

VERTEX ALGEBRAS

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The basic bibliography for this class will be [3]. The first rigorous definition of Vertex Algebras was given by Borcherds his famous 1986 paper on the monster group [1].

1. THE CALCULUS OF FORMAL DISTRIBUTIONS

Let \mathcal{U} be a vector space (over the complex numbers generally)

1.1. Definition. An \mathcal{U} -valued **formal distribution** is an expression

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^n,$$

where $a_n \in \mathcal{U}$ and z is an indeterminate. Their space is denoted by $\mathcal{U}[[z, z^{-1}]]$. Note also that $\mathcal{U}[z, z^{-1}]$ will denote the space of Laurent polynomials with values in \mathcal{U} . The linear function

$$\text{Res}_z a(z) := a_{-1},$$

is called **the residue**. Clearly satisfies

$$\text{Res}_z \partial_z a(z) = 0$$

If we consider $\mathbb{C}[z, z^{-1}]$ as space of tests functions then any \mathcal{U} -valued formal distribution induces a linear map $\mathbb{C}[z, z^{-1}] \rightarrow \mathcal{U}$ by

$$f_a(\varphi(z)) = \text{Res}_z \varphi(z) a(z).$$

Exercise 1.1. Show that all \mathcal{U} -valued linear functions on the space $\mathbb{C}[z, z^{-1}]$ are obtained uniquely in this way.

Proof. Let $f : \mathbb{C}[z, z^{-1}] \rightarrow \mathcal{U}$ be a linear map, define the formal distribution $a(z)$ by

$$(1.1.1) \quad a(z) = \sum_{n \in \mathbb{Z}} f(z^{-1-n}) z^n$$

Hence clearly we have $f(z^n) = \text{Res}_z z^n a(z)$ and by linearity existence follows. Uniqueness is checked similarly:

$$(1.1.2) \quad a(z) = \sum a_n z^n$$

Multiplying both sides of (1.1.2) by z^k and taking residues we get $f(z^k) = a_{-1-k}$ in accordance with (1.1.1) \square

Following the last exercise, it is natural to define $a_{(n)} = \text{Res}_z z^n a(z)$, then the original formal distribution is expressed as

$$(1.1.3) \quad a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-1-n}.$$

The vectors $a_{(n)}$ are called *Fourier Coefficients* of the formal distribution.

Similarly one can define formal distributions in two or more variables as

$$a(z, w, \dots) = \sum_{n, m, \dots \in \mathbb{Z}} a_{n, m, \dots} z^n w^m \dots, \quad a_{n, m, \dots} \in \mathcal{U}$$

Denote by \mathcal{A} the algebra of rational functions of the form $(z-w)^{-k}P(z, w)$ where $P(z, w) \in \mathbb{C}[z, z^{-1}, w, w^{-1}]$ and k is an integer number. Consider the following map called *expansion in the domain $|z| > |w|$* defined when $k < 0$:

$$\begin{aligned} i_{z,w} z^m w^n (z-w)^k &= z^m w^n z^k \left(1 - \frac{w}{z}\right)^k \\ &= z^{m+k} w^n \left(\frac{1}{1 - \frac{w}{z}}\right)^{-k} \\ &= z^{m+k} w^n \left(1 + \frac{w}{z} + \left(\frac{w}{z}\right)^2 + \dots\right)^{-k} \\ &= \sum_{j=0}^{\infty} \binom{j}{-1-k} z^{m-j-1} w^{j+k+n+1} \end{aligned}$$

And the obvious expansion when $k \geq 0$, we can extend linearly to the rest of \mathcal{A} . Similarly we define the *expansion in the domain $|w| > |z|$* and denote it by $i_{w,z}$.

Exercise 1.2. *These maps are homomorphisms which commute with multiplication by z^m , w^n , and with ∂_z and ∂_w .*

Proof. These maps are \mathbb{C} linear by definition. Now the following calculation

$$\begin{aligned} (i_{z,w}(z-w)^{-1})^2 &= \left(1 + \frac{w}{z} + \dots\right) \left(1 + \frac{w}{z} + \dots\right) \\ &= \left(1 + \frac{w}{z} + \dots\right)^2 \\ &= i_{z,w}(z-w)^{-2} \end{aligned}$$

is enough to prove that $i_{z,w}$ is an homomorphism.

The fact that these maps commute with multiplication by z^m and w^m is obvious.

The only non-trivial case remaining to prove in the exercise is

$$\begin{aligned} \partial_z i_{z,w}(z-w)^{-j-1} &= \\ &= \partial_z \sum_{m=0}^{\infty} \binom{m}{j} z^{-m-1} w^{m-j} \\ &= - \sum_{m=0}^{\infty} \binom{m}{j} (m+1) z^{-m-2} w^{m-j} \\ &= - \sum_{m=0}^{\infty} \frac{(m+1)!}{(j+1)!(m+1-j-1)!} (j+1) z^{-(m+1)-1} w^{m+1-j-1} \\ &= -(j+1) \sum_{m=1}^{\infty} \binom{m}{j+1} z^{-m-1} w^{m-j-1} \\ &= -(j+1) i_{w,z}(z-w)^{-j-2} \\ &= i_{z,w} \partial_z (z-w)^{-j-1} \end{aligned}$$

and similarly for ∂_w . □

1.2. Definition. The formal δ -function is defined by

$$(1.1.4a) \quad \delta(z, w) = i_{z,w} \frac{1}{z-w} - i_{w,z} \frac{1}{z-w}$$

And substituting the definition for the corresponding expansions we get

$$(1.1.4b) \quad \delta(z, w) = z^{-1} \sum_{n \in \mathbb{Z}} \left(\frac{w}{z}\right)^n$$

Differentiating equations (1.1.4a) and (1.1.4b) we get

$$(1.2.5a) \quad \frac{1}{n!} \partial_w^n \delta(z, w) = i_{z,w} \frac{1}{(z-w)^{n+1}} - i_{w,z} \frac{1}{(z-w)^{n+1}}$$

$$(1.2.5b) \quad = \sum_{j \in \mathbb{Z}} \binom{j}{n} z^{-j-1} w^{j-n}$$

respectively.

Exercise 1.3. Show there is a unique formal distribution, $\delta(z, w) \in \mathbb{C}[z^{\pm 1}, w^{\pm 1}]$ such that $\text{Res}_z \delta(z, w) \varphi(z) = \varphi(w)$ for any test function $\varphi(z) \in \mathbb{C}[z, z^{-1}]$.

Proof. Existence is proposition 1.3(6) below. Conversely, if $\delta = \sum \delta_{n,m} z^n w^m$ and we have a similar decomposition for δ' . Then we can compute

$$\text{Res}_z \delta(z, w) z^k = \sum \delta_{-1-k,m} w^m = w^k = \sum \delta'_{-1-k,m} w^m.$$

Now comparing coefficients we have $\delta_{-1-k,m} = \delta'_{-1-k,m} \forall k, m \in \mathbb{Z}$ as we wanted. \square

1.3. Proposition. The formal distribution $\delta(z, w)$ satisfies the following properties

- (1) (locality) $(z-w)^m \partial_w^n \delta(z, w) = 0$ whenever $m > n$.
- (2) $(z-w) \frac{1}{n!} \partial_w^n \delta(z, w) = \frac{1}{(n-1)!} \partial_w^{n-1} \delta(z, w)$ if $n \geq 1$.
- (3) $\delta(z, w) = \delta(w, z)$.
- (4) $\partial_z \delta(z, w) = -\partial_w \delta(w, z)$.
- (5) $a(z) \delta(z, w) = a(w) \delta(z, w)$ where $a(z)$ is any formal distribution.
- (6) $\text{Res}_z a(z) \delta(z, w) = a(w)$
- (7) $\exp(\lambda(z-w)) \partial_w^n \delta(z, w) = (\lambda + \partial_w)^n \delta(z, w)$

Proof. Properties (1)-(4) follows easily from the definitions and equations (1.2.5a). Perhaps properties (3) and (4) justifies the usual notation $\delta(z-w)$ for $\delta(z, w)$. Now because of (1) we have $(z-w) \delta(z, w) = 0$ hence $z^n \delta(z, w) = w^n \delta(z, w)$ and (5) follows by linearity. Now taking residues in property (5) we get

$$\text{Res}_z a(z) \delta(z, w) = a(w) \text{Res}_z \delta(z, w) = a(w)$$

so (6) follows. Now in order to prove (7) we expand the exponential as

$$\begin{aligned} e^{\lambda(z-w)} \partial_w^n \delta(z, w) &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} n! (z-w)^k \frac{\partial_w^n \delta(z, w)}{n!} \\ &= \sum_{k=0}^{\infty} n \frac{\lambda^k n!}{k!} \frac{\partial_w^{n-k} \delta(z, w)}{(n-k)!} \\ &= \left(\sum_{k=0}^n \binom{n}{k} \lambda^k \partial_w^{n-k} \right) \delta(z, w) \\ &= (\lambda + \partial_w)^n \delta(z, w) \end{aligned}$$

□

1.4. Definition. An \mathcal{U} -valued formal distribution is called **local** if

$$(z-w)^n a(z, w) = 0 \quad n \gg 0$$

1.5. Example. $\delta(z, w)$ is local and so are its derivatives. Also $a(z, w)$ local implies that $a(w, z)$ is local.

Exercise 1.4. Show that if $a(z, w)$ is local so are $\partial_w a(z, w)$ and $\partial_z a(z, w)$.

Proof. Of course by symmetry it is enough to prove the result for one derivative. If $(z-w)^{n-1} a(z, w) = 0$ then we have:

$$\begin{aligned} (z-w)^n \partial_z a(z, w) &= \partial_z (z-w)^n a(z, w) - a(z, w) \partial_z (z-w)^n \\ &= -n a(z, w) (z-w)^{n-1} \\ &= 0 \end{aligned}$$

proving the exercise. □

1.6. Theorem (Decomposition). Let $a(z, w)$ be a \mathcal{U} -valued local formal distribution. Then $a(z, w)$ can be uniquely decomposed as in the following finite sum

$$(1.5.1a) \quad a(z, w) = \sum_{j \in \mathbb{Z}_+} c^j(w) \frac{\partial_w^j \delta(z, w)}{j!}$$

where $c^j(w) \in \mathcal{U}[[w, w^{-1}]]$ are formal distributions given by

$$(1.5.1b) \quad c^j(w) = \text{Res}_z (z-w)^j a(z, w)$$

Proof. Let $b(z, w) = a(z, w) - \sum c^j(w) \frac{\partial_w^j \delta(z, w)}{j!}$. Clearly $b(z, w)$ is local being a finite linear combination of local formal distributions. Note that

$$\begin{aligned} \text{Res}_z (z-w)^n b(z, w) &= \\ &= \text{Res}_z (z-w)^n a(z, w) - \text{Res}_z (z-w)^n \sum c^j(w) \frac{\partial_w^j \delta(z, w)}{j!} \\ &= c^n(w) - \sum c^j(w) \frac{\partial_w^{j-n}}{(j-n)!} \text{Res}_z \delta(z, w) \\ &= c^n(w) - c^n(w) \\ &= 0 \end{aligned}$$

Now write $b(z, w) = \sum z^n b_n(w)$. Then we have $\text{Res}_z b(z, w) = b_{-1}(w) = 0$. By the above calculation we have

$$\begin{aligned} 0 &= \text{Res}_z (z - w)b(z, w) \\ &= \text{Res}_z zb(z, w) - w \text{Res}_z b(z, w) \\ &= \text{Res}_z zb(z, w) \\ &= b_{-2}(z, w) \\ &= 0 \end{aligned}$$

and iterating we have $b(z, w) = \sum_{n \in \mathbb{Z}_+} b_n z^n$. Now since b is local we have

$$(z - w)^n b(z, w) = 0 \Rightarrow b(z, w) = 0$$

as we wanted.

To show uniqueness we take residues on both sides of (1.5.1a), obtaining:

$$\text{Res}_z (z - w)^k a(z, w) = \text{Res}_z (z - w)^k \sum c^j(w) \frac{\partial_w^j \delta(z, w)}{j!} = c^k(w)$$

In accordance with (1.5.1b) . □

Given $a(z, w) \in \mathcal{U}[[z^{\pm 1}, w^{\pm 1}]]$ we may define a linear operator

$$\begin{aligned} D_a : \mathbb{C}[z, z^{-1}] &\longrightarrow \mathcal{U}[[w, w^{-1}]] \\ \varphi(z) &\mapsto \text{Res}_z \varphi(z) a(z, w) \end{aligned}$$

Exercise 1.5.

- (1) $D_{c(w)\partial_w^j \delta(z, w)} = c(w)\partial_w^j$ and in particular we have $D_{\delta(z, w)} = 1$.
- (2) $a(z, w)$ is local if and only if D_a is a finite order differential operator.
- (3) Suppose $a(z, w)$ is local, then $D_{a(w, z)} = D_{a(z, w)}^*$, where $*$ is defined by

$$\left(\sum c^j(w) \partial_w^j \right)^* = \sum (-\partial_w^j) c^j(w)$$

Proof.

$$\begin{aligned} D_{c(w)\partial_w^j \delta(z, w)} \varphi(z) &= \text{Res}_z c(w) \partial_w^j \delta(z, w) \varphi(z) = c(w) \partial_w^j \text{Res}_z \delta(z, w) \varphi(z) \\ &= c(w) \partial_w^j \varphi(w) \end{aligned}$$

proving (1). Suppose now that $a(z, w)$ is local, by the decomposition theorem 1.6 we may assume that $a(z, w) = c(w) \partial_w^j \delta(z, w)$ and in this case (1) proves that D_a is a finite order differential operator. The converse is obvious hence (2) follows. To prove (3) we may assume again by theorem 1.6 that $a(z, w) = c(w) \partial_w^j \delta(z, w)$, in this case:

$$\begin{aligned} D_{a(w, z)} &= \text{Res}_z c(z) \varphi(z) \partial_z^j \delta(w, z) \\ &= \text{Res}_z c(z) \varphi(z) \partial_z^j \delta(z, w) \\ &= \text{Res}_z c(z) \varphi(z) (-\partial_w)^j \delta(z, w) \\ &= (-\partial_w)^j \text{Res}_z c(z) \varphi(z) \delta(z, w) \\ &= (-\partial_w)^j c(w) \varphi(w) \end{aligned}$$

and the result follows by linearity. □

2. FORMAL FOURIER TRANSFORM

2.1. Definition. Given a formal distribution $a(z, w) \in \mathcal{U}[[z^{\pm 1}, w^{\pm 1}]]$ we define the **formal Fourier Transform** of $a(z, w)$ by

$$(2.1.1) \quad F_{z,w}^\lambda a(z, w) = \text{Res}_z e^{\lambda(z-w)} a(z, w)$$

Hence $F_{z,w}^\lambda$ defines a map from the formal distributions to $\mathcal{U}[[w, w^{-1}]][[\lambda]]$.

2.2. Proposition. $F_{z,w}^\lambda a(z, w) = \sum \frac{\lambda^j}{j!} c^j(w)$ where $c^j(w)$ is defined as in (1.5.1b).

Proof.

$$F_{z,w}^\lambda a(z, w) = \text{Res}_z e^{\lambda(z-w)} a(z, w) = \sum_{j \in \mathbb{Z}_+} \frac{\lambda^j}{j!} \text{Res}_z (z-w)^j a(z, w)$$

and we can recognize $c^j(w)$ in the last sum. \square

2.3. Proposition. *The formal Fourier transform satisfies the following properties:*

- (1) $F_{z,w}^\lambda \partial_z a(z, w) = -\lambda F_{z,w}^\lambda a(z, w) = [\partial_w, F_{z,w}^\lambda]$.
- (2) $F_{z,w}^\lambda a(z, w) = F_{z,w}^{-\lambda - \partial_w} a(z, w)$ provided that $a(z, w)$ is local, where

$$F_{z,w}^{-\lambda - \partial_w} a(z, w) := F_{z,w}^\mu a(z, w)|_{\mu = -\lambda - \partial_w}$$
- (3) $F_{z,w}^\lambda F_{x,w}^\mu a(z, w, x) = F_{x,w}^{\lambda + \mu} F_{z,x}^\lambda a(z, w, x)$

Proof.

$$\begin{aligned} F_{z,w}^\lambda \partial_z a(z, w) &= \text{Res}_z e^{\lambda(z-w)} \partial_z a(z, w) \\ &= -\text{Res}_z \partial_z \left(e^{\lambda(z-w)} \right) a(z, w) \\ &= -\lambda \text{Res}_z e^{\lambda(z-w)} a(z, w) \\ &= -\lambda F_{z,w}^\lambda a(z, w) \end{aligned}$$

Hence the first part of (1) follows. To prove (2) by the decomposition theorem 1.6 it suffices to prove the case

$$a(z, w) = c(w) \partial_w^j \delta(z, w),$$

but in this case we clearly have:

$$F_{z,w}^\lambda a(z, w) = F_{z,w}^\lambda c(z) \partial_z^j \delta(z, w) = \text{Res}_z e^{\lambda(z-w)} c(z) (-\partial_w)^j \delta(z, w)$$

where we have used property (4) in proposition 1.3. Now using property (7) in the same proposition we can express the last term as:

$$\text{Res}_z c(z) (-\lambda - \partial_w)^j \delta(z, w) = (-\lambda - \partial_w)^j \text{Res}_z c(z) \delta(z, w)$$

But $F_{z,w}^\mu c(z) \partial_z^j \delta(z, w)|_{\mu = -\lambda - \partial_w} = (-\lambda - \partial_w)^j c(w)$ which in turn is the RHS, hence (2) follows.

Now proving (3) is equivalent to

$$\begin{aligned} \text{Res}_z e^{\lambda(z-w)} \text{Res}_x e^{\mu(x-w)} a(z, w, x) &\stackrel{?}{=} \text{Res}_x e^{(\lambda+\mu)(x-w)} \text{Res}_z e^{\lambda(z-x)} a(z, w, x) \\ e^{\lambda(z-w) + \mu(x-w)} &\stackrel{?}{=} e^{(\lambda+\mu)(x-w) + \lambda(z-x)} \end{aligned}$$

This is clearly true. \square

Exercise 2.1. *Prove the second equality of (1) in the above proposition*

Proof.

$$\begin{aligned}
[\partial_w, F_{z,w}^\lambda]a(z, w) &= \partial_w \operatorname{Res}_z e^{\lambda(z-w)}a(z, w) - \operatorname{Res}_z e^{\lambda(z-w)}\partial_w a(z, w) \\
&= \operatorname{Res}_z \partial_w (e^{\lambda(z-w)}a(z, w)) - \operatorname{Res}_z e^{\lambda(z-w)}\partial_w a(z, w) \\
&= \operatorname{Res}_z \partial_w (e^{\lambda(z-w)})a(z, w) \\
&= -\lambda F_{z,w}^\lambda \partial_w a(z, w)
\end{aligned}$$

as we wanted. \square

A digression on Superalgebras.

2.4. Definition. A **superspace** is a vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. Similarly a **superalgebra** is a $\mathbb{Z}/2\mathbb{Z}$ -graded associative algebra, this is $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ and if $a_\alpha \in \mathcal{A}_\alpha$ then $a_\alpha a_\beta \in \mathcal{A}_{\alpha+\beta}$.

2.5. Example. $\operatorname{End}(V)$ where V is a superspace, is canonically a superalgebra, where

$$\operatorname{End}(V)_{\bar{0}} = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \operatorname{End}(V) \right\}, \quad \operatorname{End}(V)_{\bar{1}} = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \in \operatorname{End}(V) \right\}$$

2.6. Definition. If $\dim V < \infty$ then we define the **supertrace** in $\operatorname{End}(V)$ as $\operatorname{str}(A) = \operatorname{tr}(a) - \operatorname{tr}(d)$ and the **superdimension** $\operatorname{sdim} V = \dim V_{\bar{0}} - \dim V_{\bar{1}}$.

Denote by $p(a)$ the degree of the homogeneous element a in an associative superalgebra, we call this degree the **parity** of a . In an associative superalgebra we define the (super)bracket in the homogeneous elements by:

$$(2.6.1) \quad [a, b] = ab - p(a, b)ba, \quad p(a, b) = (-1)^{p(a)p(b)}.$$

Exercise 2.2. Prove that the (super)bracket defined above satisfies:

- (1) $[a, b] = -p(a, b)[b, a]$
- (2) $[a, [b, c]] = [[a, b], c] + p(a, b)[b, [a, c]]$

Proof. Multiplying equation (2.6.1) by $p(a, b)$ and noting that $p(a, b)^2 = 1$ we get

$$p(a, b)[a, b] = p(a, b)ab - ba = -[b, a]$$

And this proves (1). To prove (2) we expand each side to get

$$\begin{aligned}
[a, [b, c]] &= abc - p(a, b)p(a, c)bca - p(b, c)acb + p(b, c)p(a, c)p(a, b)cba \\
[[a, b], c] &= -p(a, c)p(b, c)cab + abc + p(a, b)p(a, c)p(b, c)cba - p(a, b)bac \\
p(a, b)[b, [a, c]] &= p(a, b)bac - p(b, c)acb - p(a, b)p(a, c)bca + p(b, c)p(a, c)cab
\end{aligned}$$

In the second equation we used (1) and parity identities as $p(a, bc) = p(a, b)p(a, c)$ which are straightforward to prove. Now it is clear that subtracting the last two equations from the first one we get the result. \square

2.7. Definition. A superspace endowed with a bracket satisfying the above properties is called a **Lie superalgebra**.

2.8. Example. $\operatorname{End}(V)$ with this bracket, where $\dim V_{\bar{0}} = m$ and $\dim V_{\bar{1}} = n$ is called $\mathfrak{gl}(m|n)$. We define a subspace as

$$\mathfrak{sl}(m|n) = \{A \in \mathfrak{gl}(m|n) \mid \operatorname{str} A = 0\}$$

Exercise 2.3. Show that $\operatorname{str}[A, B] = 0$ for every $A, B \in \mathfrak{gl}(m|n)$ hence $\mathfrak{sl}(m|n)$ is a subalgebra (even an ideal) of $\mathfrak{gl}(m|n)$.

Proof.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

hence we can express the bracket of these elements as

$$(2.8.1) \quad [A, B] = \begin{pmatrix} aa' - a'a + bc' + b'c & ab' - b'd + bd' - d'c \\ dc' - c'a + ca' - a'b & dd' - d'd + cb' + c'b \end{pmatrix}$$

And now it is clear that

$$\text{str}[A, B] = \text{tr } bc' + \text{tr } b'c - \text{tr } cb' - \text{tr } c'b = 0$$

proving the exercise. \square

Exercise 2.4. *Show that $\mathfrak{sl}(1|1)$ is nilpotent but $\mathfrak{sl}(m|n)$ is simple whenever $m > 1$ or $n > 1$ and $m \neq n$.*

Proof. From equation (2.8.1) we have $\mathfrak{sl}(1|1)' \subset \mathfrak{sl}(1|1)_0$. But $\mathfrak{sl}(1|1)_0$ is clearly a commutative algebra, hence $\mathfrak{sl}(1|1)^2 = 0$ and $\mathfrak{sl}(1|1)$ is nilpotent.

Note that in the case $m = n$ the identity matrix is in $\mathfrak{sl}(m|n)$ and hence the scalar multiples of the identity form a nontrivial ideal, hence $\mathfrak{sl}(m|m)$ is not simple.

To give a proof that in the remaining cases $\mathfrak{g} = \mathfrak{sl}(m|n)$ is simple, we proceed as in the $\mathfrak{sl}(n)$ case. Consider the subalgebra of diagonal matrices $\mathfrak{h} \subset \mathfrak{g}$. It acts on \mathfrak{g} by the adjoint representation, moreover it acts diagonally, in fact since $\mathfrak{h} \subset \mathfrak{g}_0$ we know exactly what the action of $\text{ad}(\mathfrak{h})$ is from the classical case. If we define $E_{i,j}$ to be the matrix with 1 in the i, j entry and 0 otherwise, $\epsilon_i \in \mathfrak{h}^*$ is such that $\epsilon_i(E_{jj}) = \delta_{ij}$, then we can easily see

$$[H, E_{ij}] = (\epsilon_i - \epsilon_j)(H)E_{i,j}$$

Now we have a gradation of \mathfrak{g} with respect to \mathfrak{h} given by

$$(2.8.2) \quad \mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{g}_\lambda \quad \mathfrak{g}_\lambda = \{A \in \mathfrak{g} \mid [H, A] = \lambda(H)A \forall H \in \mathfrak{h}\}$$

If $I \subset \mathfrak{g}$ is an ideal then it must be graded with respect to (2.8.2) (This is an easy consequence of the fact that the Vandermonde matrix is invertible). So we have

$$(2.8.3) \quad I = \bigoplus_{\lambda} I \cap \mathfrak{g}_\lambda$$

We claim that if I is proper then $I \cap \mathfrak{h} = \{0\}$. In fact, if this is not the case let $H \in I \cap \mathfrak{h}$.

Assumption (*): Suppose that we can find $i \neq j$ such that $H_{ii} \neq H_{jj}$.

Then we have $0 \neq [H, E_{ij}] \propto E_{ij} \in I$, hence $E_{i,j} \in I$. Now we have

$$(2.8.4) \quad [E_{i,j}, E_{j,i}] = \begin{cases} E_{ii} - E_{jj} & i, j > m \text{ or } i, j < m \\ E_{ii} + E_{jj} & \text{otherwise} \end{cases}$$

is in I . Clearly the diagonal matrices in (2.8.4) generate \mathfrak{h} so $\mathfrak{h} \subset I$ and from here is easy to see that $I = \mathfrak{g}$ a contradiction so we have proved that $I \cap \mathfrak{h} = 0$.

But now let $I \subset \mathfrak{g}$ be a proper ideal. Let $0 \neq A \in I$ be any element. Then by the gradation (2.8.3) we can write $A = \sum A_\lambda$ where only a finite number of the A_λ are different from 0 and each $A_\lambda \in I$. Now it is clear that there exist i, j such that $E_{i,j} \in I$ and from (2.8.4) we derive $I \cap \mathfrak{h} \neq 0$ which is absurd, hence $I = 0$ and \mathfrak{g} is simple.

Note that the only assumption that we have made is (*), but clearly if we cannot find such i, j then H is a multiple of the identity matrix, and this can only happen

in the $m = n$ case. The above analysis shows that I is either $\{0\}$, \mathfrak{g} or $\mathbb{F}I$, hence I is a maximal ideal and $\mathfrak{sl}(m|n)/I$ is simple.

We refer to [2, prop. 2.1.2] for further readings on Lie superalgebras. \square

2.9. Definition. Let \mathfrak{g} be a Lie superalgebra. A pair $(a(z), b(w))$ of formal \mathfrak{g} -valued formal distributions is called **local** if

$$(2.9.1) \quad [a(z), b(w)] := \sum_{m,n} [a_{(m)}, b_{(n)}] z^{-1-m} w^{-1-n} \in \mathfrak{g}[[z^{\pm 1}, w^{\pm 1}]]$$

is local.

By the decomposition theorem 1.6 we have for a local pair

$$(2.9.2) \quad [a(z), b(w)] = \sum_{j \in \mathbb{Z}_+} (a(w)_{(j)} b(w)) \frac{\partial_w^j \delta(z, w)}{j!},$$

where

$$a(w)_{(j)} b(w) = \text{Res}_z (z - w)^j [a(z), b(w)]$$

and substituting (1.2.5b) in (2.9.2) and comparing coefficients we get

$$[a_{(m)}, b_{(n)}] = \sum_{j \in \mathbb{Z}_+} \binom{m}{j} (a_{(j)} b)_{(m+n-j)}.$$

2.10. Definition. The λ -**bracket** of two \mathfrak{g} -valued formal distributions is defined by

$$(2.10.1) \quad \begin{aligned} [a_\lambda b] &= F_{z,w}^\lambda [a(z), b(w)] \\ &= \text{Res}_z e^{\lambda(z-w)} [a(z), b(w)] \\ &= \sum \frac{\lambda^j}{j!} \text{Res}_z (z - w)^j [a(z), b(w)] \\ &= \sum \frac{\lambda^j}{j!} (a_{(j)} b) \end{aligned}$$

2.11. Remark. From the last equation we can guess that the λ -bracket of a local pair is important because it gives the generating function for the products $(a_{(j)} b)$.

Note also that differentiating the expansion (1.1.3) we get $(\partial a)_{(n)} = -n a_{(n-1)}$.

2.12. Proposition. *The following are properties of the λ -bracket:*

- (1) (*Sesquilinearity*) $[\partial a_\lambda b] = -\lambda [a_\lambda b]$, $[a_\lambda \partial b] = (\partial + \lambda) [a_\lambda b]$.
- (2) (*Skew commutativity*) $[b_\lambda a] = -p(a, b) [a_{-\lambda - \partial} b]$ if (a, b) is a local pair.
- (3) $[a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda + \mu} c] + p(a, b) [b_\mu [a_\lambda c]]$.

Proof. The first equality follows from the following equation:

$$[\partial a_\lambda b] = F_{z,w}^\lambda [\partial_z a(z), b(w)] = F_{z,w}^\lambda \partial_z [a(z), b(w)] = -\lambda F_{z,w}^\lambda [a, b] = -\lambda [a_\lambda b]$$

where we have used the first property in proposition 2.3.

In order to prove (3) we have

$$\begin{aligned}
 [a_\lambda [b_\mu c]] &= F_{z,w}^\lambda [a(z), F_{x,w}^\mu [b(x), c(w)]] \\
 &= F_{z,w}^\lambda F_{x,w}^\mu [a(z), [b(x), c(w)]] && \text{by definition} \\
 &= F_{z,w}^\lambda F_{x,w}^\mu [[a(z), b(x)], c(w)] + \\
 &\quad + p(a, b) F_{z,w}^\lambda F_{x,w}^\mu [b(x), [a(z), c(w)]] && \text{by Ex. 2.2 (2)} \\
 &= F_{x,w}^{\lambda+\mu} F_{z,x}^\lambda [[a(z), b(x)], c(w)] + p(a, b) [b_\mu [a_\lambda c]] && \text{by prop 2.3 (3)} \\
 &= [[a_\lambda b]_{\lambda+\mu} c] + p(a, b) [b_\mu [a_\lambda c]]
 \end{aligned}$$

□

Exercise 2.5. Prove the second equality in (1) above and property (2).

Proof. For property (1) we have

$$\begin{aligned}
 [a_\lambda \partial b] &= F_{z,w}^\lambda [a(z), \partial_w b(w)] \\
 &= F_{z,w}^\lambda \partial_w [a(z), b(w)] \\
 &= -[\partial_w, F_{z,w}^\lambda] [a(z), b(w)] + \partial_w F_{z,w}^\lambda [a(z), b(w)] \\
 &= \lambda [a_\lambda b] + \partial_w [a_\lambda b]
 \end{aligned}$$

as we wanted. To prove (2) in the proposition we have again

$$\begin{aligned}
 [b_\lambda a] &= F_{z,w}^\lambda [b(z), a(w)] \\
 &= -p(a, b) F_{z,w}^\lambda [a(w), b(z)] \\
 &= -p(a, b) F_{z,w}^{-\lambda-\partial_w} [a(z), b(w)] \\
 &= -p(a, b) [a_{-\lambda-\partial} b]
 \end{aligned}$$

□

2.13. Definition. Let \mathcal{R} be a $\mathbb{C}[\partial]$ -module endowed with a \mathbb{C} linear map

$$\mathcal{R} \otimes \mathcal{R} \rightarrow \mathbb{C}[\lambda] \otimes \mathcal{R},$$

called λ -bracket and denoted $[a_\lambda b]$ such that properties (1)-(3) hold. Then \mathcal{R} is called a **Conformal Lie Algebra**

3. LIE CONFORMAL ALGEBRAS

Let us recall the definition of **Lie conformal algebra** given in the previous lecture.

3.1. Definition. A $\mathbb{C}[\partial]$ -module \mathcal{R} is called a Lie conformal super-algebra if it is equipped with a \mathbb{C} -bilinear map

$$(3.1.1) \quad [\]_\lambda : \mathcal{R} \otimes \mathcal{R} \mapsto \mathbb{C}[\lambda] \otimes \mathcal{R}$$

satisfying the following equations:

$$\begin{aligned}
 (3.1.2) \quad & [\partial a_\lambda b] = -\lambda [a_\lambda b], \\
 & [a_\lambda \partial b] = (\partial + \lambda) [a_\lambda b], \\
 & [b_\lambda a] = -p(a, b) [a_{(-\lambda-\partial)} b], \\
 & [a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + p(a, b) [b_\mu [a_\lambda c]].
 \end{aligned}$$

The map (3.1.1) is called a λ -**bracket**.

In what follows we will sometimes abusively omit the prefix “super-” referring to superalgebras.

Now we are going to discuss a relation between Lie conformal algebras and the so-called **formal distribution Lie superalgebras**.

Relation of Lie conformal algebras to formal distribution Lie superalgebras.

3.2. Definition. A **formal distribution Lie superalgebra** $(\mathfrak{g}, \mathcal{R})$ is a pair where \mathfrak{g} is a Lie superalgebra and \mathcal{R} is a space of \mathfrak{g} -valued formal distributions which are pairwise local, closed under all j -th products, invariant under ∂_z and the coefficients of all distributions from \mathcal{R} span the whole Lie algebra \mathfrak{g} . $(\mathfrak{g}, \mathcal{R})$ is said to be in E -relation to $(\mathfrak{g}_1, \mathcal{R}_1)$ if there exists a surjective homomorphism $\mathfrak{g} \rightarrow \mathfrak{g}_1$ inducing an isomorphism $\mathcal{R} \cong \mathcal{R}_1$. The equivalence relation between formal distribution Lie superalgebras is obtained by extending this E -relation by symmetry and transitivity.

Last time we proved that for any such pair $(\mathfrak{g}, \mathcal{R})$ \mathcal{R} is a Lie conformal superalgebra, where $\partial = \partial_z$ and $[a_\lambda b] = F_{z,w}^\lambda[a(z), b(w)]$. Now we might ask if for any Lie conformal algebra there exists a corresponding Lie superalgebra.

3.3. Proposition. *The map $(\mathfrak{g}, \mathcal{R}) \rightarrow \mathcal{R}$ establishes a bijection between equivalence classes of formal distribution Lie superalgebras and isomorphism classes of Lie conformal algebras.*

Proof.

Given a Lie superalgebra we know how to construct a Lie conformal algebra. Conversely, given a Lie conformal algebra \mathcal{R} , construct a Lie superalgebra $\text{Lie}(\mathcal{R})$ whose vector space is a quotient

$$(3.3.1) \quad \text{Lie}(\mathcal{R}) = \mathcal{R}[t, t^{-1}]/I$$

where I is a subspace spanned by the elements $\{(\bar{\partial}a)_n + n\bar{a}_{n-1} | n \in \mathbb{Z}\}$, where

$$\bar{a}_n = at^n, \quad a \in \mathcal{R}.$$

To define a Lie bracket on space (3.3.1) we introduce the following algebra structure¹ on $\mathcal{R}[t, t^{-1}]$

$$(3.3.2) \quad [\bar{a}_m, \bar{b}_n] = \sum_{j \in \mathbb{Z}_+} \binom{m}{j} \overline{(a_{(j)}b)_{m+n-j}}.$$

The following exercise shows that (3.3.2) induces an algebra structure on quotient space (3.3.1)

Exercise 3.1. *Prove that I is an ideal in the algebra $(\mathcal{R}[t, t^{-1}], [,])$.*

Proof. We mention that the first two equations in (3.1.2) are equivalent to the following identities for j -brackets

$$(3.3.3) \quad (\partial a_{(j)}b) = -j(a_{(j-1)}b), \quad (a_{(j)}\partial b) = \partial(a_{(j)}b) + j(a_{(j-1)}b).$$

¹Notice that the algebra we get on $\mathcal{R}[t, t^{-1}]$ is neither Lie nor associative.

Using equations (3.3.3) we easily show that I is indeed an ideal in $\mathcal{R}[t, t^{-1}]$. We just mention that if we substitute an element of I as the first argument in (3.3.2) we get zero identically

$$\begin{aligned}
 & [(\overline{\partial a})_m + m\bar{a}_{m-1}, \bar{b}_n] = \\
 & = \sum_{j \geq 0} \binom{m}{j} (\partial a_{(j)} b) t^{m+n-j} + m \sum_{k \geq 0} \binom{m-1}{k} (a_{(k)} b) t^{m+n-k-1} = \\
 & = - \sum_{j > 0} j \binom{m}{j} (a_{(j-1)} b) t^{m+n-j} + \sum_{k \geq 0} (m-k) \binom{m}{k} (a_{(k)} b) t^{m+n-(k+1)} \\
 & = - \sum_{j \geq 0} (m-j) \binom{m}{j} (a_{(j)} b) t^{m+n-j-1} + \sum_{k \geq 0} (m-k) \binom{m}{k} (a_{(k)} b) t^{m+n-(k+1)} \\
 & = 0.
 \end{aligned}$$

If we instead substitute an element of I as the second argument in (3.3.2) we get a non-zero sum belonging to I

$$\begin{aligned}
 & [\bar{a}_m, \overline{(\partial b)}_n + n\bar{b}_{n-1}] = \sum_{j \geq 0} \binom{m}{j} (a_{(j)} \partial b) t^{m+n-j} + n \sum_{k \geq 0} \binom{m}{k} (a_{(k)} b) t^{m+n-k-1} = \\
 & = \sum_{j \geq 0} \binom{m}{j} (\partial(a_{(j)} b) + j(a_{(j-1)} b)) t^{m+n-j} + n \sum_{k \geq 0} \binom{m}{k} (a_{(k)} b) t^{m+n-k-1} \\
 & = \dots - \sum_{j \geq 0} \binom{m}{j} (m+n-j) (a_{(j)} b) t^{m+n-j-1} + \sum_{j > 0} \binom{m}{j} j (a_{(j-1)} b) t^{m+n-j} + \\
 & \quad + \sum_{k \geq 0} n \binom{m}{k} (a_{(k)} b) t^{m+n-k-1} \\
 & = \dots + \sum_{j > 0} \binom{m}{j} j (a_{(j-1)} b) t^{m+n-j} - \sum_{j \geq 0} \binom{m}{j} (m-j) (a_{(j)} b) t^{m+n-j-1} \in I,
 \end{aligned}$$

where \dots stands for some elements of I . \square

3.4. Remark. The other two equations in (3.1.2) imply that $\mathcal{R}[t, t^{-1}]/I$ is a Lie algebra. To prove this one can either do all the calculations or to mention that we defined the algebra $\text{Lie}(\mathcal{R})$ in such a way that the anti-symmetry property and the Jacobi identity for $\text{Lie}(\mathcal{R})$ are equivalent to the last two equations in (3.1.2).

We define a Lie conformal algebra corresponding to $\text{Lie}(\mathcal{R})$ as the set

$$(3.4.1) \quad \bar{\mathcal{R}} = \left\{ \sum_{n \in \mathbb{Z}} a t^n z^{-n-1} \mid a \in \mathcal{R} \right\}.$$

By definition there is a one-to-one correspondence between elements of $\bar{\mathcal{R}}$ and elements of \mathcal{R} given by the map

$$(3.4.2) \quad a \in \mathcal{R} \mapsto \bar{a}(z) = \sum_{n \in \mathbb{Z}} a t^n z^{-n-1} \in \bar{\mathcal{R}}.$$

An obvious calculation shows that

$$\partial_z \bar{a}(z) = \overline{(\partial a)}(z) \text{ up to elements of } I,$$

the elements of $\bar{\mathcal{R}}$ are pairwise local

$$[\bar{a}(z), \bar{b}(w)] = \sum_{j \in \mathbb{Z}_+} \overline{(a_{(j)}b)}(w) \frac{\partial^j}{j!} \delta(z, w)$$

and the Lie conformal algebras \mathcal{R} and $\bar{\mathcal{R}}$ are isomorphic.

Now we have to show that if \mathcal{R} is a Lie conformal algebra corresponding to a formal distribution Lie superalgebra $(\mathfrak{g}, \mathcal{R})$ then $(\mathfrak{g}, \mathcal{R})$ is isomorphic to $(\text{Lie}\mathcal{R}, \bar{\mathcal{R}})$ as a formal distribution Lie superalgebra. For this we define a map $\phi : \text{Lie}\mathcal{R} \mapsto \mathfrak{g}$

$$(3.4.3) \quad \phi(at^m) = \text{Res}_z az^m.$$

The following calculation

$$\phi(\partial at^m + mat^{m-1}) = \text{Res}_z((\partial_z a)z^m) + \text{Res}_z(maz^{m-1}) = -ma_{m-1} + ma_{m-1} = 0$$

shows that the map ϕ is defined correctly. By definition of the formal distribution Lie superalgebra ϕ is surjective and induces isomorphism of the associated Lie conformal algebras. By construction of the Lie bracket (3.3.2) in $\text{Lie}\mathcal{R}$, ϕ is homomorphism of Lie algebras.

Thus the proposition follows.

3.5. Definition. An ideal $I \subset \mathfrak{g}$ where $(\mathfrak{g}, \mathcal{R})$ is a formal distribution Lie superalgebra, is called **irregular** if all the coefficients of $a(z) \in \mathcal{R}$ lie in I if and only if $a(z) = 0$.

It follows from the proof of prop. 3.3 that $(\text{Lie}(\mathcal{R}), \bar{\mathcal{R}})$ is the maximal formal distribution Lie superalgebra corresponding to \mathcal{R} , in the sense that any other is a quotient of the former by an irregular ideal.

3.6. Example (Virasoro Algebra). It is a Lie algebra with a basis $\{L_m, m \in \mathbb{Z}, C\}$ and the following commutation relations

$$(3.6.1) \quad [L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n} \frac{m^3 - m}{12} C,$$

where C is a central element.

This algebra is a central extension of the Lie algebra of vector fields on \mathbb{C}^\times . Namely, if we set $\bar{L}_m = -z^{m+1}\partial_z$ then we get the same formula as above but without the central element, i. e. $[\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n}$.

We are going to construct formal distributions valued in this algebra. For this we set

$$L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}.$$

Note the unusual exponent $-n-2$, hence $L(n) = L_{n-1}$. The relations (3.6.1) are equivalent to

$$(3.6.2) \quad [L(z), L(w)] = \partial_w L(w) \delta(z, w) + 2L(w) \partial_w \delta(z, w) + \frac{C}{12} \partial_w^3 \delta(z, w).$$

Hence $\{L(z), C\}$ is a local family and in terms of λ -brackets (3.6.2) is equivalent to

$$[L_\lambda L] = (\partial + 2\lambda)L + \frac{\lambda^3}{12}C, \quad [C_\lambda L] = 0.$$

In other words

$$L_{(0)}L = \partial L, \quad L_{(1)}L = 2L, \quad L_{(3)}L = \frac{C}{2}, \quad L_{(j)}C = 0, \quad L_{(j)}L = 0 \text{ otherwise.}$$

Using the λ -bracket we can recover the relations (3.6.1)

$$\begin{aligned} [L_{(m)}, L_{(n)}] &= \sum_{j \in \mathbb{Z}_+} \binom{m}{j} (L_{(j)}L)_{(m+n-j)} \\ &= (L_{(0)}L)_{(m+n)} + m(L_{(1)}L)_{(m+n-1)} + \binom{m}{3} \left(\frac{C}{2}\right)_{(m+n-3)} \\ &= (\partial L)_{(m+n)} + 2mL_{(m+n-1)} + \frac{m(m-1)(m-2)}{12}C_{(m+n-3)} \\ &= (m-n)L_{(m+n-1)} + \frac{m(m-1)(m-2)}{12}C_{(m+n-3)} \\ &= (m-n)L_{(m+n-1)} + \delta_{m+n,-2} \frac{m(m-1)(m-2)}{12}C. \end{aligned}$$

The latter gives (3.6.1) after an obvious shift $L_{(n+1)} = L_n$.

The corresponding Lie conformal algebra is given by

$$\text{Vir} = \mathbb{C}[\partial]L + \mathbb{C}C, \quad \partial C = 0.$$

Moreover, since in this case the map (3.4.3) is an isomorphism from $\text{Lie}(\text{Vir})$ to (3.6.1) the *Virasoro* algebra is the maximal formal distribution Lie algebra corresponding the Lie conformal algebra Vir .

Exercise 3.2. *Prove that*

$$\mathcal{R} = \oplus_{i \in I} \mathbb{C}[\partial]a^i + \mathbb{C}C, \quad \partial C = 0$$

as a $\mathbb{C}[\partial]$ -module if and only if $\{a_n^i, C_{-1} \mid i \in I, n \in \mathbb{Z}\}$ form a basis of $\text{Lie}(\mathcal{R})$. This characterizes the maximal formal distribution Lie algebra associated to \mathcal{R} .

Proof. \Leftarrow . If $\{a_n^i, C_{-1} \mid n \in \mathbb{Z}\}$ form a basis of $\text{Lie}(\mathcal{R})$ then $\bar{\mathcal{R}} \cong \mathcal{R}$ is generated by elements $a^i(z) = \sum_{n \in \mathbb{Z}} a_n^i z^{-n-1}$ and C by applying ∂ . To prove that the $\bar{\mathcal{R}}$ is freely generated by the action ∂ we mention that

$$\text{Res}_z a^i(z) = a_0^i$$

and

$$\text{Res}_z \frac{\partial_z^k}{(k-1)!} a^i(z) = a_{-k}^i$$

for $k > 0$. Hence if $\bar{\mathcal{R}}$ is not freely generated by ∂ the elements a_n^i would be linearly dependent. The latter contradicts to the assumption that a_n^i form a basis in $\text{Lie}\bar{\mathcal{R}}$ and the implication \Leftarrow is proved.

\Rightarrow . Let now

$$\mathcal{R} = \oplus_{i \in I} \mathbb{C}[\partial]a^i + \mathbb{C}C, \quad \partial C = 0$$

as a $\mathbb{C}[\partial]$ -module. Then the elements a_n^i and C_{-1} obviously form a generating set of $\text{Lie } \mathcal{R}$ since $\forall k > 0$

$$\frac{1}{k!}(\partial^k a^i)_n = (-1)^k \binom{n}{k} a_{n-k}^i$$

and $\forall k \neq -1$ $C_k = Ct^k = -(\partial C)t^{k+1}/(k+1) = 0$.

If some finite sum

$$\sum_{i,k} \alpha_i^k a_k^i = 0, \quad \alpha_i^k \in \mathbb{C}$$

then

$$\text{Res}_z \sum_{i,k} \alpha_i^k z^k a^i(z) = 0.$$

Hence

$$\bar{\mathcal{R}} \neq \oplus_{i \in I} \mathbb{C}[\partial]a^i + \mathbb{C}C, \quad \partial C = 0,$$

and the statement follows. \square

3.7. Example. Neveu-Schwarz (NS) Lie superalgebra. It is a Lie conformal superalgebra whose $\mathbb{C}[\partial]$ -module is

$$\text{NS} = \mathbb{C}[\partial]L \oplus \mathbb{C}[\partial]G \oplus \mathbb{C}C, \quad \partial C = 0,$$

where L and C are even and G is odd. The respective λ -brackets are defined as

$$(3.7.1) \quad \begin{aligned} [L_\lambda L] &= (\partial + 2\lambda)L + \frac{\lambda^3}{12}C \\ [L_\lambda G] &= \left(\partial + \frac{3}{2}\lambda\right)G \\ [G_\lambda G] &= L + \frac{\lambda^2}{6}C \\ [C_\lambda \cdot] &= 0 \end{aligned}$$

Exercise 3.3. Check that this is a Lie conformal superalgebra. Write

$$\begin{aligned} L(z) &= \sum_{n \in \mathbb{Z}} L_n z^{-n-2}, \\ G(z) &= \sum_{n \in \frac{1}{2} + \mathbb{Z}} G_n z^{-n-3/2} \end{aligned}$$

and derive all the brackets.

Proof. Since λ -brackets (3.7.1) are defined on generators of $\mathbb{C}[\partial]$ -module we have to check only the last two relations in (3.1.2). Since $\partial C = 0$ we get

$$\begin{aligned} [L_\lambda L] &= (\partial + 2\lambda)L + \frac{\lambda^3}{12}C = -(\partial + 2(-\lambda - \partial))L - \frac{(-\lambda - \partial)^3}{12}C = -[L_{-\lambda - \partial}L], \\ [G_\lambda G] &= L + \frac{\lambda^2}{6}C = L + \frac{(-\lambda - \partial)^2}{6}C = [G_{-\lambda - \partial}G], \end{aligned}$$

and the analogous relation for $[L_\lambda G]$ is just a definition of the λ -bracket between G and L

$$[G_\lambda L] = \left(\frac{1}{2}\partial + \frac{3}{2}\lambda\right)G$$

Thus we are left with Jacobi identity. We remark that it suffices to check it for one permutation of any three elements. While this is hard to prove directly, it follows from the symmetry of Jacobi in a Lie superalgebra. Indeed suppose we know the first three relations hold in our conformal algebra, and Jacobi holds for some permutation of any three elements. We can associate a (not necessarily Lie) algebra A to our conformal superalgebra. The associated bracket will satisfy antisymmetry and Jacobi for all elements, hence A is in fact a Lie Algebra, and Jacobi must hold for all elements in our conformal algebra as well.

