrm{Std}}^{-1}\left(\varrho_{\mathrm{Std}}(f)\varrho_{\mathrm{Std}}(g)\right), \tag{2.7}$$

so as to have all structures defined on the same space. A direct computation yields the explicit formula for the new product $\star_{\mathrm{Std}}$ on $\mathrm{Pol}(T^*\mathbb{R}^n)$:

$$f\star_{\mathrm{Std}} g = \sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{\hbar}{\mathrm{i}}\right)^r \frac{\partial^r f}{\partial p_{k_1}\cdots\partial p_{k_r}}\frac{\partial^r g}{\partial q^{k_1}\cdots\partial q^{k_r}}. \tag{2.8}$$

With this formula at hand, one may directly check the following properties:

(1) $f\star_{\mathrm{Std}} g = fg + O(\hbar)$;
(2) $f\star_{\mathrm{Std}} g - g\star_{\mathrm{Std}} f = \mathrm{i}\hbar\{f,g\} + O(\hbar^2)$;
(3) The constant function 1 satisfies $1\star_{\mathrm{Std}} f = f = f\star_{\mathrm{Std}} 1$, for all $f\in \mathrm{Pol}(T^*\mathbb{R}^n)$;
(4) $\star_{\mathrm{Std}}$ is an *associative* product.

The associativity property is evident from construction, since $\star_{\mathrm{Std}}$ is isomorphic to the composition product of differential operators. As we will see in the next section, these properties of $\star_{\mathrm{Std}}$ underlie the general notion of a *star product*.

Before presenting the precise formulation of deformation quantization, we have two final observations.

- First, we note that there are alternatives to the standard-ordering quantization (2.5). From a physical perspective, one is also interested in comparing the involutions of the algebras at the classical and quantum levels, i.e., complex conjugation of functions and adjoints of operators. Regarding the standard-ordering quantization, the (formal) adjoint of $\varrho_{\mathrm{Std}}(f)$ is not given by $\varrho_{\mathrm{Std}}(\overline{f})$. Instead, an integration by parts shows that $\varrho_{\mathrm{Std}}(f)^* = \varrho_{\mathrm{Std}}(N^2\overline{f})$, for the operator $N:\mathrm{Pol}(T^*\mathbb{R}^n)\to\mathrm{Pol}(T^*\mathbb{R}^n)$,

$$N = \exp\left(\frac{\hbar}{2\mathrm{i}}\frac{\partial^2}{\partial q^k\partial p_k}\right), \tag{2.9}$$

where exp is defined by its power series. If we pass to the *Weyl-ordering quantization map*,

$$\varrho_{\mathrm{Weyl}}:\mathrm{Pol}(T^*\mathbb{R}^n)\to\mathrm{DiffOp}(\mathbb{R}^n),\quad \varrho_{\mathrm{Weyl}}(f) := \varrho_{\mathrm{Std}}(Nf), \tag{2.10}$$

we have $\varrho_{\mathrm{Weyl}}(f)^* = \varrho_{\mathrm{Weyl}}(\overline{f})$. This quantization, when restricted to monomials on q^k, p_ℓ, agrees with the ordering prescribed by total symmetrization. Just as (2.7), the map (2.10) is a bijection, and it defines the *Weyl product* $\star_{\mathrm{Weyl}}$ on $\mathrm{Pol}(T^*\mathbb{R}^n)$ by

$$f \star_{\mathrm{Weyl}} g = \varrho_{\mathrm{Weyl}}^{-1}(\varrho_{\mathrm{Weyl}}(f)\varrho_{\mathrm{Weyl}}(g)). \tag{2.11}$$

The two products $\star_{\mathrm{Std}}$ and $\star_{\mathrm{Weyl}}$ on $\mathrm{Pol}(T^*\mathbb{R}^n)$ are related by

$$f \star_{\mathrm{Weyl}} g := N^{-1}(Nf \star_{\mathrm{Std}} Ng); \tag{2.12}$$

since $N = \mathrm{Id} + O(\hbar)$, one may directly check that $\star_{\mathrm{Weyl}}$ satisfies the same properties (1)–(4) listed above for $\star_{\mathrm{Std}}$. But $\star_{\mathrm{Weyl}}$ satisfies an additional compatibility condition relative to complex conjugation:

$$\overline{f \star_{\mathrm{Weyl}} g} = \overline{g} \star_{\mathrm{Weyl}} \overline{f}. \tag{2.13}$$

The are other possible orderings leading to products satisfying (2.13), such as the so-called *Wick ordering*, see e.g. [**28**, Sec. 5.2.3].

- The second observation concerns the difficulties in extending the quantization procedures discussed so far to manifolds other than $T^*\mathbb{R}^n$. The quantizations ϱ_{Std} and ϱ_{Weyl} are only defined for functions in $\mathrm{Pol}(T^*\mathbb{R}^n)$, i.e., polynomial in the momentum variables. On an arbitrary manifold M, however, there is no analog of this class of functions, and generally there are no natural subalgebras of $C^\infty(M)$ to be considered. From another viewpoint, one sees that the expression for $\star_{\mathrm{Std}}$ in (2.8) does not make sense for arbitrary smooth functions, as the radius of convergence in $\hbar$ is typically 0, so $\star_{\mathrm{Std}}$ does not extend to a product on $C^\infty(T^*\mathbb{R}^n)$ (and the same holds for $\star_{\mathrm{Weyl}}$). One can however interpret (2.8) as a *formal power series* in the parameter $\hbar$, i.e., as a product on $C^\infty(T^*\mathbb{R}^n)[[\hbar]]$. This viewpoint now carries over to arbitrary manifolds and leads to the general concept of deformation quantization, in which quantization is formulated in purely algebraic terms by means of associative product structures $\star$ on $C^\infty(M)[[\hbar]]$ rather than operator representations [1].

3. Deformation quantization

Let M be a smooth manifold, and let $C^\infty(M)$ denote its algebra of *complex-valued* smooth functions. We consider $C^\infty(M)[[\hbar]]$, the set of formal power series in $\hbar$ with coefficients in $C^\infty(M)$, as a module over the ring $\mathbb{C}[[\hbar]]$.