In the sector $[G.[G.G]]$ the Jacobi identity holds for the following lines of computations

$$\begin{aligned} [G_\lambda[G_\mu G]] &= [[G_\lambda G]_{\lambda+\mu} G] - [G_\mu[G_\lambda G]] \\ [G_\lambda L] &= [L_{\lambda+\mu} G] - [G_\mu L] \\ \frac{\partial + 3\lambda}{2} G &= (\partial + \frac{3}{2}(\lambda + \mu))G - \frac{1}{2}(\partial + 3\mu)G \end{aligned}$$

The following calculations prove the Jacobi identity in the sector $[L.[L.G]]$

$$\begin{aligned} [L_\lambda[L_\mu G]] - [L_\mu[L_\lambda G]] &= (\partial + \lambda + \frac{3}{2}\mu)[L_\lambda G] - (\mu \leftrightarrow \lambda) \\ &= (\lambda - \mu)\partial G + \frac{3}{2}(\lambda^2 - \mu^2)G \\ &= [(\partial + 2\lambda)L_{\lambda+\mu} G] \\ &= [[L_\lambda L]_{\lambda+\mu} G]. \end{aligned}$$

The Jacobi identity in the sector $[L.[G.G]]$ follows from

$$\begin{aligned} [(\partial + \frac{3}{2}\lambda)G_{\lambda+\mu} G] + [G_\mu(\partial + \frac{3}{2}\lambda)G] &= (\frac{\lambda}{2} - \mu)[G_{\lambda+\mu} G] + (\partial + \mu + \frac{3\lambda}{2})[G_\mu G] \\ &= (\partial + 2\lambda)L + \frac{\lambda^3}{12}C. \end{aligned}$$

We omit the proof of the Jacobi identity in the sector $[L.[L.L]]$ since the λ -bracket $[L_\lambda L]$ is the same as in the conformal Lie superalgebra Vir .

Thus the Neveu-Schwarz Lie superalgebra is indeed a Lie conformal superalgebra. Using λ -brackets (3.7.1) we get

$$\begin{aligned} [L(z), L(w)] &= \partial_w L(w)\delta(z, w) + 2L(w)\partial_w \delta(z, w) + \frac{C}{12}\partial_w^3 \delta(z, w) \\ (3.7.2) \quad [L(z), G(w)] &= \partial_w G(w)\delta(z, w) + \frac{3}{2}G(w)\partial_w \delta(z, w) \\ [G(z), G(w)] &= L(w)\delta(z, w) + \frac{C}{6}\partial_w^2 \delta(z, w) \end{aligned}$$

The brackets of the corresponding formal distribution Lie superalgebra are of the form

$$\begin{aligned} [L_m, L_n] &= (m - n)L_{m+n} + \delta(z, w)_{n, -m} \frac{m^3 - m}{12} C \\ (3.7.3) \quad [L_m, G_n] &= \left(\frac{m}{2} - n\right) G_{m+n} \\ [G(z), G(w)] &= L_{m+n} + \frac{C}{6} \left(m^2 - \frac{1}{4}\right) \delta(z, w)_{n, -m} \end{aligned}$$

where C is a central element. \square

3.8. Definition. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a (super)space. A **(super)symmetric bilinear form** $(\cdot|\cdot)$ is a map

$$(\cdot|\cdot) : V \otimes V \mapsto \mathbb{C}$$

which is zero on the subspaces $V_{\bar{0}} \otimes V_{\bar{1}}$ and $V_{\bar{1}} \otimes V_{\bar{0}}$. It is symmetric on the subspace $V_{\bar{0}} \otimes V_{\bar{0}}$ and antisymmetric on the subspace $V_{\bar{1}} \otimes V_{\bar{1}}$.

3.9. Definition. If a (super)space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is endowed with an action of a Lie (super)algebra \mathfrak{g} then a bilinear form $(\cdot|\cdot)$ is called **invariant** if for any three homogeneous elements $X \in \mathfrak{g}$, $v, w \in V$

$$(Xv, w) + p(v, X)(v, Xw) = 0.$$

3.10. Definition. If the (super)space is $V = \mathfrak{g}$ and the action of \mathfrak{g} on V is the adjoint action then an invariant bilinear form $(\cdot|\cdot)$ on \mathfrak{g} is called adjoint invariant.

3.11. Example (Kac-Moody Affinization). Let \mathfrak{g} be a finite dimensional Lie (super)algebra and $(\cdot|\cdot)$ be a (super)symmetric adjoint invariant bilinear form on \mathfrak{g} . Consider the associated loop algebra

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] = \mathfrak{g}[t, t^{-1}],$$

and define a central extension of $\tilde{\mathfrak{g}}$

$$\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] + \mathbb{C}K$$

with the following commutation relations

$$(3.11.1) \quad [at^m, bt^n] = [a, b]t^{m+n} + m\delta_{m, -n}(a|b)K,$$

where K is a central element.

The supersymmetry property and the adjoint invariance of the pairing $(\cdot|\cdot)$ implies that the bracket (3.11.1) defines a Lie superalgebra.

Consider now the following formal distributions called **currents**: for $a \in \mathfrak{g}$ let $a(z) = \sum_{n \in \mathbb{Z}} (at^n)z^{-1-n}$.

Exercise 3.4. Show that (3.11.1) is equivalent to

$$(3.11.2) \quad [a(z), b(w)] = [a, b](w)\delta(z, w) + K(a|b)\partial_w\delta(z, w), \quad a, b \in \mathfrak{g}.$$

Proof. This is shown by the following straightforward calculation

$$\begin{aligned} [a(z), b(w)] &= \sum_{m, n \in \mathbb{Z}} [at^m, bt^n]z^{-1-m}w^{-1-n} \\ &= \sum_{m, n \in \mathbb{Z}} [a, b]t^{m+n}w^{-1-m-n}z^{-1-m}w^m + \sum_{m \in \mathbb{Z}} K(a|b)mz^{-1-m}w^{m-1} \\ &= [a, b](w)\delta(z, w) + K(a|b)\partial_w\delta(z, w) \end{aligned}$$

The corresponding Lie conformal algebra is

$$\begin{aligned} \text{Curg} &= \mathbb{C}[\partial]\mathfrak{g} \oplus \mathbb{C}K, \\ [a_\lambda b] &= [a, b] + (a|b)\lambda K, \quad a, b \in \mathfrak{g}, \\ [K_\lambda a] &= 0. \end{aligned}$$

If \mathfrak{g} is commutative then Curg is called the **Conformal algebra of free bosons**. \square

3.12. Example (Clifford Affinization). Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a (super)space with a skew-supersymmetric bilinear form $\langle \cdot, \cdot \rangle$ (this means that the form is skew-symmetric in the even part, symmetric in the odd part and both parts are orthogonal).

The **Clifford Affinization** is the Lie superalgebra $\hat{A} = A[t, t^{-1}] + \mathbb{C}K$ with the following Lie bracket

$$(3.12.1) \quad [\varphi t^m, \psi t^n] = K \delta_{m, -1-n} \langle \varphi, \psi \rangle, \quad \varphi, \psi \in A.$$

The skew-symmetry of the bracket follows directly from the definition of the skew-supersymmetric bilinear form $\langle \cdot, \cdot \rangle$. The Jacobi identity of bracket (3.12.1) is trivial.

The free fermions are defined as $\varphi(z) = \sum_{m \in \mathbb{Z}} (\varphi t^m) z^{-m-1}$ where $\varphi \in A$. Then relations (3.12.1) are equivalent to

$$[a(z), b(w)] = K \langle a, b \rangle \delta(z, w)$$

and the corresponding Lie conformal superalgebra looks as follows.

$$F(A) = \mathbb{C}[\partial]A + \mathbb{C}K, \quad \partial K = 0, \quad [a_\lambda b] = \langle a, b \rangle K, \quad a, b \in A.$$

4. NORMALLY ORDERED PRODUCTS

Last time we constructed $\text{Lie}(\mathcal{R})$ for any Conformal Lie algebra \mathcal{R} . In the next exercise recall that a derivation T of a superalgebra is defined to be an homogeneous endomorphism of parity $p(T)$ such that

$$T(ab) = T(a)b + (-1)^{p(a)p(T)} aT(b)$$

Exercise 4.1. $\text{Lie}(\mathcal{R})$ has the following well-defined even derivation.

$$T(\bar{a}_n) = -n\bar{a}_{n-1} \quad (n \in \mathbb{Z})$$

Proof. It is clear that $T(I) = 0$ since $T((\bar{\partial}a)_n + n\bar{a}_{n-1}) = -n(\bar{\partial}a)_{n-1} + (n-1)\bar{a}_{n-2} \in I$. The rest of the exercise is just the definition of the lie bracket in $\text{Lie}(\mathcal{R})$:

$$\begin{aligned} [T(a_m), b_n] + [a_m, T(b_n)] &= -m[a_{m-1}, b_n] - n[a_m, b_{n-1}] = \\ &= -\sum_{j \geq 0} \binom{m-1}{j} m (a_{(j)} b)_{(m+n-j-1)} - \sum_{j \geq 0} \binom{m}{j} n (a_{(j)})_{(m+n-j-1)} \\ &= -\sum_{j \geq 0} \binom{m}{j} (m-j) (a_{(j)} b)_{(m+n-j-1)} - \sum_{j \geq 0} \binom{m}{j} n (a_{(j)} b)_{(m+n-j-1)} \\ &= -\sum_{j \geq 0} \binom{m}{j} (m+n-j) (a_{(j)} b)_{(m+n-j-1)} \\ &= T \left(\sum_{j \geq 0} \binom{m}{j} (a_{(j)} b)_{(m+n-j)} \right) \\ &= T([a_m, b_n]) \end{aligned}$$

□

Now we cite for completeness a Theorem without proof

4.1. Theorem. *A simple finitely generated, as $\mathbb{C}[\partial]$ -module, Conformal Lie algebra is isomorphic to either $\text{Vir}/\mathbb{C}\mathbb{C}$ or to $\text{Curg}/\mathbb{C}K$, where \mathfrak{g} is a simple finite dimensional Lie algebra.*

Operator Product Expansion.

4.2. Definition. The **Operator Product Expansion** for a local pair of \mathfrak{g} -valued formal distributions is defined to be

$$(4.2.1) \quad [a(z), b(w)] = \sum (a_{(j)}b)(w) \partial_w^j \delta(z, w) / j!$$

where

$$a(w)_{(j)}b(w) = \text{Res}_z [a(z), b(w)] (z - w)^j, \quad j \in \mathbb{Z}_+$$

Physicists usually write this OPE as

$$a(z)b(w) = \sum_{j \in \mathbb{Z}} \frac{a(w)_{(j)}b(w)}{(z - w)^{j+1}}$$

writing only what is called as the “regular” part of the OPE

For calculations this notation is very useful and this can be seen in the following exercise:

Exercise 4.2. *Prove that for any local pair $a(z), b(w)$ the following equation holds:*

$$a(z)b(w) = \sum_{j \in \mathbb{Z}_+} a(w)_{(j)}b(w) i_{z,w} \frac{1}{(z - w)^{j+1}} + : a(z)b(w) :$$

And the ordered product $: a(z)b(w) :$ is defined below

Proof. Recall from (1.2.5a) that:

$$\frac{\partial_w^j}{j!} \delta(z, w) = i_{z,w} \frac{1}{(z - w)^{j+1}} - i_{w,z} \frac{1}{(z - w)^{j+1}}$$

replacing this expression in (4.2.1) and taking only the negative powers of z we get (using (4.3.1)):

$$[a(z)_-, b(w)] = \sum_{j \geq 0} (a_{(j)}b)(w) i_{z,w} \frac{1}{(z - w)^{j+1}}$$

Hence we have:

$$\sum_{j \geq 0} (a_{(j)}b)(w) i_{z,w} \frac{1}{(z - w)^{j+1}} + : a(z)b(w) := a(z)b(w)$$

as we wanted. \square

4.3. Definition. Given two \mathcal{U} -valued formal distributions $a(z), b(z)$, where \mathcal{U} is an associative superalgebra, we define a \mathcal{U} -valued formal distribution in 2 indeterminates $: a(z)b(w) :$ by

$$: a(z)b(w) := a(z)_+ b(w) + p(a, b) b(w) a(z)_-$$

where we have

$$(4.3.1) \quad \begin{aligned} a(z)_+ &= \sum_{n \leq -1} a_{(n)} z^{-1-n} \\ a(z)_- &= \sum_{n \geq 0} a_{(n)} z^{-1-n} \end{aligned}$$

We remark that the partition given in (4.3.1) is the only one with the properties that

$$(4.3.2a) \quad (\partial_z a(z))_{\pm} = \partial_z(a(z)_{\pm})$$

$$(4.3.2b) \quad \text{Res}_z a(z) i_{z,w} \frac{1}{(z-w)^{n+1}} = \frac{\partial_w^n a(w)_+}{n!}$$

$$(4.3.2c) \quad \text{Res}_z a(z) i_{w,z} \frac{1}{(z-w)^{n+1}} = -\frac{\partial_w^n a(w)_-}{n!}$$

and the second and third properties are called the *Formal Cauchy Formulas*.

To prove the first Formal Cauchy Formula we calculate

$$\begin{aligned} \text{Res}_z a(z) i_{z,w} \frac{1}{(z-w)} &= \text{Res}_z \sum_n a_{(n)} z^{-1-n} z^{-1} \sum_{m \geq 0} \left(\frac{w}{z}\right)^m \\ &= \sum_{m \geq 0} a_{(-m-1)} w^m \\ &= a(w)_+ \end{aligned}$$

Differentiating both sides, we get the Cauchy Formula.

The product $: a(z)b(w) :$ is called the regular part the OPE and expanding in Taylor series we get

$$: a(z)b(w) := \sum_{j \geq 0} \frac{\partial^j a(z)b(w) :}{j!} (z-w)^j$$

And we denote

$$(4.3.3) \quad a(w)_{(-1-j)} b(w) = \frac{\partial^j a(w)b(w) :}{j!}$$

where we use the analogous formula to definition 4.3 for the normally ordered product:

$$(4.3.4) \quad \begin{aligned} : a(z)b(z) : &= a_+(z)b(z) + p(a,b)b(z)a(z)_- \\ &= \sum_{n \in \mathbb{Z}} : ab :_{(n)} z^{-1-n} \end{aligned}$$

and

$$: ab :_{(n)} = \sum_{j=-1}^{-\infty} a_{(j)} b_{(n-j-1)} + p(a,b) \sum_{j=0}^{\infty} b_{(n-j-1)} a_{(j)}$$

Completions. We shall work in the following setup: \mathfrak{g} is a Lie superalgebra with a descending \mathbb{Z} -filtration:

$$\cdots \supset \mathfrak{g}_{-1} \supset \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots$$

with $\mathfrak{g} = \cup \mathfrak{g}_i$, $\cap \mathfrak{g}_j = 0$ and $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$

4.4. Example. The Lie conformal algebra $\mathfrak{g} = \text{Vir}$ satisfies these conditions, where

$$\begin{aligned} \mathfrak{g}_j &= \sum_{i \geq j} \mathbb{C}L_i \quad \text{if } j > 0 \\ \mathfrak{g}_j &= \mathbb{C}C + \sum_{i \geq j} L_i \quad \text{if } j \leq 0 \end{aligned}$$

Exercise 4.3. Let $\mathcal{R} = \oplus \mathbb{C}[\partial]a_i + C$ where the sum is finite, prove that there exist a \mathbb{Z} -filtration as above in $\text{Lie}(\mathcal{R})$.

Proof. we know by Ex.3.2 that $\{a_n^i, C_{-1}\}_{n \in \mathbb{Z}}$ form a basis for $\text{Lie}(\mathcal{R})$. We have $\forall i, k \in I$ that there exists j_{ik} such that $a_{(j)}^i a^k = 0$ for all $j \geq j_{ik}$. Now taking $j = \max j_{ik}$ and defining $(\text{Lie}(\mathcal{R}))_n$ to be $\sum_{k \geq n-j} a_k^i$ we are done. \square

Let \mathcal{U} be an enveloping algebra of \mathfrak{g} such that $\cap \mathcal{U}\mathfrak{g}_j = 0$. For example $\mathcal{U} = \mathcal{U}(\mathfrak{g})$ satisfies the condition, where $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of \mathfrak{g} . We construct a completion $\mathcal{U}^{\text{comp}}$ of \mathcal{U} consisting of infinite series $\sum_{i \in I} u_i$ such that for each N , all but finitely many of the u_i 's lie in $\mathcal{U}\mathfrak{g}_N$.

4.5. Lemma. For any $u \in \mathcal{U}$ and $N \in \mathbb{Z}$ there exists $M \in \mathbb{Z}$ such that $\mathfrak{g}_M u \subset \mathcal{U}\mathfrak{g}_N$.

Proof. We may assume that u is a product of n elements from \mathfrak{g} . We prove the lemma by induction on n . If $n = 1$ then $u \in \mathfrak{g}$ hence it lies in some member of the filtration \mathfrak{g}_s for some s , hence defining $M = \max\{N, N - s\}$ we are done, indeed

$$\mathfrak{g}_M u = [\mathfrak{g}_M, u] + u\mathfrak{g}_M \subset \mathfrak{g}_{M+s} + \mathcal{U}\mathfrak{g}_N$$

Now if $n > 1$ we write $u = u_1 u_2$, where $u_1 \in \mathfrak{g}_s$ and u_2 is a product of $n - 1$ elements. Now we apply the inductive assumption, and it goes like

$$\mathfrak{g}_M u = \mathfrak{g}_M u_1 u_2 = [\mathfrak{g}_M, u_1] u_2 \pm u_1 [\mathfrak{g}_M, u_2] + u_1 u_2 \mathfrak{g}_M$$

Now the first and the last term clearly satisfies the lemma for M sufficiently large. The rest is included in the exercise

Exercise 4.4. Complete the proof of the lemma

We need: For every $N \in \mathbb{Z}$ and $u = u_1 u_2 \cdots u_n \in \mathcal{U}$ where $u_i \in \mathfrak{g}_{s_i}$ there exists $M \in \mathbb{Z}$ such that $\mathfrak{g}_M u \subset \mathcal{U}\mathfrak{g}_N$.

We prove this by induction on n . If $n = 1$ we can take $M = N - i_1$. For $n > 1$, we have

$$[\mathfrak{g}_M, u_1 \cdots u_n] = u_1 \cdots u_{n-1} [\mathfrak{g}_M, u_n] + [\mathfrak{g}_M, u_1 \cdots u_{n-1}] u_n$$

By induction, we can select M_1 such that $[\mathfrak{g}_{M_1}, u_{n-1}] \subset \mathcal{U}\mathfrak{g}_{N-i_n}$. Then we have $[\mathfrak{g}_M, u_1 \cdots u_{n-1}] u_n \subset \mathcal{U}\mathfrak{g}_{N-i_n} u_n \subset \mathcal{U}\mathfrak{g}_N$. Now take $M = \min\{N - i_n, M_1\}$ and we are done. \square

This lemma implies that if $\sum u_i, \sum v_j \in \mathcal{U}^{\text{comp}}$ then the product $\sum_{i,j} u_i v_j$ is in $\mathcal{U}^{\text{comp}}$ as well. Indeed, fix $N \in \mathbb{Z}$ and by the lemma, $\forall v, \exists M_v$ such that $\mathfrak{g}_{M_v} v \in \mathcal{U}\mathfrak{g}_N$. All but finitely many of the v_j are in $\mathcal{U}\mathfrak{g}_N$ and for those $u_i v_j \in \mathcal{U}\mathfrak{g}_N$. There are hence finitely many $v_j \notin \mathcal{U}\mathfrak{g}_N$, say $\{v_1, \dots, v_k\}$, and for each one finitely many u_i such that $u_i v_j \notin \mathcal{U}\mathfrak{g}_N$.

4.6. Definition. An $\mathcal{U}^{\text{comp}}$ -valued formal distribution $a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-1-n}$ is called **continuous** if for each N , $a_{(n)} \in \mathcal{U}^{\text{comp}}\mathfrak{g}_N$ for $n \gg 0$.

From now on $\mathcal{U}_c[[z, z^{-1}]]$ denotes the space of all $\mathcal{U}^{\text{comp}}$ -valued formal distributions.

4.7. Proposition. The space $\mathcal{U}_c[[z, z^{-1}]]$ is closed under all j -th products and under ∂_z .

Proof. Due to Ex.4.5 it is enough to prove for normal ordered products.

$$(4.7.1) \quad :ab :_{(n)} = \sum_{j=-1}^{-\infty} a_{(j)} b_{(n-j-1)} + p(a, b) \sum_{j=0}^{\infty} b_{(n-j-1)} a_{(j)}$$

We need to check that $:ab :_{(n)} \in \mathcal{U}^{\text{comp}}$ and that $:ab :_{(n)} \in \mathcal{U}^{\text{comp}} \mathfrak{g}_N$ for n large enough but both statements follow from the lemma above. \square

Exercise 4.5. Prove the statement for $j \geq 0$.

Proof.

$$\begin{aligned} (a_{(j)}b)_{(n)} &= \text{Res}_w \text{Res}_z w^n (z-w)^j [a(z), b(w)] \\ &= \sum_{l, k, m} \text{Res}_z \text{Res}_w \binom{j}{k} (-1)^k w^{-1+n+j-k-l} z^{-1+k-m} [a_{(m)}, b_{(l)}] \\ &= \sum_{k \geq 0} (-1)^k \binom{j}{k} [a_{(k)}, b_{(n+j-k)}] \end{aligned}$$

By the lemma, each summand is in $\mathcal{U}^{\text{comp}}$ and since the sum is finite, we see that $(a_{(j)}b)_{(n)}$ is in $\mathcal{U}^{\text{comp}}$. To prove $a_{(j)}b$ is continuous, fix an N and take m such that $b_{(n)} \in \mathcal{U}^{\text{comp}} \mathfrak{g}_N$ for all $n \geq m$. Now for each $k = 0, \dots, j$, there exists n_k such that $b_{(n)} a_{(k)} \in \mathcal{U}^{\text{comp}} \mathfrak{g}_N$ whenever $n \geq n_k$. Taking $M = \max\{m, n_0, n_1, \dots, n_j\}$, we see that $(a_{(j)}b)_{(n)} \in \mathcal{U}^{\text{comp}} \mathfrak{g}_N$ for all $n \geq M$. \square

5. WICK FORMULAS

We are still working in the setup of the last lecture. It means that \mathcal{U} is a quotient of an universal enveloping (super)algebra of a \mathbb{Z} -filtered Lie (super)algebra \mathfrak{g} .

5.1. Example. Let \mathfrak{g} be a Lie (super)algebra and $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] + \mathbb{C}K$, be its Kac-Moody affinization. Here $\hat{\mathfrak{g}}$ is filtered by the degree of t and K is a central element. $\mathcal{U} = \mathcal{U}_k := \mathcal{U}(\hat{\mathfrak{g}})/(K - k)$, $k \in \mathbb{C}$ is a quotient of an universal enveloping algebra of $\hat{\mathfrak{g}}$.

Next we consider a completion of \mathcal{U} which we denote by $\mathcal{U}^{\text{comp}}$. And as in the last lecture we denote by $\mathcal{U}_c[[z, z^{-1}]]$ the space of all continuous $\mathcal{U}^{\text{comp}}$ -valued formal distributions.

Recall that for $n \in \mathbb{Z}_+$ we have defined:

$$\begin{cases} a(w)_{(n)} b(w) = \text{Res}_z (z-w)^n [a(z), b(w)] \\ a(w)_{(-1-n)} b(w) = \frac{\partial_w^n a(w) b(w)}{n!} \end{cases},$$

where the ordered product was defined in (4.3.4). Now we can give a general formula for n^{th} -products:

5.2. Proposition.

$$(5.2.1) \quad a(w)_{(n)} b(w) = \text{Res}_z \left(i_{z,w} (z-w)^n a(z) b(w) - i_{w,z} (z-w)^n p(a, b) b(w) a(z) \right)$$

Proof. For $n \geq 0$,

$$\begin{aligned}
& \operatorname{Res}_z \left(i_{z,w}(z-w)^n a(z)b(w) - i_{w,z}(z-w)^n p(a,b)b(w)a(z) \right) = \\
& = \operatorname{Res}_z \left((z-w)^n a(z)b(w) - (z-w)^n p(a,b)b(w)a(z) \right) \\
& = \operatorname{Res}_z \left((z-w)^n (a(z)b(w) - p(a,b)b(w)a(z)) \right) \\
& = \operatorname{Res}_z((z-w)^n [a(z), b(w)]) = a(w)_{(n)} b(w)
\end{aligned}$$

For $n < 0$ we use the formal Cauchy formulas (4.3.2b) and (4.3.2c):

$$\begin{aligned}
a(w)_{(-1-n)} b(w) &= \frac{:\partial_w^n a(w)b(w):}{n!} \\
&= \left((\partial_w^n a(w))_+ b(w) + p(a,b)b(w)(\partial_w^n a(w))_- \right) / n! = \\
&= (\operatorname{Res}_z a(z) i_{z,w} \frac{1}{(z-w)^{n+1}}) b(w) - b(w) (\operatorname{Res}_z a(z) i_{w,z} \frac{1}{(z-w)^{n+1}}) \\
&= \operatorname{Res}_z \left(i_{z,w}(z-w)^{-n-1} a(z)b(w) - i_{w,z}(z-w)^{-n-1} p(a,b)b(w)a(z) \right)
\end{aligned}$$

□

Exercise 5.1. Prove the following equality called sesquilinearity:

$$\begin{aligned}
(\partial a)_{(n)} b &:= \partial_w a(w)_{(n)} b(w) = -n a_{(n-1)} b \\
\partial(a_{(n)} b) &= \partial a_{(n)} b + a_{(n)} \partial b
\end{aligned}$$

Proof. Expand the first term of (5.2.1) using the formula for a derivative of a product as following:

$$\begin{aligned}
(5.2.2a) \quad \operatorname{Res}_z i_{z,w}(z-w)^n \partial_z a(z) b(w) &= \operatorname{Res}_z \partial_z \left(i_{z,w}(z-w)^n a(z) \right) b(w) \\
&= -\operatorname{Res}_z \left(\partial_z i_{z,w}(z-w)^n \right) a(z) b(w) \\
&= -\operatorname{Res}_z \left(\partial_z i_{z,w}(z-w)^n \right) a(z) b(w) \\
&= -n \operatorname{Res}_z i_{z,w}(z-w)^{n-1} a(z) b(w)
\end{aligned}$$

Similarly we get:

$$\begin{aligned}
(5.2.2b) \quad p(a,b) \operatorname{Res}_z i_{w,z}(z-w)^n b(w) \partial_z a(z) &= \\
&= -n p(a,b) \operatorname{Res}_z i_{w,z}(z-w)^{n-1} b(w) a(z)
\end{aligned}$$

Now adding (5.2.2a) and (5.2.2b) we get:

$$\begin{aligned}
(\partial a)_{(n)} b &= \operatorname{Res}_z i_{z,w}(z-w)^n \partial_z a(z) b(w) - p(a,b) \operatorname{Res}_z i_{w,z}(z-w)^n b(w) \partial_z a(z) \\
&= -n \operatorname{Res}_z i_{z,w}(z-w)^{n-1} a(z) b(w) + n p(a,b) \operatorname{Res}_z i_{w,z}(z-w)^{n-1} b(w) a(z) \\
&= -n a_{(n-1)} b
\end{aligned}$$

Now we prove that ∂ is a derivation of all n -products:

$$\begin{aligned}
 \partial (a_{(n)}b) &= \partial_w \operatorname{Res}_z (i_{z,w}(z-w)^n a(z)b(w) - i_{w,z}(z-w)^n p(a,b)b(w)a(z)) \\
 &= \operatorname{Res}_z (-ni_{z,w}(z-w)^{n-1} a(z)b(w) + i_{z,w}(z-w)^n a(z)\partial_w b(w) + \\
 &\quad + ni_{w,z}(z-w)^{n-1} p(a,b)b(w)a(z) - i_{w,z}(z-w)^n p(a,b)\partial_w b(w)a(z)) \\
 &= -n \operatorname{Res}_z (i_{z,w}(z-w)^{n-1} a(z)b(w) - i_{w,z}(z-w)^{n-1} p(a,b)b(w)a(z)) + \\
 &\quad + \operatorname{Res}_z (i_{z,w}(z-w)^n a(z)\partial_w b(w) - i_{w,z}(z-w)^n p(a,b)\partial_w b(w)a(z)) \\
 &= (\partial a)_{(n)}b + a_{(n)}(\partial b)
 \end{aligned}$$

□

Next we have the following identity for any $n \in \mathbb{Z}$:

$$(5.2.3) \quad [a_\lambda (b_{(n)}c)] = \sum_{k \in \mathbb{Z}_+} \frac{\lambda^k}{k!} [a_\lambda b]_{(n+k)} c + p(a,b) b_{(n)} [a_\lambda c]$$

Exercise 5.2. Taking $\sum_{n \in \mathbb{Z}_+} \frac{\mu^n}{n!}$ of both sides, we get the Jacobi identity for the λ -bracket given in Proposition 2.12.

Proof. The left hand side is converted to:

$$\sum_{n \in \mathbb{Z}_+} \frac{\mu^n}{n!} [a_\lambda b_{(n)}c] = [a_\lambda \sum_{n \in \mathbb{Z}_+} \frac{\mu^n}{n!} b_{(n)}c] = [a_\lambda [b_\mu c]]$$

And the right hand side is:

$$\begin{aligned}
 &\sum_{n,k \in \mathbb{Z}_+} \frac{\mu^n}{n!} \frac{\lambda^k}{k!} [a_\lambda b]_{(n+k)} c + p(a,b) \sum_{n \in \mathbb{Z}_+} \frac{\mu^n}{n!} b_{(n)} [a_\lambda c] = \\
 &= \sum_{n,k \in \mathbb{Z}_+} \binom{n+k}{n} \frac{1}{(n+k)!} \mu^n \lambda^k [a_\lambda b]_{(n+k)} c + p(a,b) [b_\mu [a_\lambda c]] \\
 &= \sum_{m \in \mathbb{Z}_+} \frac{(\mu + \lambda)^m}{m!} [a_\lambda b]_{(m)} c + p(a,b) [b_\mu [a_\lambda c]] \\
 &= [[a_\lambda b]_{\lambda+\mu} c] + p(a,b) [b_\mu [a_\lambda c]]
 \end{aligned}$$

Now equating the with the left hand side we get the Jacobi identity. □

Exercise 5.3. Replacing b by ∂b in (5.2.3) we get the same identity for n replaced by $n-1$ if $n \leq -1$.

Proof. The left hand side of (5.2.3) is converted into:

$$[a_\lambda ((\partial b)_{(n)}c)] = [a_\lambda - (nb_{(n-1)}c)] = -n[a_\lambda (b_{(n-1)}c)]$$

On the other hand the right hand side is converted into:

$$\begin{aligned}
& \sum_{k \in \mathbb{Z}_+} \frac{\lambda^k}{k!} [a_\lambda \partial b]_{(n+k)} c + p(a, b) \partial b_{(n)} [a_\lambda c] = \\
&= \sum \frac{\lambda^k}{k!} (\lambda + \partial) [a_\lambda b]_{(n+k)} c - np(a, b) b_{(n-1)} [a_\lambda c] \\
&= \sum \frac{\lambda^{k+1}}{k!} [a_\lambda b]_{(n+k)} c + \sum \frac{\lambda^k}{k!} \partial [a_\lambda b]_{(n+k)} c - np(a, b) b_{(n-1)} [a_\lambda c] \\
&= \sum \frac{\lambda^{k+1}}{k!} [a_\lambda b]_{(n+k)} c + - \sum \frac{\lambda^k}{k!} (n+k) [a_\lambda b]_{(n-1+k)} c - np(a, b) b_{(n-1)} [a_\lambda c] \\
&= -n \left(\sum \frac{\lambda^k}{k!} [a_\lambda b]_{(n-1+k)} c + p(a, b) b_{(n-1)} [a_\lambda c] \right)
\end{aligned}$$

Now equating both sides we get the result. \square

We return to (5.2.3). From the last two exercises we see that it is enough to consider the case when $n = -1$. In this case (5.2.3) reads:

$$[a_\lambda : bc :] =: [a_\lambda b] c : + \sum_{k \geq 1} \frac{\lambda^k}{k!} [a_\lambda b]_{(k-1)} c + p(a, b) : b [a_\lambda c] :$$

Rewriting the sum we get:

$$\sum_{k \in \mathbb{Z}_+} \frac{\lambda^{k+1}}{(k+1)!} [a_\lambda b]_{(k)} c = \int_0^\lambda [[a_\lambda b]_\mu c] d\mu$$

Hence we can write this case of (5.2.3) as:

$$(5.2.4) \quad [a_\lambda : bc :] =: [a_\lambda b] c : + p(a, b) : b [a_\lambda c] : + \int_0^\lambda [[a_\lambda b]_\mu c] d\mu$$

This is known as the **non-abelian Wick formula**.

Question: Is it true that sesquilinearity and identity (5.2.3) are the only identities satisfied by arbitrary elements from $\mathcal{U}_c[[z, z^{-1}]]$?

Proof. of (5.2.3): the left hand side is:

$$\begin{aligned}
(5.2.5) \quad & [a(w)_\lambda (b(w)_{(n)} c(w))] = \text{Res}_z e^{\lambda(z-w)} [a(z), (b(w)_{(n)} c(w))] \\
&= \text{Res}_z e^{\lambda(z-w)} \text{Res}_x [a(z), b(x) c(w)] i_{x,w} (x-w)^n - \\
&\quad - p(b, c) \text{Res}_z e^{\lambda(z-w)} \text{Res}_x [a(z), c(w) b(x)] i_{w,x} (x-w)^n
\end{aligned}$$

Now using the identity $[a, bc] = [a, b]c + p(a, b)b[a, c]$ we write the first term as:

$$\begin{aligned}
 I &= \operatorname{Res}_z \operatorname{Res}_x e^{\lambda(z-w)} [a(z), b(x)] c(w) i_{x,w}(x-w)^n + \\
 &\quad + p(a, b) \operatorname{Res}_z \operatorname{Res}_x e^{\lambda(z-w)} b(x) [a(z), c(w)] i_{x,w}(x-w)^n \\
 &= \operatorname{Res}_z \operatorname{Res}_x e^{\lambda(z-w)} [a(z), b(x)] c(w) i_{x,w}(x-w)^n + \\
 &\quad + p(a, b) \operatorname{Res}_x b(x) i_{x,w}(x-w)^n \operatorname{Res}_z e^{\lambda(z-w)} [a(z), c(w)] \\
 &= \operatorname{Res}_x e^{\lambda(x-w)} i_{x,w}(x-w)^n [a(x)_\lambda b(x)] c(w) + \\
 &\quad + p(a, b) \operatorname{Res}_x b(x) [a(w)_\lambda c(w)] i_{x,w}(x-w)^n \\
 &= \operatorname{Res}_x \sum_{j \geq 0} \frac{\lambda^j}{j!} i_{x,w}(x-w)^{n+j} [a(x)_\lambda b(x)] c(w) + \\
 &\quad + p(a, b) \operatorname{Res}_x b(x) [a(w)_\lambda c(w)] i_{x,w}(x-w)^n
 \end{aligned}$$

We write the second term as:

$$\begin{aligned}
 J &= -p(b, c) \operatorname{Res}_z \operatorname{Res}_x e^{\lambda(z-w)} [a(z), c(w)] b(x) i_{w,x}(x-w)^n \\
 &= -p(b, c) \operatorname{Res}_x \operatorname{Res}_z e^{\lambda(z-w)} [a(z), c(w)] b(x) i_{w,x}(x-w)^n + \\
 &\quad + p(b, c) p(a, c) \operatorname{Res}_x \operatorname{Res}_z e^{\lambda(z-w)} c(w) [a(z), b(x)] i_{w,x}(x-w)^n \\
 &= -p(b, c) \operatorname{Res}_x [a(w)_\lambda c(w)] b(x) i_{w,x}(x-w)^n \\
 &\quad + p(b, c) p(a, c) \operatorname{Res}_x c(w) \operatorname{Res}_z e^{\lambda(z-x)} e^{\lambda(x-w)} [a(z), b(x)] i_{w,x}(x-w)^n \\
 &= -p(b, c) \operatorname{Res}_x [a(w)_\lambda c(w)] b(x) i_{w,x}(x-w)^n + \\
 &\quad + p(ab, c) \operatorname{Res}_x c(w) e^{\lambda(x-w)} \operatorname{Res}_z e^{\lambda(z-x)} [a(z), b(x)] i_{w,x}(x-w)^n \\
 &= -p(b, c) \operatorname{Res}_x [a(w)_\lambda c(w)] b(x) i_{w,x}(x-w)^n + \\
 &\quad + p(ab, c) \operatorname{Res}_x c(w) \sum_{j \geq 0} \frac{\lambda^j}{j!} [a(x)_\lambda b(x)] i_{w,x}(x-w)^{n+j}
 \end{aligned}$$

Now adding up both terms we get:

$$\begin{aligned}
I + J &= [a(w)_\lambda (b(w)_{(n)} c(w))] \\
&= \operatorname{Res}_x \sum_{j \geq 0} \frac{\lambda^j}{j!} i_{x,w} (x-w)^{n+j} [a(x)_\lambda b(x)] c(w) \\
&\quad + p(a, b) \operatorname{Res}_x b(x) [a(w)_\lambda c(w)] i_{x,w} (x-w)^n + \\
&\quad + p(ab, c) \operatorname{Res}_x \sum_{j \geq 0} \frac{\lambda^j}{j!} i_{w,x} (x-w)^{n+j} c(w) [a(x)_\lambda b(x)] \\
&\quad - p(b, c) \operatorname{Res}_x [a(w)_\lambda c(w)] b(x) i_{x,w} (x-w)^n \\
&= \operatorname{Res}_x \left(\sum_{j \geq 0} \frac{\lambda^j}{j!} (i_{x,w} (x-w)^{n+j} [a(x)_\lambda b(x)] c(w) \right. \\
&\quad \left. - c(w) [a(x)_\lambda b(x)] i_{w,x} (x-w)^{n+j} \right) + \\
&\quad + p(a, b) \operatorname{Res}_x b(x) [a(w)_\lambda c(w)] i_{x,w} (x-w)^n \\
&\quad - p(ab, c) p(a, b) \operatorname{Res}_x [a(w)_\lambda c(w)] b(x) i_{w,x} (x-w)^n \\
&= \sum_{k \in \mathbb{Z}_+} \frac{\lambda^k}{k!} [a_\lambda b]_{(n+k)} c + p(a, b) b_{(n)} [a_\lambda c]
\end{aligned}$$

□

5.3. Proposition (Quasicommutativity). *For any local pair (a, b) one has:*

$$: ab : - p(a, b) : ba := \int_{-\partial}^0 [a_\lambda b] d\lambda$$

Where the limits in the integral means first integrate formally and then replace the limits.

Proof. It follows from the locality of the pair that the following formal distribution is local:

$$a(z, w) = a(z) b(w) i_{z,w} \frac{1}{z-w} - p(a, b) b(w) a(z) i_{w,z} \frac{1}{z-w}$$

But

$$: a(w) b(w) := \operatorname{Res}_z a(z, w), \quad : b(w) a(w) := p(a, b) \operatorname{Res}_z a(w, z)$$

And now we can compute:

$$\begin{aligned}
 : a(w)b(w) : -p(a, b) : b(w)a(w) : &= \text{Res}_z (a(z, w) - a(w, z)) \\
 &= \text{Res}_z e^{\lambda(z-w)} (a(z, w) - a(w, z))|_{\lambda=0} \\
 &= F_{z,w}^\lambda a(z, w) - F_{z,w}^\lambda a(w, z)|_{\lambda=0} \\
 &= (F_{z,w}^\lambda a(z, w) - F_{z,w}^{-\lambda-\partial} a(z, w))|_{\lambda=0} \\
 &= \text{Res}_z \left(1 - e^{-\partial_w(z-w)} \right) a(z, w) \\
 &= -\text{Res}_z \sum_{n \geq 1} (-\partial_w)^n / n! (z-w)^n a(z, w) \\
 &= -\text{Res}_z \sum_{n \geq 1} (-\partial_w)^n / n! (z-w)^{n-1} [a(z), b(w)] \\
 &= -\sum_{n \geq 1} \frac{(-\partial)^n}{n!} (a_{(n-1)} b) \\
 &= \int_{-\partial}^0 [a_\lambda b] d\lambda
 \end{aligned}$$

□

Rules for calculating λ -bracket.

- **sesquilinearity:**

$$\begin{aligned}
 [\partial a_\lambda b] &= -\lambda [a_\lambda b], \\
 [a_\lambda \partial b] &= (\partial + \lambda) [a_\lambda b]
 \end{aligned}$$

- **quasicommutativity** of λ -bracket in presence of locality:

$$[a_\lambda b] = -p(a, b) [b_{-\lambda-\partial} a]$$

- **quasicommutativity** of normal ordered product in presence of locality:

$$: ab : -p(a, b) : ba := \int_{-\partial}^0 [a_\lambda b] d\lambda$$

- **non-abelian Wick formula**

$$(5.3.1) \quad [a_\lambda : bc :] = [a_\lambda b] c : + p(a, b) : b [a_\lambda c] : + \int_0^\lambda [[a_\lambda b]_\mu c] d\mu$$

Exercise 5.4. *Prove the following Dong's lemma: If a, b, c are formal distributions which are pairwise local, then $(a, b_{(n)} c)$ is a local pair as well for any $n \in \mathbb{Z}$.*

Proof. Since $a(w), b(w), c(w)$ are pairwise local $\exists r \in \mathbb{Z}$ such that:

$$(z-x)^r [a(z), b(x)] = (x-w)^r [b(x), c(w)] = (z-w)^r [a(z), c(w)] = 0$$

In other words we have following:

$$\begin{aligned}
 (z-x)^r a(z) b(x) &= p(a, b) (z-x)^r b(x) a(z) \\
 (x-w)^r c(w) b(x) &= p(c, b) (x-w)^r b(x) c(w) \\
 (z-w)^r a(z) c(w) &= p(a, c) (z-w)^r c(w) a(z)
 \end{aligned}$$

Now we want to prove that exists $m \in \mathbb{Z}$ such that:

$$(z-w)^m a(z) \operatorname{Res}_x (b(x)c(w)i_{x,w}(x-w)^n - p(b,c)c(w)b(x)i_{w,x}(x-w)^n) = \\ p(a,bc)(z-w)^m \operatorname{Res}_x (b(x)c(w)i_{x,w}(x-w)^n - p(b,c)c(w)b(x)i_{w,x}(x-w)^n) a(z)$$

We may assume that r is large enough so that $r+n > 0$. Take $m = 4r$. Then, since $z-w = (z-x) + (x-w)$:

$$(z-w)^m a(z) \operatorname{Res}_x (b(x)c(w)i_{x,w}(x-w)^n - p(b,c)c(w)b(x)i_{w,x}(x-w)^n) = \\ = \operatorname{Res}_x ((z-w)^r \left(\sum_{i=0}^{3r} \binom{3r}{i} (z-x)^i (x-w)^{3r-i} a(z) (b(x)c(w)i_{x,w}(x-w)^n - \right. \\ \left. - p(b,c)c(w)b(x)i_{w,x}(x-w)^n) \right)$$

Let:

$$X(i) = \operatorname{Res}_x (z-w)^r (z-x)^i (x-w)^{3r-i} a(z) (b(x)c(w)i_{x,w}(x-w)^n - \\ - p(b,c)c(w)b(x)i_{w,x}(x-w)^n)$$

If $i \leq r$, then $3r-i \geq 2r$ and $3r-i+n \geq r$ and:

$$X(i) = \operatorname{Res}_x (z-w)^r (z-x)^i a(z) (x-w)^{3r-i+n} (b(x)c(w) - p(b,c)c(w)b(x)) = 0$$

If $i \geq r$, then:

$$X(i) = \\ = \operatorname{Res}_x ((z-w)^r (z-x)^i a(z) (x-w)^{3r-i} (b(x)c(w)i_{x,w}(x-w)^n - \\ - p(b,c)c(w)b(x)i_{w,x}(x-w)^n)) \\ = \operatorname{Res}_x ((z-w)^r (x-w)^{3r-i} (z-x)^i a(z) b(x)c(w)i_{x,w}(x-w)^n - \\ - p(b,c)(z-x)^i (z-w)^r a(z)c(w)b(x)i_{w,x}(x-w)^n) \\ = \operatorname{Res}_x ((x-w)^{3r-i} (p(a,b)(z-w)^r (z-x)^i b(x)a(z)c(w)i_{x,w}(x-w)^n) - \\ - (p(b,c)p(a,c)(z-x)^i (z-w)^r c(w)a(z)b(x)i_{w,x}(x-w)^n)) \\ = \operatorname{Res}_x ((x-w)^{3r-i} (p(a,b)p(a,c)(z-x)^i b(x)(z-w)^r c(w)a(z)i_{x,w}(x-w)^n) - \\ - (p(b,c)p(a,c)p(a,b)(z-w)^r c(w)(z-x)^i b(x)a(z)i_{w,x}(x-w)^n)) \\ = p(a,bc) \operatorname{Res}_x ((x-w)^{3r-i} ((z-x)^i b(x)(z-w)^r - c(w)i_{x,w}(x-w)^n) a(z) - \\ (p(b,c)(z-w)^r c(w)(z-x)^i b(x)i_{w,x}(x-w)^n) a(z)) \\ = p(a,bc) \operatorname{Res}_x ((z-w)^r (z-x)^i (x-w)^{3r-i} (b(x)c(w)i_{x,w}(x-w)^n - \\ - p(b,c)c(w)b(x)i_{w,x}(x-w)^n) a(z))$$

Summing over all indices i concludes the proof. □

6. FREE (SUPER) FERMIONS

A linear algebra lemma: Let V be a finite dimensional vector space with a non-degenerate bilinear form (\cdot, \cdot) . Let $\{a_i\}$ be a basis of V and $\{a^i\}$ be the dual basis, i.e. $(a_i, a^j) = \delta_{ij}$. Then

- (1) any $v \in V$ can be written as $v = \sum_i (a_i, v) a^i = \sum_i (v, a^i) a_i$
- (2) $\Omega = \sum_i a^i \otimes a_i \in V \otimes V$ is independent of the choice of $\{a_i\}$.

- (3) if $V = \mathfrak{g}$ is a Lie superalgebra and the bilinear form is supersymmetric and invariant

$$(6.0.2) \quad ([a, b], c) = (a, [b, c]),$$

then Ω is \mathfrak{g} -invariant i.e.

$$[a, \Omega] = \sum [a, a^i] \otimes a_i + \sum_i p(a, a^i) a^i \otimes [a, a_i] = 0$$

Proof.

(1)

$$v = \sum c_i a^i = \sum c^i a_i \Rightarrow (a_i, v) = c_i, \quad c^i = (v, a^i)$$

(2) We identify $V \otimes V$ with $\text{End}(V)$, by $(a \otimes b)v = (b, v)a$, then

$$\Omega v = \sum (a_i, v) a^i = v$$

(3) Note that symmetry of the form implies $p(a_i) = p(a^i)$.

$$\begin{aligned} [a, \Omega]v &= \sum (a_i, v)[a, a^i] + \sum p(a, a^i)([a, a_i], v)a^i \\ &= [a, \sum (a_i, v)a^i] - \sum ([a_i, a], v)a^i && \text{by skew-symmetry} \\ &= [a, v] - \sum (a_i, [a, v])a^i && \text{by invariance} \\ &= [a, v] - [a, v] = 0 && \text{by (1)} \end{aligned}$$

□

6.1. Example. Free (Super)Fermions. Let A be a superspace with a non-degenerate skew-symmetric bilinear form $\langle \cdot, \cdot \rangle$. Then we can define a Lie conformal superalgebra structure on $\mathbb{C}[\partial]A + \mathbb{C}K$ by

$$\begin{aligned} [a_\lambda b] &= \langle a, b \rangle K, \quad a, b \in A, \\ [K_\lambda A] &= 0. \end{aligned}$$

The corresponding maximal formal distribution Lie algebra is the Clifford affinization of A (ref. Example 3.12). All calculations are performed in the completion of $\mathcal{U} = \mathcal{U}(\hat{A})/(K - 1)$ so that

$$[a_\lambda b] = \langle a, b \rangle, \quad a, b \in A$$

Choose dual homogeneous bases of A : $\langle \phi_i, \phi^j \rangle = \delta_{ij}$ (it follows $p(\phi_i) = p(\phi^i)$). Let

$$L = \frac{1}{2} \sum_i : \partial \phi^i \phi_i :$$

By the lemma this is independent of the choice of basis, indeed writing in components we have:

$$\begin{aligned} L_{(n)} &= \\ &= \frac{1}{2} \sum_i \sum_{j \leq -1} (\partial \phi^i)_{(j)} (\phi_i)_{(n-j+1)} + \sum_i p(\phi^i, \phi_i) \sum_{j \geq 0} (\phi_i)_{(n-j+1)} (\partial \phi^i)_{(j)} \\ &= \frac{1}{2} \sum_{j \leq -1} (-j) \sum_i (\phi^i)_{(j-1)} (\phi_i)_{(n-j+1)} + \sum_{j \geq 0} (-j) \sum_i (-1)^{p(\phi^i)} (\phi_i)_{(n-j+1)} (\phi^i)_{(j-1)}. \end{aligned}$$

If $\{\phi_i\}$, $\{\phi^i\}$ are dual bases then $\{(-1)^{p(\phi^i)+1}\phi^i\}$, $\{\phi_i\}$ are dual bases too. This shows that each summand is independent of the basis chosen. Hence L is independent of the basis chosen.

If we denote $\phi(z) = \sum_{n \in \mathbb{Z}} (\phi t^n) z^{-1-n}$ we obtain a collection of pairwise local formal distributions such that (cf. example 3.12)

$$[\phi(z), \phi_i(w)] = \langle \phi, \phi_i \rangle \delta(z, w)$$

Hence defining

$$L(z) = \frac{1}{2} \sum : \partial_z \phi^i(z) \phi_i(z) :$$

we get that all $\phi(z)$ and $L(z)$ are pairwise local. We compute the λ -brackets using the non-abelian Wick formula to get

$$(6.1.1) \quad [\phi_\lambda L] = \frac{1}{2} \sum_i : [\phi_\lambda \partial \phi^i] \phi_i : + \frac{1}{2} \sum_i p(\phi, \phi_i) : \partial \phi^i [\phi_\lambda \phi_i] : \\ + \frac{1}{2} \sum_i \int_0^\lambda [[\phi_\lambda \partial \phi^i]_\mu \phi_i] d\mu$$

The integral term is clearly zero. Then we obtain:

$$[\phi_\lambda L] = \frac{\lambda}{2} \sum \langle \phi, \phi^i \rangle \phi_i - \frac{1}{2} \sum \langle \phi_i, \phi \rangle \partial \phi^i \\ = \frac{\lambda}{2} \phi - \frac{\partial}{2} \phi \quad \text{by the lemma} \\ [L_\lambda \phi] = -[\phi_{-\lambda-\partial} L] = \left(\partial + \frac{\lambda}{2} \right) \phi \quad \text{by skew-symmetry}$$

And now for the λ -bracket of L we obtain (recall that L is even):

$$\begin{aligned}
 [L_\lambda L] &= \frac{1}{2} [L_\lambda \sum : \partial \phi^i \phi_i :] \\
 &= \frac{1}{2} \sum : [L_\lambda \partial \phi^i] \phi_i : + \frac{1}{2} \sum : \partial \phi^i [L_\lambda \phi_i] : + \frac{1}{2} \sum \int_0^\lambda [[L_\lambda \partial \phi^i]_\mu \phi_i] d\mu \\
 &= \frac{1}{2} \sum : (\partial + \lambda) [L_\lambda \phi^i] \phi_i : + \frac{1}{2} \sum : \partial \phi^i [L_\lambda \phi_i] : + \\
 &\quad + \frac{1}{2} \sum \int_0^\lambda [(\partial + \lambda) [L_\lambda \phi^i]_\mu \phi_i] d\mu \\
 &= \frac{1}{2} \sum : (\partial + \lambda) \left(\partial + \frac{\lambda}{2} \right) \phi^i \phi_i : + \frac{1}{2} \sum : \partial \phi^i \left(\partial + \frac{\lambda}{2} \right) \phi_i : + \\
 &\quad + \frac{1}{2} \sum \int_0^\lambda (\lambda - \mu) [(\partial + \frac{\lambda}{2}) \phi_\mu^i \phi_i] d\mu \\
 &= \frac{1}{2} \sum : \partial^2 \phi^i \phi_i : + \frac{1}{2} \sum \frac{3}{2} \lambda : \partial \phi^i \phi_i : + \frac{1}{4} \lambda^2 \sum : \phi^i \phi_i : + \frac{1}{2} \sum : \partial \phi^i \partial \phi_i : + \\
 &\quad + \frac{1}{2} \sum \frac{\lambda}{2} : \partial \phi^i \phi_i : + \frac{1}{2} \sum \int_0^\lambda (\lambda - \mu) [(\partial + \frac{\lambda}{2}) \phi_\mu^i \phi_i] d\mu \\
 &= \partial L + 2\lambda L + \frac{1}{4} \lambda^2 \sum : \phi^i \phi_i : + \frac{1}{2} \int_0^\lambda (\lambda - \mu) [(\partial + \frac{\lambda}{2}) \phi_\mu^i \phi_i] d\mu \\
 &= \partial L + 2\lambda L + \sum \frac{1}{2} \int_0^\lambda (\lambda - \mu) [(\partial + \frac{\lambda}{2}) \phi_\mu^i \phi_i] d\mu
 \end{aligned}$$

We have used that

$$\begin{aligned}
 \sum : \phi^i \phi_i : &= - \sum p(\phi^i) : \phi^i \phi_i : && \text{by (2) in the lemma} \\
 : \phi^i \phi_i : &= p(\phi^i) : \phi_i \phi^i : && \text{by quasi-commutativity}
 \end{aligned}$$

hence the sum $\sum_i : \phi^i \phi_i := 0$. Now it remains to calculate the last integral, we have:

$$\begin{aligned}
 \sum \frac{1}{2} \int_0^\lambda (\lambda - \mu) [(\partial + \frac{\lambda}{2}) \phi_\mu^i \phi_i] d\mu &= \frac{1}{2} \int_0^\lambda (\lambda - \mu) \left(\frac{\lambda}{2} - \mu \right) \sum \langle \phi^i, \phi_i \rangle d\mu \\
 &= - \frac{\text{sdim } A}{2} \int_0^\lambda (\lambda - \mu) \left(\frac{\lambda}{2} - \mu \right) d\mu \\
 &= - \frac{\text{sdim } A}{24} \lambda^3
 \end{aligned}$$

Hence $L(z)$ is a Virasoro formal distribution with central charge $-\frac{\text{sdim } A}{2}$.

6.2. Definition. Given a formal distribution L , we say that a formal distribution a has weight Δ_a with respect to L if

$$[L_\lambda a] = (\partial + \Delta_a \lambda) a + o(\lambda)$$

In the case when L is a Virasoro formal distribution, Δ_a is called a **conformal weight**.

6.3. Proposition. *If a and b have weights Δ_a and Δ_b with respect to an even formal distribution L then $a_{(n)} b$ has weight $\Delta_a + \Delta_b - n - 1$ with respect to L (in particular $\Delta_{:ab} = \Delta_a + \Delta_b$). Also ∂a has weight $\Delta_a + 1$.*

Exercise 6.1. *Prove the proposition*

Proof. From the general Wick formula (5.2.3) we have

$$\begin{aligned} [L_\lambda a_{(n)} b] &= \sum \frac{\lambda^k}{k!} [L_\lambda a]_{(n+k)} b + a_{(n)} [L_\lambda b] \\ &= \sum \frac{\lambda^k}{k!} (\partial a + \Delta_a \lambda a + o(\lambda))_{(n+k)} b + a_{(n)} (\partial b + \Delta_b \lambda b + o(\lambda)) \\ &= \partial a_{(n)} b + \Delta_a \lambda a_{(n)} b + \lambda \partial a_{(n+1)} b + a_{(n)} \partial b + \Delta_b \lambda a_{(n)} b + o(\lambda) \\ &= \partial(a_{(n)} b) + \lambda(\Delta_a - (n+1) + \Delta_b) a_{(n)} b + o(\lambda) \end{aligned}$$

as we wanted. The rest of the statement is obvious

$$\begin{aligned} [L_\lambda \partial a] &= (\lambda + \partial)(\partial + \Delta_a \lambda + o(\lambda)) a \\ &= \partial^2 a + \lambda(\Delta_a + 1) \partial a + o(\lambda) \end{aligned}$$

□

6.4. Example. ϕ has weight $\frac{1}{2}$ with respect to L and L has weight 2 with respect to itself.

6.5. Theorem. *Let L be an even formal distribution such that*

- (1) $[L_\lambda L] = \frac{\lambda^3}{12} c + \lambda^2 A + 2\lambda B + C$ where $c \in \mathbb{C}$, A, B, C are formal distributions.
- (2) (L, L) is a local pair.
- (3) $(L_{(1)} L) = \Delta L$ where $\Delta \in \mathbb{C} \setminus \{0\}$

Then L is a Virasoro formal distribution with central charge c , and $\Delta = 2$.

Proof. Due to (2) we have

$$\begin{aligned} [L_\lambda L] &= -[L_{-\partial-\lambda} L] \\ &= \frac{c\lambda^3}{12} - (\partial + \lambda)^2 A + 2(\partial + \lambda) B - C \end{aligned}$$

Equating the coefficients of λ^2 we get that $A = 0$. Comparing constant terms we get $C = \partial B$. Hence

$$[L_\lambda L] = \frac{c\lambda^3}{12} + (\partial + 2\lambda) B$$

In particular $L_{(1)} L = 2B = \Delta L$ by (3), so L is Virasoro. □

7. BOSONIZATION AND THE SUGAWARA CONSTRUCTION

We will continue our discussion from the last lecture with more examples. As before, let $L(z)$ be an even formal distribution. Recall that an *eigendistribution* of weight Δ_a with respect to L is a formal distribution $a(z)$, which satisfies an equation of the form:

$$[L_\lambda a] = (\partial + \Delta_a \lambda) a + o(\lambda)$$

and that in the case of L being a Virasoro distribution, Δ_a is called the *conformal weight* of a . By the key proposition 6.3, we know that for any two eigendistributions a and b of weights Δ_a and Δ_b respectively of L , their n -th product, $a_{(n)} b$, is an eigendistribution of weight $\Delta_a + \Delta_b - n - 1$ with respect to L .

Letting $\text{wt } \lambda = \text{wt } \partial = 1$ in the standard formula for the λ -bracket:

$$[a_\lambda b] = \sum_{k \in \mathbb{Z}_+} \frac{\lambda^k}{k!} (a_{(k)} b)$$

we can see that all terms in the right-hand side have weight $\Delta_a + \Delta_b - 1$.

Now, consider the following example:

7.1. Example. For the Virasoro algebra we have:

$$[L_\lambda L] = (\partial + 2\lambda)L + \frac{\lambda^3}{12}c$$

Hence, we can easily see that the weight of both sides is 3.

Returning in the general case, denote as usual a given formal distribution by $a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-\Delta_a}$. Then we can present its coefficients in an alternate form in order to obtain an improved commutator formula:

$$a(z) = \sum_{n \in -\Delta_a + \mathbb{Z}} a_n z^{-n-\Delta_a}$$

then we have $a_{(n)} = a_{n-\Delta_a+1}$ for the Fourier coefficients and, using that $\text{wt } \partial = 1$, we can see that the coefficients of the derivative satisfy the equalities $(\partial a)_n = -(n + \Delta_a)a_n$ for all n . Finally, by a simple application of proposition 6.3, the commutator formula becomes²

$$(7.1.1) \quad [a_m, b_n] = \sum_{j \in \mathbb{Z}_+} \binom{m + \Delta_a - 1}{j} (a_{(j)}b)_{m+n}$$

Namely

$$\begin{aligned} [a_m, b_n] &= [a_{(m+\Delta_a-1)}, b_{(n+\Delta_b-1)}] \\ &= \sum_j \binom{m + \Delta_a - 1}{j} (a_{(j)}b)_{(m+\Delta_a-1+n+\Delta_b-1-j)} \\ &= \sum_j \binom{m + \Delta_a - 1}{j} (a_{(j)}b)_{m+n+\Delta_a+\Delta_b-2-j-\Delta_{a_{(j)}b}+1} \\ &= \sum_j \binom{m + \Delta_a - 1}{j} (a_{(j)}b)_{m+n} \end{aligned}$$

In the next exercise we will use this formula to deal with a special type of eigendistributions. A formal distribution a is called a **primary eigendistribution** of weight Δ , if it satisfies the identity $[L_\lambda a] = (\partial + \Delta\lambda)a$.

Exercise 7.1. Let a be a primary eigendistribution of weight Δ_a with respect to L . Prove that the following identity holds for all pairs of integers n, m :

$$[L_m, a_n] = (m(\Delta - 1) - n)a_{m+n}$$

Proof. Recall that $\Delta_L = 2$. By hypothesis we have

$$L_{(0)}a = \partial a \quad L_{(1)}a = \Delta a$$

and all the other products are 0. Now the graded formula (7.1.1) implies:

$$\begin{aligned} [L_m, a_n] &= (\partial a)_{m+n} + (m + \Delta_L - 1)\Delta a_{m+n} \\ &= -(m + n + \Delta)a_{m+n} + (m + 1)\Delta a_{m+n} \\ &= [m(\Delta - 1) - n]a_{m+n} \end{aligned}$$

²Note this new form of the commutator law is graded.

□

We can use this formula to get a better impression of the behavior of the free fermions:

7.2. Example. Recall that for the free fermions φ , the λ -bracket gives $[L_\lambda \varphi] = (\partial + \lambda/2)\varphi$ (i.e. φ is a primary eigendistribution of weight $\frac{1}{2}$) and so we can express φ as:

$$\varphi(z) = \sum_{n \in 1/2 + \mathbb{Z}} \varphi_n z^{-n-1/2}$$

and use the graded formula (7.1.1) to obtain the following identities:

$$\begin{aligned} [\varphi_m, \psi_n] &= \langle \varphi, \psi \rangle \delta_{m, -n} \\ [L_m, \varphi_n] &= -\left(\frac{m}{2} + n\right) \varphi_{m+n} \end{aligned}$$

Note that in order to prove the first formula, we need to use that $\text{wt } 1 = 0$ (or, equivalently, to see that $\langle \varphi, \psi \rangle = 1 = (\varphi_{(0)} \psi)$). The second formula follows easily from the above exercise.

7.3. Example. A good motivation for calling L_0 *the energy operator* is the fact that $[L_0, a_n] = -na_n$. Indeed, using the graded commutator formula,

$$\begin{aligned} [L_0, a_n] &= \sum_{j \in \mathbb{Z}_+} \binom{1}{j} (L_{(j)} a)_n \\ &= (\partial a)_n + \Delta_a a_n \\ &= -(n + \Delta_a) a_n + \Delta_a a_n \\ &= -na_n \end{aligned}$$

Certainly, this equality can be also easily deduced from the last exercise.

Consider now the charged free fermions. Take a superspace $A = A^+ \oplus A^-$ with a non-degenerate skew-supersymmetric bilinear form $\langle \cdot, \cdot \rangle$ such that:

$$\langle A^\pm, A^\pm \rangle = 0$$

Satisfying this condition, the form $\langle \cdot, \cdot \rangle$ defines a non-degenerate pairing between A^+ and A^- . We can choose dual basis $\{\varphi^i\}$ of A^+ and $\{\varphi_i\}$ for A^- such that $\langle \varphi_i, \varphi^j \rangle = \delta_{ij}$.

Exercise 7.2. For any element φ in the Fermion space A , represented as $\varphi = \varphi^+ + \varphi^-$ for some $\varphi^+ \in A^+$ and $\varphi^- \in A^-$, the following statements hold:

(1) We can present the components φ^+ and φ^- as

$$\begin{aligned} \varphi^+ &= \sum_i \langle \varphi_i, \varphi \rangle \varphi^i \\ \varphi^- &= \sum_i \langle \varphi, \varphi^i \rangle \varphi_i \end{aligned}$$

(2) The element $\sum_i \varphi^i \otimes \varphi_i$ in the tensor product $A^+ \otimes A^-$ is independent of the choice of $\{\varphi^i\}$.

Proof. The proof is standard:

- (1) For the positive component, φ^+ , denote $\varphi^+ = \sum c_j \varphi^j$. Then under the action of the bilinear form, obtain $\langle \varphi_i, \varphi^+ \rangle = \sum_j c_j \delta_{i,j} = c_i$. The proof for φ^- goes similarly.
- (2) We can identify $A^+ \otimes A^-$ with $\text{End}(A^+)$ via the identity $(a \otimes b)v = \langle b, v \rangle a$. Therefore, we have :

$$\sum_i (\varphi^i \otimes \varphi_i) \varphi^+ = \sum_i \langle \varphi_i, \varphi^+ \rangle \varphi^i = \varphi^+$$

□

In the light of the above exercise, we can define the following distribution

$$\alpha(z) = \sum_i : \varphi^i(z) \varphi_i(z) :$$

often referred to as **Bosonization**

7.4. Proposition. *For any element $\varphi = \varphi^+ + \varphi^-$ of the Fermionic space, the Bosonization satisfies the following properties:*

- (1) $[\alpha_\lambda \varphi] = \varphi^+ - \varphi^-$.
 (2) $[\alpha_\lambda \alpha] = -(\text{sdim } A^+) \lambda$

Exercise 7.3. *Prove proposition 7.4*

Proof.

- (1) Since the triple commutator vanishes, the integral term in Wick's Formula is zero, and hence we can evaluate (using the quasicommutativity in the last step):

$$\begin{aligned} [\varphi_\lambda \alpha] &= \sum_i : [\varphi_\lambda \varphi^i] \varphi_i : + \sum_i p(\varphi, \varphi^i) : \varphi^i [\varphi_\lambda \varphi_i] : \\ &= \sum_i \langle \varphi, \varphi^i \rangle \varphi_i + \sum_i p(\varphi, \varphi^i) \langle \varphi, \varphi_i \rangle \varphi^i \\ &= \varphi^- - \varphi^+ \end{aligned}$$

$$[\alpha_\lambda \varphi] = -p(\alpha, \varphi) [\varphi_{-\lambda-\partial} \alpha] = -[\varphi_\lambda \alpha] = \varphi^+ - \varphi^-$$

- (2) Now we can use (1) to compute the λ -bracket of α with itself, again applying the Wick formula:

$$\begin{aligned} [\alpha_\lambda \alpha] &= \sum_i [\alpha_\lambda : \varphi^i \varphi_i :] \\ &= \sum_i : [\alpha_\lambda \varphi^i] \varphi_i : + \sum_i p(\alpha, \varphi^i) : \varphi^i [\alpha_\lambda \varphi_i] : + \\ &\quad + \sum_i \int_0^\lambda [[\alpha_\lambda \varphi^i]_\mu \varphi_i] d\mu \\ &= \sum_i : \varphi^i \varphi_i : - \sum_i p(\alpha, \varphi^i) : \varphi^i \varphi_i : + \\ &\quad + \sum_i \int_0^\lambda \langle \varphi^i, \varphi_i \rangle d\mu \\ &= -(\text{sdim } A^+) \lambda \end{aligned}$$

□

Now, using the first statement in the last proposition along with the fact that $\text{wt } \alpha = 1$, we can see that the coefficients of α and φ satisfy the following identities all integers m and n .

$$[\alpha_m, \varphi_n] = \varphi_{m+n}^+ - \varphi_{m+n}^-$$

in particular, setting $m = 0$,

$$[\alpha_0, \varphi_n] = \varphi_n^+ - \varphi_n^-$$

and if $\varphi \in A^\pm$, we have

$$[\alpha_0, \varphi_n] = \pm \varphi_n$$

In the last case, α_0 is called the **charge operator**.

The second statement in the proposition can be rewritten as

$$[\alpha_m, \alpha_n] = m\delta_{m,-n}(-\text{sdim } A^+) \quad \forall n, m$$

and these equalities relations give the *commutation relations of free bosons*. These relations follow easily from the graded commutator formula, (7.1.1)).

Now let $L^- = \sum : \partial \varphi^i \varphi_i :$ and $L^+ = -\sum : \varphi^i \partial \varphi_i :$. We will introduce a family of formal distributions (depending on a complex parameter m)

$$L^m = (1 - m)L^- + mL^+$$

Note that for $m = 1/2$, this is precisely the definition we had for the uncharged free fermions in lecture 6.

7.5. Proposition. *The formal distributions L^\pm and L^m have the following properties*

- (1) $[L_\lambda^\pm \varphi] = \partial \varphi + \lambda \varphi^\pm$
- (2) L^m is a Virasoro formal distribution with central charge

$$c = (6m^2 - 6m + 1) \text{sdim } A$$

- (3) If $\varphi(z) \in A^+$ (respectfully $\varphi(z) \in A^-$), then φ is a primary eigendistribution of L^m of weight m (respectfully of weight $1 - m$)

Sketch of the proof.

- (1) Is a direct implication of the the Abelian Wick Formula (5.2.4).
- (2) The first part, (that L^m is Virasoro) follows from (3). Indeed L^m has weight 2 with respect to itself. It is clearly local and from the non-Abelian Wick formula we can see that $[L_\lambda^m L^m]$ is cubic in λ with the coefficient of λ^3 being a constant; see calculation below. Hence from theorem 6.5 we get that L^m is Virasoro. We further need to calculate the λ^3 coefficient in order to obtain the central charge. We do this as follows :

$$\begin{aligned} -[L_\lambda^- L^+] &= \sum [L_\lambda^- : \varphi^i \partial \varphi_i :] \\ &= \sum : [L_\lambda^- \varphi^i] \partial \varphi_i : + \sum : \varphi^i [L_\lambda^- \partial \varphi_i] : + \\ &\quad + \sum \int [[L_\lambda^- \varphi^i]_\mu \partial \varphi_i] d\mu \\ &= \sum : (\partial \varphi_i + \lambda \varphi_i^-) \partial \varphi^i : + \sum : \varphi^i (\partial + \lambda) [L_\lambda^- \varphi_i] : + \\ &\quad + \sum \int [\partial \varphi_\mu^i \partial \varphi_i] d\mu \end{aligned}$$

Now, it is clear that the highest power of λ occurring is the third, with coefficient being a constant coming from the last summand. We have:

$$\begin{aligned} -\sum \int [\partial \varphi_\mu^i \partial \varphi_i] d\mu &= -\text{sdim } A^+ \int_0^\lambda \mu^2 d\mu \\ &= -\frac{\text{sdim } A^+}{3} \lambda^3 \end{aligned}$$

Finally, by linearity we have:

$$[L_\lambda L] = (1-m)m([L_\lambda^- L^+] + [L_\lambda^+ L^-]) + (1-m)^2[L_\lambda^- L^-] + m^2[L_\lambda^+ L^+]$$

To complete the proof, we only need to evaluate the integrals, giving the last two terms.

(3) follows from (1) since

$$[L_\lambda^m \varphi] = \partial \varphi + \lambda(m\varphi^+ + (1-m)\varphi^-)$$

□

Exercise 7.4. Complete the proof of (2) in the last proposition.

Proof. As already mentioned, we only need to find the integral terms in the corresponding Wick formulas. The term corresponding to $[L_\lambda^- L^+]$ we calculated above, and it is easy to see that the term coming from $[L_\lambda^+ L^-]$ is equal to it. The term corresponding to $[L_\lambda^+ L^+]$ is given by:

$$\begin{aligned} \sum \int_0^\lambda [[L_\lambda^+ \varphi^i]_\mu \partial \varphi_i] d\mu &= -\sum \int_0^\lambda [(\lambda + \partial) \varphi_\mu^i \partial \varphi_i] d\mu \\ &= -\sum \int_0^\lambda (\lambda - \mu) [\varphi_\mu^i \partial \varphi_i] d\mu \\ &= -\sum \int_0^\lambda (\lambda - \mu)(\partial + \mu) \langle \varphi^i, \varphi_i \rangle d\mu \\ &= (\text{sdim } A^+) \int_0^\lambda (\lambda - \mu) \mu d\mu = (\text{sdim } A^+) \frac{\lambda^3}{6} \end{aligned}$$

Similarly, for $[L_\lambda^- L^-]$ we have

$$\begin{aligned} \sum \int_0^\lambda [[L_\lambda^- \partial \varphi^i]_\mu \varphi_i] d\mu &= \sum \int_0^\lambda [(\partial + \lambda) [L_\lambda^- \varphi^i]_\mu \varphi_i] d\mu \\ &= \sum \int_0^\lambda [(\partial + \lambda) \partial \varphi_\mu^i \varphi_i] d\mu \\ &= -\sum \int_0^\lambda (\lambda - \mu) \mu \langle \varphi^i, \varphi_i \rangle d\mu \\ &= (\text{sdim } A^+) \frac{\lambda^3}{6} \end{aligned}$$

Hence we get for the 3-product the final expression:

$$(-(1-m)4m + (1-m)^2 + m^2)(\text{sdim } A^+) \frac{\lambda^3}{6} = (6m^2 - 6m + 1) \frac{\text{sdim } A \lambda^3}{12}$$

as we wanted

□

7.6. Example (Currents and the **Sugawara** construction). Let \mathfrak{g} be a finite-dimensional Lie (super) algebra with a non-degenerate (super)symmetric bilinear form (\cdot, \cdot) . Denote $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] + \mathbb{C}K$ where K is a central element and let a and b be two elements of \mathfrak{g} . Then

$$[at^m, bt^n] = [a, b]t^{m+n} + m(a, b)\delta_{m, -n}K$$

The currents are $a(z) = \sum_{n \in \mathbb{Z}} (at^n)z^{-1-n}$ and $b(z) = \sum_{n \in \mathbb{Z}} (bt^n)z^{-1-n}$. Their bracket is given by

$$[a_\lambda b] = [a, b] + \lambda(a, b)K$$

and finally all the calculations are carried out in the completion of $\mathcal{U} = \mathcal{U}(\hat{\mathfrak{g}})/(K - k)$.

As usual we choose dual bases (a_i) and (a^i) , so that $(a_i, a^j) = \delta_{i,j}$. Let

$$T(z) = \frac{1}{2} \sum_i : a^i(z) a_i(z) :$$

Then the Abelian Wick formula gives:

$$\begin{aligned} [a_\lambda T] &= \frac{1}{2} \sum_i : [a, a^i] a_i : + \frac{\lambda k}{2} \sum (a, a^i) a_i + \frac{1}{2} \sum p(a, a^i) : a^i [a, a_i] : + \\ &\quad + \frac{\lambda k}{2} \sum p(a, a_i) (a, a_i) a^i + \frac{1}{2} \sum \int_0^\lambda [[a_\lambda a^i]_\mu a_i] d\mu \\ &= \frac{1}{2} \sum (: [a, a^i] a_i : + p(a, a^i) : a^i [a, a_i] :) + \lambda k a + \\ &\quad + \frac{1}{2} \sum \int_0^\lambda [[a, a^i], a_i] d\mu + \frac{k}{2} \sum \int_0^\lambda \mu ([a, a^i], a_i) d\mu \end{aligned}$$

Note that $\sum ([a, a_i], a^i) = \sum (a_i, [a, a^i])$ is the trace of $ad(a)$ on \mathfrak{g} . On the other hand we have $\sum ([a, a_i], a^i) = \sum (-1)^{p(\phi^i)+1} ([a, a^i], a_i)$ where we used that $\{(-1)^{p(\phi^i)+1} \phi^i\}$, $\{\phi_i\}$ are dual bases. It follows that $\sum ([a, a_i], a^i) = 0$. Hence we get:

$$\begin{aligned} [a_\lambda T] &= \lambda k a + \frac{\lambda}{2} \sum [[a, a^i], a_i] \\ &= \lambda k a + \frac{\lambda}{2} \Omega(a) \\ &= (k + h^\vee) \lambda a \end{aligned}$$

where $2h^\vee$ denotes the eigenvalue of Ω on \mathfrak{g} (we have to assume that \mathfrak{g} is commutative or simple) and h^\vee is called the **dual Coxeter number**.

8. RESTRICTED REPRESENTATIONS

We will start this lecture completing the analysis on the Sugawara construction. Let \mathfrak{g} be a Lie (super) algebra with an invariant non-degenerate (super)symmetric bilinear form (\cdot, \cdot) , we choose dual bases $(a_i, a^j) = \delta_{ij}$ and we work in the Kac-Moody affinization. Introduce

$$\begin{aligned} T(z) &= \frac{1}{2} \sum : a^i(z) a_i(z) : \\ \Omega &= \sum a^i a_i \in \mathcal{U}(\mathfrak{g}) \end{aligned}$$

Last time we proved that $[a_\lambda T] = \lambda(k + 1/2\Omega)a$. Assuming that \mathfrak{g} is simple or commutative then Ω acts as a scalar operator since in the simple case the adjoint representation is irreducible and in the commutative case Ω acts as zero. Defining $2h^\vee$ to be the eigenvalue we have then

$$[a_\lambda T] = \lambda(k + h^\vee)a.$$

Assuming $k \neq h^\vee$, letting

$$L = \frac{1}{2(k + h^\vee)} \sum : a^i(z)a_i(z) :$$

we get

$$\begin{aligned} [a_\lambda L] &= \lambda a \\ [L_\lambda a] &= (\partial + \lambda)a \end{aligned}$$

so with respect to L all currents are eigendistributions of weight 1, hence L is of weight 2 with respect to itself. Using Theorem 6.5, we easily get that L is a Virasoro distribution, and $[L_\lambda L] = (\partial + 2\lambda)L + \frac{c(k)}{12}\lambda^3$. It remains to compute $c(k)$. Notice that the λ^3 term only occurs in the integral part of the nonabelian Wick formula:

$$\begin{aligned} \frac{1}{2(k + h^\vee)} \sum \int_0^\lambda [[L_\lambda a^i]_\mu a_i] d\mu &= \frac{1}{2(k + h^\vee)} \int_0^\lambda (\lambda - \mu) \sum [a_\mu^i a_i] d\mu \\ &= \frac{1}{2(k + h^\vee)} \int_0^\lambda (\lambda - \mu) \sum [[a^i, a_i] + \mu k(a^i, a_i)] d\mu \\ &= \frac{k \operatorname{sdim} \mathfrak{g}}{12(k + h^\vee)} \lambda^3 \end{aligned}$$

We have proved the following

8.1. Theorem (Sugawara Construction). *If \mathfrak{g} is simple or commutative, $2h^\vee$ is the eigenvalue of the Casimir on \mathfrak{g} , then*

$$L(z) = \frac{1}{2(k + h^\vee)} \sum : a^i(z)a_i(z) :$$

is a Virasoro formal distribution with central charge $c(k) = \frac{k \operatorname{sdim} \mathfrak{g}}{k + h^\vee}$, the currents $a(z)$ being primary of weight 1.

8.2. Remark. If $k = -h^\vee$ (critical level) then $[a_\lambda T] = 0$, hence $[T(z), a(w)] = 0$ for all $a \in \mathfrak{g}$, hence all Fourier coefficients of $T(z)$ lie in the center of the completion of $\mathcal{U}(\hat{\mathfrak{g}})$.

Exercise 8.1 (optional). *We have in general that*

$$[a_\lambda T] = \lambda(k + \frac{1}{2}\Omega)a$$

How can we construct a Virasoro element L ?

Exercise 8.2 (Modification of the Sugawara Construction). *Let $b(z)$ be an even current ($b \in \mathfrak{g}$), let $L^b = L + \partial b$, show that L^b is a Virasoro formal distribution with a central charge $c(k) - 12k(b, b)$*

Proof. We compute $[L_\lambda^b L^b] = [L + \partial b_\lambda L + \partial b]$. The sum of linear terms in λ that come from the cross terms is $2\lambda\partial b$, so $L_{(1)}^b L^b = 2(L + \partial b) = 2L^b$, and because the additional contribution to λ^3 -term only comes from $[\partial b_\lambda \partial b] = -\lambda(\partial + \lambda)[b_\lambda b] = -\lambda^3 k(b, b)$, we see L^b is a Viraroso distribution with central charge as claimed. \square

8.3. Remark. If \mathfrak{g} is commutative then

$$[a_\lambda b] = (a, b)\lambda K = (a, b)\lambda k$$

in the quotient. Assume k is not zero. In this case $h^v = 0$ and hence $c(k) = \text{sdim } \mathfrak{g}$ and the currents are called **free bosons**.

8.4. Definition. Fix a vector (super)space V . A **(quantum) field** is an $\text{End } V$ -valued formal distribution such that $a(z)v$ is a Laurent series in z for any $v \in V$. We usually write $V((z))$ for the space of Laurent series in z with coefficients in V .

Writing $a(z) = \sum a_{(n)} z^{-1-n}$ this condition becomes, for $v \in V$:

$$a_{(n)} v = 0$$

for large enough n . We define n -th products and derivatives of quantum fields as we did before.

Exercise 8.3. Show that the n -th product ($n \in \mathbb{Z}_+$) of fields

$$a(w)_{(n)} b(w) = \text{Res}_z [a(z), b(w)] (z - w)^n$$

is again a field.

Proof. We can verify this directly from the above expression.

$$\begin{aligned} a(w)_{(n)} b(w) &= \text{Res}_z [a(z), b(w)] (z - w)^n \\ &= \text{Res}_z \sum_{l, m, k} (-1)^k [a_{(m)}, b_{(l)}] z^k w^{n-k} \binom{n}{k} z^{-1-m} w^{-1-l} \\ &= \sum (-1)^k \binom{n}{k} [a_{(k)}, b_{(l)}] w^{-1-k-l-n} \\ &= \sum_{l \in \mathbb{Z}} \sum_{k \geq 0} (-1)^k \binom{n}{k} [a_{(k)}, b_{(l-k-n)}] w^{-1-l} \end{aligned}$$

So given v , we can pick N such that $b_{(l-2n)}$ for $l > N$ kills the finite set of vectors $v, a_{(0)}v, \dots, a_{(n)}v$. Then clearly the commutators above kill v and so we conclude $a(w)_{(n)} b(w)$ is a field. \square

We want to check now that

$$: a(z)b(z) := a(z)_+ b(z) + p(a, b) b(z) a(z)_-$$

is again a field. For that note that the first term is a product of a formal power series with a Laurent series and hence it will be a Laurent series. For the second term, note that $a(z)_- v$ has finitely many non-zero summands, hence multiplying by b we get a finite sum of Laurent series, hence a Laurent series when we apply any vector v . Now in general since the derivatives of fields is again a field (easy to verify) then we get

$$a(z)_{(-1-n)} b(z) = \frac{: \partial_z^n a(z) b(z) :}{n!}$$

is again a field.

We have defined before the normal ordered product of distributions for continuous $\mathcal{U}^{\text{comp}}$ -valued distributions. Recall that \mathfrak{g} is a \mathbb{Z} -filtered Lie (super)algebra.

8.5. Definition. A representation π of an algebra \mathfrak{g} in V is called **restricted** if for any $v \in V$

$$\mathfrak{g}_N v = 0 \quad N \gg 0$$

Extend the representation π to a representation of $\mathcal{U}(\mathfrak{g})$ in V and since π is restricted we can extend it to $\mathcal{U}^{\text{comp}}$. Then the image of a continuous $\mathcal{U}^{\text{comp}}$ -valued formal distribution $a(z)$ in $\text{End } V$ will be a quantum field.

A Construction of a Restricted Representation. For example, take a representation of \mathfrak{g}_0 in V_0 such that $\mathfrak{g}_1|_{V_0} = 0$, and consider the induced \mathfrak{g} -module $V = \text{Ind}_{\mathcal{U}(\mathfrak{g}_0)}^{\mathcal{U}(\mathfrak{g})} V_0 := V_0 \otimes_{\mathcal{U}(\mathfrak{g}_0)} \mathcal{U}(\mathfrak{g}_-)$. Then V is restricted, indeed since any $v \in V$ is a linear combination of elements $g_1 \dots g_k v_0$ where $g_i \in \mathfrak{g}_{n_i}$ and $v_0 \in V_0$. This element is clearly annihilated by \mathfrak{g}_N for N bigger than the sum of grades $\sum |n_i|$.

8.6. Example.

$$\begin{aligned} \hat{\mathfrak{g}} &= \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K \\ \hat{\mathfrak{g}}_0 &= \mathfrak{g}[t] + \mathbb{C}K \end{aligned}$$

Take a representation V_0 of $\hat{\mathfrak{g}}_0$ such that $\mathfrak{g}^{t^k}|_{V_0} = 0$ for $k > 0$. This is actually a \mathfrak{g} -module such that K acts as the scalar k in V_0 . Then

$$V = \text{Ind}_{\hat{\mathfrak{g}}_0}^{\hat{\mathfrak{g}}} V_0$$

is a restricted $\hat{\mathfrak{g}}$ -module and such modules are parametrized by pairs (V_0, k) .

A special case of this construction is $\mathfrak{g} = \mathbb{C}\alpha$ a one dimensional algebra with $(\alpha, \alpha) = 1$. Then $\hat{\mathfrak{g}}$ is a Lie algebra with basis $\alpha_n = \alpha t^n$ and bracket $[\alpha_m, \alpha_n] = m\delta_{m,-n}$. This is called the *oscillator algebra* or *Hisenberg algebra*. Let $V_0 = \mathbb{C}1$, such that $\alpha_n \cdot 1 = 0$ if $n > 0$ and $\alpha_0 = \mu 1$. Then $V = \text{Ind}_{\hat{\mathfrak{g}}_0}^{\hat{\mathfrak{g}}} V_0$ is an irreducible $\hat{\mathfrak{g}}$ -module isomorphic to the following representation of the oscillator algebra in $\mathbb{C}[x_1, x_2, \dots]$, where for $n > 0$, α_n acts by:

$$\alpha_n \rightarrow \frac{\partial}{\partial x_n} \quad \alpha_{-n} \rightarrow nx_n \quad \alpha_0 = \mu$$

Proof. We have a canonical map

$$\begin{aligned} \varphi : V &\longrightarrow \mathbb{C}[x_1, x_2, \dots] \\ 1 &\longrightarrow 1 \end{aligned}$$

But vectors $a_{-j_n} \dots a_{-j_1} 1$ (nondecreasing positive indices) form a formal basis of V by the PBW theorem. The image of this vector is $j_1 \dots j_n x_{j_1} \dots x_{j_n}$ so φ is an isomorphism, and the representation in the polynomial space is easily seen to be irreducible. \square

8.7. Example (Clifford Affinization). . Recall that the bracket in this example is given by

$$[\psi t^m, \varphi t^n] = \langle \psi, \varphi \rangle \delta_{m,-1-n} K$$

Let $K = 1$. For simplicity assume that the space A is purely odd, hence $\langle \cdot, \cdot \rangle$ is symmetric. Recall that in this case we have a canonical Virasoro field L with central charge $\dim A/2$, and all fermions $\varphi(z)$ have conformal weight $1/2$, hence we write

$$\varphi(z) = \sum_{n \in 1/2 + \mathbb{Z}} \varphi_n z^{-n-1/2}$$

and we get $[\varphi_m, \psi_n] = \delta_{m,-n} \langle \varphi, \psi \rangle$. We take the induced module from \hat{A}_0 (the span of all φ_n with $n > 0$ plus the center) and the module is the 1-dimensional module $\mathbb{C}|0\rangle$ where the fields act trivially and K acts as 1. We obtain the module $F(A)$.

Exercise 8.4. Show that $F(A)$ is an irreducible \hat{A} -module (The Fermionic Fock space).

Proof. Define $L(z) = \frac{1}{2} \sum_i : \partial \varphi^i(z) \psi^i(z) :$ for dual bases $\{\varphi^i\}$ and $\{\psi^i\}$ of A . This gives us L_0 . By writing out L_0 in terms of the Fourier modes of the fields,

$$L_0 = \frac{1}{2} \sum_i \sum_{n \in \frac{1}{2} + \mathbb{Z}_+} \left(\left(n + \frac{1}{2} \right) \varphi_{-n-1}^i \psi_{n+1}^i - p(\varphi^i, \psi^i) \left(n + \frac{1}{2} \right) \psi_{-n}^i \varphi_n^i \right)$$

such that $[L_0, \varphi_n] = -n\varphi_n$. From these commutation relations we see that L_0 is diagonalizable in the PBK basis and the vacuum (up to scalar) is the only vector with the lowest eigenvalue. From the explicit form of L_0 we note that any singular vector (i.e. annihilated by all positive modes) will have the same energy as the vacuum, so it must lie in $\mathbb{C}|0\rangle$. If U is a submodule of $F(A)$ and $v \in U$ is a vector of minimum energy, then it must be annihilated by all positive modes since otherwise the commutator relations with L_0 force the resulting vector to have a lower energy. Hence by above it must be proportional to $|0\rangle$. Either by construction or by quotienting by the submodule generated by $|0\rangle$ and applying the same argument, we see $|0\rangle$ generates $F(A)$ and it is irreducible. □

9. BOSON-FERMION CORRESPONDENCE

An Example of a Vertex Algebra. Before giving a formal definition of a **Vertex Algebra** we discuss an example. Recall that given a finite dimensional (super)space A with a skew-symmetric bilinear form, we have the Clifford Affinization (where $\varphi_m = \varphi t^m$, $\varphi \in A$, $m \in \mathbb{Z}$)

$$(9.0.1) \quad \hat{A} = A[t, t^{-1}] + \mathbb{C}K, \quad [\varphi_{(m)}, \psi_{(n)}] = \langle \varphi, \psi \rangle \delta_{m,-n-1} K.$$

We have the Fock space

$$F(A) = \text{Ind}_{A[t] + \mathbb{C}K}^{\hat{A}} \mathbb{C}|0\rangle, \quad A[t]|0\rangle = 0, \quad K|0\rangle = |0\rangle.$$

It is easy to see that $\varphi(z)$ is an $\text{End } F(A)$ -valued field due to the condition $\varphi_m|0\rangle = 0$ for $m \geq 0$ and the commutation relations (9.0.1). From example 8.6 and Ex.8.4 of lecture 8 we conclude that $F(A)$ is an irreducible representation of \hat{A} .

We constructed $L(z)$ such that all fields are primary fields of conformal weight $\frac{1}{2}$. As usual we write $\varphi(z) = \sum_{n \in \frac{1}{2} + \mathbb{Z}} \varphi_n z^{-n-1/2}$ and in particular we get $[L_0, \varphi_n] = -n\varphi_n$ (See example 7.3 in lecture 7).

An important special case is the example of **charged fermions** (See more generally example 7.2 and the discussion after example 7.3 in lecture 7). In this case

$A = \mathbb{C}\psi^+ + \mathbb{C}\psi^-$ is two dimensional and odd. The bilinear form is symmetric and the commutation relations are of the form

$$(9.0.2) \quad [\psi_\lambda^+ \psi^-] = 1, \quad [\psi_\lambda^\pm \psi^\pm] = 0.$$

We choose the Virasoro element $L = L^{1/2}$

$$L(z) = \frac{1}{2}(: \partial\psi^+ \psi^- : + : \partial\psi^- \psi^+ :) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},$$

since this choice makes the both primary fields ψ^+ and ψ^- to acquire the conformal weight $1/2$.

The bosonization is given by

$$(9.0.3) \quad \begin{aligned} \alpha(z) &= : \psi^+ \psi^- :, \\ [\alpha_\lambda \psi^\pm] &= \pm \psi^\pm, \\ [\alpha_\lambda \alpha] &= \lambda, \\ \alpha(z) &= \sum_{n \in \mathbb{Z}} \alpha_n z^{-1-n}. \end{aligned}$$

Due to the PBW theorem the monomials

$$(9.0.4) \quad \psi_{-n_s}^- \cdots \psi_{-n_1}^- \psi_{-m_r}^+ \cdots \psi_{-m_1}^+ |0\rangle \quad \begin{cases} 0 < m_1 < m_2 < \cdots \\ 0 < n_1 < n_2 < \cdots \end{cases}$$

form a basis in the space $F := F(A)$ of the Fock representation (note $\psi_i^\pm \psi_i^\pm$ acts by zero since $0 = [\psi_i^\pm \psi_i^\pm] |0\rangle = 2 * \psi_i^\pm \psi_i^\pm |0\rangle$. We recall that for $n > 0$, $\psi_n^\pm |0\rangle = 0$.

Exercise 9.1. *Prove that*

$$\alpha_n |0\rangle = 0, \quad \text{and} \quad L_n |0\rangle = 0 \quad \forall n \geq 0.$$

Proof. Using the bosonization formulas (9.0.3) we get

$$(9.0.5) \quad \alpha_n = \sum_{k \in \mathbb{Z} + 1/2} : \psi_k^+ \psi_{n-k}^- :, \quad L_n^\pm = - \sum_{k \in \mathbb{Z} + 1/2} (k + 1/2) : \psi_k^\pm \psi_{n-k}^\mp :,$$

where

$$: \psi_k \varphi_m := \begin{cases} \psi_k \varphi_m & \text{if } k < 0, \\ -\varphi_m \psi_k & \text{if } k > 0. \end{cases}$$

Due to the fact that for $n > 0$ $\psi_n^\pm |0\rangle = 0$ it is easy to see that (9.0.5) implies the statement of the exercise.

Using the results of examples 7.2 and 7.3 we get that

$$(9.0.6) \quad [L_n, \psi_m^\pm] = -(m + n/2) \psi_{n+m}^\pm, \quad [\alpha_n, \psi_m^\pm] = \pm \psi_{n+m}^\pm.$$

In particular,

$$(9.0.7) \quad [L_0, \psi_n^\pm] = -n \psi_n^\pm, \quad [\alpha_0, \psi_n^\pm] = \pm \psi_n^\pm.$$

This implies that the operators L_0 and α_0 are diagonal in the basis (9.0.4), and the monomial (9.0.4) has the eigenvalues $\sum m_i + \sum n_j$ and $r - s$ for L_0 and α_0 , respectively. \square

9.1. Remark. In physics literature the fields (9.0.2) are regarded as free massless fermions on a two dimensional plane. And the operators L_0 and α_0 are referred to as energy and charge operator, respectively.

In what follows we will call the the eigenvalues $\sum m_i + \sum n_j$ and $r - s$ of the monomial (9.0.4) the energy and the charge, respectively.

Thus we have the charge decomposition $F = \bigoplus_{m \in \mathbb{Z}} F^{(m)}$ and a decomposition with respect to the energy operator L_0 . To study the latter one we first note that the state of the lowest energy in $F^{(m)}$ is unique up to a constant factor and it is given by

$$(9.1.1) \quad |m \rangle = \psi_{-\frac{2|m|-1}{2}}^{\pm} \cdots \psi_{-\frac{3}{2}}^{\pm} \psi_{-\frac{1}{2}}^{\pm} |0 \rangle$$

with $+$ if $m > 0$ and $-$ if $m < 0$. Note also that each $F^{(m)}$ is invariant with respect to the oscillator algebra $\{\alpha_n\}$.

9.2. Theorem. *Each $F^{(m)}$ is irreducible with respect to the oscillator algebra.*

We define a character of the module F by

$$\text{ch } F = \text{tr}_F z^{\alpha_0} q^{L_0} = \sum_{m,j} \dim F_j^{(m)} z^m q^{\text{frac}m^2 2j},$$

where $F_j^{(m)}$ is the span of states in $F^{(m)}$ having energy $m^2/2 + j$. The spectrum of L_0 in $F^{(m)}$ is $\frac{m^2}{2} + \mathbb{Z}_+$ and

$$\dim F^{(m)} j = p(j),$$

where $p(j)$ is the number of partitions of j in a sum of positive integers. This follows from the theorem and the fact that the vectors $\alpha_{-j_n} \cdots \alpha_{-j_1} |m \rangle$ form a basis of $F^{(m)}$ (non-decreasing positive indices). Hence we have

$$\text{ch } F = \sum_{m \in \mathbb{Z}} q^{\frac{m^2}{2}} z^m \sum_{j \in \mathbb{Z}_+} p(j) q^j = \sum_{m \in \mathbb{Z}} \frac{q^{\frac{m^2}{2}} z^m}{\prod_{j=1}^{\infty} (1 - q^j)}.$$

But from the form of our basis (9.0.4) we see that

$$\text{ch } F = \prod_{j=1}^{\infty} \left(1 + q^{j-\frac{1}{2}} z\right) \left(1 + q^{j-\frac{1}{2}} z^{-1}\right).$$

Equating these two expressions for the character we get the **Jacobi triple product** identity:

$$(9.2.1) \quad \prod_{j=1}^{\infty} (1 - q^j) \left(1 + q^{j-\frac{1}{2}} z\right) \left(1 + q^{j-\frac{1}{2}} z^{-1}\right) = \sum_{m \in \mathbb{Z}} q^{\frac{m^2}{2}} z^m.$$

Exercise 9.2. *Using (9.2.1) derive the following identities:*

$$\prod_{j=1}^{\infty} (1 - w^j) = \sum_{n \in \mathbb{Z}} (-1)^n w^{\frac{3n^2+n}{2}}, \quad (\text{Euler})$$

$$\prod_{n=1}^{\infty} \frac{1 - w^n}{1 + w^n} = \sum_{m \in \mathbb{Z}} (-w)^{m^2}. \quad (\text{Gauss})$$

Proof. The Euler identity is obtained from (9.2.1) by substituting $q = w^3$ and $z = -w^{1/2}$.

To prove Gauss identity we set $q = w^2$ and $z = 1$. Then (9.2.1) gives

$$\prod_{j=1}^{\infty} (1 - w^{2j}) (1 + w^{2j-1})^2 = \sum_{m \in \mathbb{Z}} w^{m^2}.$$

Substituting $-w$ instead of w we get

$$\prod_{j=1}^{\infty} (1 - w^{2j}) (1 - w^{2j-1})^2 = \sum_{m \in \mathbb{Z}} (-w)^{m^2}.$$

The latter is equivalent to

$$\prod_{j=1}^{\infty} \frac{(1 - w^{4j}) (1 - w^{2j-1})^2}{(1 + w^{2j})} = \sum_{m \in \mathbb{Z}} (-w)^{m^2},$$

which in turn gives

$$\prod_{j=1}^{\infty} \frac{(1 - w^{2j}) (1 - w^{2j-1})^2}{(1 + w^{2j})(1 - w^{2(2j-1)})} = \sum_{m \in \mathbb{Z}} (-w)^{m^2}.$$

The latter can be rewritten as

$$\prod_{j=1}^{\infty} \frac{(1 - w^{2j}) (1 - w^{2j-1})}{(1 + w^{2j})(1 + w^{2j-1})} = \sum_{m \in \mathbb{Z}} (-w)^{m^2},$$

and since now the product runs over all natural numbers we arrive at the desired Gauss identity

$$\prod_{n=1}^{\infty} \frac{1 - w^n}{1 + w^n} = \sum_{m \in \mathbb{Z}} (-w)^{m^2}.$$

□

Exercise 9.3. *Show that*

$$(9.2.2) \quad L = \frac{1}{2} (: \partial \psi^+ \psi^- : + : \partial \psi^- \psi^+ :) = \frac{1}{2} : \alpha \alpha : .$$

Proof. One solution of the exercise is based on performing direct calculation with the coefficients of Laurent series. We start with

$$(9.2.3) \quad L_n = \frac{1}{2} \sum_{k=0}^{\infty} \alpha_{n-k} \alpha_k + \frac{1}{2} \sum_{k=1}^{\infty} \alpha_{-k} \alpha_{n+k}.$$

Our purpose is to show that L_n is given by

$$(9.2.4) \quad L_n = -\frac{1}{2} \sum_{p \in \mathbb{Z} + 1/2} (p + 1/2) : \psi_p^+ \psi_{n-p}^- : - \frac{1}{2} \sum_{p \in \mathbb{Z} + 1/2} (p + 1/2) : \psi_p^- \psi_{n-p}^+ : .$$

Using the (Abelian) Wick formula (see any book on quantum field theory)

$$\begin{aligned}
L_n &= \frac{1}{2} \sum_{p,q \in \mathbb{Z}+1/2} \sum_{k \in \mathbb{Z}} : \psi_p^+ \psi_{k-p}^- \psi_q^+ \psi_{n-k-q}^- : - \\
&\quad - \frac{1}{2} \sum_{p \in \mathbb{Z}+1/2, p > 0} \left(\sum_{k=0}^{\infty} : \frac{\partial^r}{\partial \psi_p^+} \alpha_{n-k} \frac{\partial^l}{\partial \psi_{-p}^-} \alpha_k : + \sum_{k=0}^{\infty} : \frac{\partial^r}{\partial \psi_p^-} \alpha_{n-k} \frac{\partial^l}{\partial \psi_{-p}^+} \alpha_k : + \right. \\
&\quad \left. + \sum_{k=1}^{\infty} : \frac{\partial^r}{\partial \psi_p^+} \alpha_{-k} \frac{\partial^l}{\partial \psi_{-p}^-} \alpha_{n+k} : + \sum_{k=1}^{\infty} : \frac{\partial^r}{\partial \psi_p^-} \alpha_{-k} \frac{\partial^l}{\partial \psi_{-p}^+} \alpha_{n+k} : \right) + \\
&\quad + \text{“second order contributions”}
\end{aligned}$$

where ∂^r (∂^l) denotes right (left) derivative.