3.1. Star products. A *star product* [**1**] on M is an associative product $\star$ on the $\mathbb{C}[[\hbar]]$-module $C^\infty(M)[[\hbar]]$ given as follows: for $f, g \in C^\infty(M)$,

$$f \star g = fg + \sum_{r=1}^{\infty} \hbar^r C_r(f, g), \tag{3.1}$$

where $C_r : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, $r = 1, 2, \ldots$, are bidifferential operators, and this product operation is extended to $C^\infty(M)[[\hbar]]$ by $\hbar$-linearity (and $\hbar$-adic

[1] Deformation quantization, in its most general form, completely avoids analytical issues (such as convergence properties in $\hbar$ and related operator representations); these aspects are mostly considered in particular classes of examples, see e.g. [**15**, Sec. 4] for a discussion and further references.

continuity). Additionally, we require that the constant function $1 \in C^\infty(M)$ is still a unit for $\star$:

$$1 \star f = f \star 1 = f, \quad \forall f \in C^\infty(M).$$

Since

$$f \star g = fg \mod \hbar, \quad \forall f, g \in C^\infty(M),$$

one views star products as associative, but not necessarily commutative, deformations (in the sense of [**13**]) of the pointwise product of functions on M. The $\mathbb{C}[[\hbar]]$-algebra $(C^\infty(M)[[\hbar]], \star)$ is called a *deformation quantization* of M.

Two star products $\star$ and $\star'$ on M are said to be *equivalent* if there are differential operators $T_r : C^\infty(M) \to C^\infty(M)$, $r = 1, 2, \ldots$, such that

$$T = \mathrm{Id} + \sum_{r=1}^{\infty} \hbar^r T_r \tag{3.2}$$

satisfies

$$T(f \star g) = T(f) \star' T(g). \tag{3.3}$$

We define the *moduli space of star products* on M as the set of equivalence classes of star products, and we denote it by $\mathrm{Def}(M)$.

EXAMPLE 3.1. *Formula* (2.8) *for* $\star_{\mathrm{Std}}$ *defines a star product on* $M = T^*\mathbb{R}^n$*, and the same holds for the product* $\star_{\mathrm{Weyl}}$ *given in* (2.11)*; by* (2.12)*, the operator* N *in* (2.9) *defines an equivalence between the star products* $\star_{\mathrm{Std}}$ *and* $\star_{\mathrm{Weyl}}$*.*

3.2. Noncommutativity in first order: Poisson structures. Given a star product $\star$ on M, its noncommutativity is measured, in first order, by the bilinear operation $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$,

$$\{f, g\} := \frac{1}{\mathrm{i}\hbar}(f \star g - g \star f)\Big|_{\hbar=0} = \frac{1}{\mathrm{i}}(C_1(f, g) - C_1(g, f)), \quad f, g \in C^\infty(M). \tag{3.4}$$

It follows from the associativity of $\star$ that $\{\cdot,\cdot\}$ is a *Poisson structure* on M (see e.g. [**7**, Sec.19]); recall that this means that $\{\cdot,\cdot\}$ is a Lie bracket on $C^\infty(M)$, which is compatible with the pointwise product on $C^\infty(M)$ via the Leibniz rule:

$$\{f, gh\} = \{f, g\}h + \{f, h\}g, \quad f, g, h \in C^\infty(M).$$

The Leibniz rule implies that any Poisson structure $\{\cdot,\cdot\}$ is equivalently described by a bivector field $\pi \in \mathcal{X}^2(M)$, via

$$\{f, g\} = \pi(df, dg),$$

satisfying the additional condition (accounting for the Jacobi identity of $\{\cdot,\cdot\}$) that $[\pi, \pi] = 0$, where $[\cdot,\cdot]$ is an extension to $\mathcal{X}^\bullet(M)$ of the Lie bracket of vector fields, known as the *Schouten bracket.* The pair (M, π) is called a *Poisson manifold* (see e.g. [**7**] for more on Poisson geometry). If a star product $\star$ corresponds to a Poisson structure π via (3.4), we say that $\star$ *quantizes* π, or that $\star$ is a *deformation quantization of the Poisson manifold* (M, π).

A Poisson structure π on M defines a bundle map

$$\pi^\sharp : T^*M \to TM, \ \ \alpha \mapsto i_\alpha \pi = \pi(\alpha, \cdot). \tag{3.5}$$

We say that π is *nondegenerate* if (3.5) is an isomorphism, in which case π is equivalent to a symplectic structure $\omega \in \Omega^2(M)$, defined by

$$\omega(\pi^\sharp(\alpha), \pi^\sharp(\beta)) = \pi(\beta, \alpha); \tag{3.6}$$

alternatively, the 2-form ω is defined by the condition that the map $TM \to T^*M$, $X \mapsto i_X\omega$, is inverse to (3.5).

EXAMPLE 3.2. *The star product $\star_{\mathrm{Std}}$ on $T^*\mathbb{R}^n$ quantizes the classical Poisson bracket*

$$\{f,g\} = \frac{\partial f}{\partial q^j}\frac{\partial g}{\partial p_j} - \frac{\partial g}{\partial q^j}\frac{\partial f}{\partial p_j},$$

defined by the (nondegenerate) bivector field $\pi = \frac{\partial}{\partial q^j} \wedge \frac{\partial}{\partial p_j}$. The same holds for $\star_{\mathrm{Weyl}}$.

3.3. Existence and classification of star products. A direct computation shows that if $\star$ and $\star'$ are equivalent star products, i.e., define the same element in $\mathrm{Def}(M)$, then they necessarily quantize the same Poisson structure. For a Poisson structure π on M, we denote by

$$\mathrm{Def}(M,\pi) \subset \mathrm{Def}(M)$$

the subset of equivalence classes of star products quantizing π. The central issue in deformation quantization is understanding $\mathrm{Def}(M,\pi)$, for example by finding a concrete parametrization of this space. Concretely, deformation quantization concerns the following fundamental issues:

- Given a Poisson structure π on M, is there a star product quantizing it?
- If there is a star product quantizing π, how many distinct equivalence classes in $\mathrm{Def}(M)$ with this property are there?