Any second order contribution in the Wick formula contains the factor

$$\frac{\partial^r}{\partial \psi_p^+} \frac{\partial^r}{\partial \psi_q^-} \alpha_k$$

with $k \leq 0$ and $p, q > 0$ and $p, q \in \mathbb{Z} + 1/2$. Using the expression for α_n in (9.0.5) we conclude that there are no second order contribution.

The following changes of indices show that there are no “zero order” contributions as well

$$\begin{aligned}
Z_n &= \sum_{p,q \in \mathbb{Z}+1/2} \sum_{k \in \mathbb{Z}} : \psi_p^+ \psi_{k-p}^- \psi_q^+ \psi_{n-k-q}^- : \\
&= \sum_{p,q \in \mathbb{Z}+1/2} \sum_{k \in \mathbb{Z}} : \psi_q^+ \psi_{k-q}^- \psi_p^+ \psi_{n-k-p}^- : \quad (\text{let } k = m + q - p) \\
&= \sum_{p,q \in \mathbb{Z}+1/2} \sum_{k \in \mathbb{Z}} : \psi_q^+ \psi_{m-p}^- \psi_p^+ \psi_{n-m-q}^- :
\end{aligned}$$

Since in the Abelian case we can permute the fields under the sign of the normal order as Grassmann variables we conclude that $Z_n = 0$.

Thus we are left with

$$\begin{aligned}
L_n &= -\frac{1}{2} \sum_{p \in \mathbb{Z}+1/2, p > 0} \left(\sum_{k=0}^{\infty} : \frac{\partial^r}{\partial \psi_p^+} \alpha_{n-k} \frac{\partial^l}{\partial \psi_{-p}^-} \alpha_k : + \sum_{k=0}^{\infty} : \frac{\partial^r}{\partial \psi_p^-} \alpha_{n-k} \frac{\partial^l}{\partial \psi_{-p}^+} \alpha_k : + \right. \\
&\quad \left. + \sum_{k=1}^{\infty} : \frac{\partial^r}{\partial \psi_p^+} \alpha_{-k} \frac{\partial^l}{\partial \psi_{-p}^-} \alpha_{n+k} : + \sum_{k=1}^{\infty} : \frac{\partial^r}{\partial \psi_p^-} \alpha_{-k} \frac{\partial^l}{\partial \psi_{-p}^+} \alpha_{n+k} : \right)
\end{aligned}$$

Again using the expression for α_n in (9.0.5) we derive the desired result

$$\begin{aligned}
L_n &= -\frac{1}{2} \sum_{p \in \mathbb{Z}+1/2, p > 0} \left(\sum_{k=0}^{\infty} : (\psi_{n-k-p}^- \psi_{k+p}^+ + \psi_{n-k-p}^+ \psi_{k+p}^-) : + \right. \\
&\quad \left. + \sum_{k=1}^{\infty} : (\psi_{-k-p}^- \psi_{n+k+p}^+ + \psi_{-k-p}^+ \psi_{n+k+p}^-) : \right) \\
&= -\frac{1}{2} \sum_{p \in \mathbb{Z}+1/2} (p+1/2) : \psi_p^+ \psi_{n-p}^- : - \frac{1}{2} \sum_{p \in \mathbb{Z}+1/2} (p+1/2) : \psi_p^- \psi_{n-p}^+ : .
\end{aligned}$$

There is however another simpler solution which is based on a result of the next lecture. Namely in the next lecture we will see that Clifford affinization with its Fock representation gives rise to a vertex algebra and we have the so-called state-field correspondence. In view of this correspondence it suffices to prove that

$$\frac{1}{2}(: \partial_z \psi^+(z) \psi^-(z) : + : \partial_z \psi^-(z) \psi^+(z) :) |0\rangle = \frac{1}{2} : \alpha(z) \alpha(z) : |0\rangle,$$

which is much simpler than what we do above. □

Proof. Proof Of Theorem 9.2 From the exercise 9.3 it follows that α is a primary field of conformal weight 1. Hence the commutation relations between Laurent coefficients of α and L are of the form

$$(9.2.5) \quad [L_m, \alpha_n] = -n\alpha_{n+m}.$$

Let U be a proper submodule of $F^{(m)}$. There is at least one vector (9.0.4) which is in $F^{(m)}$ but not in U .

All the vectors (9.0.4) are eigenvectors for L_0 and their eigenvalues form a discrete set bounded below. Hence there exists a vector v of the form (9.0.4) with a minimal energy e amongst the vectors (9.0.4) not belonging to U . Since α_n with $n > 0$ decreases the energy (see (9.2.5))

$$\alpha_n v \in U \quad \forall n > 0,$$

and therefore for any $n > 0$

$$\alpha_{-n} \alpha_n v \in U.$$

But this means that

$$0 = L_0 v - ev = \frac{\alpha_0^2}{2} v + \sum_{n>0} \alpha_{-n} \alpha_n v - ev,$$

and hence

$$\left(\frac{m^2}{2} - e\right)v \in U.$$

If $\frac{m^2}{2} - e \neq 0$ then $v \in U$ and this contradicts our assumption. Therefore $e = \frac{m^2}{2}$. But we know that in $F^{(m)}$ there is the only vector $|m\rangle$ up to a scalar factor with a minimal energy $m^2/2$. Thus U does not contain the vector $|m\rangle$.

Let v be some non-zero vector of U . Then

$$v = v_1 + \dots + v_k$$

where $v_i \in F^{(m)}$ and v_i are eigenvectors of L_0 with eigenvalues $e_i = m^2/2 + j_i$, $j_i \in \mathbb{Z}_+$ and $j_1 < j_2 < \dots < j_k$. The non-zero vector

$$u = (L_0 - e_2 I)(L_0 - e_3 I) \dots (L_0 - e_k I)v = (e_1 - e_2)(e_1 - e_3) \dots (e_1 - e_k)v$$

belongs to U and is a eigenvector of L_0 with the eigenvalue $e_1 = m^2/2 + j_1$. Thus there is at least one eigenvector of L_0 in U with an eigenvalue $m^2/2 + j$, $j \in \mathbb{Z}_+$. Let u_0 be such an eigenvector of L_0 in U with a minimal eigenvalue $e_0 = m^2/2 + j_0$. Then since α_n with $n > 0$ decreases the energy (see (9.2.5))

$$\alpha_n v = 0 \quad \forall n > 0.$$

The latter implies that

$$L_0 v = \frac{\alpha_0^2}{2} v + \sum_{n>0} \alpha_{-n} \alpha_n v = \frac{m^2}{2} v,$$

and hence v is proportional $|m\rangle$. This contradicts to the statement we proved above. Thus $F^{(m)}$ is an irreducible module of the oscillator algebra. \square

Boson-Fermion Correspondence. We have the fermionic Fock space F which is a module over the Clifford algebra generated by ψ^\pm . On the other hand we have the charge decomposition $F = \oplus F^{(m)}$, where $F^{(m)} \simeq B^{(m)}$ and $B^{(m)}$ denotes the unique irreducible module over the oscillator algebra $\{\alpha_n\}$ admitting a vector $|m\rangle$ such that $\alpha_n |m\rangle = 0$ for $n > 0$ and such that $\alpha_0 = mI$. Hence we have an isomorphism

$$\begin{aligned} F &\simeq B = \mathbb{C}[x_1, x_2, \dots; u, u^{-1}], \\ |m\rangle &\rightarrow u^m, \\ \alpha_j &\rightarrow \frac{\partial}{\partial x_j}, \\ \alpha_{-j} &\rightarrow j x_j, \\ \alpha_0 &\rightarrow u \frac{\partial}{\partial u}. \end{aligned}$$

This isomorphism is called the boson-fermion correspondence.

Since α preserves charge and ψ^\pm doesn't we cannot express the fermionic fields ψ^\pm in terms of the α'_n 's. But if we consider the operator of multiplication by u we get an affirmative answer.

Exercise 9.4. Using the commutation relations

$$(9.2.6) \quad [\alpha_m, \psi_n^\pm] = \pm \psi_{m+n}^\pm, \quad [\alpha_m, \psi^\pm(z)] = \pm z^m \psi^\pm(z)$$

prove that

$$\begin{aligned} \psi^\pm(z) &= u^{\pm 1} z^{\pm \alpha_0} e^{\mp \sum_{j<0} \frac{z^{-j}}{j} \alpha_j} e^{\mp \sum_{j>0} \frac{z^{-j}}{j} \alpha_j} \\ &= u^{\pm 1} : e^{\pm \int \alpha(z)} : . \end{aligned}$$

Proof. It is convenient to introduce a dual module F^* which is an induced right module of the trivial one-dimensional representation of the subalgebra \hat{A}_- of \hat{A} generated by ψ_n^+ and ψ_m^- with negative n and m . We denote the vector of the trivial module of \hat{A}_- as bra-vector $\langle 0|$. Thus due to the PBW theorem we have a basis in F^* , whose elements are the following monomials

$$(9.2.7) \quad \langle 0| \psi_{m_1}^- \dots \psi_{m_r}^- \psi_{n_1}^+ \dots \psi_{n_s}^+, \quad \begin{cases} 0 < m_1 < m_2 < \dots, \\ 0 < n_1 < n_2 < \dots \end{cases}$$

There is a unique pairing between the module F and F^* such that

$$\langle 0|0\rangle = 1,$$

and for any two vectors $v \in F^*$ and $w \in F$ and for any $n \in \mathbb{Z} + 1/2$

$$\langle v | \psi_n^\pm w \rangle = \langle v \psi_n^\pm | w \rangle .$$

It is easy to see that with this definition the basis (9.2.7) is dual to the basis (9.0.4). It is also obvious that the module F^* admits the analogous charge decomposition $F^* = \oplus F^{*(m)}$ and the analogous boson-fermion correspondence $F^{*(m)} \simeq B^{(m)}$, where $B^{(m)}$ is a right irreducible module over the oscillator algebra $O = \{\alpha_n\}$ induced from a one-dimensional representation $\mathbb{C} \langle m \rangle$ of the commutative subalgebra $O_- = \{\alpha_n, n \leq 0\}$ in which

$$\langle m | \alpha_n = m \delta_{n,0} \langle m | \quad \forall n \leq -1.$$

In this correspondence

$$(9.2.8) \quad \langle m | = \langle 0 | \psi_1^\mp \psi_{\frac{3}{2}}^\mp \dots \psi_{\frac{2|m|-1}{2}}^\mp,$$

where we put $+$ if $m > 0$ and $-$ if $m < 0$.

Thus we have that two quantum fields $q(z)$ and $q'(z)$ with values in $End(F)$ are equal if and only if for any pair of vectors v and w of the bases (9.2.7) and (9.0.4), respectively

$$(9.2.9) \quad \langle v | q(z) | w \rangle = \langle v | q'(z) | w \rangle.$$

Since for any vector v of the basis (9.0.4) there exists N such that for any collection of integers n_1, \dots, n_s with the sum $n_1 + \dots + n_s > N$

$$\alpha_{n_1} \dots \alpha_{n_s} v = 0$$

the field

$$\begin{aligned} \chi^\pm(z) &= u^{\pm 1} z^{\pm \alpha_0} e^{\mp \sum_{j < 0} \frac{z^{-j}}{j} \alpha_j} e^{\mp \sum_{j > 0} \frac{z^{-j}}{j} \alpha_j} \\ &= u^{\pm 1} : e^{\pm \int \alpha(z)} : \end{aligned}$$

is obviously a quantum field with values in $End(F)$.

Straightforward calculations show that

$$(9.2.10) \quad \langle m' | \psi^\pm(z) | m \rangle = z^{\pm m} \delta_{m \pm 1, m'}.$$

On the other hand the definition of the vectors $|m\rangle$ and $\langle m'|$ implies that

$$(9.2.11) \quad \langle m' | \chi^\pm(z) | m \rangle = z^{\pm m} \delta_{m \pm 1, m'}.$$

It is easy to derive the commutation relations between $\chi^\pm(z)$ and α_n

$$(9.2.12) \quad \begin{aligned} [\alpha_n, \chi^\pm(z)] &= u^{\pm 1} [\alpha_n, : e^{\pm \int \alpha(z)} :] = \\ &= \pm u^{\pm 1} : e^{\pm \int \alpha(z)} : \int [\alpha_n, \alpha(z)] = \pm u^{\pm 1} : e^{\pm \int \alpha(z)} : \int n z^{n-1} = \pm z^n \chi^\pm(z). \end{aligned}$$

Theorem 9.2 (and its analogue for the module F^*) implies that for each m the modules $F^{(m)}$ and $F^{*(m)}$ have the basis

$$\{\alpha_{-n_s} \dots \alpha_{-n_1} | m \rangle\}$$

and the basis

$$\{\langle m | \alpha_{n_1} \dots \alpha_{n_s}\},$$

respectively. Hence to prove the statement we have to show that for all combinations of integers $n_0, n_1, \dots, n_s, m_0, m_1, \dots, m_r$

$$\begin{aligned} & \langle n_0 | \alpha_{n_1} \dots \alpha_{n_s} \chi^\pm(z) \alpha_{-m_1} \dots \alpha_{-m_r} | m_0 \rangle = \\ & = \langle n_0 | \alpha_{n_1} \dots \alpha_{n_s} \psi^\pm(z) \alpha_{-m_1} \dots \alpha_{-m_r} | m_0 \rangle . \end{aligned}$$

This can be easily proved by induction on numbers of right and left oscillator modes. The base of the induction is given by (9.2.10) and (9.2.11) and the step is based on (9.2.6) and (9.2.12). Thus the statement is proved. \square

9.3. Remark. It is easy to see that $x = \log u$ is a canonically conjugated operator to α_0 . In string theory x plays a role of a coordinate of the string center of mass. The fact that α_0 has a discrete spectrum \mathbb{Z} means that the respective string is compactified on a circle in the direction of the coordinate x . (See any book on string theory.)

10. DEFINITION OF VERTEX ALGEBRA

10.1. Definition. A **Vertex Algebra** $(V, |0\rangle, Y)$ is the following data:

- V is a vector (super)space (*space of states*).
- $|0\rangle \in V$ is called the *vacuum vector*.
- Y is a linear parity preserving map from V to the space of $\text{End } V$ -valued fields (*the state field correspondence*). Given $a \in V$ we call $Y(a, z)$ a **vertex operator**.

Such that the following axioms hold

- (Vacuum Axioms) $Y(|0\rangle, z) = \text{Id}_V$, $Y(a, z)|0\rangle = a + T(a)z + T_2(a)z^2 + \dots$ where $T, T_i \in \text{End } V$.
- (Translation Covariance Axioms) $[T, Y(a, z)] = \partial_z Y(a, z)$ and T is called the *infinitesimal translation operator*.
- (Locality) $\{Y(a, z), a \in V\}$ is a local system of fields.

10.2. Remark. $Y(a, z) = \sum a_{(n)} z^{-1-n}$ where $a_{(n)} \in \text{End } V$ and we can define n -products of elements of V by

$$a_{(n)} b = a_{(n)}(b)$$

And the axioms can be rewritten in terms of the n -products:

Exercise 10.1.

- (*Vacuum Axioms*) $|0\rangle_{(n)} a = \delta_{n,-1} a$, ($n \in \mathbb{Z}$), $a_{(n)} |0\rangle = \delta_{n,-1} a$ ($n \in \mathbb{Z}_+ \cup \{-1\}$).
- (*Translation Covariance Axioms*) $[T, a_{(n)}] = -n a_{(n-1)}$ ($n \in \mathbb{Z}$).

And we also have that $a_{(n)} b = 0$ for $n \gg 0$ since the vertex operators are fields. We can write the locality axiom in terms of n -products.

Proof. For the vacuum axioms:

$$Y(|0\rangle, z)a = a = \sum_{n \in \mathbb{Z}} |0\rangle_{(n)} a z^{-1-n} \Leftrightarrow |0\rangle_{(n)} a = a \delta_{n,-1}$$

and the second set:

$$Y(a, z)|0\rangle = a + Ta + T_1 a z + \dots \Rightarrow a_{(n)} |0\rangle = 0 \text{ if } n \geq 0 \text{ and } a_{(-1)} |0\rangle = a$$

For the translation invariance axioms:

$$\begin{aligned} [T, Y(a, z)] &= \sum [T, a_{(n)}] z^{-1-n} \\ &= - \sum (n+1) a_{(n)} z^{-n-2} = \\ &= - \sum n a_{(n-1)} z^{-1-n} \\ \Leftrightarrow [T, a_{(n)}] &= -n a_{(n-1)} \end{aligned}$$

□

10.3. Definition. An **Holomorphic** vertex algebra is a vertex algebra where the vertex operators $Y(a, z)$ are formal power series in z .

10.4. Example. Let V have a structure of an unital commutative associative (super)algebra with an even derivation T . Let

$$|0\rangle = 1,$$

$$\begin{aligned} Y(a, z)b &= (e^{zT}a)b = \sum_{k \geq 0} \frac{z^k}{k!} (T^k a)b = \sum a_{(n)} b z^{-1-n} \\ \Rightarrow a_{(-1-n)}b &= \left(\frac{T^n a}{n!} \right) b \end{aligned}$$

This is a vertex algebra and $a_{(n)}b = 0$ for $n \geq 0$. And

$$a_{(-1-n)}b = \left(\frac{T^n a}{n!} \right) b \quad n \geq 0$$

Exercise 10.2. Check that this is a vertex algebra and moreover that every holomorphic vertex algebra is obtained in this way. For this define the product

$$\begin{aligned} ab &= Y(a, z)b|_{z=0} \\ T(a) &= \partial_z Y(a, z)|_{z=0} |0\rangle \end{aligned}$$

Proof. for the vacuum axioms:

$$\begin{aligned} Y(|0\rangle, z)b &= (e^{zT}1)b = b \\ Y(a, z)|0\rangle &= (e^{zT}a) = a + Taz + \dots \end{aligned}$$

Translation invariance:

$$\begin{aligned} [T, Y(a, z)]b &= T(e^{zT}a)b - (e^{zT}a)Tb \\ &= (Te^{zT}a)b \\ &= \partial_z (e^{zT}a)b \\ &= \partial_z Y(a, z)b \end{aligned}$$

Locality is obvious from commutativity. The converse follows easily from results of lecture 11. □

10.5. Theorem.

- (1) Let V be a space of continuous $\mathcal{U}^{\text{comp}}$ -valued formal distributions containing 1, ∂_z -invariant and closed under all n -products ($n \in \mathbb{Z}$). Suppose that V is a local system. Then V is a vertex algebra with vacuum vector 1 and n -product is the one for formal distributions. Also, $Ta(z) = \partial_z a(z)$.

- (2) *The same holds if V is a space of $\text{End } \mathcal{U}$ -valued fields, satisfying the same properties.*

Exercise 10.3. *A better setup to define normal ordered products is the following: \mathcal{U} is an associative unital (super)algebra with descending filtration*

$$\mathcal{U} \supset \mathcal{U}_0 \supset \mathcal{U}_1 \supset \dots$$

such that for any $u \in \mathcal{U}$ and $n \in \mathbb{Z}_+$ there exist $m \in \mathbb{Z}_+$ such that $\mathcal{U}_m u \subset \mathcal{U}_n$. Then $\mathcal{U}^{\text{comp}}$ consist of all infinite series $\sum u_\alpha$ such that all but finitely many u_α lie in \mathcal{U}_n for each $n \in \mathbb{Z}_+$. Check that this is an associative (super)algebra that contains \mathcal{U} . A formal distribution is continuous if $a_{(n)} \in \mathcal{U}\mathcal{U}_N$ for $n \gg 0$ and every $N \in \mathbb{Z}_+$. Show that one can define n -products as before.

Proof. The assumptions made here are what we previously got as a result of 4.5. Note that the proof of proposition 4.7 is still valid in this context. \square

Representing $V = \cup V_j$ with $\dim V_j = j$ we get a filtration

$$(\text{End } V)_j = \{a \in \text{End } V, \quad aV_j = 0\}$$

Hence we have (2) in the theorem as a special case of (1).

Proof of Thm 10.5. We have for $n \geq 0$ that

$$\begin{aligned} 1_{(-1)}a(z) &=: 1a(z) = a(z) \\ 1_{(-1-n)}a(z) &=: \frac{\partial^n}{n!}1a(z) = 0 \\ 1_{(n)}a(w) &= \text{Res}_z[1, a(w)](z-w)^n = 0 \end{aligned}$$

These are the first set of axioms. To check the second set

$$\begin{aligned} a(z)_{(n)}1 &= 0 \\ a(z)_{(-1-n)}1 &= 0 \\ a(z)_{(-1-n)}1 &=: \frac{\partial^n a(z)}{n!}1 := \frac{\partial^n a(z)}{n!} \\ Ta(z) &= a(z)_{(-2)}1 = \partial_z a(z) \end{aligned}$$

Now we have to check the translation axiom $[T, a_{(n)}]b = -na_{(n-1)}b$.

$$\partial_z(a(z)_{(n)}b(z)) - a(z)_{(n)}\partial_z b(z) = -na(z)_{(n-1)}b(z)$$

But this is

$$(\partial_z a(z))_{(n)}b(z) = -na(z)_{(n-1)}b(z)$$

which we know is true. Hence only locality is left. For that

$$\begin{aligned}
 Y(a(w), x)b(w) &= \sum_{n \in \mathbb{Z}} (a(w)_{(n)}b(w))x^{-1-n} = \\
 (10.5.1) \quad &= \text{Res}_z \left[(a(z)b(w))i_{z,w} \left(\sum_{n \in \mathbb{Z}} x^{-1-n}(z-w)^n \right) - \right. \\
 &\quad \left. - p(a, b)b(w)a(z)i_{w,z} \left(\sum_{n \in \mathbb{Z}} x^{-1-n}(z-w)^n \right) \right] \\
 &= \text{Res}_z \left((a(z)b(w))i_{z,w}\delta(z-w, x) - p(a, b)b(w)a(z)i_{w,z}\delta(z-w, x) \right)
 \end{aligned}$$

Exercise 10.4. Prove that the commutator is given by

$$\begin{aligned}
 &[Y(a(z), x), Y(b(w), y)]c(w) = \\
 &= \text{Res}_{z_1} \text{Res}_{z_2} ([a(z_1), b(z_2)]c(w))i_{z_1,w}i_{z_2,w} - \\
 &\quad - p(a, c)p(b, c)c(w)[a(z_1), b(z_2)]i_{w,z_1}i_{w,z_2}\delta(z_1-w, x)\delta(z_2-w, y)
 \end{aligned}$$

Proof. From (10.5.1) we get an expression for the product:

$$\begin{aligned}
 Y(a(z), z)Y(b(w), y)c(w) &= \text{Res}_{z_1} \text{Res}_{z_2} \left\{ a(z_1)b(z_2)c(w)i_{z_1,w}i_{z_2,w} - \right. \\
 &\quad - p(a, b)p(a, c)b(z_2)c(w)a(z_1)i_{w,z_1}i_{z_2,w} - p(b, c)a(z_1)c(w)b(z_2)i_{z_1,w}i_{w,z_2} + \\
 &\quad \left. + p(a, b)p(a, c)p(b, c)c(w)b(z_2)a(z_1)i_{w,z_1}i_{w,z_2} \right\} \delta(z_1-w, x)\delta(z_2-w, y)
 \end{aligned}$$

and a similar expression for $Y(b(w), y)Y(a(z), x)c(w)$.

Subtracting we get the desired commutator:

$$\begin{aligned}
 &\left[Y(a(z), x), Y(b(w), y) \right] c(w) = \\
 &= \text{Res}_{z_1} \text{Res}_{z_2} \left\{ [a(z_1)b(z_2)c(w) - \right. \\
 &\quad - p(a, b)b(z_2)a(z_1)c(w)] \times i_{z_1,w}i_{z_2,w} - p(a, b)p(a, c)b(z_2)c(w)a(z_1)i_{w,z_1}i_{z_2,w} + \\
 &\quad + p(a, b)p(a, c)b(z_2)c(w)a(z_1)i_{z_2,w}i_{w,z_1} - p(b, c)a(z_1)c(w)b(z_2)i_{z_1,w}i_{w,z_2} + \\
 &\quad + p(a, b)p(a, b)p(b, c)a(z_1)c(w)b(z_2)i_{w,z_2}i_{z_1,w} - p(b, c)p(a, c)c(w)[a(z_1)b(z_2) - \\
 &\quad \left. - p(a, b)b(z_2)a(z_1)]i_{w,z_2}i_{w,z_1} \right\} \delta(z_1-w, x)\delta(z_2-w, y) \\
 &= \text{Res}_{z_1} \text{Res}_{z_2} \left\{ [a(z_1), b(z_2)]c(w)i_{z_1,w}i_{z_2,w} - \right. \\
 &\quad \left. - p(a, c)p(b, c)c(w)[a(z_1), b(z_2)]i_{w,z_1}i_{w,z_2} \right\} \delta(z_1-w, x)\delta(z_2-w, y)
 \end{aligned}$$

□

Also we know that $(z_1 - z_2)^n [a(z_1), b(z_2)] = 0$. We claim that

$$(x - y)^n [Y(a(z), x), Y(b(w), y)]c(w) = 0 \quad n \gg 0$$

Indeed $x - y = (z_1 - z_2) - ((z_1 - w) - x) + ((z_2 - w) - y)$ hence all terms in the expansion of $(x - y)^n$ different from $(z_1 - z_2)^n$ contain either $((z_1 - w) - x)$ or $((z_2 - w) - y)$ so we clearly get the result from the exercise. \square

10.6. Corollary. *Any identity that holds for a general vertex algebra automatically holds for a local system of continuous formal distributions, or of fields.*

Proof. Given such a system say F , we take the closure \bar{F} . This is the minimal space which contains 1 and is closed under n -products. By Dong's lemma it is again local hence we can apply 10.5 And \bar{F} is a vertex algebra. \square

10.7. Lemma. *Let V be a vector space and let $|0\rangle \in V$ be a vector, and let $a(z), b(z)$ be $\text{End } V$ -valued fields such that $a_{(n)}|0\rangle = 0$ and $b_{(n)}|0\rangle = 0$ for all $n \geq 0$. Then for all $N \in \mathbb{Z}$, $a_{(N)}b|0\rangle$ contains no negative powers of z and the constant term is $a_{(N)}b_{(-1)}|0\rangle$ i.e.*

$$a(z)_{(N)}b(z)|0\rangle \Big|_{z=0} = a_{(N)}b_{(-1)}|0\rangle$$

Proof. For $N \geq 0$ we have

$$\begin{aligned} a(w)_{(N)}b(w)|0\rangle &= \text{Res}_z [a(z), b(w)](z-w)^N |0\rangle \\ &= \text{Res}_z \sum_{m,n} [a_{(m)}, b_{(n)}] z^{-1-m} w^{-1-n} \sum_{j=0}^N \binom{N}{j} z^j (-w)^{N-j} |0\rangle \\ &= \sum_{n \in \mathbb{Z}} \sum_{j=0}^N \binom{N}{j} (-1)^{N-j} [a_{(j)}, b_{(n)}] w^{N-n-j-1} |0\rangle \\ &= \sum_{n < 0} \sum_{j=0}^N \binom{N}{j} (-1)^{N-j} w^{N-n-j-1} a_{(j)} b_{(n)} |0\rangle \end{aligned}$$

Hence $N - n - j - 1 \geq 0$ and the constant term occurs when $j = N$ and $n = -1$ and it is equal to $a_{(N)}b_{(-1)}|0\rangle$. This proves the case when $N \geq 0$ now for the other cases:

$$\begin{aligned} a(z)_{(-1-N)}b(z)|0\rangle &= \frac{\partial_z^N}{N!} a(z)_+ b(z)|0\rangle + p(a, b) b(z) \frac{\partial_z^N a(z)_-}{N!} |0\rangle \\ &= \frac{\partial_z^N a(z)_+}{N!} b(z)_+ |0\rangle \end{aligned}$$

This shows that the expansion only contains non-negative powers of z , the rest of the statement is obvious. \square

11. UNIQUENESS AND n -PRODUCT THEOREMS

11.1. Lemma. *Let A be a linear operator on a vector space \mathcal{U} then there exist a unique solution of the differential equation*

$$\frac{df(z)}{dz} = Af(z), \quad f \in \mathcal{U}[[z]],$$

for any initial condition $f(0) = f_0$.

Exercise 11.1. *Prove the lemma.*

Proof.

$$\begin{aligned} f(z) &= \sum_{i \in \mathbb{Z}_+} a_i z^i \quad (a_i \in \mathcal{U}), \quad \frac{df(z)}{dz} = Af(z), \quad f(0) = f_0, \quad f \in \mathcal{U}[[z]] \\ \Leftrightarrow \sum_{i \neq 0 \in \mathbb{Z}_+} i a_i z^{i-1} &= \sum_{i \in \mathbb{Z}_+} A a_i z^i, \quad a_0 = f_0 \\ \Leftrightarrow a_{i+1} &= \frac{A a_i}{i+1}, \quad a_0 = f_0 \end{aligned}$$

But the last condition is clearly a recursion which determines a unique sequence of coefficients of a power series for f . Hence a solution of the given differential equation exists and is unique. \square

11.2. Proposition. *Let V be a vertex algebra. Then it follows*

- (1) $Y(a, z)|0 \rangle = e^{zT} a.$
- (2) $e^{zT} Y(a, w) e^{-zT} = i_{w,z} Y(a, z+w).$
- (3) $(Y(a, z)_{(n)} Y(b, z))|0 \rangle = Y(a_{(n)} b, z)|0 \rangle.$

Proof. In all three cases both sides are formal powers in z with coefficients in V , $(\text{End } V)[[w, w^{-1}]]$ and V , respectively. Indeed this follows easily from the vacuum axioms and lemma 10.7. Also, in all three cases the initial conditions are equal. Indeed the only non-trivial case is (3) but we see by lemma 10.7 that:

$$(Y(a, z)_{(n)} Y(b, z))|0 \rangle |_{z=0} = a_{(n)}(b_{(-1)}|0 \rangle) = a_{(n)} b|0 \rangle$$

Now by Ex 11.1 it remains to show that both sides satisfy the same differential equation. Denote the RHS by $X(z)$, then it satisfies the following differential equations:

- (1) $\frac{dX(z)}{dz} = TX(z).$
- (2) $\frac{dX(z)}{dz} = [T, X(z)]$ by the translation axiom.
- (3) $\frac{dX(z)}{dz} = TX(z)$ by the translation axiom and $T|0 \rangle = 0.$

But now we only have to check that the LHS satisfy the same differential equations, in each case:

- (1) Is clear by the translation invariance axiom.
- (2) $TX(z) - X(z)T = \frac{dX(z)}{dz}.$

\square

Exercise 11.2. *Check that the LHS of (3) satisfies the differential equation:*

$$\frac{dX(z)}{dz} = TX(z)$$

Proof.

$$\begin{aligned}
& TY(a, w)_{(n)}Y(b, w)|0 \rangle = \\
& = \text{Res}_z \left\{ TY(a, z)Y(b, w)|0 \rangle i_{z, w}(z - w)^n - \right. \\
& \quad \left. - p(a, b)TY(b, w)Y(a, z)|0 \rangle i_{w, z}(z - w)^n \right\} \\
& = \text{Res}_z \left\{ [T, Y(a, z)]Y(b, w)|0 \rangle i_{z, w}(z - w)^n + \right. \\
& \quad + Y(a, z)TY(b, w)|0 \rangle i_{z, w}(z - w)^n - \\
& \quad - p(a, b)[T, Y(b, w)]Y(a, z)|0 \rangle i_{z, w}(z - w)^n - \\
& \quad \left. - p(a, b)Y(b, w)TY(a, z)|0 \rangle i_{w, z}(z - w)^n \right\} \\
& = \text{Res}_z \left\{ [T, Y(a, z)]Y(b, w)|0 \rangle i_{z, w}(z - w)^n - \right. \\
& \quad - p(a, b)Y(b, w)[T, Y(a, z)]|0 \rangle i_{w, z}(z - w)^n + \\
& \quad + Y(a, z)[T, Y(b, w)]|0 \rangle i_{z, w}(z - w)^n - \\
& \quad \left. - p(a, b)[T, Y(b, w)]Y(a, z)|0 \rangle i_{w, z}(z - w)^n \right\} \\
& = [T, Y(a, z)]_{(n)}Y(b, w)|0 \rangle + Y(a, w)_{(n)}[T, Y(b, w)]|0 \rangle \\
& = \partial Y(a, w)_{(n)}Y(b, w)|0 \rangle + Y(a, w)_{(n)}\partial Y(b, w)|0 \rangle \\
& = \partial(Y(a, w)_{(n)}Y(b, w)|0 \rangle)
\end{aligned}$$

□

11.3. Theorem (Uniqueness Theorem). *Let V be a vertex algebra and let $B(z)$ be an End V -valued field such that:*

- (1) $(B(z), Y(a, z))$ is a local pair for all $a \in V$.
- (2) $B(z)|0 \rangle = Y(b, z)|0 \rangle$

Then $B(z) = Y(b, z)$.

Proof. Consider the difference $B_1(z) = B(z) - Y(b, z)$, then

- (1) $(B_1(z), Y(a, z))$ is a local pair by the locality axiom.
- (2) $B_1(z)|0 \rangle = 0$

Hence it is enough to prove that $B_1(z) = 0$. Now by locality we have

$$(z - w)^N B_1(z)Y(a, w)|0 \rangle = \pm(z - w)^N Y(a, w)B_1(z)|0 \rangle = 0$$

Hence $(z - w)^N B_1(z)e^{wT}a = 0$ by prop. 11.2(1). Letting $w = 0$ we get $z^N B_1(z)a = 0$ hence $B_1(z)a = 0$ for all $a \in V$, so $B_1(z) = 0$. □

11.4. Theorem (n^{th} -product Theorem).

$$Y(a_{(n)}b, z) = Y(a, z)_{(n)}Y(b, z)$$

Proof. We apply the uniqueness theorem. Let $B(z)$ be the RHS. Then (1) holds by Dong's lemma, and (2) holds by Proposition by 11.2(3). □

11.5. Corollary.

- (1) $Y(a_{(-1)}b, z) =: Y(a, z)Y(b, z) :$
- (2) $Y(Ta, z) = \partial_z Y(a, z)$
- (3) *The OPE* $[Y(a, z), Y(b, z)] = \sum_{j \in \mathbb{Z}} Y(a_{(j)}b, w) \partial_w^j \delta(z - w) / j!$

Proof.

- (1) Follows from the theorem letting $n = -1$.
- (2) Follows from the theorem letting $b = |0 \rangle$ and $n = -2$. Indeed we have:

$$Y(Ta, z) = Y(a_{(-2)}|0 \rangle, z) = Y(a, z)_{(-2)}I =: \partial Y(a, z)I := \partial Y(a, z)$$

- (3) By the decomposition theorem due to the locality axiom we have:

$$[Y(a, z), Y(b, z)] = \sum_{j \in \mathbb{Z}} Y(a, w)_{(j)}Y(b, w) \partial_w^j \delta(z - w) / j!$$

And now the result follows from the theorem. □

11.6. Remark. Any vertex algebra V is an example of a vertex algebra of $\text{End } V$ -valued fields $a \rightarrow Y(a, z)$ on V , by the n -product theorem.

11.7. Theorem (Skew-symmetry or Quasi-symmetry).

$$Y(a, z)b = p(a, b)e^{zT}Y(b, -z)a$$

Proof. By the locality axiom we have that:

$$(z - w)^N Y(a, z)Y(b, w)|0 \rangle = (z - w)^N p(a, b)Y(b, w)Y(a, z)|0 \rangle \quad N \gg 0.$$

Now we use (1) in prop. 11.2 and we get for $N \gg 0$:

$$\begin{aligned} (z - w)^N Y(a, z)e^{wT}b &= (z - w)^N p(a, b)Y(b, w)e^{zT}a \\ &= (z - w)^N p(a, b)e^{zT}e^{-zT}Y(b, w)e^{zT}a \\ &= (z - w)^N p(a, b)e^{zT}i_{w, z}Y(b, w - z)a \end{aligned}$$

The LHS is a formal power series in w and the right hand side too since for $N \gg 0$ we have $b_{(N)}a = 0$. Hence now we can let $w = 0$ and now we can multiply by z^{-N} to get the answer. □

Exercise 11.3.

- (1) Show that (2) in Corollary 11.5 means that $(Ta)_{(n)}b = -na_{(n-1)}b$ for all $n \in \mathbb{Z}$.
- (2) Show that Skew-symmetry means that:

$$a_{(n)}b = -p(a, b)(-1)^n \sum_{j \geq 0} \frac{(-T)^j}{j!} (b_{(n+j)}a)$$

and show that for $n = -1$ this is quasi-commutativity for the normally ordered product, i.e.:

$$a_{(-1)}b - p(a, b)b_{(-1)}a = \int_{-T}^0 [a_\lambda b] d\lambda$$

where $[a_\lambda b] = \sum \frac{\lambda^j}{j!} a_{(j)}b$.

Proof. (1)

$$\begin{aligned} Y(Ta, z) &= \partial_z Y(a, z) \\ \Leftrightarrow \sum_{n \in \mathbb{Z}} (Ta)_{(n)} z^{-n-1} &= \partial_z \sum_{n \in \mathbb{Z}} a_{(n-1)} z^{-n} = \sum_{n \in \mathbb{Z}} -na_{(n-1)} z^{-n-1} \\ \Leftrightarrow (Ta)_{(n)} &= -na_{(n-1)}, \quad \forall n \in \mathbb{Z} \end{aligned}$$

Hence:

$$(2) \quad (Ta)_{(n)} b = -na_{(n-1)} b$$

$$\begin{aligned} Y(a, z)b &= p(a, b)e^{zT}Y(b, -z)a \\ \rightarrow \sum a_{(n)}bz^{-n-1} &= p(a, b)\left(\sum_{j \geq 0} \frac{z^j T^j}{j!}\right)\left(\sum b_{(k)}a(-z)^{-k-1}\right) \end{aligned}$$

Taking coefficients at z^{-n-1} in both sides we get:

$$\begin{aligned} a_{(n)}b &= p(a, b) \sum_{j \geq 0} \frac{T^j}{j!} (-1)^{-n-j-1} b_{(n+j)}a \\ &= -p(a, b)(-1)^n \sum_{j \geq 0} \frac{(-T)^j}{j!} (b_{(n+j)}a) \end{aligned}$$

(3) For $n = -1$ we get

$$\begin{aligned} :ab: &= p(a, b) :ba: + p(a, b) \sum_{k \geq 0} \frac{(-T)^{k+1}}{(k+1)!} b_{(k)}a \\ \Leftrightarrow :ba: - p(a, b) :ab: &= - \sum_{k \geq 0} \frac{(-T)^{k+1}}{(k+1)!} b_{(k)}a = \int_{-T}^0 [b_\lambda a] d\lambda \end{aligned}$$

□

Exercise 11.4. Prove a special case of the (-1) -product formula, called quasi-associativity:

$$\begin{aligned} (a_{(-1)}b)_{(-1)}c - a_{(-1)}(b_{(-1)}c) &= \\ &= \sum_{j \in \mathbb{Z}_+} a_{(-j-2)}(b_{(j)}c) + p(a, b) \sum_{j \in \mathbb{Z}_+} b_{(-j-2)}(a_{(j)}c) \\ &= \left(\int_0^{-T} d\lambda a \right)_{(-1)} [b_\lambda c] + p(a, b) \left(\int_0^{-T} d\lambda b \right)_{(-1)} [a_\lambda c], \end{aligned}$$

where we first expand λ -bracket into powers of λ , then integrate and finally take -1^{st} -products.

Proof. Let $A = Y(a, z), B = Y(b, z)$ be the corresponding formal distributions. We will use the simple formula (4.7.1) for the normal product :

$$\begin{aligned} :A(z)B(z): &= A_+(z)B(z) + p(A, B)B(z)A(z)_- \\ &= \sum_{n \in \mathbb{Z}} :AB:_{(n)} z^{-1-n} \end{aligned}$$

Where:

$$: AB :_{(n)} = \sum_{j=-1}^{-\infty} a_{(j)} b_{(n-j-1)} + p(a, b) \sum_{j=0}^{\infty} b_{(n-j-1)} a_{(j)}$$

It is clear that:

$$\begin{aligned} : AB :_{(-1)} &= \sum_{j=-1}^{-\infty} a_{(j)} b_{(-1-j-1)} + p(a, b) \sum_{j=0}^{\infty} b_{(-1-j-1)} a_{(j)} \\ &= \sum_{j=-1}^{-\infty} a_{(j)} b_{(-2-j)} + p(a, b) \sum_{j=0}^{\infty} b_{(-2-j)} a_{(j)} \end{aligned}$$

Hence:

$$\begin{aligned} : AB :_{(-1)} - a_{(-1)} b_{(-1)} &= \sum_{j=-2}^{-\infty} a_{(j)} b_{(-2-j)} + p(a, b) \sum_{j=0}^{\infty} b_{(-2-j)} a_{(j)} \\ &= \sum_{t=-2-j=0}^{\infty} a_{(-2-t)} b_{(t)} + p(a, b) \sum_{j=0}^{\infty} b_{(-2-j)} a_{(j)} \\ &= \sum_{j=0}^{\infty} a_{(-2-j)} b_{(j)} + p(a, b) \sum_{j=0}^{\infty} b_{(-2-j)} a_{(j)} \end{aligned}$$

From the product formula we deduce that:

$$\begin{aligned} (a_{(-1)} b)_{(-1)} - a_{(-1)} b_{(-1)} &= Y(a_{(-1)} b, z)_{(-1)} - a_{(-1)} b_{(-1)} \\ &= (Y(a, z)_{(-1)} Y(b, z))_{(-1)} - a_{(-1)} b_{(-1)} \\ &=: AB :_{(-1)} - a_{(-1)} b_{(-1)} \\ &= \sum_{j=0}^{\infty} a_{(-2-j)} b_{(j)} + p(a, b) \sum_{j=0}^{\infty} b_{(-2-j)} a_{(j)} \end{aligned}$$

Applying both sides to c we obtain the desired result. Finally,

$$\begin{aligned} \left(\int_0^{-T} d\lambda a \right)_{(-1)} [b_\lambda c] &= \left(\int_0^{-T} d\lambda a \right)_{(-1)} \left(\sum_{j=0}^{j=\infty} \frac{\lambda^j}{j!} (b_{(j)} c) \right) \\ &= \sum_{j=0}^{j=\infty} \left(\left(\int_0^{-T} \frac{\lambda^j}{j!} d\lambda \right) a \right)_{(-1)} (b_{(j)} c) \\ &= \left(\sum_{j=0}^{j=\infty} \frac{(-T)^{j+1}}{(j+1)!} a \right)_{(-1)} (b_{(j)} c) \end{aligned}$$

Now applying $j+1$ times Exercise 11.3(1) to the term $\left(\frac{(-T)^{j+1}}{(j+1)!} a \right)_{(-1)} (b_{(j)} c)$ we get:

$$(-1)^{j+1} \left(\frac{T^{j+1}}{(j+1)!} a \right)_{(-1)} (b_{(j)} c) = (-1)^j \left(\frac{T^j}{(j)!} a \right)_{(-2)} (b_{(j)} c) = \dots = a_{(-j-2)} (b_{(j)} c)$$

□

11.8. Theorem. *Let V be a vertex algebra, then V is a Lie conformal (super)algebra with $\partial = T$ and*

$$[a_\lambda b] = \sum_{j \in \mathbb{Z}^+} \frac{\lambda^j}{j!} (a_{(j)} b)$$

Proof. We know that $[T, a_{(n)}]b = -na_{(n-1)}b$ by translation invariance. Also we know that $(Ta)_{(n)}b = -na_{(n-1)}b$ from Ex 11.3(1). Hence we have sesquilinearity. Skewsymmetry of the Lie conformal algebra is (2) in Ex 11.3 for $n \geq 0$. The Jacobi identity follows from the n -product theorem. \square

To give a conceptual proof of the above theorem, we introduce the **formal Fourier transform** of a formal distribution in one indeterminate:

$$F_z^\lambda a(z) = \text{Res}_z e^{\lambda z} a(z)$$

Its properties are summarized in:

- (1) $F_z^\lambda \partial_z a(z) = -\lambda F_z^\lambda a(z)$.
- (2) $F_z^{\lambda+\mu} [a(z)_\lambda b(z)] = [F_z^\lambda a(z), F_z^\mu b(z)]$
- (3) $F_z^\lambda (e^{zT} a(z)) = F_z^{\lambda+T} a(z)$.
- (4) $F_z^\lambda a(-z) = -F_z^{-\lambda} a(z)$.

Proof. (1)

$$\partial_z e^{\lambda z} a(z) = \lambda e^{\lambda z} a(z) + e^{\lambda z} \partial_z a(z)$$

Taking Res_z of both parts we get:

$$0 = \lambda F_z^\lambda a(z) + F_z^\lambda \partial_z a(z)$$

(2)

$$\begin{aligned} F_z^{\lambda+\mu} [a(z)_\lambda b(z)] &= \text{Res}_z e^{(\lambda+\mu)z} \text{Res}_x e^{\lambda(x-z)} (a(x)b(z) - p(a, b)b(z)a(x)) \\ &= \text{Res}_z \text{Res}_x (e^{\lambda x} a(x) e^{\mu z} b(z) - ip(a, b) e^{\lambda x} a(x) e^{\mu z} b(z)) \\ &= \text{Res}_x e^{\lambda x} a(x) \text{Res}_z e^{\mu z} b(z) - p(a, b) \text{Res}_z e^{\mu z} b(z) \text{Res}_x e^{\lambda x} a(x) \\ &= [F^\lambda a(z), F^\mu b(z)] \end{aligned}$$

$$(3) F_z^\lambda (e^{zT} a(z)) = \text{Res}_z e^{\lambda z} e^{zT} a(z) = \text{Res}_z e^{(\lambda+T)z} a(z) = F^{\lambda+T} a(z)$$

- (4) Denote $G(z) = e^{\lambda z} a(-z) + e^{-\lambda z} a(z)$. Then $G(z) = G(-z)$, hence $G(z)$ is an even formal distribution and its coefficients at odd powers are zero. In particular $\text{Res}_z G(z) = 0 \Rightarrow \text{Res}_z e^{\lambda z} a(-z) + \text{Res}_z e^{-\lambda z} a(z) = 0 \Rightarrow F_z^\lambda a(-z) + F_z^{-\lambda} a(z) = 0$.

\square

The fundamental property is that the Fourier transforms a vertex algebra to a Lie conformal (super)algebra.

$$F_z^\lambda (Y(a, z)b) = [a(z)_\lambda b(z)]$$

by definition of the λ -bracket. Applying F_z^λ to both sides of the skew-symmetry property of V produces skew-symmetry of the λ -bracket.

The n -product Theorem for $n \geq 0$ can be written as:

$$Y([a_\lambda b], z)c = [Y(a, z)_\lambda Y(b, z)]c$$

And now applying $F_z^{\lambda+\mu}$ to both sides we get the Jacobi identity.

12. EXISTENCE THEOREM

Exercise 12.1. The OPE for vertex operators $Y(a, z)$ and $Y(b, z)$ is given by

$$(12.0.1a) \quad [a_{(m)}, Y(b, w)] = \sum_{j \in \mathbb{Z}^+} \binom{m}{j} Y(a_{(j)}b, w) w^{m-j}$$

$$(12.0.1b) \quad = Y(e^{-wT} a_{(m)} e^{wT} b, w) \quad m \geq 0$$

Proof.

$$\begin{aligned} [a_{(m)}, Y(b, w)] &= \sum_{n \in \mathbb{Z}} [a_{(m)}, b_{(n)}] w^{-1-n} \\ &= \sum_{n \in \mathbb{Z}} \sum_{j \in \mathbb{Z}^+} \binom{m}{j} (a_{(j)}b)_{(m+n-j)} w^{-1-n} \\ &= \sum_{j \in \mathbb{Z}^+} \binom{m}{j} w^{m-j} \sum_{n \in \mathbb{Z}} (a_{(j)}b)_{(m+n-j)} w^{-1-(m+n-j)} \\ &= \sum_{j \in \mathbb{Z}^+} \binom{m}{j} w^{m-j} Y(a_{(j)}b, w) \end{aligned}$$

Note now that by Dong's lemma $[a_{(m)}, Y(b, w)]$ is local with $Y(c, z)$ for all c . To apply the uniqueness theorem we need to check the action on the vacuum vector, for that:

$$\begin{aligned} [a_{(m)}, Y(b, w)]|0\rangle &= a_{(m)}e^{wT}b - p(a, b)Y(b, w)a_{(m)}|0\rangle \\ &= a_{(m)}e^{wT}b \quad \text{since } m \geq 0 \end{aligned}$$

On the other hand:

$$\begin{aligned} Y(e^{-wT} a_{(m)} e^{wT} b, w)|0\rangle &= e^{wT} e^{-wT} a_{(m)} e^{wT} b \\ &= a_{(m)} e^{wT} b \end{aligned}$$

And the result follows from the uniqueness theorem. \square

Exercise 12.2. Derive from (12.0.1a) that

$$[a_\lambda Y(b, w)c] = p(a, b)Y(b, w)[a_\lambda c] + e^{\lambda w}Y([a_\lambda b], w)c$$

Hence $a_{(0)}$ is a derivation of all n -th products ($n \in \mathbb{Z}$). And this last equation is the non-abelian Wick Formula (in presence of locality).

Proof. Summing over m we get

$$\begin{aligned} \sum_{m \in \mathbb{Z}^+} a_{(m)} Y(b, w) c \frac{\lambda^m}{m!} &= \\ &= \sum_{m, j \in \mathbb{Z}^+} \binom{m}{j} w^{m-j} Y(a_{(j)}b, w) c \frac{\lambda^m}{m!} + \sum_{m \in \mathbb{Z}^+} p(a, b) Y(b, w) a_{(m)} c \frac{\lambda^m}{m!} \\ &= \sum_{m, j \in \mathbb{Z}^+} \frac{w^{m-j} \lambda^{m-j}}{(m-j)!} Y\left((a_{(j)}b) \frac{\lambda^j}{j!}, w\right) c + p(a, b) Y(b, w) [a_\lambda c] \\ &= e^{\lambda w} Y([a_\lambda b], w) + p(a, b) Y(b, w) [a_\lambda c] \end{aligned}$$

It is straightforward to check that we would get the same equation if we use the non-abelian Wick formula. \square

Exercise 12.3. Give an explicit expression for

$$(a_{(-1)}b)_{(-1)}c - a_{(-1)}(b_{(-1)}c)$$

Note now that the RHS is (super)symmetric in a and b . Deduce that $a_{(-1)}b - p(a, b)b_{(-1)}a$ is a Lie (super)algebra bracket.

Show also that the free boson ($[\alpha_\lambda\alpha] = \lambda$) is not associative

$$:: \alpha\alpha : \alpha : - : \alpha : \alpha\alpha :: \neq 0$$

Proof. In Ex 11.4 we proved that

$$(12.0.2) \quad (a_{(-1)}b)_{(-1)}c - a_{(-1)}(b_{(-1)}c) = \sum_j a_{(-j-2)}(b_{(j)}c) + p(a, b)(a \leftrightarrow b)$$

It is clear that the RHS is (super)symmetric in a and b . To check that $a_{(-1)}b - p(a, b)b_{(-1)}a$ is a Lie (super)algebra bracket we need to check only the Jacobi identity (skew-commutativity is obvious). For this we expand (all the products are (-1) -products):

$$\begin{aligned} [a, [b, c]] &= a(bc) - p(c, b)a(cb) - p(a, b)p(a, c)(bc)a + p(a, b)p(a, c)p(b, c)(cb)a \\ [b, [a, c]] &= b(ac) - p(c, a)b(ca) - p(a, b)p(b, c)(ac)b + p(a, b)p(b, c)p(a, c)(ca)b \\ [[a, b], c] &= -p(a, c)p(c, b)[c, [a, b]] = p(c, a)p(c, b)p(a, b)[c, [b, a]] \\ [[a, b], c] &= p(a, b)p(b, c)p(c, a)c(ba) - p(b, c)p(c, a)c(ab) - p(a, b)(ba)c + (ab)c \end{aligned}$$

Now we subtract the fourth equation and add the second (times $p(a, b)$) to the first equation to find:

$$\begin{aligned} &\underbrace{[a(bc) - (ab)c]}_I + \underbrace{p(a, b)p(b, c)p(c, a)[(cb)a - c(ba)]}_{II} - \underbrace{p(c, a)[a(cb) - (ac)b]}_{III} - \\ &- \underbrace{p(a, b)p(a, c)[(bc)a - b(ca)]}_{III} + \underbrace{p(a, b)[(ba)c - b(ac)]}_I + \underbrace{p(b, c)p(c, a)[c(ab) - (ca)b]}_{II} \end{aligned}$$

and since the associator is (super)symmetric in a and b we see that the terms with equal labels cancel each other, proving Jacobi Identity.

In the particular case of the Free boson applying directly (12.0.2) we get:

$$: \alpha\alpha : \alpha : - : \alpha : \alpha\alpha :: = 2 \sum \alpha_{(j-2)}(\alpha_{(j)}\alpha) = 2\alpha_{(-1)}1 = 2\alpha \neq 0.$$

□

12.1. Definition.

- (1) A **subalgebra** of a vertex algebra is a subspace \mathcal{U} , which contains $|0\rangle$ and is closed under all n -products.
- (2) An **Ideal** of a vertex algebra is a subspace J such that $|0\rangle \notin J$, $TJ \subset J$ and $V_{(n)}J \subset J$ for all $n \in \mathbb{Z}$.
- (3) An **Homomorphism** $\varphi : U \rightarrow V$ is a linear map such that $\varphi(a_{(n)}b) = \varphi(a)_{(n)}\varphi(b)$ for all n and $\varphi(|0\rangle_U) = |0\rangle_V$.
- (4) The **Tensor Product** of two vertex algebras V_1 and V_2 is the vertex algebra $V_1 \otimes V_2$ with $|0\rangle = |0\rangle_1 \otimes |0\rangle_2$ and

$$Y(a_1 \otimes a_2, z) = Y(a_1, z) \otimes Y(a_2, z) = \sum_{m, n \in \mathbb{Z}} (a_{1(m)} \otimes a_{2(n)})z^{-m-n-2}$$

Comments:

- If J is an ideal then also $J_{(n)}V \subset J$ for all $n \in \mathbb{Z}$ and this follows from the skew-symmetry of the n -th product and the fact that J is T -invariant.
- The kernel of a given morphism is an ideal.
- $a_{1(m)} \otimes a_{2(m)} \in \text{End}(V_1 \otimes V_2)$ is defined by

$$(a_{1(m)} \otimes a_{2(n)})(v_1 \otimes v_2) = p(a_2, v_1)a_{1(m)}v_1 \otimes a_{2(n)}v_2$$

$$\text{and } p(a \otimes b) = p(a) + p(b).$$

Exercise 12.4. Check that the tensor product defined above is a vertex algebra with $T = T_1 \otimes 1 + 1 \otimes T_2$.

Proof. We have to check several axioms.

vacuum axioms:

$$\begin{aligned} Y(|0 \rangle_1 \otimes |0 \rangle_2, z) &= Id_1 \otimes Id_2 = Id \\ Y(a \otimes b, z)|0 \rangle &= Y(a, z) \otimes Y(b, z)|0 \rangle \otimes |0 \rangle \\ &= e^{zT} a \otimes e^{zT} b \\ &= a \otimes b + a \otimes Tbz + Ta \otimes bz + o(z) \\ &= a \otimes b + T(a \otimes b)z + o(z) \end{aligned}$$

trans. invariance:

$$\begin{aligned} [T, Y(a \otimes b, z)] &= [T_1 \otimes 1, Y(a, z) \otimes Y(b, z)] + [1 \otimes T_2, Y(a, z) \otimes Y(b, z)] \\ &= [T_1, Y(a, z)] \otimes Y(b, z) + Y(a, z) \otimes [T_2, Y(b, z)] \\ &= \partial_z Y(a, z) \otimes Y(b, z) + Y(a, z) \otimes \partial_z Y(b, z) \\ &= \partial_z (Y(a, z) \otimes Y(b, z)) \\ &= \partial_z Y(a \otimes b, z) \end{aligned}$$

locality: expanding $[Y(a_1 \otimes b_1, z), Y(a_2 \otimes b_2, w)](v_1 \otimes v_2)$ we get:

$$\begin{aligned} &(Y(a_1, z) \otimes Y(b_1, z))(Y(a_2, w) \otimes Y(b_2, w))v_1 \otimes v_2 - \\ &\quad -p(a_1 + b_1, a_2 + b_2)(Y(a_2, w) \otimes Y(b_2, w))(Y(a_1, z) \otimes Y(b_1, z))v_1 \otimes v_2 \\ &= (Y(a_1, z) \otimes Y(b_1, z))p(b_2, v_1)Y(a_2, w)v_1 \otimes Y(b_2, w)v_2 - \\ &\quad -p(a_1 + b_1, a_2 + b_2)(Y(a_2, w) \otimes Y(b_2, w))p(b_1, v_1)Y(a_1, z)v_1 \otimes Y(b_1, z)v_2 \\ &= p(b_1, v_1)p(b_1, a_2)p(b_2, v_1)Y(a_1, z)Y(a_2, w)v_1 \otimes Y(b_1, z)Y(b_2, w)v_2 - \\ &\quad -p(a_1 + b_1, a_2 + b_2)p(b_1, b_1)p(b_2, a_1)p(b_2v_1)Y(a_2, w)Y(a_1, z)v_1 \otimes (b_2, w)Y(b_1, z)v_2 \\ &= (b_1, v_1)p(b_1, a_2)p(b_2, v_1)Y(a_1, z)Y(a_2, w)v_1 \otimes Y(b_1, z)Y(b_2, w)v_2 - \\ &\quad -p(a_1, a_2)p(b_1, a_2)p(b_1, b_2)p(b_1, v_1)p(b_2v_1)Y(a_2, w)Y(a_1, z)v_1 \otimes Y(b_2, w)Y(b_1, z)v_2 \\ &= p(b_1, v_1)p(b_1, a_2)p(b_2, v_1)Y(a_1, z)Y(a_2, w)v_1 \otimes Y(b_1, z)Y(b_2, w)v_2 - \\ &\quad -p(b_1, v_1)p(a_2, a_1)p(b_1, a_2)p(b_2, v_1)Y(a_2, w)Y(a_1, z)v_1 \otimes Y(b_1, z)Y(b_2, w)v_2 + \\ &\quad +p(b_1, v_1)p(a_2, a_1)p(b_2, v_1)p(b_1, a_2)Y(a_2, w)Y(a_1, z)v_1 \otimes Y(b_1, z)Y(b_2, w)v_2 - \\ &\quad -p(b_1, v_1)p(a_1, a_2)p(b_1, a_2)p(b_2v_1)p(b_1, b_2)Y(a_2, w)Y(a_1, z)v_1 \otimes Y(b_2, w)Y(b_1, z)v_2 \\ &= p(b_2, v_1)p(b_1, a_2)p(b_2, v_1)[Y(a_1, z), Y(a_2, w)]v_1 \otimes Y(b_1, z)Y(b_2, w)v_2 + \\ &\quad +p(a_2, a_1)p(b_2, v_1)p(b_1, a_2)Y(a_2, w)Y(a_1, z)v_1 \otimes [Y(b_1, z), Y(b_2, w)]v_2 \end{aligned}$$

Now it is clear that multiplying by $(z - w)^n$ we get zero for sufficiently large n . \square

12.2. Remark. An equivalent definition of a vertex algebra: quadruple $(V, |0\rangle, Y(a, z), T)$ where V is a vector space, $|0\rangle \in V$, $Y(a, z)$ is an $\text{End } V$ -valued field such that $a \rightarrow Y(a, z)$ is linear. Finally $T \in \text{End } V$ such that

- **Vacuum axioms** $Y(|0\rangle, z) = \text{Id}_V$, $Y(a, z)|0\rangle|_{z=0} = a$, $T|0\rangle = 0$,
- **Translation invariance** $[T, Y(a, z)] = \partial_z Y(a, z)$,
- **locality** $(z-w)^N [Y(a, z), Y(b, w)] = 0$ for some $N \gg 0$.

Is clear that the old definition implies the new one since $Ta = a_{(-2)}|0\rangle$, hence we get $T|0\rangle = |0\rangle_{(-2)}|0\rangle = 0$.

To prove the other implication we need to show that $Ta = a_{(-2)}|0\rangle$. But we have

$$TY(a, z)|0\rangle - Y(a, z)T|0\rangle = \partial_z Y(a, z)|0\rangle$$

Hence $Ta + O(z) = a_{(-2)}|0\rangle + O(z)$ so $Ta = a_{(-2)}|0\rangle$.

12.3. Theorem (Existence Theorem). *Let V be a vector space, $|0\rangle \in V$, and $T \in \text{End } V$. Let*

$$\mathcal{F} = \{a^j(z) = \sum a_{(n)}^j z^{-1-n}\}_{j \in J}$$

be a collection of $\text{End } V$ -valued fields such that the following properties hold:

- (1) $a^j(z)|0\rangle|_{z=0} = a^j \in V$, and $T|0\rangle = 0$.
- (2) $[T, a^j(z)] = \partial_z a^j(z)$.
- (3) All pairs $(a^i(z), a^j(z))$ are local.
- (4) The following vectors span V

$$a_{(n_s)}^{j_s} \dots a_{(n_1)}^{j_1} |0\rangle$$

Then the formula:

$$(12.3.1) \quad Y(a_{(n_s)}^{j_s} \dots a_{(n_1)}^{j_1} |0\rangle, z) = a^{j_s}(z)_{(n_s)} (\dots a^{j_2}(z)_{(n_2)} (a^{j_1}(z)_{(n_1)} \text{Id}_V) \dots)$$

defines a structure of a vertex algebra on V with vacuum vector $|0\rangle$, infinitesimal translation operator T , such that

$$(12.3.2) \quad Y(a^j, z) = a^j(z).$$

Such a structure of vertex algebra is unique.

Proof. Uniqueness is clear since for any vertex algebra we have $Y(a_{(n)}b, z) = Y(a, z)_{(n)}Y(b, z)$.

To prove existence we may assume that the identity operator is in \mathcal{F} . We claim that the following linear map is injective

$$\begin{aligned} \varphi : \text{span } \mathcal{F} &\rightarrow V, \\ a^j(z) &\rightarrow a^j. \end{aligned}$$

Indeed, suppose that $a(z) = \sum c_j a^j(z)$, and $\varphi a(z) = 0$, hence

$$a(z)|0\rangle|_{z=0} = 0$$

Now we state the following lemma which is important in its own:

12.4. Lemma. *Let V be a vector space, $|0\rangle \in V$ a vector and $\{a^j(z)\}$ be a family of $\text{End } V$ -valued fields satisfying (4) above. Let $a(z)$ be an $\text{End } V$ -valued field such that*

- (1) all $(a^j(z), a(z))$ are local pairs.
- (2) $a(z)|0\rangle = 0$.

then $a(z) = 0$

Proof.

$$\begin{aligned} & \prod_i (z_i - w)^{N_i} a(w) a^{j_1}(z_1) \dots a^{j_s}(z_s) |0 \rangle = \\ & = \pm \prod_i (z_i - w)^{N_i} a^{j_1}(z_1) \dots a^{j_s}(z_s) a(w) |0 \rangle \\ & = 0 \end{aligned}$$

But since all the products are fields, the LHS is a Laurent series in

$$V((z_s)) \dots ((z_1))((w))$$

Which doesn't have zero divisors, hence we can cancel $\prod (z_i - w)^{N_i}$. Taking now the coefficients of $z_1^{-1-n_1} \dots z_s^{-1-n_s}$ we get that

$$a(w)(a_{(n_1)}^{j_1} \dots a_{(n_s)}^{j_s} |0 \rangle) = 0$$

and due to (4) we have then $a(w) = 0$. \square

Going back to the proof of the theorem, we have $a(z)|0 \rangle|_{z=0} = 0$. It is easy to see that $\varphi \partial a(z) = T \varphi a(z)$ and this together with (2) implies $a^j(z)|0 \rangle = e^{Tz} a^j$, hence $a(z)|0 \rangle = 0$ and now by the lemma we have $a(z) = 0$.

Now we have φ injective and using (4) we choose a basis of V consisting of vectors $a_{(n_s)}^{j_s} \dots a_{(n_1)}^{j_1} |0 \rangle$ that includes the vectors $|0 \rangle$ and a^j such that $a^j(z)$ is a basis for the span of \mathcal{F} . Define the vertex operators by (12.3.1) for this basis. This gives us a state-field correspondence, and we have to check that it satisfies all the three axioms in the definition of vertex algebra. Locality is clear by Dong's lemma. The vacuum axioms follow easily from lemma 10.7. Translation invariance holds since $\text{ad } T$ and ∂_z are derivations of all n -th products and both annihilate Id_V therefore they are equal.

To finish the proof, note that injectivity of φ implies

$$a(z)|0 \rangle = \sum c_{j_1, \dots, j_s, n_1, \dots, n_s} a_{(n_s)}^{j_s} \dots a_{(n_1)}^{j_1} |0 \rangle$$

if and only if

$$a(z) = \sum c_{j_1, \dots, j_s, n_1, \dots, n_s} a^{j_s}(z)_{(n_s)} (\dots a^{j_2}(z)_{(n_2)} (a^{j_1}(z)_{(n_1)} \text{Id}_V) \dots)$$

Since (12.3.1) holds for basis vectors by definition, and both sides are linear, it holds in general. The same argument applies to (12.3.2). \square

12.5. Example. Vertex Algebra of the Free Boson. The vector space is $V = \mathbb{C}[x_1, x_2, \dots]$, $|0 \rangle = 1$. Let

$$\alpha(z) = \sum_{n \in \mathbb{Z}} \alpha_n z^{-1-n}$$

where for $n > 0$ we let $\alpha_n = \frac{\partial}{\partial x_n}$ and $\alpha_{-n} = n x_n$, $\alpha_0 = 0$. Let for $j_1, \dots, j_n \geq 1$.

$$Y(x_{j_1} x_{j_2} \dots x_{j_n}, z) = \frac{: \partial_z^{j_1-1} \alpha(z) \dots \partial_z^{j_n-1} \alpha(z) :}{j_1! j_2! \dots j_n!}$$

where for any fields $a_1(z), \dots, a_n(z)$, we define

$$: a_1(z) \cdots a_n(z) := a_1(z) : a_2(z) : \dots : a_{n-1}(z) a_n(z) : \dots ::$$

This is a vertex algebra. To show this we apply the existence theorem. Let $\mathcal{F} = \{\alpha(z)\}$; we only need to find T such that $[T, \alpha(z)] = \partial_z \alpha(z)$.

Exercise 12.5. $T = \sum_{n \geq 2} n x_n \frac{\partial}{\partial x_{n-1}}$ has this property.

Proof. for $m > 0$ we have

$$\begin{aligned} [T, \alpha_m]f &= \sum_{n \geq 2} n x_n \frac{\partial^2 f}{\partial x_{n-1} \partial x_m} - \sum_{n \geq 2} n \frac{\partial}{\partial x_m} \left(x_n \frac{\partial f}{\partial x_{n-1}} \right) \\ &= - \sum_{n \geq 2} n \delta_{n,m} \frac{\partial f}{\partial x_{n-1}} \\ &= -m \frac{\partial}{\partial x_{m-1}} f \\ &= -m \alpha_{m-1} f \end{aligned}$$

And also for $m > 0$.

$$\begin{aligned} [T, \alpha_{-m}]f &= \sum_{n \geq 2} n x_n \frac{\partial}{\partial x_{n-1}} (x_m f) - \sum_{n \geq 2} n x_n x_m \frac{\partial f}{\partial x_{n-1}} \\ &= \sum_{n \geq 2} n \delta_{m,n-1} x_n f \\ &= (m+1) x_{m+1} f \\ &= -(-m-1) \alpha_{-m-1} f \end{aligned}$$

□

13. EXAMPLES OF VERTEX ALGEBRAS

Applications of the existence Theorem. Recall that a formal distribution Lie (super)algebra is a pair $(\mathfrak{g}, \mathcal{F})$ where \mathfrak{g} is a Lie (super)algebra and \mathcal{F} is a local family of \mathfrak{g} -valued formal distributions, whose coefficients span \mathfrak{g} . It is called **regular** if there exist a derivation T of \mathfrak{g} such that $Ta(z) = \partial_z a(z)$. Recall also that \mathfrak{g}_- is defined to be the span of all $a_{(n)}$ with $n \geq 0$ and $a \in \mathcal{F}$, it is a subalgebra of \mathfrak{g} called the **annihilation subalgebra** since we have

$$[a_{(m)}, b_{(n)}] = \sum_{j \in \mathbb{Z}_+} \binom{m}{j} (a_{(j)} b)_{(m+n-j)}$$

Moreover $T\mathfrak{g}_- \subset \mathfrak{g}_-$ since $Ta_{(n)} = -na_{(n-1)}$.

Exercise 13.1. Let T be a derivation of the Lie (super)algebra \mathfrak{g} . Show that

- (1) T extends uniquely to a derivation T of $\mathcal{U}(\mathfrak{g})$, and that $T(1) = 0$.
- (2) The Center of \mathfrak{g} is T -invariant.

Proof. (1) Since T is a derivation, it satisfies the identity

$$T([a, b]) = [T(a), b] + (-1)^{p(a)p(T)} [a, T(b)]$$

for any pair of elements $a, b \in \mathfrak{g}$. We only need to define T on the basis elements of $\mathcal{U}(\mathfrak{g})$, $a_{i_1} \cdot a_{i_2} \cdot \dots \cdot a_{i_n}$. For $n > 0$ we do it inductively (by n) by :

$$T(ba_i) = T(b)a_i + (-1)^{p(b)p(T)} bT(a_i)$$

For $n = 0$, observe that we must have $T(1 \cdot 1) = T(1) + T(1)$, that is $T(1) = 0$. This much is forced on us by requiring that T be a derivation of

$\mathcal{U}(\mathfrak{g})$, hence if T exists, it is unique. Now, by linearity, we can see that for any two elements $b, c \in \mathcal{U}(\mathfrak{g})$, we have

$$T(bc) = T(b)c + (-1)^{p(b)p(T)}bT(c)$$

from which the commutation relation follows trivially.

- (2) Pick an arbitrary element a of the center of \mathfrak{g} . By definition, it commutes with all elements of \mathfrak{g} , or, put it another way, $b \in \mathfrak{g}$ we have $[a, b] = 0$. In particular, for all $b \in \mathfrak{g}$, we have $[a, T(b)] = 0$. Now, by the definition of a derivation we obtain :

$$[T(a), b] = T([a, b]) - (-1)^{p(a)p(T)}[a, T(b)] = T(0) = 0$$

so $T(a)$ commutes with all elements of \mathfrak{g} and therefore is in the center. \square

13.1. Theorem. For \mathfrak{g} regular, the space $V(\mathfrak{g}, \mathcal{F}) = \mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})\mathfrak{g}_-$ carries a unique structure of a vertex algebra such that $|0\rangle$ (defined as the image of 1 in V) is the vacuum vector, T is the derivation of $\mathcal{U}(\mathfrak{g})$ which extends T acting on \mathfrak{g} and

$$Y(a_{(-1)}|0\rangle) = a(z)$$

acting on $V(\mathfrak{g}, \mathcal{F})$, $a(z) \in \mathcal{F}$.

Exercise 13.2. For each $a(z) \in \mathcal{F}$, the corresponding $\text{End } V(\mathfrak{g}, \mathcal{F})$ -valued formal distribution is a field.

Proof. Let $a(z) = \sum a_{(i)}z^{-1-i}$. Since \mathfrak{g}_- is generated by all $a_{(i)}$ for $i \geq 0$, passing to the quotient the image of $a(z)$ will be

$$\overline{a(z)} = \sum_{i < 0} \overline{a_{(i)}}z^{-1-i}$$

which shows that the image of $a(z)$ is a field (it contains only summands of positive degree). \square

Proof of 13.1. It follows easily from theorem 12.3 and Ex.13.2 \square

13.2. Remark. Let C be a central subalgebra of \mathfrak{g} which is T -invariant, and $\lambda : C \rightarrow \mathbb{C}$ be a linear function such that $\lambda(TC) = 0$ Then we define the ideal

$$(13.2.1) \quad I^\lambda = \text{span}\{(c - \lambda(c))V(\mathfrak{g}, \mathcal{F}) \mid c \in C\}$$

and the quotient vertex algebra $V^\lambda(\mathfrak{g}, \mathcal{F}) = V(\mathfrak{g}, \mathcal{F})/I^\lambda$.

13.3. Example (Kac-Moody Affinization).

$$\begin{aligned} \hat{\mathfrak{g}} &= \mathfrak{g}[t, t^{-1}] + \mathbb{C}K \\ [at^m, bt^n] &= [a, b]t^{m+n} + m\delta_{m, -n}(a, b) \\ \mathcal{F} &= \{a(z) = \sum at^n z^{-1-n}, K\} \\ T &= -\partial_t \quad \text{on } \mathfrak{g}[t, t^{-1}], T(K) = 0 \\ Ta_{(n)} &= -\partial_t(at^n) = -na_{(n-1)} \end{aligned}$$

Hence $Ta(z) = \partial_z a(z)$ and we get $V(\hat{\mathfrak{g}}, \mathcal{F})$ is the so called **Universal affine Vertex Algebra**. If we take $C = \mathbb{C}K$, $\lambda(C) = k \in \mathbb{C}$ (called the **level**) then we get that $V^k(\mathfrak{g}) := V^\lambda(\hat{\mathfrak{g}}, \mathcal{F})$ is the **universal affine Vertex Algebra of level k** .

13.4. Remark. Recall that $\mathfrak{g} = \mathfrak{g}_- + \mathfrak{g}_+$ and \mathfrak{g}_+ is also a T -invariant subalgebra, hence $V(\mathfrak{g}, \mathcal{F}) = \mathcal{U}(\mathfrak{g}_+)|0\rangle$. We see then that the elements

$$a_{(-1-j_1)}^1 \cdots a_{(-1-j_s)}^s |0\rangle$$

span $V(\mathfrak{g}, \mathcal{F})$ and the state field correspondence looks as follows for $j_i \geq 0$.

$$(13.4.1) \quad Y(a_{(-1-n_1)}^{j_1} \cdots a_{(-1-n_s)}^{j_s} |0\rangle, z) = \frac{:\partial_z^{n_1} a^{j_1}(z) \cdots \partial_z^{n_s} a^{j_s}(z):}{n_1! \cdots n_s!}$$

13.5. Proposition. *Provided that \mathfrak{g} is reductive and (\cdot, \cdot) is non-degenerate invariant and provided that $k \neq -h_i^v$ (the dual Coxeter number of the i -th simple component) then $V^k(\mathfrak{g})$ has a unique maximal ideal $I^k(\mathfrak{g})$ so that $V_k(\mathfrak{g}) = V^k(\mathfrak{g})/I^k(\mathfrak{g})$ is a simple Vertex Algebra.*

Proof. Let $L(z) = \sum L^{\mathfrak{g}_i}(z)$ where each summand is given by the Sugawara construction. We proved that

$$[L_\lambda a] = (\partial + \lambda)a, \quad a \in \mathcal{F}$$

and in particular that $[L_0, a_{(n)}] = -na_{(n)}$. Also we have $L_0|0\rangle = L_{(1)}|0\rangle = 0$. Hence the vector

$$a_{(-1-j_1)}^1 \cdots a_{(-1-j_s)}^s |0\rangle$$

is an eigenvector of L_0 with eigenvalues $\sum (j_s + 1) \geq 0$. So L_0 is a diagonal operator with non-negative eigenvalues. Hence any ideal is \mathbb{Z}_+ -graded and doesn't contain $|0\rangle$ we have that the sum of proper ideals is again a proper ideal, the proposition follows easily by standard arguments. \square

13.6. Example (Virasoro). here we have the usual commutation relations:

$$\begin{aligned} [L_\lambda L] &= (\partial + 2\lambda)L + \frac{\lambda^3}{12}C \\ L(z) &= \sum L_n z^{-n-2} \\ \mathcal{F} &= \{L(z), C\} \\ T &= \text{ad } L_{-1} \end{aligned}$$

We have in this case the **universal Virasoro Vertex Algebra** $V(\text{Vir}, \mathcal{F})$. Also we get V^c which is the **universal Virasoro Vertex Algebra with central charge c** , and the simple algebra $V_c = V^c/I^c$ the unique maximal ideal.

13.7. Example (Clifford Affinization).

$$\begin{aligned} \hat{A} &= A[t, t^{-1}] + \mathbb{C}K \\ [at^m, at^n] &= \delta_{m, -n-1} \langle a, b \rangle K \\ \mathcal{F} &= \{a(z) = \sum at^n z^{-1-n}, K\} \\ T &= -\partial_t \quad \text{on } A[t, t^{-1}], \quad T(K) = 0 \\ Ta_{(n)} &= -\partial_t(at^n) = -na_{(n-1)} \end{aligned}$$

And we get $V(\hat{A}, \mathcal{F})$, the **universal v.a. of free fermions**. Take $C = \mathbb{C}K$, $\lambda(K) = 1$ we get a vertex algebra defined by $F(A)$ (**Vertex Algebra of Free Fermions**)

Exercise 13.3. *Provided that the skew-symmetric form is non-degenerate, show that $F(A)$ is simple.*

Proof. Any ideal I of $F(A)$ will be an invariant \hat{A} -submodule, and this implies $I = F(A)$ by Ex.8.4 \square

Exercise 13.4. Write down the explicit state-field correspondence for V^c .

Proof. This is just a reformulation of (13.4.1):

$$Y(L_{j_1} \dots L_{j_k} c^j, z) = \frac{1}{(-j_1 - 2)!} \dots \frac{1}{(-j_k - 2)!} : \partial^{-j_1-2} L(z) \dots \partial^{-j_k-2} L(z) : c^j \text{Id}.$$

\square

14. POISSON VERTEX ALGEBRAS

14.1. Definition. Let V be a vertex algebra, a vector $\nu \in V$ is called a **conformal vector** if

- (1) $Y(\nu, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ where L_n form the Virasoro algebra with central charge c .
- (2) $L_{-1} = T$.
- (3) L_0 is diagonalizable.

14.2. Remark. $L_0 a = \Delta a$ for some $\Delta \in \mathbb{C}$ is equivalent to say that $Y(a, z)$ is an eigenfield with respect to $Y(\nu, z)$ of conformal weight Δ , since

$$[\nu_\lambda a] = \nu_{(0)} a + \lambda \nu_{(1)} a + o(\lambda) = T a + \lambda \Delta a + o(\lambda).$$

14.3. Example. Let \mathfrak{g} be a finite dimensional simple Lie (super)algebra. $V^k(\mathfrak{g})$ the corresponding vertex algebra of level $k \neq -h^\vee$. Then

$$\nu = \frac{1}{2(k + h^\vee)} \sum a_{(-1)}^i a_{i(-1)} |0\rangle$$

is a conformal vector (Here $\{a_i\}, \{a^i\}$ are dual bases of \mathfrak{g}). Indeed

$$Y(\nu, z) = \frac{1}{2(k + h^\vee)} \sum : a^i(z) a_i(z) :$$

And this is the Virasoro element given in the Sugawara construction with the central charge $c = c_k = \frac{k \text{ sdim } \mathfrak{g}}{k + h^\vee}$. We proved that $[L_\lambda a] = (\partial + \lambda)a$, and hence $L_{(0)} a = \partial a$ and $L_{(1)} a = a$. Hence $L_{(0)} = L_{-1} = T$. Also we proved that L_0 is diagonalizable with eigenvalues in \mathbb{Z}_+ .

Exercise 14.1.

- (1) For the Virasoro vertex algebra V^c , $\nu = L_{-2}|0\rangle$ is a conformal vector and $Y(\nu, z) = L(z)$.
- (2) For $F(A)$ we have $\nu = \frac{1}{2} \sum \varphi_{(-2)}^i \varphi_{i(-1)} |0\rangle$, with central charge $c = -\frac{\text{sdim } A}{2}$.

Proof.

- (1) Using the explicit formula for the state-field correspondence we know that $Y(\nu, z) = Y(L_{(-1)}|0\rangle, z) = L(z)$ as claimed, and we have seen that L_n form the Virasoro algebra and that $[L_\lambda L] = (\partial + 2\lambda)L + \frac{c}{12}\lambda^3$. So $L_{-1} = T$ and $[L_0, L_m] = -mL_m$ so L_0 is diagonalizable.
- (2) Again, $Y(\nu, z) = Y(\frac{1}{2} \sum \varphi_{(-2)}^i \varphi_{i(-1)} |0\rangle, z) = \frac{1}{2} \sum_i : \partial \varphi^i(z) \varphi_i(z) :$ which we know is a Virasoro distribution with central charge $c = -\frac{\text{sdim } A}{2}$ and $L_{(0)} \varphi = \partial \varphi$ by Proposition 7.5 and $[L_0, \varphi_n] = -n\varphi_n$. So the properties of $L_{-1} = L_{(0)}$ and L_0 are as claimed.

□

Last time to any regular formal distribution Lie (super)algebra $(\mathfrak{g}, \mathcal{F})$ we associated the corresponding universal vertex algebra $V(\mathfrak{g}, \mathcal{F}) = \mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})\mathfrak{g}_-$. But given a Lie conformal (super)algebra R we have canonically associated a maximal formal distribution Lie (super)algebra $(\text{Lie}R, R)$, and it is regular. Recall that

$$(14.3.1) \quad \text{Lie}R = (\text{Lie}R)_+ \oplus (\text{Lie}R)_-$$

as vector spaces and this decomposition is T -invariant.

14.4. Definition. The **Universal Enveloping** vertex algebra of a Lie conformal (super)algebra R is

$$V(R) := V(\text{Lie}R, R)$$

14.5. Example. The previous constructions for the Virasoro algebras and the current algebras are examples of this. The construction for the free fermions is similar.

Due to the decomposition (14.3.1) we get the corresponding decomposition for the enveloping algebras, i.e.

$$\mathcal{U}(\text{Lie}R) = \mathcal{U}((\text{Lie}R)_+) \otimes \mathcal{U}((\text{Lie}R)_-)$$

where $(\text{Lie}R)_+$ and $(\text{Lie}R)_-$ are subalgebras. Hence $V(R) = \mathcal{U}((\text{Lie}R)_+)$ is the universal enveloping (super)algebra. For example if $R = \text{Cur}(\mathfrak{g})$ then $V(R) = \mathcal{U}(\mathfrak{g}[t^{-1}]t^{-1} \oplus \mathbb{C}K)$.

Note we have the map $\varphi : R \rightarrow V(R)$ sending $a \rightarrow a_{(-1)}|0\rangle$ and we know this map is injective (cf. proof of theorem 12.3).

Exercise 14.2. Prove that the map φ is a homomorphism of Lie conformal (super)algebras.

Proof. By the lemma 12.4, we know that in $V(R) = V(\text{Lie}R, \bar{R})$, $a = b$ if and only if $Y(a, z) = Y(b, z)$. By definition of $V(R)$, $Y(\varphi(a), z) = Y(a_{(-1)}|0\rangle, z) = \bar{a}(z) \in \bar{R}$ for any field $a \in R$. The fact that φ is a homomorphism of Lie conformal superalgebras follows from the fact that the map $R \rightarrow \bar{R}$ is. Indeed we have $\partial a(z) = \partial \bar{a}(z)$ and $[a_\lambda b](z) = [\bar{a}(z)_\lambda \bar{b}(z)]$. □

Proof. First it is clear that φ is a module homomorphism since $\varphi(\partial a) = (\partial a)_{(-1)}|0\rangle$, and $Y((\partial a)_{(-1)}|0\rangle, z) = \partial a(z) = \partial Y(a_{(-1)}|0\rangle, z)$. The fact that λ -bracket structures are compatible is due to our construction of Lie bracket on $\text{Lie}R$, or on \bar{R} ; it is shown in the proof of 3.3 □

14.6. Theorem. $V(R)$ has the following universality property: any homomorphism from R to a vertex algebra V , viewed as a Lie conformal (super)algebra, extends uniquely to a vertex algebra homomorphism $V(R) \rightarrow V$.

Proof. Any $\varphi : R \rightarrow V$ induces a map $\tilde{\varphi} : \text{Lie}R \rightarrow \text{End} V$, $\tilde{\varphi}(a_n) = \varphi(a)_{(n)}$. Hence, by the universal property of universal enveloping algebra, $\tilde{\varphi}$ extends to homomorphism $\mathcal{U}(\text{Lie}R) \rightarrow \text{End} V$. This gives us a map $\psi : \mathcal{U}(\text{Lie}R) \rightarrow V$, $\psi(u) = \tilde{\varphi}(u)|0\rangle$. Since $\psi((\text{Lie}R)_-) = 0$, we get

$$\psi : \mathcal{U}(\text{Lie}R)/\mathcal{U}(\text{Lie}R)(\text{Lie}R)_- = V(R) \rightarrow V$$

such that

- (1) $\psi(R_{(-1)}V(R)) \simeq \psi(R)_{(-1)}\psi(V(R))$
- (2) $\psi([a_\lambda b]) = [\psi(a)_\lambda \psi(b)]$

(3) commutes with T.

Hence ψ is a homomorphism of vertex algebras because of quasi-commutativity, quasi-associativity, and the Wick formula. \square

Recall that any vertex algebra is a Lie (super)algebra with respect to the bracket $[a, b] = a_{(-1)}b - p(a, b)b_{(-1)}a$. But due to anti-commutativity we get that this bracket is $\int_{-T}^0 [a_\lambda b]$. Hence $R \subset V(R)$ is a lie subalgebra with respect to this bracket. Denote this algebra by R_{Lie} (with bracket $[a, b] = \int_{-T}^0 [a_\lambda b] d\lambda$). But we have an injective map $\varphi : R \rightarrow V(R)$ given by $a \rightarrow a_{(-1)}|0\rangle$ so this extends to an isomorphism of vector spaces

$$\mathcal{U}(R_{\text{Lie}}) \simeq V(R) \simeq \mathcal{U}((\text{Lie}R)_+)$$

Note the parallel to the Lie algebra case: Any homomorphism from \mathfrak{g} to an associative algebra U viewed as a Lie algebra (with bracket $[a, b] = ab - ba$) extends uniquely to a homomorphism of associative algebras $\mathcal{U}(\mathfrak{g}) \rightarrow U$.

Exercise 14.3.

- (1) $R_{\text{Lie}} \simeq (\text{Lie}R)_+$ by the map $a \rightarrow a_{(-1)}|0\rangle$ which is an isomorphism of Lie (super)algebras.
- (2) The product $R \cdot \mathcal{U}(R_{\text{Lie}})$ coincides with the (-1) st product in $V(R)$.

Proof. (1) The fact that φ gives an isomorphism of vector spaces follows by exercise 14.2. We check what happens to the Lie bracket $[a, b] = \int_{-T}^0 [a_\lambda b] d\lambda$ in R , which in $V(R)$ can be written as $:ab : -p(a, b) :ba :$. One way to proceed is to notice that by acting on vacuum, this is just a special case (lhs) of Borcherds formula ($m, n \in \mathbb{Z}$)

$$[a_{(m)}, a_{(n)}] = \sum_{j \in \mathbb{Z}_+} \binom{m}{j} (a_{(j)}b)_{(m+n-j)}$$

with $m = n = -1$, and the right hand side precisely coincides with the Lie bracket structure on $(\text{Lie}R)_+$ constructed in Proposition 3.3.

- (2) This is easily seen on the fields level, since if $a \in R$ and $B \in \mathcal{U}(R_{\text{Lie}})$, then the field corresponding to the expression aB is

$$Y(aB, z) =: Y(a, z)Y(B, z) := Y(a, z)_{(-1)}Y(B, z) = Y(a_{(-1)}B, z)$$

Hence the product coincides with (-1) st product in $V(R)$. \square

Recall some general properties of vertex algebras.

- We have the λ -bracket $[a_\lambda b] = \sum \lambda^j (a_{(j)}b)/j!$. And with respect to this product V is a Lie conformal (super)algebra.
- We have the (-1) -product $a_{(-1)}b =: ab :$ with the following properties:
 - (1) $:|0\rangle a := a$.
 - (2) $:ab : -p(a, b) :ba := \int_{-T}^0 [a_\lambda b] d\lambda$ (quasi-commutativity).
 - (3) $:(:ab :)c : - :a(:bc :) := \left(\int_0^T d\lambda a \right) [b_\lambda c] + p(a, b)(a \leftrightarrow b)$ (quasi-associativity)
 - (4) T is an even derivation of normal ordered products.
- $[a_\lambda :bc :] =: [a_\lambda b]c : + p(a, b) :b[a_\lambda c] : + \int_0^T [[a_\lambda b]_\mu c] d\mu$ (quasi-Leibniz).

14.7. Definition. A **Poisson** vertex algebra is one with “the quantum corrections” removed, i.e. it is a Lie conformal (super)algebra (the bracket denoted $\{\lambda\}$) with a product, which is a unital commutative associative (super)algebra, such that

- (1) ∂ is an even derivation of the product.
- (2) $\{a_\lambda bc\} = \{a_\lambda b\}c + p(a, b)b\{a_\lambda c\}$ (left Leibniz rule)..

Exercise 14.4. *Derive the analogous formula for the Right Leibniz rule.*

Proof.

$$\begin{aligned}
[(a_{(-1)}b)_\lambda c] &= \sum_{n \in \mathbb{Z}_+} \frac{\lambda^n}{n!} (a_{(-1)}b)_{(n)} c \\
&= \sum_{n \in \mathbb{Z}_+} \text{Res}_z \frac{\lambda^n z^n}{n!} (a_{(-1)}b)_{(n)} z^{-1-n} c \\
&= \text{Res}_z e^{\lambda z} Y(a_{(-1)}b, z) c \\
&= \text{Res}_z e^{\lambda z} : Y(a, z) Y(b, z) : c \\
&= \text{Res}_z e^{\lambda z} (Y(a, z)_+ Y(b, z) c + p(a, b) Y(b, z) Y(a, z)_- c)
\end{aligned}$$

Using the identities

$$Y(a, z)_+ Y(b, z) c = (e^{zT} a)_{(-1)} (Y(b, z) c)$$

and

$$Y(a, z)_- c = (a_{-\partial_z} c) z^{-1}$$

and integrating by parts the first term is

$$\text{Res}_z (e^{T\partial_\lambda} e^{\lambda z} a)_{(-1)} (Y(b, z) c) = (e^{T\partial_\lambda} a)_{(-1)} [b_\lambda c]$$

The remainder gives:

$$\begin{aligned}
p(a, b) \text{Res}_z Y(b, z) [a_{\lambda - \partial_z} c] (e^{\lambda z} z^{-1}) &= \\
&= p(a, b) \text{Res}_z Y(b, z) [a_{\lambda - \partial_z} c] \left(z^{-1} + \int_0^\lambda e^{\mu z} d\mu \right) \\
&= p(a, b) \text{Res}_z (e^{\partial_z \partial_\lambda} Y(b, z)) [a_\lambda c] z^{-1} + p(a, b) \int_0^\lambda \text{Res}_z Y(b, z) [a_{\lambda - \mu} c] e^{\mu z} d\mu \\
&= p(a, b) (e^{T\partial_\lambda} b)_{(-1)} [a_\lambda c] + p(a, b) \int_0^\lambda [b_\mu [a_{\lambda - \mu} c] d\mu
\end{aligned}$$

Throwing away the quantum correction and combining with above we get:

$$\{ab_\lambda c\} = (e^{T\partial_\lambda} a) \{b_\lambda c\} + p(a, b) (e^{T\partial_\lambda} b) \{a_\lambda c\}.$$

□

14.8. Definition. A quantization of the Poisson vertex algebra V_0 is a family of vertex algebras V_\hbar such that the product on V_0 is the limit of the (-1) -st product in V_\hbar as $\hbar \rightarrow 0$, and

$$\{a_\lambda b\} = \lim_{\hbar \rightarrow 0} \frac{[a_\lambda b]_\hbar}{\hbar}$$

We say that V_0 is the quasi-classical limit of V_\hbar .

14.9. Remark. In $V(R) : a(bc) := abc$ if $a, b, c \in R$ by Ex. 14.3, but this is not true in general. For example : $(ab)c := abc + \text{quantum corrections}$.

14.10. Example. Let R be a Lie conformal (super)algebra. Consider the following family of Lie conformal (super)algebras

$$R_{\hbar} = R, \quad [a_{\lambda}b]_{\hbar} = \hbar[a_{\lambda}b]$$

And now consider the following family of vertex algebras

$$V_{\hbar} = V(R_{\hbar}) = \mathcal{U}(R_{\hbar\text{Lie}})$$

But

$$\mathcal{U}(R_{\hbar\text{Lie}}) = T(R)/(a \otimes b - p(a, b)b \otimes a - \int_{-T}^0 [a_{\lambda}b]_{\hbar} d\lambda)$$

Hence we have $\lim_{\hbar \rightarrow 0} V_{\hbar} = T(R)/(a \otimes b - p(a, b)b \otimes a) = S(R)$. And the Poisson bracket is

$$\{a_{\lambda}b\} = \lim_{\hbar \rightarrow 0} \frac{[a_{\lambda}b]_{\hbar}}{\hbar} = [a_{\lambda}b] \quad a, b \in R$$

but by Ex 14.3 we know that the product of R and $\mathcal{U}(R_{\text{Lie}})$ is the usual product in $V(R)$, hence when $\hbar \rightarrow 0$ we get the usual product in $S(R)$.

We can conclude that the quasi-classical limit of V_{\hbar} is $S(R)$ with λ -bracket given by $\{a_{\lambda}b\} = [a_{\lambda}b]$ if $a, b \in R$ and extended to $S(R)$ by Leibniz rule and skew-commutativity.

15. INFINITE-DIMENSIONAL HAMILTONIAN SYSTEMS

Let \mathcal{M} be a N -dimensional manifold with x being local coordinates. Consider a space \mathcal{S} of C^{∞} vector-valued functions $\vec{u}(x) = (u_1(x), \dots, u_r(x))$ on \mathcal{M} whose partial derivatives tend to zero when x goes to infinity. The most general Poisson bracket we are going to consider is given by:

$$(15.0.1) \quad \{u_i(x), u_j(y)\} = B_{i,j}(\vec{u}(x), \vec{u}'(x), \dots)\delta(x - y) = B_{ij}[u](x, y),$$

where B_{ij} are differential operators with coefficients being polynomials in u_i and finite number of partial derivatives

$$u_i^{(\alpha)}(x) = \partial^{(\alpha)} u_i(x) = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} u_i(x).$$

δ is a distribution defined by

$$\int_{\mathcal{M}} \delta(x - y)f(y)dy = f(x).$$

We can extend (15.0.1) by linearity and Leibniz rule to arbitrary differential polynomials, i.e. polynomials in the $u_i^{(\alpha)}$'s. Namely, if P and Q are differential polynomials the bracket $\{P, Q\}$ looks as

$$(15.0.2) \quad \{P, Q\} = \sum_{\alpha, \beta, i, j} \frac{\partial P}{\partial u_i^{(\alpha)}} \frac{\partial Q}{\partial u_j^{(\beta)}} \partial_x^{\alpha} \partial_y^{\beta} B_{ij}[u](x, y).$$

Exercise 15.1. Prove that (15.0.2) satisfies Leibniz rule (and therefore (15.0.1) implies (15.0.2)).

Proof. The statement is a simple corollary of the properties of derivative. If P_1 , P_2 and Q are differential polynomials

$$\begin{aligned} \{P_1 P_2, Q\} &= \sum_{\alpha, \beta, i, j} \frac{\partial(P_1 P_2)}{\partial u_i^{(\alpha)}} \frac{\partial Q}{\partial u_j^{(\beta)}} \partial_x^\alpha \partial_y^\beta B_{ij}[u](x, y) \\ &= P_1 \sum_{\alpha, \beta, i, j} \frac{\partial P_2}{\partial u_i^{(\alpha)}} \frac{\partial Q}{\partial u_j^{(\beta)}} \partial_x^\alpha \partial_y^\beta B_{ij}[u](x, y) + \sum_{\alpha, \beta, i, j} \frac{\partial P_1}{\partial u_i^{(\alpha)}} \frac{\partial Q}{\partial u_j^{(\beta)}} \partial_x^\alpha \partial_y^\beta B_{ij}[u](x, y) P_2 \\ &= P_1 \{P_2, Q\} + \{P_1, Q\} P_2. \end{aligned}$$

We omit checking the Leibniz rule for the second argument since we assume that bracket (15.0.1) is skew-symmetric and the skew-symmetry property is thoroughly discussed in the next exercise. \square

Exercise 15.2. *The skew-symmetry is equivalent to the fact that $B_{ij}^* = -B_{ji}$ where $*$ is the usual anti-involution on differential operators such that $*(\partial_i) = -\partial_i$ and $*(u_i^{(\alpha)}) = u_i^{(\alpha)}$.*

Proof. It is very convenient to introduce the following notation

$$(15.0.3) \quad u_i^f = \int_{\mathcal{M}} dx u^i(x) f(x),$$

where $f(x)$ is a smooth function on \mathcal{M} with a compact support. We call f test function. Then (15.0.1) can be rewritten as

$$(15.0.4) \quad \{u_i^f, u_j^g\} = \int_{\mathcal{M}} f(x) (B_{ij}(u^{(\alpha)}) g)(x)$$

With equation (15.0.4) the statement of the exercise easily follows from the rule of integration by parts, properties of the functions $u_i \in \mathcal{S}$ and definition of the test functions f and g . \square

Warning! We should be careful with the definition of the involution $*$ when B_{ij} are not constants but polynomials in u 's and their partial derivatives. For example if $B = u(x) \partial_x$

$$B^* = -(\partial_x u(x)) - u(x) \partial_x.$$

If the requirement of skew-symmetry is satisfied and (15.0.2) defines a Lie algebra on the functionals we consider B_{ij} is called a **Hamiltonian operator**.

The basic quantities are **local functionals**³:

$$(15.0.5) \quad I_P = \int_{\mathcal{M}} P(x) dx, \quad P \in A = \mathbb{C}[u_i, \partial^\alpha u_i].$$

Extending (15.0.2) by linearity gives

$$\begin{aligned} \{I_P, I_Q\} &= \int_{\mathcal{M}} \int_{\mathcal{M}} \sum \frac{\partial P(x)}{\partial u_i^{(\alpha)}} \frac{\partial Q(y)}{\partial u_j^{(\beta)}} \partial_x^\alpha \partial_y^\beta \{u_i(x), u_j(y)\} dx dy \\ &= \sum_{i, j} \int_{\mathcal{M}} \int_{\mathcal{M}} \frac{\delta P(x)}{\delta u_i} \frac{\delta Q(y)}{\delta u_j} \{u_i(x), u_j(y)\} dx dy. \end{aligned}$$

³Also called Hamiltonian functions

In the second line we integrate by parts assuming that $u_i \in \mathcal{S}$ and denote

$$\frac{\delta P(x)}{\delta u_i} = \sum_{\alpha \in \mathbb{Z}_+^n} (-\partial_x)^\alpha \frac{\partial P}{\partial u_i^{(\alpha)}}(x).$$

Now we can integrate by parts again and using the definition of the Hamiltonian in (15.0.1) we get

$$(15.0.6) \quad \{I_P, I_Q\} = \sum_{i,j} \int_{\mathcal{M}} \frac{\delta P(x)}{\delta u_i} \left(B_{ij} \frac{\delta Q(x)}{\delta u_j} \right) dx.$$

From now on, assume $N = 1$ and let $\partial = \frac{d}{dx}$, so that

$$\frac{\delta P}{\delta u_i} = \sum_{\alpha \in \mathbb{Z}^+} (-\partial)^\alpha \frac{\partial P}{\partial u_i^{(\alpha)}}$$

and $u_i^{(\alpha)} = \partial^\alpha u_i$.