The main result on existence and classification of star products on Poisson manifolds follows from Kontsevich's formality theorem [**18**], that we briefly recall.

Let $\mathcal{X}^2(M)[[\hbar]]$ denote the space of formal power series in $\hbar$ with coefficients in bivector fields. A *formal Poisson structure* on M is an element $\pi_\hbar \in \hbar\mathcal{X}^2(M)[[\hbar]]$,

$$\pi_\hbar = \sum_{r=1}^{\infty} \hbar^r \pi_r, \quad \pi_r \in \mathcal{X}^2(M),$$

such that

$$[\pi_\hbar, \pi_\hbar] = 0, \tag{3.7}$$

where $[\cdot,\cdot]$ denotes the $\hbar$-bilinear extension of the Schouten bracket to formal power series. It immediately follows from (3.7) that

$$[\pi_1, \pi_1] = 0,$$

i.e., π_1 is an ordinary Poisson structure on M. So we view $\pi_\hbar$ as a formal deformation of π_1 in the realm of Poisson structures.

A formal Poisson structure $\pi_\hbar$ defines a bracket $\{\cdot,\cdot\}_\hbar$ on $C^\infty(M)[[\hbar]]$ by

$$\{f,g\}_\hbar = \pi_\hbar(df, dg).$$

Two formal Poisson structures $\pi_\hbar$ and $\pi'_\hbar$ are *equivalent* if there is a formal diffeomorphism $T = \exp(\sum_{r=1}^{\infty} \hbar^r X_r) : C^\infty(M)[[\hbar]] \to C^\infty(M)[[\hbar]]$, where each $X_r \in \mathcal{X}^1(M)$ is a vector field, satisfying

$$T\{f,g\}_\hbar = \{Tf, Tg\}'_\hbar.$$

(Here the exponential exp is defined by its formal series, and it gives a well-defined formal power series in $\hbar$ since $\sum_{r=1}^{\infty} \hbar^r X_r$ starts at order $\hbar$.) We define the *moduli space of formal Poisson structures* on M as the set of equivalence classes of formal

Poisson structures, and we denote it by $\mathrm{FPois}(M)$. One may readily verify that two equivalent formal Poisson structures necessarily agree in first order of $\hbar$, i.e., deform the same Poisson structure. So, given a Poisson structure π on M, we may consider the subset

$$\mathrm{FPois}(M, \pi) \subset \mathrm{FPois}(M)$$

of equivalence classes of formal Poisson structures deforming π.

We can now state Kontsevich's theorem [**18**].

THEOREM 3.3. *There is a one-to-one correspondence*

$$\mathcal{K}_* : \mathrm{FPois}(M) \xrightarrow{\sim} \mathrm{Def}(M), \quad [\pi_\hbar] = [\hbar\pi_1 + \dots] \mapsto [\star], \tag{3.8}$$

such that $\frac{1}{\mathrm{i}\hbar}[f, g]_\star\big|_{\hbar=0} = \pi_1(df, dg)$.

For a given star product $\star$ on M, the element in $\mathrm{FPois}(M)$ corresponding to $[\star]$ under (3.8) is called its *characteristic class*, or its *Kontsevich class*.

Theorem 3.3 answers the existence and classification questions for star products as follows:

- Any Poisson structure π on M may be seen as a formal Poisson structure $\hbar\pi$. So it defines a class $[\hbar\pi] \in \mathrm{FPois}(M)$, which is quantized by any star product $\star$ such that $[\star] = \mathcal{K}_*([\hbar\pi_1])$.
- For any Poisson structure π on M, the map (3.8) restricts to a bijection

$$\mathrm{FPois}(M, \pi) \xrightarrow{\sim} \mathrm{Def}(M, \pi). \tag{3.9}$$

This means that the distinct classes of star products quantizing π are in one-to-one correspondence with the distinct classes of formal Poisson structures deforming π.

REMARK 3.4.

(a) *Theorem 3.3 is a consequence of a much more general result, known as Kontsevich's formality theorem* [**18**]*; this theorem asserts that, for any manifold M, there is an L_∞-quasi-isomorphism from the differential graded Lie algebra (DGLA) $\mathcal{X}(M)$ of multivector fields on M to the DGLA $\mathcal{D}(M)$ of multidifferential operators on M, and moreover the first Taylor coefficient of this L_∞-morphism agrees with the natural map $\mathcal{X}(M) \to \mathcal{D}(M)$ (defined by viewing vector fields as differential operators). It is a general fact that any L_∞-quasi-isomorphism between DGLAs induces a one-to-one correspondence between equivalence classes of Maurer-Cartan elements. Theorem 3.3 follows from the observation that the Maurer-Cartan elements in $\mathcal{X}(M)[[\hbar]]$ are formal Poisson structures, whereas the Maurer-Cartan elements in $\mathcal{D}(M)[[\hbar]]$ are star products.*

(b) *We recall that the L_∞-quasi-isomorphism from $\mathcal{X}(M)$ to $\mathcal{D}(M)$, also called a* formality*, is not unique, and the map* (3.8) *may depend upon this choice (see e.g.* [**12**] *for more details and references). Just as in* [**3**]*, for the purposes of these notes, we will consider the specific global formality constructed in* [**10**]*. The specific properties of the global formality that we will need are explicitly listed in* [**3**, Sec. 2.2].