Taking an algebraic point of view, we think of $\int P(x)dx$ as the image of $P(x)$ in the quotient space $A/\sum \partial_i A = \bar{A}$. We need to prove the following:

15.1. Lemma. *The bracket given in (15.0.6) is well defined on \bar{A} .*

Exercise 15.3. *Prove the lemma.*

Proof. This is a standard exercise from the course of classical mechanics or classical field theory.

On the set of local functionals (15.0.5) we have

$$\partial = \sum_{\alpha \geq 1, i} \partial^\alpha u_i \frac{\partial}{\partial u_i^{\alpha-1}}.$$

Hence

$$\left[\frac{\partial}{\partial u_i^\alpha}, \partial \right] = \frac{\partial}{\partial u_i^{\alpha-1}}.$$

The latter gives us

$$\begin{aligned} \frac{\delta}{\delta u_i} \partial &= \sum_{\beta \geq 0} (-\partial)^\beta \frac{\partial}{\partial u_i^\beta} \partial \\ &= \sum_{\beta \geq 0} (-\partial)^\beta (\partial)^{\beta+1} \frac{\partial}{\partial u_i^\beta} + \sum_{\beta \geq 1} (-\partial)^\beta \frac{\partial}{\partial u_i^{\beta-1}} = 0 \end{aligned}$$

□

15.2. Remark. $\frac{\delta}{\delta u}$ is not a derivation as the following example shows:

$$\frac{\delta}{\delta u} (u')^2 = (-2u')' = -2''$$

but $\frac{\delta}{\delta u} u' = 0$.

Given a Hamiltonian function $h = \int P(x)dx$ we can write the corresponding Hamiltonian system of PDE on $u_i = u_i(t, x)$:

$$\dot{u}_i = \{u_i, h\}.$$

Using formula (15.0.2) we get

$$\dot{u}_i(x) = \int \sum \frac{\partial P}{\partial u_j^\alpha}(y) \partial_y^\alpha B_{ij}^\beta(u_k^\gamma(x)) \partial_x^\beta \delta(x-y) dy.$$

Integration by parts yields

$$\dot{u}_i(x) = \int \sum \frac{\delta P}{\delta u_j}(y) B_{ij}^\beta(u_k^\gamma(x)) \partial_x^\beta \delta(x-y) dy.$$

Integrating by parts once again and getting rid of $\delta(x-y)$ -function we get

$$(15.2.1) \quad \dot{u}_i(x) = \sum B_{ij} \frac{\delta P(x)}{\delta u_j}.$$

Moreover, if h_1 is another Hamiltonian function such that $\{h, h_1\} = 0$, then due to the Leibniz identity we get from the evolution equation $\dot{u}_i = \{h, u_i\}$ that $\dot{h}_1 = \{h, h_1\} = 0$ hence h_1 is an integral of motion.

Relation to Poisson Vertex Algebras. On the unital commutative associative algebra A , we have the Poisson bracket given by (15.0.1).

$$\{u_i(x), u_j(y)\} = B_{ij}(u_k^\alpha(x)) \delta(x-y).$$

Applying the Fourier transform

$$F(\varphi(x, y))(x) := \int e^{-\lambda(x-y)} \varphi(x, y) dy$$

to the Hamiltonian operator

$$B_{ij}[u](x, y) = \sum_{\alpha \geq 1} B_{ij}^\alpha(u_k^\beta(x)) \partial_x^\alpha \delta(x-y)$$

we get

$$\{u_i \lambda u_j\}(x) = B_{ij}(u_k^\beta, \lambda) = \sum_{\alpha \geq 1} B_{ij}^\alpha(u_k^\beta(x)) \lambda^\alpha$$

We may think of $u_i(x)$ as “formal distributions” and then the machinery that we have developed so far is applicable and we get a structure of a Lie conformal algebra on A with the λ -bracket $\{\lambda\}$. It obeys Leibniz rule and we have $T = \partial$ hence A is a Poisson vertex algebra.

A natural general setup is to take for A an arbitrary Poisson v.a. then the Hamiltonian functions are simply elements of A/TA , and the commutator in (15.0.6) reads:

$$\{I_{u_i}, I_{u_j}\} = \int B_{ij}(u_i, \dots) 1 dx$$

And this is equivalent to replace $\lambda = 0$ in the λ -bracket. Hence the commutator on A/TA is defined by ($a, b \in A/TA$)

$$(15.2.2) \quad [a, b] = [a_\lambda b]_{\lambda=0} \pmod{TA}$$

15.3. Theorem. $B_{ij} = \sum_{\alpha \geq 1} B_{i,j}^\alpha(u_i^\beta) \partial^\alpha$ is a Hamiltonian operator on $\mathbb{C}[u_i^\beta]$ if and only if the λ -bracket $[u_i \lambda u_j] = \sum_{\alpha \geq 1} B_{ij}^\alpha \lambda^\alpha$ defines a structure of a Poisson v.a.

The statement is proved analogously as the Jacobi identity for the Lie conformal (super)algebra induced by a formal distribution Lie super-algebra.

15.4. Theorem. *Let R be a Lie conformal algebra and define on R a bracket by (15.2.2). Then*

- (1) *ad a is a derivation of this bracket.*
- (2) *$\{\partial R, R\} = 0$.*
- (3) *∂R is a 2 – sided ideal with respect to (15.2.2).*
- (4) *$R/\partial R$ with the induced bracket is a Lie algebra.*

Exercise 15.4. *Prove this theorem.*

Proof. Let us recall axioms of the Lie conformal (super)algebra introduced in the lecture 2.

$$\begin{aligned}
 (15.4.1) \quad & \text{(Sesquilinearity)} \quad [\partial a_\lambda b] = -\lambda[a_\lambda b], \quad [a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b], \\
 & \text{(Skew – commutativity)} \quad [b_\lambda a] = -p(a, b)[a_{(-\lambda-\partial)}b], \\
 & \text{(Jacobi identity)} \quad [a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu}c] + p(a, b)[b_\mu [a_\lambda c]].
 \end{aligned}$$

Setting $\lambda = \mu = 0$ in Jacobi identity and in the first equation of sesquilinearity (15.4.1) we get the first two statements of the theorem. From the second equation of sesquilinearity we find that

$$[a_\lambda \partial b] |_{\lambda=0} \in \partial R .$$

Finally, setting $\lambda = 0$ in skew-commutativity equation (15.4.1) we get

$$([a_\lambda b] + p(a, b)[b_\lambda a]) |_{\lambda=0} \in \partial R .$$

Thus the last statement of the theorem also follows.

For a general Poisson v.a. A , given a Hamiltonian function $h \in \bar{A}$ we can write the corresponding Hamiltonian system ($u \in A$)⁴:

$$\dot{u} = -\{h, u\} .$$

If $h_1 \in \bar{A}$ is an involution with h , i.e. $\{h, h_1\} = 0$ then $\dot{h}_1 = 0$ in \bar{A} . Thus the space of all integral of motion is the centralizer of h with respect to to the bracket in the space \bar{A} . □

16. BI-HAMILTONIAN SYSTEMS

Last time we worked in the polynomial algebra $A = \mathbb{C}[u_i, u_i, \dots]$ using the notation $u_i^{(\alpha)}$, $i \in I$, $\alpha \in \mathbb{Z}_+$, $\partial = \frac{d}{dx} : u^{(\alpha)} \rightarrow u^{(\alpha)}$. We also defined $\bar{A} = A/\partial A$ whose elements are the Hamiltonian functions. Given $P \in A$ its image in \bar{A} is denoted by \bar{P} , $\int P dx$ or I_P . The variational derivatives are linear operators $\frac{\delta}{\delta u_i}$ on A defined by $\frac{\delta P}{\delta u_i} = \sum_{\alpha \in \mathbb{Z}_+} (-\partial)^\alpha \frac{\partial P}{\partial u_i^{(\alpha)}}$. Let $B = (B_{ij})$, $i, j \in I$ is a matrix with coefficients $B_{ij} = B_{ij}(u, u', \dots, \frac{\partial}{\partial x})$ which are differential operators with coefficients in A . Given such a matrix we define a bracket on A extending by the ordinary Leibniz rule the formula $\{u_i(x), u_j(y)\} = B_{ij}(y)\delta(x-y)$. B is called a Hamiltonian operator if this bracket satisfies one of the equivalent definitions:

- (1) $\{u_i, u_j\} = B(u, u', \dots, \lambda)$ is a Poisson vertex algebra.
- (2) $\{, \}$ satisfies the Jacobi identity.

⁴Note that the bracket is well defined in virtue of the second statement of theorem 15.4

Exercise 16.1. Extend the theory to the case of $x = (x_1, \dots, x_N)$ by taking $\bar{A} = A / \sum \partial_i A$, and

$$\frac{\delta P}{\delta u_i} = \sum_{\alpha \in \mathbb{Z}_+^N} (-\partial)^\alpha \frac{\partial P}{\partial u_i^{(\alpha)}}$$

Let $\frac{\delta P}{\delta u}$ be the vector $(\frac{\delta P}{\delta u_1}, \dots, \frac{\delta P}{\delta u_r})^t$. Denote the space of such vectors by $\frac{\delta A}{\delta u}$ and we define the standard bilinear form on this space by

$$(16.0.2) \quad (\xi, \eta) = \sum_i \xi_i \eta_i.$$

We have the induced Lie algebra bracket on \bar{A}

$$\{h, g\}_B = \int \left(B \frac{\delta h}{\delta u}, \frac{\delta g}{\delta u} \right),$$

provided that B is a Hamiltonian operator. Since the bracket $\{, \}_B$ is skew-symmetric we see that the operator B is skew-symmetric on the space $\frac{\delta A}{\delta u}$ with respect to the bilinear form (16.0.2):

$$(16.0.3) \quad (B\xi, \eta) = -(\xi, B\eta)$$

We have also the evolution equations $\dot{u}_i = \{h, u_i\}$ and $h_i \in \bar{A}$ is an integral of motion ($\dot{h}_i = 0$) if $\{h, h_1\} = 0$.

16.1. Example. $\{u_i \lambda u_j\} = B_{ij}(\lambda)$, it is clearly a Poisson vertex algebra when $B_{ij}(-\lambda) = -B_{ji}(\lambda)$ is a Lie conformal algebra ($B_{ij} \in \mathbb{C}[\lambda]$) therefore the bracket

$$\{h, g\} = \left(B \left(\frac{\partial}{\partial x} \right) \frac{\delta h}{\delta u} \right) \frac{\delta g}{\delta u},$$

is a Lie bracket. In the special case of one dimension we have $B_{11} = \partial_x$ and $\{h, g\} = \int \frac{d}{dx} \left(\frac{\delta h}{\delta u} \right) \frac{\delta g}{\delta u}$ is a Lie algebra bracket, called the Gardner-Fadeev-Zakkarov bracket.

16.2. Example. $r = 1$, and

$$\{u_\lambda u\} = (\partial + 2\lambda)u + \lambda^3 + a\lambda \quad (a \in \mathbb{C})$$

Is a Virasoro Lie conformal algebra (we just added a trivial cocycle). The corresponding bracket is given by

$$\{h, g\} = \int \left(\left(\underbrace{u' + 2u \frac{d}{dx} + \left(\frac{d}{dx} \right)^3}_H + a \underbrace{\frac{d}{dx}}_K \right) \frac{\delta h}{\delta u} \right) \frac{\delta g}{\delta u}$$

Both H and K are Hamiltonian operators, $\alpha H + \beta K$ is too. We say that (H, K) is a **bi-hamiltonian Pair**.

16.3. Example. \mathfrak{g} is a Lie algebra with invariant bilinear form $(,)$, $p \in \mathfrak{g}$, c_{ij}^k are the structure constants: $[u_i, u_j] = \sum_k c_{ij}^k u_k$. Then

$$\{u_i \lambda u_j\} = \sum_k c_{ij}^k u_k + a\lambda(u_i, u_j) + (p, [u_i, u_j])$$

where $a \in \mathbb{C}$ is a Lie conformal algebra bracket. The corresponding Hamiltonian operator is

$$B_{ij} = \underbrace{\sum_k c_{ij}^k u_k + (p, [u_i, u_j])}_H + \underbrace{a(u_i, u_j)}_K \frac{d}{dx}$$

Then (H, K) is a bi-hamiltonian pair (Drinfeld-Sokolov).

16.4. Example (Lennard Scheme). Let H and K be two skew-symmetric operators in the sense of (16.0.3). Suppose there exists a sequence of Hamiltonian functions $h_\alpha \in \bar{A}$, $\alpha = 0, 1, \dots, n$ such that $K(\frac{\delta h_\alpha}{\delta u}) = H(\frac{\delta h_{\alpha-1}}{\delta u})$ for all α . Then $\{h_\alpha, h_\beta\}_H = 0 = \{h_\alpha, h_\beta\}_K$ for all α, β . Hence all the h_α are integral of motion for

$$\dot{u}_i = \{h_\alpha, u_i\}_K = \{h_{\alpha-1}, u_i\}_H$$

Proof. Let $\xi_\alpha = \frac{\delta h_\alpha}{\delta u}$, then $K\xi_\alpha = H\xi_{\alpha-1}$ and we may assume $\alpha < \beta \leq h$ and we get

$$\{h_\alpha, h_\beta\}_H = (H\xi_\alpha, \xi_\beta) = (K\xi_{\alpha+1}, \xi_\beta) = -(\xi_{\alpha+1}, K\xi_\beta) = -\{h_\beta, h_{\alpha+1}\}_K$$

Similarly $\{h_\alpha, h_\beta\}_K = \{h_\alpha, h_{\beta-1}\}_H$ and after a finite number of steps we get $\{h_\alpha, h_\beta\}_H = \{h_\gamma, h_\gamma\}_{H,K} = 0$ for $\alpha \leq \gamma \leq \beta$. \square

16.5. Remark. Writing $\xi(x, z) = \sum_{\alpha \in \mathbb{Z}_+} \xi_\alpha(x) z^{-\alpha}$. Then the condition $K\xi_\alpha = H\xi_{\alpha-1}$ is equivalent to the condition:

$$(H - zK)\xi(x, z) = 0$$

and this is a differential equation on ξ , we solve it to get $\xi(x, z) = \sum_{\alpha} \xi_\alpha z^{-\alpha}$ but now we need to find h_α such that $\xi_\alpha = \frac{\delta h_\alpha}{\delta u}$.

16.6. Example.

$$H = u' + 2u \frac{d}{dx} + \left(\frac{d}{dx} \right)^3$$

$$K = \frac{d}{dx}$$

Then we get the differential equation $\xi^{(3)} + (2u - z)\xi' + u'\xi = 0$, where primes means differentials with respect to x . We can reduce the order of this equation multiplying by ξ to get

$$f(z) = \xi\xi'' - \frac{1}{2}\xi'^2 + (u - \frac{1}{2}z)\xi^2$$

we look for a solution of this equation of the form $\xi = 1 + \sum_{j \geq 1} \xi_j z^{-j}$. The coefficient of z in the RHS is $-1/2$ hence $f(z) = -z/2$ plus negative powers of z hence $f(z) = -z/2$. We plug this scheme in the equation to get (comparing coefficients of z^0) $\xi_1 = u$, (comparing coefficients of z^{-1}) $\xi_2 = u'' + \frac{3}{2}u^2$.

Exercise 16.2. Continue in this line of reasoning to get $\xi_3 = u^{(4)} + \frac{5}{2}u^3 + \frac{5}{2}u'^2 + 5uu''$

Taking ξ_1 and ξ_2 as above, the coefficient of z^{-2} on the RHS will be

$$\begin{aligned}
& [z^{-2}] \left[(1 + uz^{-1})(u''z^{-1} + (u^{(4)} + \right. \\
& \quad \left. + 3u'^2 + 3uu'')z^{-2}) - \frac{1}{2}(u'z^{-1} + u^{(3)}z^{-2} + 3uu'z^{-2})^2 + \right. \\
& \quad \left. + (u - \frac{1}{2}z)(1 + uz^{-1} + (u'' + \frac{3}{2}u^2)z^{-2} + \xi_3 z^{-3})^2 \right] \\
& = u^{(4)} + 3u'^2 + 3uu'' + uu'' - \frac{1}{2}u'^2 + u^3 + 2uu'' + 3u^3 - uu'' - \text{frac}32u^3 - \xi_3 \\
& = u^{(4)} + \frac{5}{2}u^3 + \frac{5}{2}u'^2 + 5uu' - \xi_3
\end{aligned}$$

Exercise 16.3. *More generally prove that $\xi_k = u^{(2k-2)}$ plus polynomials of differential degrees less or equal than $2k - 2$. Compute also h_3 (see below)*

Now we want to find h such that $\frac{\delta h_\alpha}{\delta k} = \xi_\alpha$ we get then

$$\begin{aligned}
h_0 &= u, \\
h_1 &= \frac{1}{2}u^2, \\
h_2 &= \frac{1}{2}u^3 - \frac{1}{2}u'^2
\end{aligned}$$

The equation corresponding to h_2 with respect to $\{, \}_K$ is

$$\dot{u} = \frac{d}{dx} \frac{\delta h_2}{\delta u} = \left(\frac{3}{2}u^2 + u'' \right)' = 3uu' + u^{(3)}$$

which is the KdV system and we proved that the first integral of motion are:

$$\int u, \quad \int u^2, \quad \int u^3 - u'^2$$

We need to make sure that such an h exists, for that we state the following

16.7. Theorem. *Provided that (K, H) is bi-hamiltonian and K is non-degenerate in the sense that $KMK = 0 \Rightarrow M = 0$ for any operator M . Suppose that $K\xi = H\eta$ and $K\zeta = H\xi$, with $\xi, \eta \in \frac{\delta A}{\delta u}$. Then $\zeta \in \frac{\delta A}{\delta u}$.*

Proof. In the conditions for the Lennard Scheme we know that $K\xi_\alpha = H\xi_{\alpha-1}$ and $K\xi_{\alpha+1} = H\xi_\alpha$. Hence $\xi_{\alpha-1}, \xi_\alpha \in \frac{\delta A}{\delta u}$ implies that $\xi_{\alpha+1} \in \frac{\delta A}{\delta u}$, so it suffices to check that $\xi_0, \xi_1 \in \frac{\delta A}{\delta u}$. \square

Given ξ the formula for h is

$$h = \int_0^1 d\sigma(\xi(\sigma u), u)$$

16.8. Example. for $\xi = u'' + \frac{3}{2}u^2$ we get

$$\begin{aligned}
h &= \int \sigma uu'' + \sigma^2 \frac{3}{2}u^3 d\sigma \\
&= \frac{1}{2}uu'' + \frac{1}{2}u^3
\end{aligned}$$

17. LATTICE VERTEX ALGEBRAS I - TRANSLATION INVARIANCE

Exercise 17.1. Consider the following one-dimensional algebra $R = \mathbb{C}[\partial_1, \partial_2]\omega$ with $\lambda = (\lambda_1, \lambda_2)$ -bracket:

$$[\omega_\lambda \omega] = \lambda_2 \partial_1 \omega - \lambda_1 \partial_2 \omega + \alpha_1 \lambda_1 + \alpha_2 \lambda_2, \quad \alpha_i \in \mathbb{C}$$

Show that this is a Lie conformal algebra corresponding to the Lie algebra of divergenceless vector fields on $S^1 \times S^1$ with the corresponding Hamiltonian operator:

$$B\omega = \frac{\partial \omega}{\partial x} \frac{\partial}{\partial y} - \frac{\partial \omega}{\partial y} \frac{\partial}{\partial x} + \alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y}$$

Proof. Let \mathfrak{g} be the Lie algebra of polynomial divergenceless vector fields on the 2-torus. Parameterize torus $S^1 \times S^1$ by angle coordinates $(e^{\pi i \theta_1}, e^{\pi i \theta_2})$. Let $x = e^{\pi i \theta_1}$ and $y = e^{\pi i \theta_2}$. The vector fields on the torus then will be of the form $f \partial_x + g \partial_y$. By Green's theorem each divergenceless vector field is given by the following formula: $V(f) = \partial_y f \partial_x - \partial_x f \partial_y$, for some polynomial function f . It is easy to see that, the Lie bracket for such fields is given then as follows:

$$[V(f), V(g)] = V(\partial_x g \partial_y f - \partial_y g \partial_x f)$$

A function on $S^1 \times S^1$ can be thought of as a smooth double-periodic function on \mathbb{R}^2 and from Fourier analysis we know that functions $e^{n\pi i \theta_1} e^{m\pi i \theta_2} = x^n y^m$ comprise a basis for the space of such polynomial functions. Hence we can choose the following vector fields to be a basis of our Lie algebra: $a_{n,m} = V(x^n y^m) = mx^n y^{m-1} \partial_x - nx^{n-1} y^m \partial_y$. Then:

$$\begin{aligned} [a_{n,m}, a_{n',m'}] &= [V(x^n y^m), V(x^{n'} y^{m'})] \\ &= V(n' x^{n'-1} y^{m'} m x^n y^{m-1} - n x^{n-1} y^m m' x^{n'} y^{m'-1}) \\ &= (n'm - nm') a_{n+n'-1, m+m'-1} \end{aligned}$$

Consider now a central extension $\widehat{\mathfrak{g}}$ of \mathfrak{g} given by a basis $\{L_{n,m}\}_{n,m \in \mathbb{Z}}, \alpha_1, \alpha_2$, where α_1 and α_2 are central elements, and commutation relations:

$$\begin{aligned} [L_{n_1, m_1}, L_{n_2, m_2}] &= (n_2 m_1 - n_1 m_2) L_{n_1+n_2-1, m_1+m_2-1} + \\ &\quad + \alpha_1 n_1 \delta_{n_1, -n_2} \delta_{m_1+m_2, -1} + \alpha_2 m_1 \delta_{n_1+n_2, -1} \delta_{m_1, -m_2} \end{aligned}$$

We need to check that this central extension still defines a Lie algebra:

(1) *Skew-symmetry.* This is clear since:

$$\begin{aligned} [L_{n_1, m_1}, L_{n_2, m_2}] + [L_{n_2, m_2}, L_{n_1, m_1}] &= (n_2 m_1 - n_1 m_2) L_{n_1+n_2-1, m_1+m_2-1} + \\ &\quad + \alpha_1 n_1 \delta_{n_1, -n_2} \delta_{m_1+m_2, -1} + \alpha_2 m_1 \delta_{n_1+n_2, -1} \delta_{m_1, -m_2} + \\ &\quad + (n_1 m_2 - n_2 m_1) L_{n_2+n_1-1, m_2+m_1-1} + \\ &\quad + \alpha_1 n_2 \delta_{n_2, -n_1} \delta_{m_2+m_1, -1} + \alpha_2 m_2 \delta_{n_2+n_1, -1} \delta_{m_2, -m_1} \\ &= \alpha_1 (n_1 + n_2) \delta_{n_1+n_2, 0} \delta_{m_1+m_2, -1} + \alpha_2 (m_1 + m_2) \delta_{n_1+n_2, -1} \delta_{m_1+m_2, 0} \\ &= 0 \end{aligned}$$

(2) *Jacobi identity.* The proof is a tedious computation:

$$\begin{aligned}
& [L_{n_1, m_1}, [L_{n_2, m_2}, L_{n_3, m_3}]] + [L_{n_2, m_2}, [L_{n_3, m_3}, L_{n_1, m_1}]] + [L_{n_3, m_3}, [L_{n_1, m_1}, L_{n_2, m_2}]] = \\
& = [L_{n_1, m_1}, (n_3 m_2 - m_3 n_2) L_{n_2+n_3-1, m_2+m_3-1} + \alpha_1 A_1 + \alpha_2 B_1] + \\
& \quad + [L_{n_2, m_2}, (n_1 m_3 - m_1 n_3) L_{n_1+n_3-1, m_1+m_3-1} + \alpha_1 A_2 + \alpha_2 B_2] + \\
& \quad + [L_{n_3, m_3}, (n_2 m_1 - m_2 n_1) L_{n_1+n_2-1, m_1+m_2-1} + \alpha_1 A_3 + \alpha_2 B_3] \\
& = - \sum_{\sigma \in A_3} \alpha_1 n_{\sigma_1} (n_{\sigma_3} m_{\sigma_2} - m_{\sigma_3} n_{\sigma_2}) \delta_{n_{\sigma_1}+n_{\sigma_2}+n_{\sigma_3}, 1} \delta_{m_{\sigma_1}+m_{\sigma_2}+m_{\sigma_3}, 2} - \\
& \quad - \sum_{\sigma \in A_3} \alpha_2 m_{\sigma_1} (n_{\sigma_3} m_{\sigma_2} - m_{\sigma_3} n_{\sigma_2}) \delta_{n_{\sigma_1}+n_{\sigma_2}+n_{\sigma_3}, 0} \delta_{m_{\sigma_1}+m_{\sigma_2}+m_{\sigma_3}, 1} \\
& = \alpha_1 (n_1 n_2 m_3 - n_1 n_2 m_1 + n_2 n_1 m_1 - n_2 n_3 m_2 + n_2 n_3 m_1 - n_2 n_3 m_2) \times \\
& \quad \times \delta_{n_1+n_2+n_3, 1} \delta_{m_1+m_2+m_3, 0} + \\
& \quad + \alpha_2 (m_1 m_2 n_3 - m_1 m_2 n_1 + m_2 m_1 n_1 - m_2 m_3 n_2 + m_2 n_3 n_1 - m_2 m_3 n_2) \times \\
& \quad \times \delta_{n_1+n_2+n_3, 0} \delta_{m_1+m_2+m_3, 1} \\
& = 0
\end{aligned}$$

Define a formal distribution $\omega = L(z_1, z_2) = \sum_{n, m \in \mathbb{Z}} L_{n, m} z_1^{-n-1} z_2^{-m-1}$.

We have:

$$\begin{aligned}
& [L(z_1, z_2), L(w_1, w_2)] = \\
& = \sum z_1^{-n_1-1} z_2^{-m_1-1} w_1^{-n_2-1} w_2^{-m_2-1} [L_{n_1, m_1}, L_{n_2, m_2}] \\
& = z_1^{-n_1-1} z_2^{-m_1-1} w_1^{-n_2-1} w_2^{-m_2-1} (m_1 n_2 - n_1 m_2) L_{n_1+n_2-1, m_1+m_2-1} + \\
& \quad + \sum \alpha_1 n_1 z_1^{-n_1-1} z_2^{-m_1-1} w_1^{n_1-1} w_2^{m_1} + \sum \alpha_2 m_1 z_1^{-m_1-1} z_2^{-m_2-1} w_1^{m_1} w_2^{m_2-1} \\
& = (\sum k z_2^{-k-1} w_2^{k-1}) (\sum (-n-1) w_1^{-n-2} w_2^{-m-1} L_{m, n}) (\sum z_1^{-l-1} w_1^l) - \\
& \quad - (\sum k z_1^{-k-1} w_1^{k-1}) (\sum (-m-1) w_1^{-n-1} w_2^{-m-2} L_{m, n}) (\sum z_2^{l-1} w_2^l) + \\
& \quad + \alpha_1 (\sum k z_1^{-k-1} w_1^{k-1}) (\sum z_2^{-l-1} w_2^l) + \alpha_2 (\sum k z_2^{-k-1} w_2^{k-1}) (\sum z_1^{-l-1} w_1^l) \\
& = \partial_{w_2} \delta(z_2, w_2) \partial_{w_1} L(w_1, w_2) \delta(z_1, w_1) - \partial_{w_1} \delta(z_1, w_1) \partial_{w_2} L(w_1, w_2) \delta(z_2, w_2) + \\
& \quad + \alpha_1 \partial_{w_1} \delta(z_1, w_1) \delta(z_2, w_2) + \alpha_2 \partial_{w_2} \delta(z_2, w_2) \delta(z_1, w_1)
\end{aligned}$$

Now apply the Fourier transform to get:

$$\begin{aligned}
[\omega_\lambda \omega] & = F_{z_1, w_1}^{\lambda_1} F_{z_2, w_2}^{\lambda_2} (\partial_{w_2} \delta(z_2, w_2) \partial_{w_1} L(w_1, w_2) \delta(z_1, w_1) - \\
& \quad - \partial_{w_1} \delta(z_1, w_1) \partial_{w_2} L(w_1, w_2) \delta(z_2, w_2) + \alpha_1 \partial_{w_1} \delta(z_1, w_1) \delta(z_2, w_2) \\
& \quad + \alpha_2 \partial_{w_2} \delta(z_2, w_2) \delta(z_1, w_1)) = \\
& = \lambda_2 \partial_1 \omega - \lambda_1 \partial_2 \omega + \alpha_1 \lambda_1 + \alpha_2 \lambda_2
\end{aligned}$$

Since we obtained the λ -bracket from the commutation relations of the \mathfrak{g} -valued formal distribution, this bracket defines a Lie conformal algebra (c.f. Lecture 2).

The fact that B is a Hamiltonian operator follows from theorem 15.3. Indeed, the (λ_1, λ_2) -bracket gives a Lie conformal algebra on R . By example 14.10, the quasi-classical limit of $V(R_\hbar)$ is $S(R)$ with λ -bracket from R extended to $S(R)$ by Leibnitz rule and skew-commutativity. Hence on $S(R)$ our λ -bracket defines a

structure of a Poisson vertex algebra, and the associated operator

$$B\omega = \frac{\partial\omega}{\partial x} \frac{\partial}{\partial y} - \frac{\partial\omega}{\partial y} \frac{\partial}{\partial x} + \lambda_1 \frac{\partial}{\partial x} + \lambda_2 \frac{\partial}{\partial y}$$

is Hamiltonian. □

Lattice Vertex Algebras. We have constructed a vertex algebra corresponding to any Lie conformal algebra. In particular, given a finite dimensional Lie algebra \mathfrak{g} with a non-degenerate symmetric bilinear form, we can construct its Kac-Moody affinization $\hat{\mathfrak{g}}$ and the corresponding Lie conformal algebra $\text{Cur}\mathfrak{g} = \mathbb{C}[\partial]\mathfrak{g} + \mathbb{C}K$. We get then the universal affine vertex algebra $V(\text{Cur}\mathfrak{g})$ and $V^k(\mathfrak{g}) = V(\text{Cur}\mathfrak{g})/(K - k)$ the universal vertex algebra of level k .

A very special case is a commutative Lie algebra \mathfrak{h} with a non-degenerate symmetric bilinear form. We have $V^1(\mathfrak{h})$ and the corresponding $R = \mathbb{C}[\partial]\mathfrak{h} + \mathbb{C}K$. We know also that the quasi-classical limit of $V(R)$ is $S(R)$. We can view then the vertex algebra $V^1(\mathfrak{h})$ as a quantization of the Poisson algebra of functions $\text{Map}(S^1, \mathfrak{h}) = \mathfrak{h}[t, t^{-1}]$. An important open problem is how to quantize the algebra of functions $\text{Map}(S^1, M)$, where M is an arbitrary manifold. by **quantization** we would understand a family of non-commutative associative algebras A_{\hbar} , such that $\lim_{\hbar \rightarrow 0} A_{\hbar} = A$.

The simplest case after the flat space \mathfrak{h} is \mathfrak{h}/Q where Q is a lattice of rank equal to $\dim \mathfrak{h}$. In this case we have:

$$\text{map}(S^1, \mathfrak{h}/Q) = \text{map}(S^1, \mathfrak{h}) \times Q$$

And the space of functions is then:

$$\mathcal{F}(\text{map}(S^1, \mathfrak{h})) \otimes \mathbb{C}[Q]$$

Hence its quantization is:

$$V^1(\mathfrak{h}) \otimes \mathbb{C}_{\epsilon}[Q]$$

We have $V^1(\mathfrak{h}) = S(\mathfrak{h}[t^{-1}]t^{-1} \oplus K)$ with the vacuum vector $|0\rangle = 1$. For each $h \in \mathfrak{h}$ we have the field $h(z) = \sum_n h_n z^{-1-n}$. Here h_n are operators on $V^1(\mathfrak{h})$ defined as follows: $h_0 = 0$, h_n is the operator of multiplication by ht^n , if $n < 0$. And for $n > 0$, h_n is the derivation of the symmetric algebra defined by (recall $[h_m, h'_n] = m\delta_{m,-n}(h, h')$)

$$h_n(h't^{-m}) = n\delta_{n,m}(h, h')$$

We then obtain:

$$\begin{aligned} [h_m, h'_n] &= m\delta_{m,-n}(h, h') \\ \Rightarrow [h(z), h'(w)] &= (h, h')\partial_w \delta(z - w) \\ \Rightarrow [h_\lambda h'] &= (h, h')\lambda \end{aligned}$$

The vectors $h_{-1-n_1}^1 \dots h_{-1-n_s}^s |0\rangle$ span the space $V^1(\mathfrak{h})$ and the state fields correspondence is given by:

$$Y\left((h^1 t^{-1-n_1}) \dots (h^s t^{-1-n_s}), z\right) = \frac{\partial_z^{n_1} h^1(z) \dots \partial_z^{n_s} h^s(z)}{n_1! \dots n_s!}$$

This makes $V^1(\mathfrak{h})$ a simple vertex algebra.

Now we want to discuss $\mathbb{C}_\epsilon[Q]$. First of all, $\mathbb{C}[Q]$ is a commutative unital algebra with basis e^α ($\alpha \in Q$) and product $e^\alpha e^\beta = e^{\alpha+\beta}$, such that $e^0 = 1$. Take an arbitrary function $\epsilon : Q \times Q \rightarrow \{\pm 1\}$, and define the new product:

$$e^\alpha e^\beta = \epsilon(\alpha, \beta) e^{\alpha+\beta}$$

We now want $\mathbb{C}_\epsilon[Q]$ to be an associative algebra with a unit element $1 = e^0$.

Exercise 17.2.

(1) $\mathbb{C}_\epsilon[Q]$ is associative if and only if for all $\alpha, \beta, \gamma \in Q$:

$$\epsilon(\alpha, \beta)\epsilon(\alpha + \beta, \gamma) = \epsilon(\alpha, \beta + \gamma)\epsilon(\beta, \gamma)$$

i.e. $\epsilon(\alpha, \beta)$ is a 2-cocycle.

(2) e^0 is a multiplicative unit if and only if $\epsilon(\alpha, 0) = \epsilon(0, \alpha) = 1$.

Proof.

$\mathbb{C}_\epsilon[Q]$ is associative

$$\Leftrightarrow \forall \alpha, \beta, \gamma \in Q, (e^\alpha e^\beta) e^\gamma = e^\alpha (e^\beta e^\gamma)$$

$$\Leftrightarrow \epsilon(\alpha, \beta) e^{\alpha+\beta} e^\gamma = e^\alpha (\epsilon(\beta, \gamma) e^{\beta+\gamma})$$

$$\Leftrightarrow \epsilon(\alpha, \beta)\epsilon(\alpha + \beta, \gamma) e^{\alpha+\beta+\gamma} = \epsilon(\beta, \gamma)\epsilon(\alpha, \beta + \gamma) e^{\alpha+\beta+\gamma}$$

$$\Leftrightarrow \epsilon(\alpha, \beta)\epsilon(\alpha + \beta, \gamma) = \epsilon(\alpha, \beta + \gamma)\epsilon(\beta, \gamma)$$

e^0 is a unit in Q

$$\Leftrightarrow e^0 e^\alpha = e^\alpha = e^\alpha e^0, \quad \forall \alpha \in Q$$

$$\Leftrightarrow \epsilon(0, \alpha) e^\alpha = e^\alpha = \epsilon(\alpha, 0) e^\alpha$$

$$\Leftrightarrow \epsilon(\alpha, 0) = 1 = \epsilon(0, \alpha), \quad \forall \alpha \in Q$$

□

We let $V_Q = S(\mathfrak{h}[t^{-1}]t^{-1}) \otimes \mathbb{C}_\epsilon[Q]$ and we want to extend the structure of vertex algebra from S to V_Q . For this it will be enough to extend the action of the operators h_n to V_Q . Define:

$$h_n(s \otimes e^\alpha) = h_n(s) \otimes e^\alpha, \quad n \neq 0$$

$$h_0(s \otimes e^\alpha) = (\alpha, h)s \otimes e^\alpha$$

We still have:

$$[h_m, h'_n] = m\delta_{m,-n}(h, h')$$

$$\Rightarrow [h(z), h'(w)] = (h, h')\partial_w \delta(z - w)$$

$$\Rightarrow [h_\lambda h'] = (h, h')\lambda$$

We need to construct the fields, for this let:

$$Y(1 \otimes e^\alpha, z) := \Gamma_\alpha(z)$$

such that V_Q extends the structure on S by $Y(v \otimes 1, z) = Y(v, z)$. We take $1 \otimes 1$ for the vacuum vector, and we get:

$$Y(s \otimes e^\alpha, z) =: Y(s, z)\Gamma_\alpha(z) :$$

where we take $s \otimes e^\alpha = (s \otimes 1)(1 \otimes e^\alpha)$. The dependence of the definition on this choice is illustrated in the following exercise:

Exercise 17.3. Denote by e^α the operator of multiplication by $1 \otimes e^\alpha$ in V_Q . Then $e^\alpha h_n = h_n e^\alpha$ if $n \neq 0$ and $e^\alpha h_0 = h_0 e^\alpha - (\alpha, h)e^\alpha$.

Proof. For $n \neq 0, \forall s \otimes e^\beta \in V_Q$:

$$\begin{aligned} (e^\alpha h_n - h_n e^\alpha)(s \otimes e^\beta) &= e^\alpha(h_n(s \otimes e^\beta)) - h_n(e^\alpha(s \otimes e^\beta)) \\ &= e^\alpha(h_n(s) \otimes e^\beta) - h_n(s \otimes \epsilon(\alpha, \beta)e^{\alpha+\beta}) \\ &= h_n(s) \otimes \epsilon(\alpha, \beta)e^{\alpha+\beta} - h_n(s) \otimes \epsilon(\alpha, \beta)e^{\alpha+\beta} = 0 \end{aligned}$$

For $n = 0, \forall s \otimes e^\beta \in V_Q$:

$$\begin{aligned} (e^\alpha h_0 - h_0 e^\alpha)(s \otimes e^\beta) &= e^\alpha(h_0(s \otimes e^\beta)) - h_0(e^\alpha(s \otimes e^\beta)) \\ &= e^\alpha((\beta, h)s \otimes e^\beta) - h_0(s \otimes \epsilon(\alpha, \beta)e^{\alpha+\beta}) \\ &= (\beta, h)s \otimes \epsilon(\alpha, \beta)e^{\alpha+\beta} - (\alpha + \beta, h)s \otimes \epsilon(\alpha, \beta)e^{\alpha+\beta} \\ &= -(\alpha, h)e^\alpha(s \otimes e^\beta) \end{aligned}$$

□

The OPE tells us:

$$[h(z), \Gamma_\alpha(w)] = [Y\left((ht^{-1})|0 \rangle, z\right), \Gamma_\alpha(w)] = \sum_{n \geq 0} Y\left((ht^{-1})_{(n)}e^\alpha, w\right) \partial_w^n \delta(z-w)/n!$$

This gives us (by the Exercise 17.3):

$$(ht^{-1})_{(n)}e^\alpha = h_n e^\alpha |0 \rangle = [h_n, e^\alpha] |0 \rangle$$

Hence we get:

$$(17.0.1) \quad [h(z), \Gamma_\alpha(w)] = (\alpha, h)\Gamma_\alpha(w)\delta(z-w) \Leftrightarrow [h_n, \Gamma_\alpha(z)] = (\alpha, h)z^n \Gamma_\alpha(z)$$

17.1. Lemma. Formula (17.0.1) is equivalent to

$$\Gamma_\alpha(z) = e^\alpha e^{-\sum_{j < 0} \frac{z^{-j}}{j} \alpha_j} e^{-\sum_{j > 0} \frac{z^{-j}}{j} \alpha_j} A_\alpha(z)$$

where A_α commutes with all the h'_n 's.

Proof. Let $X_\alpha(z) = e^{\sum_{j < 0} \frac{z^{-j}}{j} \alpha_j} (e^\alpha)^{-1} \Gamma_\alpha(z) e^{\sum_{j > 0} \frac{z^{-j}}{j} \alpha_j}$. Then $X_\alpha(z)$ commutes with all the h_n . Indeed, if $n > 0$ we have:

$$\begin{aligned} [h_n, X_\alpha] &= (e^\alpha)^{-1} [h_n, e^{\sum_{j < 0} \frac{z^{-j}}{j} \alpha_j} \Gamma_\alpha(z) e^{\sum_{j > 0} \frac{z^{-j}}{j} \alpha_j} + \\ &\quad + (e^\alpha)^{-1} e^{\sum_{j < 0} \frac{z^{-j}}{j} \alpha_j} (\alpha, h) z^n \Gamma_\alpha(z) e^{\sum_{j > 0} \frac{z^{-j}}{j} \alpha_j} \\ &= 0 \end{aligned}$$

□

Exercise 17.4. Check the last statement when $n \leq 0$.

Proof. If $n=0$, we have:

$$\begin{aligned} [h_0, X_\alpha] &= e^{\sum_{j < 0} \frac{z^{-j}}{j} \alpha_j} (-\alpha, h) (e^\alpha)^{-1} \Gamma_\alpha(z) e^{\sum_{j > 0} \frac{z^{-j}}{j} \alpha_j} + \\ &\quad + e^{\sum_{j < 0} \frac{z^{-j}}{j} \alpha_j} (e^\alpha)^{-1} (\alpha, h) \Gamma_\alpha(z) e^{\sum_{j > 0} \frac{z^{-j}}{j} \alpha_j} \\ &= 0 \end{aligned}$$

If $n < 0$, h_n commutes with all α_j for $j < 0$, in particular it commutes with $e^{\sum_{j<0} \frac{z^{-j}}{j} \alpha_j}$. Hence:

$$\begin{aligned} [h_n, X_\alpha] &= \\ &= h_n (e^\alpha)^{-1} e^{\sum_{j<0} \frac{z^{-j}}{j} \alpha_j} \Gamma_\alpha(z) e^{\sum_{j>0} \frac{z^{-j}}{j} \alpha_j} - (e^\alpha)^{-1} e^{\sum_{j<0} \frac{z^{-j}}{j} \alpha_j} \Gamma_\alpha(z) e^{\sum_{j>0} \frac{z^{-j}}{j} \alpha_j} h_n \\ &= (e^\alpha)^{-1} e^{\sum_{j<0} \frac{z^{-j}}{j} \alpha_j} h_n \Gamma_\alpha(z) e^{\sum_{j>0} \frac{z^{-j}}{j} \alpha_j} - (e^\alpha)^{-1} e^{\sum_{j<0} \frac{z^{-j}}{j} \alpha_j} \Gamma_\alpha(z) e^{\sum_{j>0} \frac{z^{-j}}{j} \alpha_j} h_n \\ &= (e^\alpha)^{-1} (e^{\sum_{j<0} \frac{z^{-j}}{j} \alpha_j} [h_n, \Gamma_\alpha(z)] e^{\sum_{j>0} \frac{z^{-j}}{j} \alpha_j} + e^{\sum_{j<0} \frac{z^{-j}}{j} \alpha_j} \Gamma_\alpha(z) [h_n, e^{\sum_{j>0} \frac{z^{-j}}{j} \alpha_j}]) \end{aligned}$$

But,

$$\begin{aligned} e^{\sum_{j<0} \frac{z^{-j}}{j} \alpha_j} [h_n, \Gamma_\alpha(z)] e^{\sum_{j>0} \frac{z^{-j}}{j} \alpha_j} + e^{\sum_{j<0} \frac{z^{-j}}{j} \alpha_j} \Gamma_\alpha(z) [h_n, e^{\sum_{j>0} \frac{z^{-j}}{j} \alpha_j}] &= \\ = e^{\sum_{j<0} \frac{z^{-j}}{j} \alpha_j} (\alpha, h) z^n \Gamma_\alpha(z) e^{\sum_{j>0} \frac{z^{-j}}{j} \alpha_j} + e^{\sum_{j<0} \frac{z^{-j}}{j} \alpha_j} \Gamma_\alpha(z) [h_n, e^{\sum_{j>0} \frac{z^{-j}}{j} \alpha_j}] &= \\ = 0 \end{aligned}$$

since

$$[h_n, e^{\sum_{j>0} \frac{z^{-j}}{j} \alpha_j}] = -(\alpha, h) z^n e^{\sum_{j>0} \frac{z^{-j}}{j} \alpha_j}$$

□

17.2. Corollary. $A_\alpha(z)(s \otimes e^\beta) = a_{\alpha\beta}(z)s \otimes e^\beta \in \mathbb{C}((z))$

Proof.

$$\begin{aligned} h_0 A_\alpha(z)s \otimes e^\beta &= (\beta, h) A_\alpha(z)s \otimes e^\beta \quad \forall h \Rightarrow \\ \Rightarrow A_\alpha(z)s \otimes e^\beta &= a_{\alpha,\beta}(z)s \otimes e^\beta. \end{aligned}$$

□

We have $\Gamma_\alpha(z)|0\rangle = e^\alpha A_\alpha(z)(1 + z\alpha_{-1} + o(z))$. By the vacuum axioms:

$$\Gamma_\alpha(z)|0\rangle|_{z=0} = e^\alpha \Rightarrow a_{\alpha,0}(z) = 1 + c_\alpha z + o(z)$$

and $\Gamma_\alpha(z)|0\rangle = 1 \otimes e^\alpha + (c_\alpha 1 \otimes e^\alpha + \alpha_{-1} \otimes e^\alpha)z + o(z)$. Hence (since $Ta = a_{(-2)}|0\rangle$), $T(1 \otimes e^\alpha) = c_\alpha(1 \otimes e^\alpha) + \alpha_{-1} \otimes e^\alpha$. Since $Y(Ta, z) = \partial_z Y(a, z)$ and $Y(a_{(-1)}b, z) =: Y(a, z)Y(b, z)$: we must have:

$$\partial_z \Gamma_\alpha(z) = c_\alpha \Gamma_\alpha(z) + : \alpha(z) \Gamma_\alpha(z) :$$

Exercise 17.5. Applying lemma 17.1 in the last equation we get a differential equation on $A_\alpha(z)$:

$$(17.2.1) \quad \partial_z A_\alpha(z) = z^{-1} \alpha_0 A_\alpha(z) + c_\alpha A_\alpha(z)$$

and all its solutions are of the form $Ce^{c_\alpha z} z^{\alpha_0}$ where $z^{\alpha_0}(s \otimes e^\beta) = z^{(\alpha,\beta)}(s \otimes e^\beta)$.

Proof. By lemma:

$$\begin{aligned}
 \partial_z \Gamma_\alpha(z) &= e^\alpha \left(\sum_{j < 0} z^{-j-1} \alpha_j \right) e^{-\sum_{j < 0} \frac{z^{-j}}{j} \alpha_j} e^{-\sum_{j > 0} \frac{z^{-j}}{j} \alpha_j} A_\alpha(z) + \\
 &\quad + e^\alpha e^{-\sum_{j < 0} \frac{z^{-j}}{j} \alpha_j} \left(\sum_{j > 0} \alpha_j z^{-1-j} \right) e^{-\sum_{j > 0} \frac{z^{-j}}{j} \alpha_j} A_\alpha(z) + \\
 &\quad + e^\alpha e^{-\sum_{j < 0} \frac{z^{-j}}{j} \alpha_j} e^{-\sum_{j > 0} \frac{z^{-j}}{j} \alpha_j} \partial_z A_\alpha(z) \\
 c_\alpha \Gamma_\alpha(z) &= c_\alpha e^\alpha e^{-\sum_{j < 0} \frac{z^{-j}}{j} \alpha_j} e^{-\sum_{j > 0} \frac{z^{-j}}{j} \alpha_j} A_\alpha(z) \\
 : \alpha(z) \Gamma_\alpha(z) &:= \alpha(z)_+ \Gamma_\alpha(z) + \Gamma_\alpha(z) \alpha(z)_-
 \end{aligned}$$

Noting that $\alpha(z)_- = \sum_{j > 0} \alpha_j z^{-1-j} + \alpha_0 z^{-1}$ and $\alpha(z)_+ = \sum_{j > 0} \alpha_j z^{-1-j}$ and adding up above equalities and canceling by $e^\alpha e^{-\sum_{j < 0} \frac{z^{-j}}{j} \alpha_j}$ on the left and $e^{-\sum_{j > 0} \frac{z^{-j}}{j} \alpha_j}$ on the right, we get the desired differential equation:

$$\partial_z A_\alpha(z) = z^{-1} \alpha_0 A_\alpha(z) + c_\alpha A_\alpha(z)$$

Multiply both sides of this ODE by $z^{-\alpha_0}$:

$$\begin{aligned}
 (\partial_z A_\alpha(z)) z^{-\alpha_0} &= z^{-1} \alpha_0 A_\alpha(z) z^{-\alpha_0} + c_\alpha A_\alpha(z) z^{-\alpha_0} \iff \\
 (\partial_z A_\alpha(z)) z^{-\alpha_0} - A_\alpha(z) z^{-1} \alpha_0 z^{-\alpha_0} &= c_\alpha A_\alpha(z) z^{-\alpha_0}
 \end{aligned}$$

Note that $\forall s \otimes e^\beta \in V_Q$, we have:

$$\begin{aligned}
 \partial_z (z^{-\alpha_0} (s \otimes e^\beta)) &= \partial_z (z^{-(\alpha, \beta)} (s \otimes e^\beta)) \\
 &= -(\alpha, \beta) z^{-(\alpha, \beta)-1} (s \otimes e^\beta) \\
 &= -z^{-1} z^{-(\alpha, \beta)} (\alpha, \beta) (s \otimes e^\beta) \\
 &= -z^{-1} z^{-(\alpha, \beta)} \alpha_0 (s \otimes e^\beta) \\
 &= -z^{-1} \alpha_0 z^{-\alpha_0} (s \otimes e^\beta)
 \end{aligned}$$

Hence $\partial_z (z^{-\alpha_0}) = -z^{-1} \alpha_0 z^{-\alpha_0}$. So for $B(z) = A_\alpha(z) z^{-\alpha_0}$ we get the following differential equation:

$$\partial_z B(z) = c_\alpha B(z)$$

Let $C(z) = e^{-c_\alpha z} B(z)$. Then above equation transforms into simple equation for $C(z)$:

$$\partial_z C(z) = 0$$

Its solutions are $C(z) = C$, C -constant. Hence all solutions to the initial differential equation are:

$$C e^{c_\alpha z} z^{\alpha_0},$$

where C is a constant. □

17.3. Corollary. Q must be an integral lattice, i.e. $(\alpha, \beta) \in \mathbb{Z}$ for all $\alpha, \beta \in Q$.

18. LATTICE VERTEX ALGEBRAS II - LOCALITY

We are constructing the vertex algebra $V_Q = V^1(\mathfrak{h}) \otimes \mathbb{C}_\epsilon[Q]$, where Q is an integral lattice in \mathfrak{h} (i.e. $(a, b) \in \mathbb{Z}$ for all $a, b \in Q$). Recall

$$\begin{aligned} Y((ht^{-1}), z) &= h(z) = \sum_{n \in \mathbb{Z}} h_n z^{-1-n} \\ T(ht^{-1}) &= ht^{-2} \\ z^{\alpha_0}(s \otimes e^\beta) &= z^{(\alpha, \beta)}(s \otimes e^\beta) \end{aligned}$$

and

$$\begin{aligned} (18.0.1) \quad Y(1 \otimes e^\alpha, z) &= \Gamma_\alpha(z) \\ &= e^{c_\alpha z} e^\alpha z^{\alpha_0} e^{-\sum_{j < 0} \frac{z^{-j}}{j} \alpha_j} e^{-\sum_{j > 0} \frac{z^{-j}}{j} \alpha_j}, \quad c_\alpha \in \mathbb{C} \\ T(1 \otimes e^\alpha) &= c_\alpha(1 \otimes e^\alpha) + \alpha_{-1} \otimes e^\alpha \end{aligned}$$

we must have:

$$T(s \otimes e^\alpha) = T(s) \otimes e^\alpha + (s \otimes 1)T(1 \otimes e^\alpha)$$

Recall that the requirements to apply the existence theorem are:

vacuum axiom holds for $h(z)$ by example 13.3 and for $\Gamma_\alpha(z)$ by the remarks above Ex. 17.5.

trans. invariance holds for $h(z)$ by example 13.3 and we need to check it for $\Gamma_\alpha(z)$.

locality holds for pairs $(h(z), h'(z))$ by example 13.3 and for pairs $(h(z), \Gamma_\alpha(z))$ by (17.0.1).

completeness obviously holds.