In general, not much is known about the space $\mathrm{FPois}(M, \pi)$, which parametrizes $\mathrm{Def}(M, \pi)$, according to (3.9). An exception is when the Poisson structure π is nondegenerate, i.e., defined by a symplectic structure $\omega \in \Omega^2(M)$. In this case, any formal Poisson structure $\pi_\hbar = \hbar\pi + \sum_{r=2}^{\infty} \hbar^r \pi_r$ is equivalent (similarly to (3.6)) to a formal series $\frac{1}{\hbar}\omega + \sum_{r=0}^{\infty} \hbar^r \omega_r$, where each $\omega_r \in \Omega^2(M)$ is closed (see e.g. [**15**, Prop. 13]); moreover, two formal Poisson structures $\pi_\hbar$, $\pi'_\hbar$ deforming the same Poisson structure π, and corresponding to $\frac{1}{\hbar}\omega + \sum_{r=0}^{\infty} \hbar^r \omega_r$ and $\frac{1}{\hbar}\omega + \sum_{r=0}^{\infty} \hbar^r \omega'_r$, define the same class in $\mathrm{FPois}(M,\pi)$ if and only if, for all $r \geq 1$, ω_r and ω'_r are cohomologous. As a result, $\mathrm{FPois}(M,\pi)$ is in bijection with $H^2_{dR}(M,\mathbb{C})[[\hbar]]$. But in order to keep track of the symplectic form ω, one usually replaces $H^2_{dR}(M,\mathbb{C})[[\hbar]]$ by the affine space $\frac{[\omega]}{\hbar} + H^2_{dR}(M,\mathbb{C})[[\hbar]]$ and considers the identification

$$\frac{1}{\hbar}[\omega] + H^2_{dR}(M,\mathbb{C})[[\hbar]] \cong \mathrm{FPois}(M,\pi). \tag{3.10}$$

By (3.9), the map $\mathcal{K}_*$ induces a bijection

$$\frac{1}{\hbar}[\omega] + H^2_{dR}(M,\mathbb{C})[[\hbar]] \xrightarrow{\sim} \mathrm{Def}(M,\pi), \tag{3.11}$$

which gives an explicit parametrization of star products on the symplectic manifold (M,ω). The map (3.11) is proven in [**3**, Sec. 4] to coincide with the known classification of symplectic star products (see e.g. [**15, 16**] for an exposition with original references), which is intrinsic and prior to Kontsevich's general result. The element $c(\star) \in \frac{1}{\hbar}[\omega] + H^2_{dR}(M,\mathbb{C})[[\hbar]]$ corresponding to a star product $\star$ on (M,ω) under (3.11) is known as its *Fedosov-Deligne characteristic class*. In particular, if $H^2_{dR}(M) = \{0\}$, all star products quantizing a fixed symplectic structure on M are equivalent to one another. For star products satisfying the additional compatibility condition (2.13), a classification is discussed in [**23**].

We now move to the main issue addressed in these notes: characterizing star products on a manifold M which are *Morita equivalent* in terms of their characteristic classes. We first recall basic facts about Morita equivalence.

4. Morita equivalence reminder

In this section, we will consider k-algebras (always taken to be associative and unital), where k is a commutative, unital, ground ring; we will be mostly interested in the cases $k = \mathbb{C}$ or $\mathbb{C}[[\hbar]]$.

Morita equivalence aims at characterizing a k-algebra in terms of its representation theory, i.e., its category of modules. Let us consider unital k-algebras $\mathcal{A}$, $\mathcal{B}$, and denote their categories of left modules by ${}_\mathcal{A}\mathfrak{M}$ and ${}_\mathcal{B}\mathfrak{M}$. In order to compare ${}_\mathcal{A}\mathfrak{M}$ and ${}_\mathcal{B}\mathfrak{M}$, we observe that any $(\mathcal{B},\mathcal{A})$-bimodule X (which we may also denote by ${}_\mathcal{B}X_\mathcal{A}$, to stress the left $\mathcal{B}$-action and right $\mathcal{A}$-action) gives rise to a functor ${}_\mathcal{A}\mathfrak{M} \to {}_\mathcal{B}\mathfrak{M}$, defined on objects by tensor product:

$$V \mapsto X \otimes_\mathcal{A} V.$$

We call ${}_\mathcal{B}X_\mathcal{A}$ *invertible* if there is an $(\mathcal{A},\mathcal{B})$-bimodule ${}_\mathcal{A}Y_\mathcal{B}$ such that $X \otimes_\mathcal{A} Y \cong \mathcal{B}$ as $(\mathcal{B},\mathcal{B})$-bimodules, and $Y \otimes_\mathcal{B} X \cong \mathcal{A}$ as $(\mathcal{A},\mathcal{A})$-bimodules. In this case, the functor ${}_\mathcal{A}\mathfrak{M} \to {}_\mathcal{B}\mathfrak{M}$ defined by ${}_\mathcal{B}X_\mathcal{A}$ is an equivalence of categories.

We say that two unital k-algebras $\mathcal{A}$ and $\mathcal{B}$ are *Morita equivalent* if there exists an invertible bimodule ${}_\mathcal{B}X_\mathcal{A}$. Note that if $\mathcal{A}$ and $\mathcal{B}$ are isomorphic algebras, through an isomorphism $\psi : \mathcal{B} \to \mathcal{A}$, then they are necessarily Morita equivalent: $\mathcal{A}$ itself may be viewed as an invertible $(\mathcal{B},\mathcal{A})$-bimodule, with right $\mathcal{A}$-action given by algebra multiplication on the right, and left $\mathcal{B}$-action given by left multiplication via ψ, $(b,a) \mapsto \psi(b)a$. One readily verifies that Morita equivalence is a reflexive and symmetric relation; to see that it is transitive, hence an equivalence relation for unital k-algebras, the key observation is that if Y is a $(\mathcal{C}, \mathcal{B})$-bimodule and X is a $(\mathcal{B}, \mathcal{A})$-bimodule, then the tensor product

$$Y \otimes_{\mathcal{B}} X$$

is a $(\mathcal{C}, \mathcal{A})$-bimodule, which is invertible provided X and Y are.

For any unital k-algebra $\mathcal{A}$, the set of isomorphism classes of invertible $(\mathcal{A}, \mathcal{A})$-bimodules has a natural group structure with respect to bimodule tensor product; we denote this group of "self-Morita equivalences" of $\mathcal{A}$ by $\mathrm{Pic}(\mathcal{A})$, and call it the *Picard group* of $\mathcal{A}$.