So we need to check

- (1) $[T, \Gamma_\alpha(z)] = \partial_z \Gamma_\alpha(z)$.
- (2) Locality of pairs $(\Gamma_\alpha(z), \Gamma_\beta(z))$.

Recall that we denoted by e^α the operator of multiplication by $1 \otimes e^\alpha$.

Exercise 18.1.

$$\begin{aligned} [T, e^\alpha](s \otimes e^\beta) &= \epsilon(\alpha, \beta) s((\alpha t^{-1}) + c_{\alpha+\beta} - c_\beta) \otimes e^{\alpha+\beta} \\ &= \epsilon(\alpha, \beta) s(\alpha t^{-1}) \otimes e^{\alpha+\beta} + e^\alpha s \otimes (B_\alpha - B_0) e^\beta \end{aligned}$$

where $B_\alpha(e^\beta) = c_{\alpha+\beta} e^\beta$.

Proof.

$$\begin{aligned} [T, e^\alpha](s \otimes e^\beta) &= \epsilon(\alpha, \beta) T(s \otimes e^{\alpha+\beta}) - e^\alpha \left(T(s) \otimes e^\beta + (s \otimes 1) \times \right. \\ &\quad \left. \times (c_\beta(1 \otimes e^\beta) + \beta t^{-1} \otimes e^\beta) \right) \\ &= \epsilon(\alpha, \beta) \left(T(s) \otimes e^{\alpha+\beta} + c_{\alpha+\beta}(s \otimes 1)(1 \otimes e^{\alpha+\beta}) + \right. \\ &\quad \left. + (s \otimes 1)(\alpha t^{-1} + \beta t^{-1}) \otimes e^{\alpha+\beta} - T(s) \otimes e^{\alpha+\beta} - \right. \\ &\quad \left. - c_\beta s \otimes e^{\alpha+\beta} - s(\beta t^{-1}) \otimes e^{\alpha+\beta} \right) \\ &= \epsilon(\alpha, \beta) s(c_{\alpha+\beta} + (\alpha t^{-1}) - c_\beta) \otimes e^{\alpha+\beta} \end{aligned}$$

Hence the first part follows. For the second part we continue

$$\begin{aligned} [T, e^\alpha](s \otimes e^\beta) &= \epsilon(\alpha, \beta)s(c_{\alpha+\beta} + (\alpha t^{-1}) - c_\beta) \otimes e^{\alpha+\beta} \\ &= \epsilon(\alpha, \beta)s(\alpha t^{-1}) \otimes e^{\alpha+\beta} + e^\alpha s \otimes (c_{\alpha+\beta}e^\beta - c_\beta e^\beta) \\ &= \epsilon(\alpha, \beta)s(\alpha t^{-1}) \otimes e^{\alpha+\beta} + e^\alpha s \otimes (B_\alpha - B_0)e^\beta \end{aligned}$$

□

Note that since $s \in V^1(\mathfrak{h}) = \mathcal{U}(\hat{\mathfrak{h}}_+)$ and \mathfrak{h} is commutative, we can write this as

$$[T, e^\alpha] = e^\alpha(\alpha_{-1} + B_\alpha - B_0)$$

Now we compute $[T, \Gamma_\alpha]$ using (18.0.1) and $[T, h_n] = -nh_{n-1}$ we claim that:

$$\begin{aligned} [T, e^a] &= T(a)e^a \\ [T, -\sum_{j<0} \frac{z^{-j}}{j} \alpha_j] &= \sum_{j<0} \alpha_{j-1} z^{-j} \\ [T, -\sum_{j>0} \frac{z^{-j}}{j} \alpha_j] &= \sum_{j>0} \alpha_{j-1} z^{-j} \\ [T, z^{\alpha_0}] &= 0 \quad \text{since } [T, \alpha_0] = 0 \end{aligned}$$

Indeed, we see inductively that $[T, a^n] = n[T, a]a^{n-1}$ and hence it follows that

$$[T, e^a] = \sum_{k \geq 0} \frac{1}{k!} [T, a^k] = \sum_{k \geq 0} \frac{1}{k!} k [T, a] a^{k-1} = [T, a] e^a$$

The following two equalities follow from this one. For the last one we write explicitly:

$$\begin{aligned} Tz^{\alpha_0} s \otimes e^\beta - z^{\alpha_0} T s \otimes e^\beta &= \\ &= z^{(\alpha, \beta)} T s \otimes e^\beta - z^{\alpha_0} (T(s) \otimes e^\beta + c_\beta s \otimes e^\beta + s\beta_{-1} \otimes e^\beta) \\ &= z^{(\alpha, \beta)} T(s) \otimes e^\beta + z^{(\alpha, \beta)} c_\beta s \otimes e^\beta + z^{(\alpha, \beta)} s\beta_{-1} \otimes e^\beta - \\ &\quad - z^{(\alpha, \beta)} T(s) \otimes e^\beta - z^{(\alpha, \beta)} c_\beta s \otimes e^\beta - z^{(\alpha, \beta)} s\beta_{-1} \otimes e^\beta \\ &= 0 \end{aligned}$$

Exercise 18.2. *Deduce from this that*

$$\begin{aligned} [T, \Gamma_\alpha(z)](s \otimes e^\beta) &= \left(c_\alpha \Gamma_\alpha(z) + : \alpha(z) \Gamma_\alpha(z) : + \right. \\ &\quad \left. + (B_\alpha - B_0) \Gamma_\alpha(z) + (c_{2\alpha+\beta} - 2c_{\alpha+\beta} + c_\beta) \Gamma_\alpha(z) \right) (s \otimes e^\beta) \end{aligned}$$

Proof. We know that $\text{ad}(T)$ is a derivation hence using the above formulas we have:

$$\begin{aligned}
[T, \Gamma_\alpha(z)] &= \\
&= [T, e^{c_\alpha z} e^\alpha z^{\alpha_0} e^{-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j} e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j}] \\
&= [T, e^{c_\alpha z}] e^\alpha z^{\alpha_0} e^{-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j} e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j} + \\
&\quad + e^{c_\alpha z} [T, e^\alpha] z^{\alpha_0} e^{-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j} e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j} + \\
&\quad + e^{c_\alpha z} e^\alpha z^{\alpha_0} [T, e^{-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j}] e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j} \\
&\quad + e^{c_\alpha z} e^\alpha z^{\alpha_0} e^{-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j} [T, e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j}] \\
&= c_\alpha \Gamma_\alpha(z) + e^{c_\alpha z} e^\alpha ((\alpha t^{-1}) \\
&\quad + (B_\alpha - B_0)) z^{\alpha_0} e^{-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j} e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j} + \\
&\quad + e^{c_\alpha z} e^\alpha z^{\alpha_0} \left(\sum_{j<0} \alpha_{j-1} z^{-j} \right) e^{-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j} e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j} \\
&\quad + e^{c_\alpha z} e^\alpha z^{\alpha_0} e^{-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j} \left(\sum_{j>0} \alpha_{j-1} z^{-j} \right) e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j} \\
&= c_\alpha \Gamma_\alpha(z) + e^{c_\alpha z} e^\alpha (B_\alpha - B_0) z^{\alpha_0} e^{-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j} e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j} + : \alpha(z) \Gamma_\alpha(z) :
\end{aligned}$$

Now note that we have

$$[B_\gamma, e^\alpha] e^\beta = (c_{\gamma+\alpha+\beta} - c_{\gamma+\beta}) e^\alpha e^\beta$$

And this commutes with e^α , hence we can commute the term $(B_\alpha - B_0)$ to the left to get:

$$\begin{aligned}
[T, \Gamma_\alpha(z)](s \otimes e^\beta) &= \left(c_\alpha \Gamma_\alpha(z) + : \alpha(z) \Gamma_\alpha(z) : + \right. \\
&\quad \left. + (B_\alpha - B_0) \Gamma_\alpha(z) - (c_{2\alpha+\beta} - 2c_{\alpha+\beta} + c_\beta) \Gamma_\alpha(z) \right) (s \otimes e^\beta)
\end{aligned}$$

□

But last time we calculated

$$\partial_z \Gamma_\alpha(z) = c_\alpha \Gamma_\alpha(z) + : \alpha(z) \Gamma_\alpha(z) :$$

Hence (1) holds if and only if $B_\alpha - B_0 = (c_{2\alpha+\beta} - 2c_{\alpha+\beta} + c_\beta) \text{Id}$. Applying the expression to e^β , we get $c_{\alpha+\beta} - c_\beta = c_{2\alpha+\beta} - 2c_{\alpha+\beta} + c_\beta$, hence

$$c_{2\alpha+\beta} = 3c_{\alpha+\beta} - 2c_\beta$$

This last equation applied to the cases $\beta = 0$ and $\beta = -2\alpha$ implies (noting obviously that $c_0 = 0$):

$$0 = 2c_0 = 3c_\alpha - c_{2\alpha}$$

$$0 = 3c_\alpha - 2c_{2\alpha}$$

And from this we get $c_\alpha \equiv 0$.

Now to check (2) we need to compute $\Gamma_\alpha(z)\Gamma_\beta(w)$:

$$\begin{aligned} \Gamma_\alpha(z)\Gamma_\beta(w) &= e^\alpha z^{\alpha_0} e^{-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j} e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j} \times \\ &\quad \times e^\beta w^{\beta_0} e^{-\sum_{j<0} \frac{w^{-j}}{j} \beta_j} e^{-\sum_{j>0} \frac{w^{-j}}{j} \beta_j} \end{aligned}$$

Exercise 18.3. $e^\alpha z^{\beta_0} = z^{-(\alpha,\beta)} z^{\beta_0} e^\alpha$.

Proof.

$$\begin{aligned} z^{-(\alpha,\beta)} z^{\beta_0} e^\alpha s \otimes e^\gamma &= \epsilon(\alpha, \gamma) z^{-(\alpha,\beta)} z^{\beta_0} s \otimes e^{\alpha+\gamma} \\ &= \epsilon(\alpha, \gamma) z^{-(\alpha,\beta)} z^{(\alpha+\gamma,\beta)} s \otimes e^{\alpha+\gamma} \\ &= e^\alpha z^{(\gamma,\beta)} s \otimes e^\gamma \\ &= e^\alpha z^{\beta_0} s \otimes e^\gamma \end{aligned}$$

□

18.1. Lemma. *Let A and B be operators on a vector space V , such that $[A, B] = C$ commutes with both of them. Then*

$$e^{zA} e^{wB} = e^{zwC} e^{wB} e^{zA}$$

Proof. We rewrite

$$e^{zA} e^{wB} e^{-zA} = e^{zwC} e^{wB}$$

and now both sides satisfy the same differential equation

$$\frac{d}{dz} X(z) = wCX(z)$$

Indeed, it is straightforward to prove by induction that $B^n A = AB^n - nCB^{n-1}$, after this we get (calling $X(z)$ the LHS)

$$\begin{aligned} \frac{dX(z)}{dz} &= AX - XA = AX - e^{zA} \left(\sum_{n \geq 0} \frac{w^n}{n!} B^n A \right) e^{-zA} \\ &= AX - e^{zA} \left(\sum_{n \geq 0} \frac{w^n}{n!} AB^n \right) e^{-zA} + e^{zA} \left(\sum_{n \geq 0} \frac{w^{n-1}}{(n-1)!} wCB^{n-1} \right) e^{-zA} \\ &= AX - AX + wCX \end{aligned}$$

On the other hand the RHS obviously satisfies the same equation, and clearly at $z = 0$ both sides are equal. □

By the lemma we have

$$e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j} e^{-\sum_{j<0} \frac{w^{-j}}{j} \beta_j} = e^{-\sum_{j<0} \frac{w^{-j}}{j} \beta_j} e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j} e^{-(\alpha,\beta) \sum_{j>0} \frac{1}{j} \left(\frac{w}{z}\right)^j}$$

where we have used that $[\alpha_j, \beta_{-j}] = j(\alpha, \beta)$. Now recalling that $-\sum_{j>0} \frac{x^j}{j} = \log(1-x)$ if $|x| < 1$, we get

$$e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j} e^{\sum_{j>0} \frac{w^j}{j} \beta_{-j}} = e^{-\sum_{j<0} \frac{w^{-j}}{j} \beta_j} e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j} i_{z,w} \left(1 - \frac{w}{z}\right)^{(\alpha,\beta)}$$

Hence we get the important formula:

$$\begin{aligned} \Gamma_\alpha(z)\Gamma_\beta(w) &= \epsilon(\alpha, \beta) i_{z,w}(z-w)^{(\alpha, \beta)} \times \\ &\quad \times \underbrace{e^{\alpha+\beta} z^{\alpha_0} w^{\beta_0} e^{-\sum_{j<0} \left(\frac{z^{-j}}{j} \alpha_j + \frac{w^{-j}}{j} \beta_j\right)} e^{-\sum_{j>0} \left(\frac{z^{-j}}{j} \alpha_j + \frac{w^{-j}}{j} \beta_j\right)}}_{\Gamma_{\alpha, \beta}} \\ \Gamma_\beta(w)\Gamma_\alpha(z) &= (-1)^{(\alpha, \beta)} \epsilon(\beta, \alpha) i_{w,z}(z-w)^{(\alpha, \beta)} \Gamma_{\alpha, \beta}(z, w) \end{aligned}$$

Note that we want $\epsilon(\alpha, \beta) = (-1)^{(\alpha, \beta)} \epsilon(\beta, \alpha)$ in particular we get if $\alpha = \beta$ that $(-1)^{(\alpha, \alpha)} = 1$. Therefore to deal with non-even lattices we are forced to introduce the following structure of a super-space on V_Q :

$$p(s \otimes e^\alpha) = p((\alpha, \alpha))$$

Then the necessary and sufficient condition of locality of $(\Gamma_\alpha(z), \Gamma_\beta(z))$ is given by

$$(18.1.1) \quad \epsilon(\alpha, \beta) = (-1)^{(\alpha, \beta) + (\alpha, \alpha)(\beta, \beta)} \epsilon(\beta, \alpha)$$

then we have

$$[\Gamma_\alpha(z), \Gamma_\beta(w)] = \epsilon(\alpha, \beta) \left(i_{z,w}(z-w)^{(\alpha, \beta)} - i_{w,z}(z-w)^{(\alpha, \beta)} \right) \Gamma_{\alpha, \beta}(z, w)$$

And finally

$$[\Gamma_\alpha(z), \Gamma_\beta(w)] = \begin{cases} 0 & (\alpha, \beta) \geq 0 \\ \epsilon(\alpha, \beta) \frac{\partial_w^{-(\alpha, \beta) + 1} \delta(z-w)}{(-(\alpha, \beta) + 1)!} \Gamma_{\alpha, \beta}(z, w) & (\alpha, \beta) < 0 \end{cases}$$

Summarizing we have proved:

18.2. Theorem. *Given a finite dimensional commutative Lie algebra \mathfrak{h} , a non-degenerate $(,)$, a lattice Q, ϵ a 2-cocycle of Q with $\epsilon(\alpha, 0) = \epsilon(0, \alpha) = 1$, the space $V_Q = V^1(\mathfrak{h}) \otimes \mathbb{C}_\epsilon[Q]$ is a space of states of a vertex algebra with vacuum vector $|0\rangle = 1 \otimes 1$ and $Y(ht^{-1}, z) = h(z) = \sum_n h_n z^{-1-n}$ if and only if Q is an integral lattice, V_Q has parity $p(s \otimes e^\alpha) = p((\alpha, \alpha))$ and ϵ satisfies the condition (18.1.1). Then we have $Y(1 \otimes e^\alpha, z) = \Gamma_\alpha(z)$ given by*

$$(18.2.1) \quad \begin{aligned} Y(1 \otimes e^\alpha, z) &= \Gamma_\alpha(z) \\ &= e^{\alpha} z^{\alpha_0} e^{-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j} e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j} \\ T(1 \otimes e^\alpha) &= \alpha_{-1} \otimes e^\alpha \end{aligned}$$

and, more generally;

$$Y((h^1 t^{-1-n_1}) \dots (h^r t^{-1-n_r}) \otimes e^\alpha, z) = \frac{: \partial_z^{n_1} h^1(z) \dots \partial_z^{n_r} h^r(z) \Gamma_\alpha(z) :}{n_1! \dots n_r!}$$

Construction of the Virasoro Field. Take dual bases $\{a_i\}, \{a^i\}$ and let

$$\begin{aligned} L(z) &= \frac{1}{2} \sum_i : a_i(z) a^i(z) := Y(\nu, z) \\ \nu &= \frac{1}{2} \sum_i (a_i)_{-1} (a^i)_{-1} |0\rangle \end{aligned}$$

We know that this is a Virasoro field with central charge $c = \dim \mathfrak{h}$ and all $h(z)$ are primary of conformal weight 1. We also know that $[h_\lambda \Gamma_\alpha] = (\alpha, h) \Gamma_\alpha$ or $[\Gamma_{\alpha\lambda} h] = -(\alpha, h) \Gamma_\alpha$. Then we can compute:

$$\begin{aligned}
 [\Gamma_{\alpha\lambda}L] &= \frac{1}{2}[\Gamma_{\alpha\lambda} \sum_i : a_i a^i :] \\
 &= \frac{1}{2} \sum_i : [\Gamma_{\alpha\lambda} a_i] a^i : + : a_i [\Gamma_{\alpha\lambda} a^i] : + \int_0^\lambda [[\Gamma_{\alpha\lambda} a_i]_\mu a^i] d_\mu \\
 &= -\frac{1}{2} \sum_i \left[(\alpha, a_i) : \Gamma_\alpha a^i : + (\alpha, a^i) : a_i \Gamma_\alpha : \right] + \frac{1}{2} \sum \int_0^\lambda (\alpha, a_i) (\alpha, a^i) \Gamma_\alpha d_\mu \\
 &= -\frac{1}{2} : \alpha \Gamma_\alpha : - \frac{1}{2} : \Gamma_\alpha \alpha : + \frac{(\alpha, \alpha)}{2} \lambda \Gamma_\alpha
 \end{aligned}$$

By skew-symmetry we get:

$$[L_\lambda \Gamma_\alpha] = \frac{1}{2} : \alpha \Gamma_\alpha : + \frac{1}{2} : \Gamma_\alpha \alpha : + (\partial + \lambda) \frac{(\alpha, \alpha)}{2} \Gamma_\alpha$$

but $: \Gamma_\alpha \alpha := \alpha \Gamma_\alpha + \int_{-T}^0 [\Gamma_{\alpha\lambda} \alpha] d\lambda$ so we get

$$: \Gamma_\alpha \alpha := \alpha \Gamma_\alpha - (\alpha, \alpha) \partial \Gamma_\alpha$$

Hence

$$[L, \Gamma_\alpha] := \alpha \Gamma_\alpha + (\alpha, \alpha) \frac{\lambda}{2} \Gamma_\alpha$$

now recalling that $: \alpha \Gamma_\alpha := \partial \Gamma_\alpha$ we get

$$[L, \Gamma_\alpha] = \partial \Gamma_\alpha + \left(\lambda \frac{(\alpha, \alpha)}{2} \right) \Gamma_\alpha$$

So ν is a conformal vector and Γ_α is a primary field of conformal weight $(\alpha, \alpha)/2$.

$$[L, \Gamma_\alpha] = \partial \Gamma_\alpha + \lambda \frac{(\alpha, \alpha)}{2} \Gamma_\alpha \Rightarrow T = L_{-1}$$

19. LATTICE VERTEX ALGEBRAS III - UNIQUENESS

19.1. Remark. V^k with $k = -h^\nu$ has no conformal vectors (?).

Exercise 19.1 (Correction of a correction in Lecture 16). *Any Hamiltonian operator B is skew-symmetric in the usual sense of matrix differential operators. For example the Virasoro Hamiltonian operator is*

$$\begin{aligned}
 B &= 2u\partial + u' \\
 B^* &= 2(-\partial)u + u' \\
 &= -2u' - 2u\partial + u' \\
 &= -u' - 2u\partial \\
 &= -B
 \end{aligned}$$

Proof. Follows directly from Ex. 15.2 □

Exercise 19.2. $V_{Q,\epsilon}$ is isomorphic to the charged free fermions if $Q = \mathbb{Z}$, $(a, b) = ab$ and $\epsilon = 1$.

Proof. This is a restatement of Ex. 9.4, defining $\varphi : V_{\mathbb{Z},1} \rightarrow F(A)$ by

$$\begin{aligned} h_n &\rightarrow \alpha_n, \\ e^m &\rightarrow |m\rangle, \end{aligned}$$

then by the uniqueness theorem we only need to check $\Gamma_{\pm 1} \rightarrow \psi^{\pm}$ which is the statement of Ex 9.4. \square

Recall that $V_{Q,\epsilon} = V^1(\mathfrak{h}) \otimes C_{\epsilon}[Q]$ where $\epsilon : Q \times Q \rightarrow \{\pm 1\}$ satisfies:

- (1) $\epsilon(a, 0) = \epsilon(0, a) = 1$,
- (2) $\epsilon(a, b)\epsilon(a+b, c) = \epsilon(a, b+c)\epsilon(b, c)$,
- (3) $\epsilon(a, b) = \epsilon(b, a)(-1)^{(a,b)+(a,a)(b,b)}$

Existence and uniqueness of ϵ . if we replace e^{α} by $\epsilon_{\alpha}e^{\alpha}$ with $\epsilon_{\alpha} = \pm 1$ and $\epsilon_0 = 1$, we get:

$$(\epsilon_{\alpha}e^{\alpha})(\epsilon_{\beta}e^{\beta}) = \epsilon(\alpha, \beta)(\epsilon_{\alpha}\epsilon_{\beta}\epsilon_{\alpha+\beta}^{-1})\epsilon_{\alpha+\beta}e^{\alpha+\beta}$$

and this is in turn by definition

$$\epsilon(\alpha, \beta) \simeq \epsilon_1(\alpha, \beta) = \epsilon_{\alpha}\epsilon_{\beta}\epsilon_{\alpha+\beta}^{-1}\epsilon(\alpha, \beta)$$

which by definition is an equivalent cocycle, we take then by definition $H^2(Q, \pm 1)$ to be the group of all 2-cocycles modulo the trivial 2-cocycles.

A different interpretation is given by noting that any 2-cocycle defines a central extension:

$$1 \rightarrow \{\pm 1\} \rightarrow \tilde{Q} \rightarrow Q \rightarrow 1$$

given by

$$\begin{aligned} \tilde{Q} &= \{\pm 1\} \times Q \\ (x, a) \cdot (y, b) &= (xy\epsilon(a, b), a + b) \end{aligned}$$

Exercise 19.3. \tilde{Q} is a group if and only if $\epsilon(\alpha, \beta)$ is a 2-cocycle. Prove also that these groups are isomorphic if and only if the corresponding cocycles are equivalent

Proof. Consider a central extension

$$0 \rightarrow Q \rightarrow \tilde{Q} \rightarrow \{\pm 1\} \rightarrow 0$$

where the group operation in \tilde{Q} extends that of Q . This means that if we choose a splitting

$$\tilde{Q} \xrightarrow{\simeq} Q \times \{\pm 1\}$$

Then the operation in \tilde{Q} is of the form

$$(a, \epsilon_a)(b, \epsilon_b) = (a + b, \epsilon(a, b)\epsilon_a\epsilon_b),$$

for some function $\epsilon : Q \times Q \rightarrow \{\pm 1\}$. Clearly since the operation in \tilde{Q} extends that of Q , we have:

$$(a, 1) = (a, 1)(a, 1) = (a, \epsilon(a, 0)) \Rightarrow \epsilon(a, 0) = 1$$

and similarly $\epsilon(0, a) = 1$. We also have

$$\begin{aligned} [(a, 1)(b, 1)](c, 1) &= (a + b, \epsilon(a, b))(c, 1) = (a + b + c, \epsilon(a, b)\epsilon(a + b, c)) \\ (a, 1)[(b, 1)(c, 1)] &= (a, 1)(b + c, \epsilon(b, c)) = (a + b + c, \epsilon(a, b + c)\epsilon(b, c)) \end{aligned}$$

and from associativity in \tilde{Q} we get

$$\epsilon(a, b + c)\epsilon(b, c) = \epsilon(a, b)\epsilon(a + b, c)$$

hence ϵ is by definition a 2-cocycle.

Suppose now that we have two different central extensions \tilde{Q} and \tilde{Q}' , together with an isomorphism $\phi : \tilde{Q} \xrightarrow{\sim} \tilde{Q}'$ preserving the subspace Q , i. e. choose two splittings $\tilde{Q} \simeq Q \times \{\pm 1\}$ and $\tilde{Q}' \simeq Q \times \{\pm\}$. This gives us two cocycles ϵ and ϵ' . We have for $a, b \in Q$ the inclusion $a \rightarrow (a, 1)$ and $b \rightarrow (b, 1)$ in \tilde{Q} now under ϕ these two map to (a, ϵ_a) and (b, ϵ_b) (recall that ϕ preserves the subspace Q). Their product is in \tilde{Q}' :

$$(19.1.1) \quad (a + b, \epsilon'(a, b)\epsilon_a\epsilon_b).$$

On the other hand, since ϕ is an isomorphism we have

$$\phi(a, 1)(b, 1) = \phi(a + b, \epsilon(a, b)) = (a + b, \epsilon(a, b)\epsilon_{a+b})$$

this is equal then to (19.1.1), hence we get (comparing second coordinates)

$$\epsilon(a, b) = \epsilon'(a, b)\epsilon_{a+b}^{-1}\epsilon_a\epsilon_b$$

and the two cocycles are then equivalent. □

Exercise 19.4. Given a central extension \tilde{Q} , we define the map $B : Q \times Q \rightarrow \{\pm 1\}$ as follows:

$$B(a, b) = \hat{a}\hat{b}\hat{a}^{-1}\hat{b}^{-1} \in \{\pm 1\}$$

where \hat{a}, \hat{b} are their preimages of $a, b \in Q$ in \tilde{Q} respectively. Prove that B is an unique invariant, i.e. that it does not depend on the choice of ϵ .

Proof. Choosing one splitting $\tilde{Q} \simeq Q \times \{\pm 1\}$ given by ϵ we have

$$B(a, b) = (a, 1)(b, 1)(a, 1)^{-1}(b, 1)^{-1}$$

and this is expanding and taking only the second coordinate:

$$(19.1.2) \quad \epsilon(a, b)\epsilon(a, -a)^{-1}\epsilon(b, -b)^{-1}\epsilon(-a, -b)\epsilon(a + b, -a - b).$$

Choosing a different splitting given by a cocycle ϵ' and expanding is straightforward to expand and get the same expression (19.1.2) so B is an invariant. Uniqueness follows by an easy (but long) calculation to check that $B(a, b)$ is itself a 2-cocycle, hence given the invariant B we construct a central extension \tilde{Q} , and we have just seen that two different central extensions give rise to two different invariants B . □

Note that in our case we get

$$\begin{aligned} (1, e^\alpha)(1, e^\beta)(1, e^\alpha)^{-1}(1, e^\beta)^{-1} &= (1, e^\alpha)(1, e^\beta)((1, e^\beta)(1, e^\alpha))^{-1} \\ &= \epsilon(\alpha, \beta)e^{\alpha+\beta}(\epsilon(\beta, \alpha)e^{\alpha+\beta})^{-1} \\ &= \epsilon(\alpha, \beta)\epsilon(\beta, \alpha)^{-1} \\ &= (-1)^{(\alpha, \beta) + (\alpha, \alpha)(\beta, \beta)} \end{aligned}$$

The invariant in our case is then $B(\alpha, \beta) = (-1)^{(\alpha, \beta) + (\alpha, \alpha)(\beta, \beta)}$. So we cannot have a quasiclassical limit of $V_{Q, \epsilon}$ if B is non-trivial. Also, by the proved uniqueness of B we get that $V_{Q, \epsilon}$ is independent of ϵ .

To prove existence we construct $\epsilon(\alpha, \beta)$ by demanding

(1) (bimultiplicativity)

$$\begin{aligned}\epsilon(\alpha + \alpha', \beta) &= \epsilon(\alpha, \beta)\epsilon(\alpha', \beta) \\ \epsilon(\alpha, \beta + \beta') &= \epsilon(\alpha, \beta)\epsilon(\alpha, \beta')\end{aligned}$$

(2) $\epsilon(\alpha, \alpha) = (-1)^{\frac{1}{2}((\alpha, \alpha) + (\alpha, \alpha)^2)}$.

it is easy to show that this conditions guarantee that ϵ is a 2-cocycle. And also writing

$$\begin{aligned}\epsilon(\alpha + \beta, \alpha + \beta) &= \epsilon(\alpha, \alpha)\epsilon(\beta, \beta)\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) \\ (-1)^{\frac{1}{2}((\alpha + \beta, \alpha + \beta) + (\alpha + \beta, \alpha + \beta)^2)} &= (-1)^{\frac{1}{2}((\alpha, \alpha) + (\alpha, \alpha)^2)}(-1)^{\frac{1}{2}((\beta, \beta) + (\beta, \beta)^2)}\epsilon(\alpha, \beta)\epsilon(\beta, \alpha)\end{aligned}$$

We see that the third property for ϵ is satisfied. Now to construct such an ϵ we choose a basis α_i of Q and define

$$\begin{aligned}\epsilon(\alpha_i, \alpha_i) &= (-1)^{\frac{1}{2}((\alpha_i, \alpha_i) + (\alpha_i, \alpha_i)^2)} \\ \epsilon(\alpha_i, \alpha_j)\epsilon(\alpha_j, \alpha_i) &= (-1)^{(\alpha_i, \alpha_j) + (\alpha_i, \alpha_i)(\alpha_j, \alpha_j)}\end{aligned}$$

And now we extend to Q by bimultiplicativity.

Exercise 19.5. Check that condition (2) is satisfied for all α

Proof. First, see that if condition (2) holds for α , it also holds for $-\alpha$. Indeed, by bimultiplicativity, we have $\epsilon(-\alpha, -\alpha).\epsilon(-\alpha, \alpha) = \epsilon(-\alpha, 0) = 1$, so $\epsilon(-\alpha, -\alpha) = \epsilon(-\alpha, \alpha)$. Using bimultiplicativity one more time, $\epsilon(-\alpha, \alpha).\epsilon(\alpha, \alpha) = \epsilon(0, \alpha) = 1$ and finally, $\epsilon(\alpha, \alpha) = \epsilon(-\alpha, \alpha) = \epsilon(-\alpha, -\alpha)$ and considering the bilinear form, we see that $(-\alpha, -\alpha) = (\alpha, \alpha)$ which proves our claim. Similarly, we can see that the defining equation

$$\epsilon(\alpha_i, \alpha_j)\epsilon(\alpha_j, \alpha_i) = (-1)^{(\alpha_i, \alpha_j) + (\alpha_i, \alpha_i)(\alpha_j, \alpha_j)}$$

holds if we substitute α_i with $-\alpha_i$ and/or α_j with $-\alpha_j$. Therefore, we can change the sign of the basis elements without violating the defining relations.

Now, let $\alpha = \sum m_i \alpha_i$. By the above remark, we can think that $m_i \geq 0$ for all i . Now, let α and β be two elements with non-negative coordinates. By induction on the number of components of α , we can prove that

$$\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{(\alpha, \beta) + (\alpha, \alpha)(\beta, \beta)}$$

and use this relation to run induction on the sum of the non-zero coordinates towards proving (2). In fact, if this sum equals 1, the statement is true by definition. If the statement is true for α and for β , we can see that it holds for $\alpha + \beta$ as well as follows:

$$\begin{aligned}\epsilon(\alpha + \beta, \alpha + \beta) &= \epsilon(\alpha + \beta, \alpha).\epsilon(\alpha + \beta, \beta) \\ &= \epsilon(\alpha, \alpha)\epsilon(\alpha, \beta)\epsilon(\beta, \alpha)\epsilon(\beta, \beta) \\ &= (-1)^{\frac{1}{2}((\alpha, \alpha) + (\alpha, \alpha)^2)}(-1)^{(\alpha, \beta) + (\alpha, \alpha)(\beta, \beta)}(-1)^{\frac{1}{2}((\beta, \beta) + (\beta, \beta)^2)} \\ &= (-1)^{\frac{1}{2}((\alpha, \alpha) + (\alpha, \alpha)^2 + 2(\alpha, \beta) + 2(\alpha, \alpha)(\beta, \beta) + (\beta, \beta) + (\beta, \beta)^2)} \\ &= (-1)^{\frac{1}{2}((\alpha + \beta) + (\alpha + \beta)^2)}\end{aligned}$$

□

The state-fields correspondence is given by:

$$Y((h^1 t^{-1-n_1}) \dots (h^r t^{-1-n_r}) \otimes e^\alpha, z) = \frac{: \partial_z^{n_1} h^1(z) \dots \partial_z^{n_r} h^r(z) \Gamma_\alpha(z) :}{n_1! \dots n_r!}$$

where $h(z) = \sum_n h_n z^{-1-n}$ and these operators are the usual ones except for

$$h_0(s \otimes e^\alpha) = (\alpha, h)s \otimes e^\alpha$$

$$\begin{aligned} \Gamma_\alpha(z) &= e^\alpha z^{\alpha_0} e^{-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j} e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j} \\ [h(z), h'(w)] &= (h, h') \partial_w \delta(z, w) \\ [h(z), \Gamma_\alpha(w)] &= (\alpha, h) \Gamma_\alpha(w) \delta(z, w) \\ [\Gamma_\alpha(z), \Gamma_\beta(w)] &= \epsilon(\alpha, \beta) (i_{z,w}(z-w)^{(\alpha, \beta)} - i_{w,z}(z-w)^{(\alpha, \beta)}) \Gamma_{\alpha, \beta}(z, w) \end{aligned}$$

19.2. Corollary.

- (1) $[\Gamma_\alpha(z), \Gamma_\beta(w)] = 0$ if $(\alpha, \beta) \geq 0$.
- (2) $[\Gamma_\alpha(z), \Gamma_\beta(w)] = \epsilon(\alpha, \beta) \delta(z, w) \Gamma_{\alpha+\beta}(w)$ if $(\alpha, \beta) = -1$.
- (3) $[\Gamma_\alpha(z), \Gamma_{-\alpha}(w)] = \epsilon(\alpha, -\alpha) (\alpha(w) \delta(z, w) + \partial_w \delta(z, w))$ if $(\alpha, \alpha) = 2$.

Proof. Recall that $\delta(z, w) f(z, w) = \delta(z, w) f(w, w)$. Also $\Gamma_{\alpha, \beta}(w, w) = \Gamma_{\alpha+\beta}(w)$, so (2) follows. Now $\partial_z f(z, w) \delta(z, w) - f(z, w) \partial_w \delta(z, w) = -\partial_w \delta(z, w) f(w, w)$ and thus

$$(19.2.1) \quad f(z, w) \partial_w \delta(z, w) = \partial_z f(z, w)|_{z=w} \delta(z, w) + f(w, w) \partial_w \delta(z, w)$$

Therefore, to check (3) we apply (19.2.1) to

$$[\Gamma_\alpha(z), \Gamma_{-\alpha}(w)] = \epsilon(\alpha, -\alpha) \partial_w \delta(z, w) \Gamma_{\alpha, -\alpha}(z, w)$$

and get the RHS of (3). Here we note that $\Gamma_{\alpha, -\alpha}(w, w) = 1$ and since

$$\begin{aligned} \Gamma_{\alpha, -\alpha}(z, w) &= \\ &= z^{\alpha_0} w^{-\alpha_0} \exp\left(-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j - \frac{w^{-j}}{j} \alpha_j\right) \exp\left(-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j - \frac{w^{-j}}{j} \alpha_j\right) \end{aligned}$$

we get

$$\begin{aligned} \partial_z \Gamma_{\alpha, -\alpha}(z, w) &= \\ &= \alpha_0 z^{-1} \Gamma_{\alpha, -\alpha} + \left(\sum_{j<0} z^{-j-1} \alpha_j\right) \Gamma_{\alpha, -\alpha} + \Gamma_{\alpha, -\alpha} \left(\sum_{j>0} z^{-j-1} \alpha_j\right) \end{aligned}$$

and then

$$\partial_z \Gamma_{\alpha, -\alpha}(z, w)|_{z=w} = \alpha_0 w^{-1} + \left(\sum_{j<0} w^{-j-1} \alpha_j\right) + \left(\sum_{j>0} w^{-j-1} \alpha_j\right) = \alpha(w)$$

□

Consider now the special case when $Q \subset \mathfrak{h}$ is a root lattice, simply laced, i.e. A, D, E cases, so that the set of roots is $\Delta = \{\alpha \in Q | (\alpha, \alpha) = 2\}$ then Q is an even

integral lattice. We have the fields

$$\begin{aligned} h(z) &= \sum_n h_n z^{-1-n} & h \in \mathfrak{h} \\ \Gamma_\alpha(z) &= \sum_n e_n^\alpha z^{-1-n} & \alpha \in \Delta \end{aligned}$$

Then the conditions above are translated to

- (1) $[h_m, h'_n] = m\delta_{m,-n}(h, h')$
- (2) $[h_m, e_n^\alpha] = (\alpha, h)e_{m+n}^\alpha$
- (3) $[e_m^\alpha, e_n^\beta] = 0$ if $(\alpha, \beta) \geq 0$.
- (4) $[e_m^\alpha, e_n^\beta] = \epsilon(\alpha, \beta)e_{m+n}^{\alpha+\beta}$ if $(\alpha, \beta) = -1$.
- (5) $[e_m^\alpha, e_n^{-\alpha}] = \epsilon(\alpha, -\alpha)(\alpha_{m+n} + m\delta_{m,-n})$

And these are the commutation relations for the affine Lie algebra of level 1: $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] + \mathbb{C}1$ and we have $\mathfrak{g} = \mathfrak{h} + \bigoplus_{\alpha \in \Delta} \mathbb{C}e_0^\alpha$ is a simple Lie algebra with root system Δ .

Exercise 19.6. *The invariant bilinear form on \mathfrak{g} is the extension from \mathfrak{h} by letting $(h, e_0^\alpha) = 0$, $(e_0^\alpha, e_0^\beta) = 0$ if $\alpha + \beta \neq 0$ and $(e^\alpha, e^{-\alpha}) = -1$. Check that this bilinear form is invariant.*

Proof. By definition, the bilinear form is invariant, i.e. for any triple of elements $a, b, c \in \mathfrak{g}$, we have $([a, b], c) = (a, [b, c])$. By linearity, it suffices to consider the cases when each of a, b, c is in \mathfrak{h} or in $\mathbb{C}e^\alpha$. We have to consider a number of cases.

- (1) All a, b, c are elements of \mathfrak{h} . Then having that \mathfrak{h} is commutative,

$$([a, b], c) = (0, c) = 0 = (a, 0) = (a, [b, c])$$

- (2) Two of the elements a, b, c are in \mathfrak{h} and the third is in $\mathbb{C}e^\alpha$. There are three possibilities:
 - (a) $a \in \mathbb{C}e_0^\alpha$. Then we have $[a, b]$ is a multiple of (h, e_0^α) which is by definition 0 on the one hand, and $[b, c] = 0$ on the other which shows that both LHS and RHS equal 0.
 - (b) $b \in \mathbb{C}e_0^\alpha$. Then, again both LHS and RHS are equal to 0. Similar to above, $[a, b]$ and $[b, c]$ are multiples of (h, e_0^α) .
 - (c) $c \in \mathbb{C}e_0^\alpha$. This case is exactly like the first one.
- (3) Two elements are in e_0^α and e_0^β and the third is in \mathfrak{h} . Here we have again three subcases:
 - (a) $a \in \mathfrak{h}$. Then we have $(a, [e^\alpha, e^\beta])$ for the LHS and $([a, e^\alpha], e^\beta)$ for the RHS. Then, considering equalities (3), (4), and (5) above, we can see that again LHS=RHS.
 - (b) $b \in \mathfrak{h}$. Then since $[e_0^\alpha, h]$ is proportional to e_0 and $(e_0, h) = 0$, so both LHS and RHS are equal to 0.
 - (c) $c \in \mathfrak{h}$. This case is exactly like the first one.
- (4) $a = e^\alpha$, $b = e^\beta$, and $c = e^\gamma$.
 - (a) $\alpha + \beta + \gamma \neq 0$. Then obviously both LHS and RHS equal 0. The rest is similar to these ones.

□

20. BORCHERDS IDENTITY

We will show that $V_{Q,\epsilon}$ is simple.

20.1. Theorem. *Let V be a vertex algebra, $a, b, c \in V$ then the following Borcherds identity holds: for any rational function $F(z, w)$ with poles only at $z = 0, w = 0, z = w$ one has*

$$\begin{aligned} \operatorname{Res}_{z=w} Y(Y(a, z-w)b, w)i_{w, z-w}F(z, w) &= \\ &= \operatorname{Res}_z (Y(a, z)Y(b, w)i_{z, w}F(z, w) - p(a, b)Y(b, w)Y(a, z)i_{w, z}F(z, w)), \end{aligned}$$

Proof. It suffices to prove this formula for $F = z^m(z-w)^n w^l$. For this F the identity becomes (multiplying by w^{-l}):

$$\begin{aligned} (20.1.1) \quad \sum_{j \geq 0} \binom{m}{j} Y(a_{(n+j)}b, w)w^{m-j} &= \\ &= \sum_{j \geq 0} (-1)^j \binom{n}{j} (a_{(m+n-j)}Y(b, w)w^j - p(a, b)(-1)^n Y(b, w)a_{(m+j)}w^{n-j}) \end{aligned}$$

in particular this identity is independent of l so it suffices to consider only 2 cases:

Case 1 $F(z, w) = z^m$ where $m \in \mathbb{Z}$.

Case 2 $F(z, w) = (z-w)^{-n-1}$ where $n \in \mathbb{Z}_+$.

since any $F(z, w)$ can be written as a linear combination of these expressions with coefficients Laurent polynomials in w . But now case 1 of formula (20.1.1) is

$$[a_{(m)}, Y(b, w)] = \sum_{j \in \mathbb{Z}_+} \binom{m}{j} Y(a_{(j)}b, w)w^{m-j}$$

which is the OPE formula we have proved in Ex 12.1. In the second case the formula reads:

$$\frac{\partial_w^n Y(a, w)Y(b, w)}{n!} = Y(a_{(-1-n)}b, w)$$

which is the n -th product formula for $n < 0$ □

The coefficient of w^{-1-k} of (20.1.1) is given by:

$$\begin{aligned} \sum_{j \in \mathbb{Z}_+} \binom{m}{j} (a_{(n+j)}b)_{(m+k-j)} c &= \sum_{j \in \mathbb{Z}_+} (-1)^j \binom{n}{j} \left(a_{(m+n-j)}(b_{(k+j)}c) - \right. \\ (20.1.2) \quad &\quad \left. - p(a, b)(-1)^n b_{(n+k-j)}(a_{(m+j)}c) \right) \end{aligned}$$

Therefore the following is immediate:

20.2. Corollary. *If $a(z), b(z), c(z)$ are pairwise local fields then they satisfy (20.1.2)*

Borcherds identity can in fact be written in a somewhat better form as follows.

20.3. Proposition. *The Borcherds identity is equivalent to the following:*

$$\begin{aligned} (20.3.1) \quad Y(a, z)Y(b, w)i_{z, w}(z-w)^n - p(a, b)Y(b, w)Y(a, z)i_{w, z}(z-w)^n &= \\ &= \sum_{j \in \mathbb{Z}_+} Y(a_{(j+n)}b, w)\partial_w^j \delta(z-w)/j! \end{aligned}$$

Proof. Take $F(z, w) = (z - w)^n \delta(x - z) = (z - w)^n \sum_{m \in \mathbb{Z}} z^m x^{-1-m}$ where x is a parameter. In order to substitute $F(z, w)$ in Borcherds identity we calculate:

$$\text{Res}_{z-w} (z - w)^{-1-j} i_{w, z-w} \delta(x - z) = \partial_w^j \delta(x - w) / j!$$

which follows from differentiating by w j -times of:

Exercise 20.2.

$$\text{Res}_{z-w} (z - w)^{-1} i_{w, z-w} \delta(x - z) = \delta(x - w)$$

□

Proof.

$$\begin{aligned} \delta(x - z) &= x^{-1} \sum_{n \in \mathbb{Z}} \left(\frac{z}{x} \right)^n \\ &= \sum x^{-1-n} ((z - w) + w)^n \\ &= \sum x^{-1-n} w^n \left(1 + \frac{z - w}{w} \right)^n \end{aligned}$$

But it is clear that $\text{Res}_{z-w} (z - w)^{-1} i_{w, z-w} \left(1 + \frac{z-w}{w} \right)^n = 1$ for all n and hence

$$\text{Res}_{z-w} (z - w)^{-1} i_{w, z-w} \delta(x - z) = \sum x^{-1-n} w^n = \delta(x - w)$$

□

Exercise 20.3. Take the generating series of n -th product axioms

$$Y(a, z)_{(n)} Y(b, z) = Y(a_{(n)} b, z)$$

we get:

$$Y(Y(a, z) b, -w) = i_{z, w} Y(a, z - w) Y(b, -w) - p(a, b) Y(b, w) \sum_{j \in \mathbb{Z}_+} \frac{\partial_w^j \delta(z, w)}{j!} a_{(j)}$$

Proof.

$$\begin{aligned} \sum_{n \in \mathbb{Z}} Y(a_{(n)} b, -w) z^{-n-1} &= \\ &= \sum_{n \in \mathbb{Z}} Y(a, -w)_{(n)} Y(b, -w) z^{-n-1} \\ &= \sum_{n \in \mathbb{Z}} \text{Res}_y [Y(a, y) Y(b, -w) i_{y, w} (y + w)^n \\ &\quad - p(a, b) Y(b, -w) Y(a, y) i_{w, y} (y + w)^n] z^{-n-1} \end{aligned}$$

The two sums on the right hand side can be computed by making use of straightforward calculations similar to Ex. 20.2, namely,

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \text{Res}_y f(y) i_{y, w} (y + w)^n z^{-n-1} &= i_{z, w} f(z - w) \\ \sum_{n \in \mathbb{Z}} i_{w, y} (y + w)^n z^{-n-1} &= \sum_{j \in \mathbb{Z}_+} y^j \partial_w^{(j)} \delta(z, w) \end{aligned}$$

we get:

$$Y(Y(a, z)b, -w) = i_{z,w}Y(a, z-w)Y(b, -w) - p(a, b)Y(b, w) \sum_{j \in \mathbb{Z}_+} \frac{\partial_w^j \delta(z, w)}{j!} a_{(j)}$$

□

But applying these operators to c and multiplying by $(z-w)^n$ for n large enough the second term vanishes so we get the associativity formula:

$$(z-w)^n Y(Y(a, z)b, -w)c = (z-w)^n Y(a, z-w)Y(b, -w)c$$

which physicists often write without the $(z-w)^n$ factor.

Comparing with usual associativity formula $L(L(a)b)c = L(a)(L(b)c)$ where L is the operator of left multiplication, we see the above formula is a parameter dependent analogue.

Exercise 20.4. An equivalent definition of a vertex algebra is given by two axioms:

- (1) $Y(|0\rangle, z) = I, \quad a_{(-1)}|0\rangle = a.$
- (2) Borchers identity.