The main characterization of invertible bimodules is given by Morita's theorem [**21**] (see e.g. [**19**, Sec. 18]):

THEOREM 4.1. *A $(\mathcal{B}, \mathcal{A})$-bimodule X is invertible if and only if the following holds: as a right $\mathcal{A}$-module, $X_{\mathcal{A}}$ is finitely generated, projective, and full, and the natural map $\mathcal{B} \to \mathrm{End}(X_{\mathcal{A}})$ is an algebra isomorphism.*

In other words, the theorem asserts that a bimodule ${}_{\mathcal{B}}X_{\mathcal{A}}$ is invertible if and only if the following is satisfied: there exists a projection $P \in M_n(\mathcal{A})$, $P^2 = P$, for some $n \in \mathbb{N}$, so that, as a right $\mathcal{A}$-module, $X_{\mathcal{A}} \cong P\mathcal{A}^n$; additionally, $X_{\mathcal{A}}$ being full means that the ideal in $\mathcal{A}$ generated by the entries of P agrees with $\mathcal{A}$; furthermore, the left $\mathcal{B}$-action on X identifies $\mathcal{B}$ with $\mathrm{End}_{\mathcal{A}}(P\mathcal{A}^n) = PM_n(\mathcal{A})P$.

A simple example of Morita equivalent algebras is $\mathcal{A}$ and $M_n(\mathcal{A})$ for any $n \geq 1$; in this case, an invertible bimodule is given by the free $(M_n(\mathcal{A}), \mathcal{A})$-bimodule $\mathcal{A}^n$. Amongst commutative algebras, Morita equivalence boils down to algebra isomorphism; nevertheless, the Picard group of a commutative algebra is generally larger than its group of algebra automorphisms, as illustrated by the next example.

EXAMPLE 4.2. *Let $\mathcal{A} = C^\infty(M)$, equipped with the pointwise product. By the smooth version of Serre-Swan's theorem, see e.g.* [**22**, Thm. 11.32]*, finitely generated projective modules $X_{\mathcal{A}}$ are given by the space of smooth sections of vector bundles $E \to M$,*

$$X_{\mathcal{A}} = \Gamma(E).$$

Writing $E = P\mathcal{A}^n$ for a projection P, we see that $\mathrm{tr}(P) = \mathrm{rank}(E)$, so the module $X_{\mathcal{A}}$ is full whenever E has nonzero rank, in which case $\Gamma(E)$ is an invertible $(\Gamma(\mathrm{End}(E)), C^\infty(M))$-bimodule. We conclude that all the algebras Morita equivalent to $C^\infty(M)$ are (isomorphic to one) of the form $\Gamma(\mathrm{End}(E))$. In particular, for any line bundle $L \to M$, $\Gamma(L)$ defines a self Morita equivalence of $C^\infty(M)$, since $\mathrm{End}(L)$ is the trivial line bundle $M \times \mathbb{C}$, so $\Gamma(\mathrm{End}(L)) \cong C^\infty(M)$. Recall that the set of isomorphism classes of complex line bundles over M forms a group (under tensor product), denoted by $\mathrm{Pic}(M)$, which is isomorphic to the additive group $H^2(M, \mathbb{Z})$.

To obtain a complete description of the Picard group of the algebra $C^\infty(M)$, recall that any automorphism of $C^\infty(M)$ is realized by a diffeomorphism $\varphi : M \to M$ via pullback,

$$f \mapsto \varphi^* f = f \circ \varphi,$$

so the group of algebra automorphisms of $C^\infty(M)$ is identified with $\mathrm{Diff}(M)$. Putting these ingredients together, one verifies that

$$\mathrm{Pic}(C^\infty(M)) = \mathrm{Diff}(M) \ltimes \mathrm{Pic}(M) = \mathrm{Diff}(M) \ltimes H^2(M, \mathbb{Z}),$$

where the semi-direct product is with respect to the action of $\mathrm{Diff}(M)$ *on line bundles (or integral cohomology classes) by pullback.*

5. Morita equivalence of star products

We now address the issue of describing when two star products on a manifold M define Morita equivalent $\mathbb{C}[[\hbar]]$-algebras. The main observation in this section is that Morita equivalence can be described as orbits of an action on $\mathrm{Def}(M)$. Let us start by describing when two star products define isomorphic $\mathbb{C}[[\hbar]]$-algebras.

5.1. Isomorphic star products. Any equivalence T between star products $\star$ and $\star'$, in the sense of Section 3.1, is an algebra isomorphism (by definition, $T = \mathrm{Id} + O(\hbar)$, so it is automatically invertible as a formal power series). But not every isomorphism is an equivalence. In general, a $\mathbb{C}[[\hbar]]$-linear isomorphism between star products $\star$ and $\star'$ on M is of the form

$$T = \sum_{r=0}^{\infty} \hbar^r T_r,$$

where each $T_r : C^\infty(M) \to C^\infty(M)$ is a differential operator, such that (3.3) holds (c.f. [**15**], Prop. 14 and Prop. 29); note that this forces $T_0 : C^\infty(M) \to C^\infty(M)$ to be an isomorphism of commutative algebras (relative to the pointwise product), but not necessarily the identity. In particular, there is a diffeomorphism $\varphi : M \to M$ such that

$$T_0 = \varphi^*.$$

If we consider the natural action of the diffeomorphism group $\mathrm{Diff}(M)$ on star products: $\star \mapsto \star_\varphi$, where

$$f \star_\varphi g = (\varphi^{-1})^*(\varphi^* f \star \varphi^* g), \quad \varphi \in \mathrm{Diff}(M), \tag{5.1}$$

we see that it descends to an action of $\mathrm{Diff}(M)$ on $\mathrm{Def}(M)$,

$$\mathrm{Diff}(M) \times \mathrm{Def}(M) \to \mathrm{Def}(M), \quad (\varphi, [\star]) \mapsto [\star_\varphi], \tag{5.2}$$

in such a way that two star products $\star$, $\star'$ define isomorphic $\mathbb{C}[[\hbar]]$-algebras if and only if their classes in $\mathrm{Def}(M)$ lie on the same $\mathrm{Diff}(M)$-orbit.

Remark 5.1. *Similarly, given a Poisson structure* π *on* M *and denoting by*

$$\mathrm{Diff}_\pi(M) \subseteq \mathrm{Diff}(M)$$

the group of Poisson automorphisms of (M, π)*, we see that the action* (5.2) *restricts to an action of* $\mathrm{Diff}_\pi(M)$ *on* $\mathrm{Def}(M, \pi)$ *whose orbits characterize isomorphic star products quantizing* π*.*

We will see that there is a larger group acting on $\mathrm{Def}(M)$ whose orbits characterize Morita equivalence.

5.2. An action of $\mathrm{Pic}(M)$**.** By Morita's characterization of invertible bimodules in Theorem 4.1, the first step in describing Morita equivalent star products is understanding, for a given star product $\star$ on M, the right modules over $(C^\infty(M)[[\hbar]], \star)$ that are finitely generated, projective, and full.

One obtains modules over $(C^\infty(M)[[\hbar]], \star)$ by starting with a classical finitely generated projective module over $C^\infty(M)$, defined by a vector bundle $E \to M$, and then performing a deformation-quantization type procedure: one searches for bilinear operators $R_r : \Gamma(E) \times C^\infty(M) \to \Gamma(E)$, $r = 1, 2, \ldots$, so that

$$s \bullet f := sf + \sum_{r=1}^{\infty} \hbar^r R_r(s, f), \quad s \in \Gamma(E),\ f \in C^\infty(M), \tag{5.3}$$

defines a right module structure on $\Gamma(E)[[\hbar]]$ over $(C^\infty(M)[[\hbar]], \star)$; i.e.,

$$(s \bullet f) \bullet g = s \bullet (f \star g).$$

One may show [**4**] that the deformation (5.3) is always unobstructed, for any choice of $\star$; moreover, the resulting module structure on $\Gamma(E)[[\hbar]]$ is unique, up to a natural notion of equivalence. Also, the module $(\Gamma(E)[[\hbar]], \bullet)$ over $(C^\infty(M)[[\hbar]], \star)$ is finitely generated, projective, and full, and any module with these properties arises in this way.

The endomorphism algebra $\mathrm{End}(\Gamma(E)[[\hbar]], \bullet)$ may be identified, as a $\mathbb{C}[[\hbar]]$-module, with $\Gamma(\mathrm{End}(E))[[\hbar]]$, so it induces an associative product $\star'$ on $\Gamma(\mathrm{End}(E))[[\hbar]]$, deforming the (generally noncommutative) algebra $\Gamma(\mathrm{End}(E))$. For a line bundle $L \to M$, since $\Gamma(\mathrm{End}(L)) \cong C^\infty(M)$, it follows that $\star'$ defines a new star product on M. The equivalence class $[\star'] \in \mathrm{Def}(M)$ is well-defined, i.e., it is independent of the specific module deformation $\bullet$ or identification $(C^\infty(M)[[\hbar]], \star') \cong \mathrm{End}(\Gamma(L)[[\hbar]], \bullet)$, and it is completely determined by the isomorphism class of L in $\mathrm{Pic}(M)$. The construction of $\star'$ from $\star$ and L gives rise to a canonical action [**2**]

$$\Phi : \mathrm{Pic}(M) \times \mathrm{Def}(M) \to \mathrm{Def}(M), \quad (L, [\star]) \mapsto \Phi_L([\star]). \tag{5.4}$$

Additionally, for any Poisson structure π, this action restricts to a well defined action of $\mathrm{Pic}(M)$ on $\mathrm{Def}(M, \pi)$.

5.3. Morita equivalence as orbits. Let us consider the semi-direct product

$$\mathrm{Diff}(M) \ltimes \mathrm{Pic}(M),$$

which is nothing but the Picard group of $C^\infty(M)$, see Example 4.2. By combining the actions of $\mathrm{Diff}(M)$ and $\mathrm{Pic}(M)$ on $\mathrm{Def}(M)$, described in (5.2) and (5.4), one obtains a $\mathrm{Diff}(M) \ltimes \mathrm{Pic}(M)$-action on $\mathrm{Def}(M)$, which leads to the following characterization of Morita equivalent star products, see [**2**]:

THEOREM 5.2. *Two star products $\star$ and $\star'$ on M are Morita equivalent if and only if $[\star]$, $[\star']$ lie in the same $\mathrm{Diff}(M) \ltimes \mathrm{Pic}(M)$-orbit:*

$$[\star'] = \Phi_L([\star_\varphi])$$

Similarly (see Remark 5.1), two star products on M quantizing the same Poisson structure π are Morita equivalent if and only if the lie in the same orbit of $\mathrm{Diff}_\pi(M) \ltimes \mathrm{Pic}(M)$ on $\mathrm{Def}(M, \pi)$.

Our next step is to transfer the actions of $\mathrm{Diff}(M)$ and $\mathrm{Pic}(M) = H^2(M, \mathbb{Z})$ to $\mathrm{FPois}(M)$ via $\mathcal{K}_*$ in (3.8); i.e, we will find explicit actions of $\mathrm{Diff}(M)$ and $H^2(M, \mathbb{Z})$ on $\mathrm{FPois}(M)$ making $\mathcal{K}_*$ equivariant with respect to $\mathrm{Diff}(M) \ltimes H^2(M, \mathbb{Z})$.

6. B-field action on formal Poisson structures

There is a natural way in which Poisson structures may be modified by closed 2-forms. In the context of formal Poisson structures, this leads to a natural action of the abelian group $H^2_{dR}(M, \mathbb{C})[[\hbar]]$ on $\mathrm{FPois}(M)$,

$$H^2_{dR}(M, \mathbb{C})[[\hbar]] \times \mathrm{FPois}(M) \to \mathrm{FPois}(M), \tag{6.1}$$

that we will refer to as the B*-field action*, to be discussed in this section.

6.1. B**-field transformations of Poisson structures.** A convenient way to describe how closed 2-forms may "act" on Poisson structures is to take a broader perspective on Poisson geometry, following [**9, 26**], see also [**14**]. The starting point is considering, for a manifold M, the direct sum $TM \oplus T^*M$. This bundle is naturally equipped with two additional structures: a symmetric, nondegenerate, fibrewise pairing, given for each $x \in M$ by

$$\langle (X,\alpha),(Y,\beta)\rangle = \beta(X) + \alpha(Y), \quad X, Y \in T_xM,\ \alpha,\beta \in T_x^*M, \tag{6.2}$$

as well as a bilinear operation on $\Gamma(TM \oplus T^*M)$, known as the *Courant bracket*, given by

$$[\![(X,\alpha),(Y,\beta)]\!] := ([X,Y], \mathcal{L}_X\beta - i_Y d\alpha), \tag{6.3}$$

for $X, Y \in \mathcal{X}^1(M)$ and $\alpha, \beta \in \Omega^1(M)$. The Courant bracket extends the usual Lie bracket of vector fields, but it is not a Lie bracket itself.

One may use (6.2) and (6.3) to obtain an alternative description of Poisson structures. Specifically, Poisson structures on M are in one-to-one correspondence with subbundles $L \subset TM \oplus T^*M$ satisfying the following conditions:

(a) $L \cap TM = \{0\}$,
(b) $L = L^\perp$ (i.e., L is self orthogonal with respect to the pairing (6.2)), and
(c) $\Gamma(L)$ is involutive with respect to the Courant bracket.

For a Poisson structure π, the bundle $L \subset TM \oplus T^*M$ corresponding to it is (see (3.5))

$$L = \mathrm{graph}(\pi^\sharp) = \{(\pi^\sharp(\alpha),\alpha) \mid \alpha \in T^*M\}.$$

Indeed, (a) means that L is the graph of a bundle map $\rho : T^*M \to TM$, while (b) says that $\rho^* = -\rho$, so that $\rho = \pi^\sharp$ for $\pi \in \mathcal{X}^2(M)$; finally, (c) accounts for the condition $[\pi,\pi] = 0$. In general, subbundles $L \subset TM \oplus T^*M$ satisfying only (b) and (c) are referred to as *Dirac structures* [**9**]; Poisson structures are particular cases also satisfying (a).

We will be interested in the group of bundle automorphisms of $TM \oplus T^*M$ which preserve the pairing (6.2) and the Courant bracket (6.3); we refer to such automorphisms as *Courant symmetries.* Any diffeomorphism $\varphi : M \to M$ naturally lifts to a Courant symmetry, through its natural lifts to TM and T^*M. Another type of Courant symmetry, known as B*-field (or gauge) transformation*, is defined by closed 2-forms [**26**]: any $B \in \Omega^2_{cl}(M)$ acts on $TM \oplus T^*M$ via

$$(X,\alpha) \overset{\tau_B}{\mapsto} (X, \alpha + i_X B). \tag{6.4}$$

The full group of Courant symmetries turns out to be exactly $\mathrm{Diff}(M) \ltimes \Omega^2_{cl}(M)$.

For a Poisson structure π on M, the B-field transformation (6.4) takes $L = \mathrm{graph}(\pi^\sharp)$ to the subbundle

$$\tau_B(L) = \{(\pi^\sharp(\alpha), \alpha + i_{\pi^\sharp(\alpha)}B) \,|\, \alpha \in T^*M\} \subset TM \oplus T^*M,$$

and since τ_B preserves (6.2) and (6.3), $\tau_B(L)$ automatically satisfies (b) and (c) (i.e., it is a Dirac structure). It follows that $\tau_B(L)$ determines a new Poisson structure π^B, via $\tau_B(L) = \mathrm{graph}((\pi^B)^\sharp)$, if and only if $\tau_B(L) \cap TM = \{0\}$, which is equivalent to the condition that

$$\mathrm{Id} + B^\sharp\pi^\sharp : T^*M \to T^*M \ \text{ is invertible,} \tag{6.5}$$

where $B^\sharp : TM \to T^*M$, $B^\sharp(X) = i_X B$. In this case, the Poisson structure π^B is completely characterized by

$$(\pi^B)^\sharp = \pi^\sharp \circ (\mathrm{Id} + B^\sharp \pi^\sharp)^{-1}. \tag{6.6}$$

In conclusion, given a Poisson structure π and a closed 2-form B, if the compatibility condition (6.5) holds, then (6.6) defines a new Poisson structure π^B. A simple case is when π is nondegenerate, hence equivalent to a symplectic form ω; then condition (6.5) says that $\omega + B$ is nondegenerate, and π^B is the Poisson structure associated with it.

6.2. Formal Poisson structures and the B-field action. The whole discussion about B-field transformations carries over to formal Poisson structures – and even simplifies in this context. Given a formal Poisson structure

$$\pi_\hbar = \hbar\pi_1 + \hbar^2\pi_2 + \ldots \in \hbar\mathcal{X}^2(M)[[\hbar]]$$

and any $B \in \Omega^2_{cl}(M)[[\hbar]]$, then $B^\sharp\pi^\sharp_\hbar = O(\hbar)$, where here $\pi^\sharp_\hbar$ and $B^\sharp$ are the associated (formal series of) bundle maps. Hence $(\mathrm{Id} + B^\sharp\pi^\sharp_\hbar)$ is automatically invertible as a formal power series (i.e., (6.5) is automatically satisfied),

$$(\mathrm{Id} + B^\sharp\pi^\sharp_\hbar)^{-1} = \sum_{n=0}^{\infty}(-1)^n(B^\sharp\pi^\sharp_\hbar)^n,$$

and the same formula as (6.6) defines an action of the abelian group $\Omega^2_{cl}(M)[[\hbar]]$ on formal Poisson structures: $\pi_\hbar \mapsto \pi^B_\hbar$, where

$$(\pi^B_\hbar)^\sharp = \pi^\sharp_\hbar \circ (\mathrm{Id} + B^\sharp\pi^\sharp_\hbar)^{-1}.$$

There are two key observations concerning this action: First, the B-field transformations of equivalent formal Poisson structures remain equivalent; second, the B-field transformation by an exact 2-form $B = dA$ does not change the equivalence class of a formal Poisson structure. This leads to the next result [**3**, Prop. 3.10]:

Theorem 6.1. *The action of $\Omega^2_{cl}(M)[[\hbar]]$ on formal Poisson structures descends to an action*

$$H^2_{dR}(M, \mathbb{C})[[\hbar]] \times \mathrm{FPois}(M) \to \mathrm{FPois}(M),\ [\pi_\hbar] \mapsto [\pi^B_\hbar] \tag{6.7}$$

This action is the identity in first order of $\hbar$:

$$\pi_\hbar = \hbar\pi_1 + O(\hbar) \implies \pi^B_\hbar = \hbar\pi_1 + O(\hbar).$$

So, for any Poisson structure $\pi \in \mathcal{X}^2(M)$, the action (6.7) restricts to

$$H^2_{dR}(M, \mathbb{C})[[\hbar]] \times \mathrm{FPois}(M, \pi) \to \mathrm{FPois}(M, \pi).$$

When π is symplectic, so that we have the identification (3.10), this action is simply

$$[\omega_\hbar] \mapsto [\omega_\hbar] + [B], \tag{6.8}$$

for $[\omega_\hbar] \in \frac{1}{\hbar}[\omega] + H^2_{dR}(M, \mathbb{C})[[\hbar]]$ and $[B] \in H^2_{dR}(M, \mathbb{C})[[\hbar]]$.

7. Morita equivalent star products via Kontsevich's classes

Our final goal is to present the description of Morita equivalent star products in terms of their Kontsevich classes, i.e., by means of the correspondence

$$\mathcal{K}_* : \mathrm{FPois}(M) \xrightarrow{\sim} \mathrm{Def}(M)$$

of Thm. 3.3. By Thm. 5.2, Morita equivalent star products are characterized by lying on the same orbit of $\mathrm{Diff}(M) \ltimes \mathrm{Pic}(M)$ on $\mathrm{Def}(M)$. In order to find the corresponding action on $\mathrm{FPois}(M)$, we treat the actions of $\mathrm{Diff}(M)$ and $\mathrm{Pic}(M) \cong H^2(M, \mathbb{Z})$ independently.

The group $\mathrm{Diff}(M)$ naturally acts on formal Poisson structures: for $\varphi \in \mathrm{Diff}(M)$,

$$\pi_\hbar = \sum_{r=1}^{\infty} \hbar^r \pi_r \overset{\varphi}{\mapsto} \varphi_* \pi_\hbar = \sum_{r=1}^{\infty} \hbar^r \varphi_* \pi_r,$$

and this action induces an action of $\mathrm{Diff}(M)$ on $\mathrm{FPois}(M)$, with respect to which $\mathcal{K}_*$ is equivariant [**11**]. This accounts for the classification of isomorphic star products in terms of their Kontsevich classes, which is (the easy) part of the classification up to Morita equivalence.

The less trivial part is due to the action (5.4) of $\mathrm{Pic}(M)$. In this respect, the main result asserts that the transformation Φ_L on $\mathrm{Def}(M)$ defined by a line bundle $L \in \mathrm{Pic}(M)$ corresponds to the B-field transformation on $\mathrm{FPois}(M)$ by a curvature form of L, see [**3**, Thm. 3.11]:

Theorem 7.1. *The action $\Phi : \mathrm{Pic}(M) \times \mathrm{Def}(M) \to \mathrm{Def}(M)$ satisfies*

$$\Phi_L([\star]) = \mathcal{K}_*([\pi_\hbar^B]),$$

where B is a closed 2-form representing the cohomology class $2\pi i c_1(L)$, where $c_1(L)$ is the Chern class of $L \to M$.

A direct consequence is that flat line bundles (which are the torsion elements in $\mathrm{Pic}(M)$) act trivially on $\mathrm{Def}(M)$ under (5.4).

The theorem establishes a direct connection between the B-field action (6.7) on formal Poisson structures and algebraic Morita equivalence: If a closed 2-form $B \in \Omega^2_{cl}(M)$ is 2πi-integral, then B-field related formal Poisson structures quantize under $\mathcal{K}_*$ to Morita equivalent star products.

The following classification results for Morita equivalent star products in terms of their characteristic classes readily follow from Thm. 7.1:

- Two star products $\star$ and $\star'$, with Kontsevich classes $[\pi_\hbar]$ and $[\pi'_\hbar]$, are Morita equivalent if and only if there is a diffeomorphism $\varphi : M \to M$ and a closed 2-form B, whose cohomology class is 2πi-integral, such that

$$[\pi'_\hbar] = [(\varphi_* \pi_\hbar)^B]. \tag{7.1}$$

- If $\star$ and $\star'$ quantize the same Poisson structure π, so that $[\pi_\hbar]$, $[\pi'_\hbar] \in \mathrm{FPois}(M, \pi)$, then the result is analogous: $\star$ and $\star'$ are Morita equivalent if and only if (7.1) holds, but now $\varphi : M \to M$ is a Poisson automorphism.
- Assume that $\star$ and $\star'$ quantize the same nondegenerate Poisson structure, defined by a symplectic form ω. In this case, by (6.8), Theorem 7.1 recovers the characterization of Morita equivalent star products on symplectic manifolds obtained in [**5**], in terms of Fedosov-Deligne classes

(3.11): $\star$ and $\star'$ are Morita equivalent if and only if there is a symplectomorphism $\varphi : M \to M$ for which the difference $c(\star) - c(\star'_\varphi)$ is a 2πi-integral class in $H^2_{dR}(M, \mathbb{C})$ (viewed as a subspace of $H^2_{dR}(M, \mathbb{C})[[\hbar]] = H^2_{dR}(M, \mathbb{C}) + \hbar H^2_{dR}(M, \mathbb{C})[[\hbar]]$).

REMARK 7.2. *These notes treat star-product algebras simply as associative unital* $\mathbb{C}[[\hbar]]$*-algebras. But these algebras often carry additional structure: one may consider star products for which complex conjugation is an algebra involution (e.g. the Weyl star product, see* (2.13)*), and use the fact that* $\mathbb{R}[[\hbar]]$ *is an ordered ring to obtain suitable notions of positivity on these algebras (e.g. positive elements, positive linear functionals). One can develop refined notions of Morita equivalence for star products, parallel to strong Morita equivalence of* C^**-algebras, taking these additional properties into account, see e.g.* [**6**]. *An overview of these more elaborate aspects of Morita theory for star products can be found e.g. in* [**29**].

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