Proof. We already know (1) and (2) are implied by the usual axioms. We show the converse. First, from Proposition 20.3 locality is immediate. Now a special case of Theorem 20.1 with $F = 1$ and $b = |0\rangle$ gives $a_{(j)}|0\rangle = 0$ for all $j \geq 0$. Also, by defining T on V by $Ta = a_{(-2)}|0\rangle$, the Borchers identity in the form of (20.1.2) with $c = |0\rangle$ and $m = 0, k = -2$ we get $(a_{(n)}b)_{(-2)}|0\rangle = a_{(n)}b_{(-2)}|0\rangle - na_{(n-1)}b_{-1}|0\rangle$ so

$$[T, a_{(n)}]b = (a_{(n)}b)_{(-2)}|0\rangle - a_{(n)}b_{(-2)}|0\rangle = -na_{(n-1)}b_{-1}|0\rangle = -na_{(n-1)}b$$

and hence translation invariance follows.

□

Representation Theory of Vertex algebras.

20.4. Definition. a **representation** of a unital associative algebra A is a linear map $A \rightarrow \text{End}(M)$ (we denote it by $a \rightarrow a^M$) where M is a linear vector space, such that

- (1) $1^M = \text{Id}_M.$
- (2) $(ab)^M = a^M b^M$

20.5. Definition. A **representation** of a vertex algebra V is a vector space M (a V -module M) and a linear map $V \rightarrow (\text{End } M)[[z, z^{-1}]]$ which maps a to a field $Y^M(a, z) = \sum_n a_{(n)}^M z^{-1-n}$ with $a_{(n)}^M \in \text{End } M$ such that

- (1) $Y^M(|0\rangle, z) = \text{Id}_M.$
- (2) $Y^M(a_{(n)}b, z) = Y^M(a, z)_{(n)}Y^M(b, z),$ where $n \in \mathbb{Z}.$
- (3) $\{Y^M(a, z)\}_{a \in V}$ is a local system.

20.6. Remark. Taking $b = |0\rangle, n = -2$ in (20.1.2) we get $Y^M(Ta, z) = \partial_z Y^M(a, z).$

Exercise 20.5. An equivalent definition of V -module is given by

- (1) $Y^M(|0\rangle, z) = \text{Id}_M$
- (2) $Y^M(a, z)Y^M(b, w)i_{z,w}(z-w)^n - p(a, b)Y^M(b, w)Y^M(a, z)i_{w,z}(z-w)^n = \sum_{j \in \mathbb{Z}} Y^M(a_{(j+n)}b, w)\partial_w^j \delta(z-w)/j!$

Proof. Assume the original definition of vertex algebra representation. From locality we have decomposition of commutators of fields $[Y^M(a, z), Y^M(b, w)]$ as a finite sum involving delta functions, and the coefficients must be given as $Y(a_{(n)}b, w)$ due to the n -th product formula for $n \geq 0$. Hence we have the usual OPE formula, which, together with the n -th product formula for $n < 0$ implies the Borcherds identity, and vice versa. Hence the two definitions are equivalent. \square

21. REPRESENTATIONS OF VERTEX ALGEBRAS

Equivalent Definition of a Module over a Vertex Algebra

Let V be a vertex algebra. A vector space M is called a V -module if there is a linear map from V to a space of $\text{End } M$ -valued fields, $a \rightarrow Y^M(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)}^M z^{-1-n}$ such that

- (1) $Y^M(|0 \rangle, z) = \text{Id}_M$.
- (2) $Y^M(a_{(n)}b, z) = Y^M(a, z)_{(n)}Y^M(b, z)$.
- (3) $\{Y^M(a, z)\}_{a \in V}$ is a local system.

In the last lecture we stated that these conditions are equivalent to the following ones

- (1) Vacuum axiom $Y^M(|0 \rangle, z) = \text{Id}_M$.
- (2) Borcherds identity

$$\begin{aligned} \text{Res}_{z-w} Y^M(Y(a, z-w)b, w) i_{w, z-w} F(z, w) &= \\ &= \text{Res}_z (Y^M(a, z) Y^M(b, w) i_{z, w} F - p(a, b) Y^M(b, w) Y^M(a, z) i_{w, z} F) \end{aligned}$$

for any rational function $F(z, w)$ with poles only at $z = 0$, $w = 0$, $z = w$.

Exercise 21.1. Check that these two sets of conditions are equivalent to

- (1) $Y^M(|0 \rangle, z) = \text{Id}_M$.
- (2) $Y^M(Ta, z) = \partial_z Y^M(a, z)$.
- (3) $[Y^M(a, z), Y^M(b, w)] = \sum_{j \in \mathbb{Z}_+} Y(a_{(j)}b, w) \partial_w^j \delta(z, w) / j!$.
- (4) $:Y^M(a, z)Y^M(b, z): := Y^M(a_{(-1)}b, z)$.

Proof. The equivalence of the first two definitions is proved in Ex 20.5. Thus it suffices to prove the equivalence of the above definition to the first one. Let us prove that the conditions of the third definition follows from the first one. The first condition is the same. The second condition follows from Borcherds identity (see remark 20.6). Condition 3 follows from the fact that $Y^M(a, z)$ form a local system and that the map Y^M preserves j -th products. Finally, condition 2 in the first definition implies the last condition in the third one.

The converse implication is also easy. Condition 3 of the third definition implies that the system $\{Y^M(a, z)\}_{a \in V}$ is a local and that the map Y^M preserves j -th products for $j \geq 0$. The following calculation shows that Y^M preserves j -th products for $j < 0$ as well

$$\begin{aligned}
 Y^M(a, z)_{(-1-n)}Y^M(b, z) &= \frac{: \partial^n Y^M(a, z) Y^M(b, z) :}{n!} \\
 &= \frac{: Y^M(T^n a, z) Y^M(b, z) :}{n!} \\
 &= Y^M\left(\frac{: (T^n a) b :}{n!}, z\right) \\
 &= Y^M(a_{(-1-n)}b, z), \quad n > 0.
 \end{aligned}$$

Here we use conditions 2 and 4 of the third definition. □

21.1. Example (Abstract example). Let V be a local system of $\text{End } M$ -valued fields, ∂_z -invariant, containing Id_M and closed under all n -products of fields ($n \in \mathbb{Z}$). We know that V is a vertex algebra but then defining

$$Y^M(a, z) = a \in V$$

makes M into a V -module.

21.2. Definition. A V -module M is called **conformal** if there exist an operator T^M such that $[T^M, Y^M(a, z)] = \partial_z Y^M(a, z)$.

In particular if V has a conformal vector ν then $T^M = L_{-1}^M$ is such an operator.

Exercise 21.2. A conformal V -module M is a module over V , viewed as a Lie conformal algebra.

Proof. Since the map Y^M preserves all n -th products we have that the λ -bracket in the Lie conformal algebra V turns to the λ -bracket of the Lie conformal algebra defined naturally on the space of the local system $\{Y^M(a, z)\}_{a \in V}$ by the respective n -th products. Since we also have the operation $[T^M, \cdot]$ on the space of the local system $\{Y^M(a, z)\}_{a \in V}$ and the identity

$$[T^M, Y^M(a, z)] = \partial_z Y^M(a, z)$$

M can indeed be viewed as a module over the Lie conformal algebra. □

21.3. Theorem. Let $(\mathfrak{g}, \mathcal{F})$ be a maximal formal distribution Lie (super)algebra associated to a Lie conformal (super)algebra $R = \bar{\mathcal{F}}$. Let M be a \mathfrak{g} -module such that

- (1) for any $v \in M$ and $a \in \mathcal{F}$ we have $a_{(n)}v = 0$ for $n \gg 0$ (restricted module).
- (2) There exists $T^M \in \text{End } M$ such that $[T^M, a^M(z)] = \partial_z a^M(z)$, where $a(z) \in \mathcal{F}$.

Let $V(R) = \mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})\mathfrak{g}_-$ be the universal enveloping vertex algebra of R . Then the \mathfrak{g} -module structure of M extends uniquely to a $V(R)$ -module $V(\mathfrak{g}, M)$. Moreover, these are all the conformal $V(R)$ -modules.

Proof. By the existence theorem the fields $a(z)$ (they are fields due to (1)) generate a vertex algebra $V(\mathfrak{g}, M)$. Then the representation of \mathfrak{g} in the vector space M give rise to an homomorphism (of Lie conformal (super)algebras) $\bar{\mathcal{F}} \rightarrow V(\mathfrak{g}, M)$. Now by the universal property of the enveloping vertex algebra we get that this morphism lifts to an homomorphism $V(R) \rightarrow V(\mathfrak{g}, M)$ which is a conformal representation of $V(R)$ in M .

Exercise 21.3. *Prove the last statement of the theorem.*

Proof. If M is a conformal module of $V(R)$ conditions 1 and 2 of the theorem are satisfied since $a \in V(R)$. Hence M has the desired structure of the restricted \mathfrak{g} -module which in turn generates the $V(R)$ -module structure by the first part of the proof.

We mention that for a representation of the vertex algebra associated to a formal distribution Lie (super)algebra $(\mathfrak{g}, \mathcal{F})$ we have the following formula for the map Y^M

$$Y^M(a_{(-1-n_1)}^1 \cdots a_{(-1-n_s)}^s |0\rangle, z) = \frac{:\partial_z^{n_1} a^{1M}(z) \cdots \partial_z^{n_s} a^{sM}(z):}{n_1! \cdots n_s!},$$

which follows from the vacuum axiom and the fact that Y^M preserves n -th products. \square

21.4. Example. $\mathfrak{g} = \text{Vir} = \{L_n, C\}_{n \in \mathbb{Z}}$. Let $\text{Vir}_{(0)} = \sum_{n \geq 0} \mathbb{C}L_n + \mathbb{C}C$, let also

$$M(h, c) = \text{Ind}_{\text{Vir}_{(0)}}^{\text{Vir}} \mathbb{C}_{(h, c)}$$

where

$$\begin{aligned} L_n \cdot 1 &= 0 & n > 0 \\ L_0 \cdot 1 &= h \cdot 1 \\ C \cdot 1 &= c \cdot 1 \end{aligned}$$

This is a V^c -module (after dividing by $(C - c)$). Thus $M(h, c)$ are modules over V^c , the universal enveloping vertex algebra of the Virasoro algebra of central charge c . It is clear that L_0 acts on $M(h, c)$ diagonally on the basis

$$L_{-n_1} \cdots L_{-n_s} |h, c\rangle$$

with eigenvalue $h + n_1 + \cdots + n_s \in h + \mathbb{Z}_+$ where $|h, c\rangle$ is the image of 1. $M(h, c)$ obviously decomposes into a direct sum of finite dimensional eigenspaces of L_0 . Then the general technique of highest weight representations allows us to conclude that $M(h, c)$ contains a unique maximal submodule J . Hence we get $L(h, c) = M(h, c)/J$, the irreducible V^c -module.

Fundamental Question: When is $L(h, c)$ a V_c -module?

21.5. Example (Ising module). $c = \frac{1}{2}$. In this case $L(h, \frac{1}{2})$ is a $V_{1/2}$ -module if and only if $h = 0, \frac{1}{16}, \frac{1}{2}$.

Exercise 21.4. *Prove that if $L(h, \frac{1}{2})$ is a $V_{1/2}$ -module then $h = 0, \frac{1}{16}$ or $\frac{1}{2}$.*

Proof. Let us try to find a singular vector of the vertex algebra $V^{1/2}$. Simple calculations show that there are no singular vectors with $c = 1/2$ if the energy of the vector is $e = 2, 3, 4$, and 5. We are looking for such a vector of the form

$$(21.5.1) \quad v = (\alpha L_{-6} + \beta L_{-4} L_{-2} + \gamma L_{-3}^2 + \sigma L_{-2}^3) |0\rangle.$$

Since L_1 and L_2 generates the whole subalgebra of the Virasoro algebra spanned by the generators L_n for $n > 0$ the statement

$$L_n v = 0, \quad \forall n > 0$$

is equivalent to the pair of equations

$$(21.5.2) \quad L_1 v = 0, \quad L_2 v = 0.$$

Direct computation reduces (21.5.2) to the following homogeneous linear system on α, β, γ and σ

$$(21.5.3) \quad \begin{cases} 7\alpha + 4\gamma + 6\sigma = 0 \\ 5\beta + 8\gamma + 9\sigma = 0 \\ 31\alpha + \beta + 40\gamma = 0 \\ 8\beta + 33\sigma = 0. \end{cases}$$

The determinant of the respective matrix vanishes and a nontrivial solution for (21.5.3) is found to be

$$\alpha = 9, \quad \beta = 22, \quad \gamma = -\frac{31}{4}, \quad \sigma = -\frac{16}{3}.$$

Thus

$$(21.5.4) \quad v = (9L_{-6} + 22L_{-4}L_{-2} - \frac{31}{4}L_{-3}^2 - \frac{16}{3}L_{-2}^3)|0 \rangle$$

is a singular vector of $V^{1/2}$.

If $M = L(h, \frac{1}{2})$ is $V_{1/2}$ -module then

$$Y^M(v) = 0.$$

By definition of a module over a vertex algebra and by definition of $-n$ -th products for $n > 0$ we have

$$(21.5.5) \quad Y^M(v) = \frac{3}{8} \partial_z^4 L(z) + 11 : \partial_z^2 L(z) L(z) : - \frac{31}{4} : \partial_z L(z) \partial_z L(z) : - \frac{16}{3} : L^3(z) :,$$

where $: L^3(z) :$ is just $L(z)_{(-1)}(L(z)_{(-1)}L(z))$.

The operator (21.5.5) is of conformal weight 6. Hence the lowest degree in z in the expression

$$(21.5.6) \quad Y^M(v)|h, 1/2 \rangle = \left(\frac{3}{8} \partial_z^4 L(z) + 11 : \partial_z^2 L(z) L(z) : - \frac{31}{4} : \partial_z L(z) \partial_z L(z) : - \frac{16}{3} : L^3(z) : \right) |h, 1/2 \rangle$$

is -6 .

A rather long computation shows that

$$z^6 Y^M(v)|h, 1/2 \rangle \Big|_{z=0} = \left(3h^2 - \frac{16h^3}{3} - \frac{h}{6} \right) |h, 1/2 \rangle.$$

The latter expression vanishes if and only if $h = 0$ or $1/2$ or $1/16$ and the statement follows. \square

21.6. Remark. In the following lectures we will give explicit realizations of $V_{1/2}$ -modules which are the $V^{1/2}$ -modules $L(h, 1/2)$ for $h = 0, 1/2$ and $1/16$. It would imply the respective converse statement.

21.7. Definition. Let V be a vertex algebra with a conformal vector ν so that $Y(\nu, z) = \sum L_n z^{-n-2}$. A V -module is called a **positive energy module** if L_0 is diagonalizable. The eigenspaces of L_0 are finite dimensional, and the real part of its spectrum is bounded below.

21.8. Proposition. *All positive energy irreducible V^c -modules are $L(h, c)$.*

Proof. Let M be such a module. Let $\nu \in V^c$ be an eigenvector of L_0 with eigenvalue h' . Then, since M is irreducible, it is irreducible as a module over the Virasoro algebra Vir . Then $M \simeq \mathcal{U}(\text{Vir})\nu$ hence the eigenvalues of $L_0 \subset h' + \mathbb{Z}$. This is a discrete set, hence we have a minimal eigenvalue h , M_h is the corresponding eigenspace. Then $\mathcal{U}(\text{Vir})v$ is a submodule of M , with v the only vector with eigenvalue h . Hence $\dim M_h = 1$ and hence M is a quotient of $M(h, c)$. Now this implies that $M = L(h, c)$. \square

Exercise 21.5.

- (1) Consider the Clifford affinization \hat{A} and let $F^1(A)$ be the corresponding vertex algebra of free fermions. Show that $F(A)$ is the only positive energy irreducible module.
- (2) Let $V^1(\mathfrak{h})$ be the vertex algebra of free bosons, show that all positive energy irreducible $\hat{\mathfrak{h}}$ -modules are parameterized by \mathfrak{h}^* as follows:

$$L(\mu) = \text{Ind}_{\hat{\mathfrak{h}}[\mathfrak{t}] + \text{CK}}^{\hat{\mathfrak{h}}} \mathbb{C}_\mu \quad \mu \in \mathfrak{h}^*$$

where $(\alpha^n)\mathbb{C}_\mu = 0$ for $n > 0$ and $\alpha 1 = \mu(\alpha)1$, $K = 1$.

Proof. Let M an irreducible module of the vertex algebra of free fermions. Then M is irreducible representation of the Lie (super)algebra of free fermions with the commutation relations

$$(21.8.1) \quad [\varphi_m, \psi_n] = \delta_{m, -n} \langle \varphi, \psi \rangle, \quad n, m \in \mathbb{Z}_+ + 1/2.$$

Since M is a positive energy module of the Virasoro algebra there exists an eigenvalue h of L_0 with minimal real part. Let v be the corresponding eigenvector. Then due to the commutation relations of L and φ (see for example lecture 9) we have

$$L_0 \varphi_n v = (h - n)v,$$

and hence

$$\varphi_n v = 0, \quad \forall n > 0.$$

This means that there is a morphism of \hat{A} -modules $\sigma : F(A) \rightarrow M$. The morphism σ is surjective since M is irreducible and injective since $F(A)$ is irreducible. Hence $M = F(A)$.

Furthermore, $F(A)$ is indeed a positive energy module of the Virasoro algebra. In lecture 9 we had

$$[L_0, \varphi_n^i] = -n\varphi_n^i.$$

Hence the vectors

$$\varphi_{-n_1}^{i_1} \dots \varphi_{-n_s}^{i_s} |0\rangle$$

are eigenvectors of L_0 with eigenvalue $n_1 + \dots + n_s$. These vectors generate the module $F(A)$ therefore real part the spectrum of L_0 is bounded below by 0 and every eigenspace of L_0 is finite dimensional.

If L is a positive energy irreducible $\hat{\mathfrak{h}}$ -module then there exists an eigenvector v of L_0 whose eigenvalue h has a minimal real part. In lecture 7 we had

$$[L_0, \alpha_n] = -n\alpha_n,$$

which implies that

$$L_0\alpha_nv = (h - n)\alpha_nv.$$

Hence $\alpha_nv = 0$ for any $n > 0$. Furthermore

$$L_0\alpha_0v = h\alpha_0v = \mu(\alpha)v$$

for some $\mu \in \mathfrak{h}^*$ since α_0 is a central element of \mathfrak{h} and L an irreducible $\hat{\mathfrak{h}}$ -module. Thus as in the case of the Lie algebra of free fermions we have nontrivial morphism of modules $L(\mu)$ and L . Since both modules $L(\mu)$ and L are irreducible by Schur lemma we get that $L(\mu) \sim L$.

Finally, by analogous line of arguments as for $F(A)$ we get that $L(\mu)$ is indeed a positive energy module for the Virasoro algebra for any $\mu \in \mathfrak{h}^*$. \square

21.9. Example (affine algebras). Consider $V^k(\mathfrak{g})$ where \mathfrak{g} is simple and $k \neq -h^\vee$. For a finite dimensional irreducible \mathfrak{g} -module F let $M(F, k) = \text{Ind}_{\mathfrak{g}[t] + \mathbb{C}K}^{\hat{\mathfrak{g}}} F$ where $\mathfrak{g}t^n|_F = 0$ for $n > 0$, $K = k$. Let $L(F, k) = M(F, k)/\text{the unique maximal submodule}$.

Exercise 21.6. Show that $M(F, k)$ is a positive energy module, $L(F, k)$ is a positive energy irreducible module. And these are all positive energy irreducible $V^k(\mathfrak{g})$ -module ($k \neq -h^\vee$).

Proof. From the Sugawara construction (see lecture 8) we have

$$L_0 = \frac{a_0^i(a_i)_0}{2(k + h^\vee)} + \frac{1}{2(k + h^\vee)} \sum_{m>0} (a_{-m}^i(a_i)_m + (a_i)_{-m}a_m^i)$$

Hence the subspace $F \subset M(F, k)(L(F, k))$ is a eigenspace of L_0 with an eigenvalue $C/(2(k + h^\vee))$ where C is the value of the quadratic Casimir operator of \mathfrak{g} in F .

From lecture 8 we also know that the λ -bracket between L and $a \in \hat{\mathfrak{g}}$ looks as

$$[L_\lambda a] = (\partial + \lambda)a.$$

This gives the following commutation relation

$$(21.9.1) \quad [L_0, a_n^i] = -na_n^i.$$

Both $\hat{\mathfrak{g}}$ -modules $M(F, k)$ and $L(F, k)$ are spanned by vectors

$$a_{-n_1}^{i_1} \dots a_{-n_s}^{i_s} v, \quad v \in F.$$

Hence due to (21.9.1), the spectrum of L_0 is $C/(2(k + h^\vee)) + \mathbb{Z}_+$ and the respective eigenspaces are obviously finite dimensional. Thus $M(F, k)$ and $L(F, k)$ are positive energy modules for any F and $k \neq -h^\vee$. $L(F, k)$ is irreducible by construction.

Let further M be a positive energy irreducible $\hat{\mathfrak{g}}$ -module. There exists an eigenvalue h of L_0 with minimal real part. Let F be the respective eigenspace of L_0 . F is finite dimensional by definition of the positive energy modules. Relation (21.9.1) implies that $\mathfrak{g}t^n|_F = 0$ for $n > 0$ and the subalgebra \mathfrak{g} corresponding to the zero modes preserves F .

If $F_1 \subset F$ is a proper \mathfrak{g} -submodule of F then we have a proper submodule in M generated by action of $\hat{\mathfrak{g}}$ on the vectors of F_1 . Thus F is an irreducible finite

dimensional \mathfrak{g} -module and we have a nontrivial morphism from $L(F, k)$ to M which is an isomorphism by Schur's lemma. \square

22. REPRESENTATIONS OF LATTICE VERTEX ALGEBRAS

We are studying representations of the lattice vertex algebra $V_Q = V^1(\mathfrak{h}) \otimes \mathbb{C}_\epsilon[Q]$ where $Q \subset \mathfrak{h}$ is an integral lattice. V_Q is generated by the fields $h(z) = \sum_{n \in \mathbb{Z}} h_n z^{-1-n}$ (where $h \in \mathfrak{h}$), and $\Gamma_\alpha(z) = e^\alpha z^{\alpha_0} e^{-\sum_{j < 0} \frac{z^{-j}}{j} \alpha_j} e^{-\sum_{j > 0} \frac{z^{-j}}{j} \alpha_j}$.

In order to construct a representation of V_Q consider the dual lattice

$$\mathfrak{h} \supset Q^* = \{\gamma \mid (\gamma, Q) \subset \mathbb{Z}\} \supset Q$$

22.1. Example. $Q = \mathbb{Z}$, $(a, b) = mab$ for some fixed non-zero $m \in \mathbb{Z}$. Hence we get $Q^* = m^{-1}\mathbb{Z}$ hence we get a lattice but not always integral.

Consider the decomposition $Q^* = \coprod_{\mu \in Q^* \bmod Q} (\mu + Q)$ and the space

$$V_{Q^*} = V^1(\mathfrak{h}) \otimes \mathbb{C}_{\epsilon^*}[Q^*]$$

where $\mathbb{C}_{\epsilon^*}[Q^*]$ is a $\mathbb{C}_\epsilon[Q]$ -module defined by

$$(22.1.1) \quad e^\alpha(e^\lambda) = \epsilon^*(\alpha, \lambda)e^{\lambda+\alpha}$$

so that each subspace $\mathbb{C}_{\epsilon^*}[\mu + Q] = \bigoplus_{\lambda \in \mu + Q} \mathbb{C}e^\lambda$ is a submodule. We want (22.1.1) to be a representation of the associative algebra $\mathbb{C}_\epsilon[Q]$

$$e^\alpha(e^\beta e^\lambda) = (e^\alpha e^\beta)e^\lambda$$

and we want then

$$(22.1.2) \quad \epsilon^*(\alpha, \beta + \lambda)\epsilon^*(\beta, \lambda) = \epsilon(\alpha, \beta)\epsilon^*(\alpha + \beta, \lambda)$$

A construction of ϵ^* fixing a coset representative, let

$$\epsilon^*(\alpha, \mu + \beta) = \epsilon(\alpha, \beta) \quad \alpha, \beta \in Q$$

Now we can check (22.1.2) since for $\lambda = \mu + \gamma$, $\gamma \in Q$ then (22.1.2) becomes:

$$\epsilon(\alpha, \beta + \gamma)\epsilon(\beta, \gamma) = \epsilon(\alpha, \beta)\epsilon(\alpha, \beta, \gamma)$$

which is the cocycle condition.

Exercise 22.1. By rescaling e^λ ($\lambda \in Q^*$), any cocycle satisfying (22.1.2) can be transformed to any other one.

Construction of a representation of V_Q in $V_{Q^*} = M$. we start defining h_n^M in the same way as before: for $n \neq 0$, h_n acts on the first factor so we get the same operators as before. For $n = 0$ we define $h_0(s \otimes e^\lambda) = (\lambda, h)s \otimes e^\lambda$. We let $Y^M(ht^{-1}, z) = h^M(z)$ for every $h \in \mathfrak{h}$. Now we let for $\alpha \in Q$

$$\begin{aligned} Y^M(1 \otimes e^\alpha, z) &= \Gamma_\alpha^M(z) \\ &= e^\alpha z^{\alpha_0^M} e^{-\sum_{j < 0} \frac{z^{-j}}{j} \alpha_j^M} e^{-\sum_{j > 0} \frac{z^{-j}}{j} \alpha_j^M} \end{aligned}$$

where in the RHS, e^α acts on $\mathbb{C}_{\epsilon^*}[Q^*]$.

In general we get:

$$(22.1.3) \quad Y^M((h^1 t^{-1-n_1}) \dots (h^r t^{-1-n_r}) \otimes e^\alpha, z) = \frac{:\partial_z^{n_1} h^{1M}(z) \dots \partial_z^{n_r} h^{rM}(z) \Gamma_\alpha(z):}{n_1! \dots n_r!}$$

22.2. Proposition.

- (1) Equation (22.1.3) defines a structure of a V_Q -module in M .
- (2) $V_{\mu+Q} = V^1(\mathfrak{h}) \otimes \mathbb{C}_{\epsilon^*}[\mu + Q]$ is an irreducible submodule of M and

$$(22.2.1) \quad M = \bigoplus_{\mu \in Q^* \pmod Q} V_{\mu+Q}$$

Proof. See Remark before theorem 23.3. First of all note that all the fields we have defined so far are pairwise local. We need to check that the n -th product satisfies:

$$Y^M(a_{(n)}b, z) = Y^M(a, z)_{(n)}Y^M(b, z) \quad a, b \in V_Q$$

And this is true without the superscript M . But this products are expressed via $h'_n s, e^{\alpha'} s$ (resp. the corresponding operators on M).

Another argument: Consider the vertex algebra V generated by the End M -valued fields $h^M(z)$ and $\Gamma_{\alpha}^M(z)$, where $h \in \mathfrak{h}$ and $\alpha \in Q$. Then we have to check that it satisfies the uniqueness conditions $V = V^1(\mathfrak{h}) \otimes \mathbb{C}_{\epsilon}[Q]$ We need to check then that the span of the fields (22.1.3) is a v.a.

Now since $e^{\alpha}\mathbb{C}_{\epsilon^*}[\mu + Q] \subset \mathbb{C}_{\epsilon^*}[\mu + Q]$ we have that $V_{\mu+Q}$ is a submodule of M and it is clear that M is a direct sum of these submodules. The only thing to check is that these are irreducible. Let \mathcal{U} be a non-zero $V_{\mu+Q}$ sub-module. Let $u \in \mathcal{U}$ be a non-zero element with

$$u = \sum_i s_i \otimes e^{\lambda_i}, \quad s_i \in S(\mathfrak{h}[t^{-1}]t^{-1})$$

such that the λ_i are pairwise distinct. Therefore applying $h_0^j(u) = \sum_i (\lambda_i, h)^j s_i \otimes e^{\lambda_i}$ and noting that the Vandermonde matrix is invertible, we get that each $s_i \otimes e^{\lambda_i} \in \mathcal{U}$. Now since $S(\mathfrak{h}[t^{-1}]t^{-1})$ is irreducible we conclude that $1 \otimes e^{\lambda} \in \mathcal{U}$ for some $\lambda \in \mu + Q$. But then $(1 \otimes e^{\lambda_1}) \in \mathcal{U}$ for any $\lambda_1 = \lambda + \beta, \beta \in Q$, just apply

$$\Gamma_{\beta}^M(z)(1 \otimes e^{\lambda}) = z^{(\lambda, \beta)} e^{\lambda + \beta}(1 + o(z))$$

and note that all the coefficients of z lie in \mathcal{U} hence $(1 \otimes e^{\lambda + \beta}) \in \mathcal{U}$. Now applying the lowering operators we get $\mathcal{U} = V_{\mu+Q}$. \square

Exercise 22.2. Show that all V_Q -modules $V_{\mu+Q}$ for $(\mu \in Q^* \pmod Q)$ are non-isomorphic. Thus we constructed Q^*/Q irreducible V_Q -modules.

Proof. Comparing the action of h_0 we get $(\lambda, h) = (\mu, h) \forall h$, which implies the result. \square

Exercise 22.3. The vector $(h_{-n_1}^{1M}) \dots (h_{-n_r}^{rM}) \otimes e^{\lambda}$ is an eigenvector of L_0 with eigenvalue $\frac{1}{2}(\lambda, \lambda) + n_1 + \dots + n_r$. Conclude that, provided that Q is a positive lattice (i.e. $(\alpha, \alpha) > 0$ except for $\alpha = 0$), all modules $V_{\mu+Q}$ are positive energy modules. Next time we will prove that any p.e. module over V_Q (Q positive definite) is a direct sum of $V_{\mu+Q}$ s.

Proof. This follows from the construction of the Virasoro field in Lecture 18 and Ex. 13.4. \square

22.3. Example. If Q is a root lattice of type A, D or E it is easy to describe the decomposition of Q^* :

$$Q^* = Q \amalg \left(\amalg \omega_i + Q \right)$$

where the product is over the special vertices i and ω_i are the fundamental weights.

22.4. Corollary. V_Q is simple, hence $V_Q \cong V_1(q)$.

22.5. Corollary. Let $(,)$ be normalized with the condition that $(\alpha, \alpha) = 2$ for all roots $(\mathfrak{g} = A, D, E)$. The positive energy irreducible $\hat{\mathfrak{g}}$ -modules are parameterized by a pair (k, λ) where k is the level and $\lambda \in \mathfrak{h}$

We know that V_Q is isomorphic to $V_1(\mathfrak{g})$ hence as a result we constructed $|Q^*/Q|$ positive energy modules of $V_1(\mathfrak{g})$. We proved that these are all p.e. modules of $V_1(\mathfrak{g})$. For any $k \in \mathbb{Z}_+$ all irreducible positive energy modules of $V_k(\mathfrak{g})$ appear in tensor products of those for $k = 1$ (see exercise 24.1) And this gives a complete description of $V_k(\mathfrak{g})$ -modules where $k \in \mathbb{Z}_+$, these are $L(k, F(\lambda))$, with the condition $(\lambda, \theta) \leq k$ where θ is the highest root (see theorem 24.1).

23. RATIONAL VERTEX ALGEBRAS

A digression on the complete OPE

In this lecture we will talk about the complete operator product expansion. If (a, b) is a local pair of End V -valued fields, one has the complete OPE:

$$(23.0.1) \quad a(z)b(w) = \sum_{j \geq 0} (a(w)_{(j)}b(w))i_{z,w}(z-w)^{-1-j} + :a(w)b(w):$$

23.1. Definition. An End V -valued formal distribution in two indeterminates is called a **field** (in z and w) if for all $v \in V$, $a(z, w)v \in V((z, w)) = V[[z, w]][z^{-1}, w^{-1}]$

Exercise 23.1. $:a(z)b(w):$ is a field if $a(z)$ and $b(z)$ are.

Proof. Recall that $:a(z)b(w) := a(z)_+b(w) + p(a, b)b(w)a(z)_-$. For each $v \in V$, choose $M \in \mathbb{Z}$ such that $a_{(m)}v = 0$ for all $m \geq M$. For each integer i from 0 to $|M| - 1$ let $N_i \in \mathbb{Z}$ be such an integer that $b_{(n)}(a_{(i)}v) = 0$ for all $n \geq N_i$. Finally, let N be such an integer that $b_{(n)}v = 0$ for all $n \geq N$. Clearly, then $a(z)_+b(w)v$ has no negative powers of z and only up to the $|N|$ negative power of w . At the same time, $b(w)a(z)_-v$ has only finite number of negative powers of z and hence $b(w)$ applied to each of them, lies in $V((z, w))$.

So $:a(z)b(w):$ is indeed a field. \square

Exercise 23.2. Any field $a(z, w)$ has a Taylor expansion up to any order $N > 0$:

$$(23.1.1) \quad a(z, w) = \sum_{j=0}^{N-1} c^j(w)(z-w)^j + (z-w)^N d_N(z, w)$$

where $c^j(w) = \partial_z^j a(z, w)|_{z=w}/j!$ and $d_N(z, w)$ is a field. Moreover, this Taylor expansion determines $c^j(w)$ uniquely.

Proof. First we show uniqueness.

If we differentiate in z both sides k times we get:

$$\partial_z^k a(z, w) = \sum_{j=k}^{N-1} j(j-1)\dots(j-k+1)c^j(w)(z-w)^{j-k} + (z-w)^{N-k} d'_N(z, w)$$

Plugging in $z = w$ we get:

$$\partial_z^k a(z, w)|_{z=w} = k!c^k(w)$$

Uniqueness is proved.

Now we show existence. It will be enough to show that for any field $a(z, w)$ there exist fields $c(w)$ and $d(z, w)$ such that:

$$a(z, w) = c(w) + (z - w)d(z, w)$$

The general case then follows trivially by induction. But the above equation suggests to take $c(w) = a(w, w)$. This will clearly be a field as $a(z, w)$ is a field. Note that for the simple case $a(z, w) = z^{-n}w^m, n > 0$ we have:

$$\begin{aligned} d(z, w) &= \frac{a(z, w) - a(w, w)}{z - w} \\ &= \frac{w^m}{z - w} \left(\frac{1}{z^n} - \frac{1}{w^n} \right) \\ &= w^m \frac{w^n - z^n}{(z - w)z^n w^n} \\ &= -w^{m-n} z^{-n} (w^{n-1} + \dots + z^{n-1}) \end{aligned}$$

and $d(z, w)$ is easily seen to be a field. Same goes for a case when $a(z, w) = z^n w^m, n > 0$. In this case:

$$d(z, w) = \frac{a(z, w) - a(w, w)}{z - w} = w^m \frac{z^n - w^n}{z - w} = w^m (z^{n-1} + \dots + w^{n-1})$$

is also a field. When $a(z, w)$ is a power series in z and w , $d(z, w)$ is also a power series. Therefore, since $a(z, w)$ is a finite sum of simple fields above and a power series, $d(z, w)$ is a finite sum of fields and a power series. Hence it is a field. \square

Now applying (23.1.1) to (23.0.1) we get:

$$: a(z)b(w) := \sum_{j=0}^{N-1} (a(w)_{(-1-j)} b(w)) (z - w)^j + (z - w)^N d_N(z, w)$$

And this gives us the complete OPE:

$$(23.1.2) \quad a(z)b(w) = \sum_{j \geq -N} (a(w)_{(j)} b(w)) i_{z,w} (z - w)^{-1-j} + (z - w)^N d_N(z, w)$$

After some preliminary work we will apply this to

$$\Gamma_\alpha(z)\Gamma_\beta(w) = \epsilon(\alpha, \beta) i_{z,w} (z - w)^{(\alpha, \beta)} \Gamma_{\alpha, \beta}(z, w)$$

where:

$$\Gamma_{\alpha, \beta} = e^{\alpha + \beta} z^{\alpha_0} w^{\beta_0} e^{-\sum_{j < 0} \binom{-j}{j} \alpha_j + \frac{w^{-j}}{j} \beta_j} e^{-\sum_{j > 0} \binom{-j}{j} \alpha_j + \frac{w^{-j}}{j} \beta_j}$$

is a field in z and w . Note also that:

$$\begin{aligned} \Gamma_{\alpha, \beta}(z, w)|_{z=w} &= \Gamma_{\alpha + \beta}(w) \\ \partial_z \Gamma_{\alpha, \beta}(z, w)|_{z=w} &=: \alpha \Gamma_{\alpha + \beta}(w) : \end{aligned}$$

Exercise 23.3.

$$\begin{aligned} \partial_z \Gamma_{\alpha, \beta}(z, w) &=: \alpha(z) \Gamma_{\alpha, \beta}(z, w) : \\ \frac{\partial_z^n \Gamma_{\alpha, \beta}(z, w)}{n!} &= \sum_{k_1 + 2k_2 + \dots = n} c_n(k_1, k_2, \dots) : \alpha(z)^{k_1} (\partial_z \alpha(z))^{k_2} \dots \Gamma_{\alpha, \beta}(z, w) : \end{aligned}$$

where c_n are some numbers.⁵

Proof. Differentiate $\Gamma_{\alpha,\beta}(z, w)$ using the formula for derivative of products:

$$\begin{aligned}
\partial_z \Gamma_{\alpha,\beta}(z, w) &= e^{\alpha+\beta} \partial_z z^{\alpha_0} w^{\beta_0} e^{-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j + \frac{w^{-j}}{j} \beta_j} e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j + \frac{w^{-j}}{j} \beta_j} + \\
&+ e^{\alpha+\beta} z^{\alpha_0} w^{\beta_0} \left(\partial_z e^{-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j + \frac{w^{-j}}{j} \beta_j} \right) e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j + \frac{w^{-j}}{j} \beta_j} + \\
&+ e^{\alpha+\beta} z^{\alpha_0} w^{\beta_0} e^{-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j + \frac{w^{-j}}{j} \beta_j} \left(\partial_z e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j + \frac{w^{-j}}{j} \beta_j} \right) = \\
&= e^{\alpha+\beta} \alpha_0 z^{\alpha_0-1} w^{\beta_0} e^{-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j + \frac{w^{-j}}{j} \beta_j} e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j + \frac{w^{-j}}{j} \beta_j} + \\
&+ e^{\alpha+\beta} z^{\alpha_0} w^{\beta_0} \left(\sum_{j<0} z^{-j-1} \alpha_j \right) e^{-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j + \frac{w^{-j}}{j} \beta_j} e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j + \frac{w^{-j}}{j} \beta_j} + \\
&+ e^{\alpha+\beta} z^{\alpha_0} w^{\beta_0} e^{-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j + \frac{w^{-j}}{j} \beta_j} \left(\sum_{j>0} z^{-j-1} \alpha_j \right) e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j + \frac{w^{-j}}{j} \beta_j} = \\
&= (\alpha_0 z^{-1} + \sum_{j>0} z^{-j-1} \alpha_j) \Gamma_{\alpha,\beta}(z, w) + \Gamma_{\alpha,\beta}(z, w) \left(\sum_{j<0} z^{-j-1} \alpha_j \right) \\
&= \alpha(z)_+ \Gamma_{\alpha,\beta}(z, w) + \Gamma_{\alpha,\beta}(z, w) \alpha(z)_- =: \alpha(z) \Gamma_{\alpha,\beta}(z, w) :
\end{aligned}$$

The proof of the second part goes by induction. For $n = 1$ clearly the statement holds. Assume that statement holds for some n . Then to prove that it holds for $n + 1$ it will be enough to show that for all tuples (l_i) , such that $\sum l_i = n$:

$$\begin{aligned}
\partial_z : \alpha(z)^{l_1} (\partial_z \alpha(z))^{l_2} \dots \Gamma_{\alpha,\beta}(z, w) &:= \\
= \sum_{k_1+2k_2+\dots=n+1} b_n(k_1, k_2, \dots) : \alpha(z)^{k_1} (\partial_z \alpha(z))^{k_2} \dots \Gamma_{\alpha,\beta}(z, w) :
\end{aligned}$$

for some numbers $b_n(k_1, k_2, \dots)$. We use the fact that ∂_z is a derivation of normal ordered product.

$$\begin{aligned}
\partial_z : \alpha(z)^{l_1} (\partial_z \alpha(z))^{l_2} \dots \Gamma_{\alpha,\beta}(z, w) &:= \\
= \left(\sum_{l_i \geq 1} l_i : \alpha(z)^{k_1} (\partial_z \alpha(z))^{k_2} \dots \partial_z (\partial_z^i \alpha(z)) (\partial_z^i \alpha(z))^{l_i-1} \dots \Gamma_{\alpha,\beta}(z, w) : \right) + \\
&+ : \alpha(z)^{l_1} (\partial_z \alpha(z))^{l_2} \dots \partial_z \Gamma_{\alpha,\beta}(z, w) : \\
= \left(\sum_{l_i \geq 1} l_i : \alpha(z)^{l_1} (\partial_z \alpha(z))^{l_2} \dots (\partial_z^i \alpha(z))^{l_i-1} (\partial_z^{i+1} \alpha(z))^{l_{i+1}+1} \dots \Gamma_{\alpha,\beta}(z, w) : \right) + \\
&+ : \alpha(z)^{l_1+1} (\partial_z^\alpha(z))^{l_2} \dots \Gamma_{\alpha,\beta}(z, w) : \\
= \sum_{k_1+2k_2+\dots=n+1} b_n(k_1, k_2, \dots) : \alpha(z)^{k_1} (\partial_z \alpha(z))^{k_2} \dots \Gamma_{\alpha,\beta}(z, w) :
\end{aligned}$$

⁵It can be shown $c_n = (k_1! k_2! \dots (1!)^{k_1} (2!)^{k_2} \dots)^{-1}$.

□

Now we have:

$$\Gamma_\alpha(z)\Gamma_\beta(w) = \epsilon(\alpha, \beta) \sum_{n \in \mathbb{Z}_+} \sum_{\substack{k_i \in \mathbb{Z}_+ \\ k_1 + 2k_2 + \dots = n}} c_n(k_1, k_2, \dots) i_{z,w}(z-w)^{(\alpha, \beta) + n} \times \\ \times : \alpha(w)^{k_1} (\partial_w \alpha(w))^{k_2} \dots \Gamma_{\alpha+\beta}(w) :$$

up to arbitrary order.

23.2. Remark. Note that exactly the same OPE hold for the $\Gamma_\alpha^M(z)$, hence the n -th product identity for $\Gamma_\alpha(z)$ corresponds to n -th products of $\Gamma_\alpha^M(z)$. More generally, to construct a V_Q -module structure on $M = V^1(\mathfrak{h}) \otimes \mathbb{C}_{\epsilon^*}[Q^*]$:

$$Y^M((h^1 t^{-1-n_1}) \dots (h^{s_t} t^{-1-n_s}) \otimes e^\alpha, z) = \frac{: \partial_z^{n_1} h^1(z) \dots \partial_z^{n_s} h^s(z) \Gamma_\alpha^M(z) :}{n_1! \dots n_s!}$$

we consider the space spanned by all these fields. This space is clearly ∂_z invariant. Moreover the complete OPE implies that this space is closed under all n -th products. So this is a vertex algebra, hence M is a module over V_Q . This in turn completes the proof of Proposition 22.2.

Recall that $M = \bigoplus_{\mu \in Q^* \bmod Q} V_{\mu+Q}$, where $V_{\mu+Q}$ are irreducible V_Q -modules. All of them are positive energy modules provided that Q is a positive definite lattice.

23.3. Theorem. Assume Q is a positive definite integral lattice. Then:

- (1) The V_Q -modules $V_{\mu+Q}$, $\mu \in Q^* \bmod Q$ are all positive energy irreducible V_Q -modules.
- (2) Any positive energy V_Q -module is a direct sum of a finite number of them.

23.4. Definition. A vertex algebra with a Virasoro element is called **rational** if it has finitely many positive energy irreducible modules and any positive energy module is completely reducible.

Proof of 23.3. Let M be a V_Q -module, then we have End M -valued fields $h^M(z) = \sum_{n \in \mathbb{Z}} h_n^M z^{-1-n}$ and $\Gamma_\alpha^M(z)$ such that:

- (1) $[h_m^M, h_n^{M'}] = m \delta_{m, -n} (h, h')$.
- (2) $[h_m^M, \Gamma_\alpha^M(z)] = (\alpha, h) z^m \Gamma_\alpha^M(z)$.
- (3)

$$\Gamma_\alpha(z)\Gamma_\beta(w) = \epsilon(\alpha, \beta) \sum_{n \in \mathbb{Z}_+} \sum_{\substack{k_i \in \mathbb{Z}_+ \\ k_1 + 2k_2 + \dots = n}} c_n(k_1, k_2, \dots) i_{z,w}(z-w)^{(\alpha, \beta) + n} \times \\ \times : \alpha(w)^{k_1} (\partial_w \alpha(w))^{k_2} \dots \Gamma_{\alpha+\beta}(w) :$$

- (4) $\partial_z \Gamma_\alpha^M(z) =: \alpha^M(z) \Gamma_\alpha^M(z) :$, since $\partial_z Y^M(e^\alpha, z) = Y^M(Te^\alpha, z) =: \alpha^M \Gamma_\alpha^M(z) :$

The same argument as we used for V_Q shows that (2) implies:

$$Y^M(1 \otimes e^\alpha, z) := \Gamma_\alpha^M(z) = e^{-\sum_{j < 0} \frac{z^{-j}}{j} \alpha_j^M} e^{-\sum_{j > 0} \frac{z^{-j}}{j} \alpha_j^M} A_\alpha^M(z),$$

where $A_\alpha^M(z)$ commutes with all h_m^M , ($m \in \mathbb{Z}$). As for V_Q , formula (4) gives a differential equation for $A_\alpha^M(z)$:

$$z \partial_z A_\alpha^M(z) = A_\alpha^M(z) \alpha_0^M$$

The same argument as before shows that the only solution for this equation is $a^\alpha z^{\alpha_0^M}$ where a^α is some fixed element of $\text{End } M$. α_0^M commutes with L_0^M , since $[\alpha_0, L_0] = 0$. Since eigenspaces of L_0 are finite dimensional we conclude that α_0^M is a direct sum of operators in a finite dimensional space. Hence $\alpha_0^M = S + N$ where N is the nilpotent part and S is the diagonal part (semisimple). Now we have

$$z^{\alpha_0^M} = z^S z^N = z^S e^{(\log z)N}$$

and this is not a series in z unless $N = 0$, hence α_0^M is diagonal with integer eigenvalues. We have then:

$$\Gamma_\alpha^M(z) = a^\alpha z^{\alpha_0^M} e^{-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j^M} e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j^M}$$

Exercise 23.4. Derive from (2) (respectively, (3)) that:

- (5) $[h_n^M, a^\alpha] = \delta_{n,0}(\alpha, h)e^\alpha$.
(6) $a^\alpha a^\beta = \epsilon(\alpha, \beta)a^{\alpha+\beta}$.

Proof. If $n = 0$, h_0^M commutes with all α_j , hence from (2):

$$\begin{aligned} [h_0^M, \Gamma_\alpha^M(z)] &= [h_0^M, a^\alpha] z^{\alpha_0^M} e^{-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j^M} e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j^M} \\ &= (\alpha, h) a^\alpha z^{\alpha_0^M} e^{-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j^M} e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j^M} \end{aligned}$$

Canceling by $z^{\alpha_0^M} e^{-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j^M} e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j^M}$ we get $[h_0^M, a^\alpha] = (\alpha, h)a^\alpha$.

If say $n > 0$, then h_n commutes with all α_j for all $j > 0$, in particular h_n commutes with $e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j^M}$, hence canceling by $e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j^M}$ we can rewrite (2) as:

$$(23.4.1) \quad [h_n^M, a^\alpha z^{\alpha_0^M} e^{-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j^M}] = (\alpha, h) z^n a^\alpha z^{\alpha_0^M} e^{-\sum_{j<0} \frac{z^{-j}}{j} \alpha_j^M}$$

Now observe that $e^{-\sum_{j>0} \frac{z^{-j}}{j} \alpha_j^M} = 1 + c_1 z^1 + c_2 z^2 + \dots$. Compare the coefficients at the $z^{\alpha_0^M}$ in both sides of (23.4.1) to get:

$$[h_n^M, a^\alpha] = 0$$

Similarly, $[h_n^M, a^\alpha] = 0$ when $n < 0$.

For the second part we note that we can proceed as for $\text{End } V$ -valued fields to see that if we denote:

$$M_{\alpha,\beta}(z) = a^{\alpha+\beta} z^{\alpha_0^M} w^{\beta_0^M} e^{-\sum_{j<0} \left(\frac{z^{-j}}{j} \alpha_j^M + \frac{w^{-j}}{j} \beta_j^M \right)} e^{-\sum_{j>0} \left(\frac{z^{-j}}{j} \alpha_j^M + \frac{w^{-j}}{j} \beta_j^M \right)}$$

then by the same argument that we used for V_Q , the field

$$N_{\alpha,\beta}(z) = \epsilon(\alpha, \beta) i_{z,w}(z-w)^{(\alpha,\beta)} M_{\alpha,\beta}$$

gives the same OPE as the field $\Gamma_\alpha(z)\Gamma_\beta(z)$:

$$\begin{aligned} N_{\alpha,\beta}(z) &= \epsilon(\alpha, \beta) \sum_{n \in \mathbb{Z}_+} \sum_{\substack{k_i \in \mathbb{Z}_+ \\ k_1 + 2k_2 + \dots = n}} c_n(k_1, k_2, \dots) i_{z,w}(z-w)^{(\alpha,\beta)+n} \times \\ &\quad \times : \alpha(w)^{k_1} (\partial_w \alpha(w))^{k_2} \dots N_{\alpha+\beta}(w) : \end{aligned}$$

Therefore, $\Gamma_\alpha^M(z)\Gamma_\beta^M(z) = \epsilon(\alpha, \beta)i_{z,w}(z-w)^{(\alpha,\beta)}M_{\alpha,\beta}(z)$. Hence:

$$\begin{aligned} a^\alpha a^\beta z^{\alpha_0^M} w^{\beta_0^M} i_{z,w}(z-w)^{(\alpha,\beta)} e^{-\sum_{j<0} \left(\frac{z^{-j}}{j} \alpha_j^M + \frac{w^{-j}}{j} \beta_j^M\right)} e^{-\sum_{j>0} \left(\frac{z^{-j}}{j} \alpha_j^M + \frac{w^{-j}}{j} \beta_j^M\right)} = \\ = \epsilon(\alpha, \beta)i_{z,w}(z-w)^{(\alpha,\beta)}M_{\alpha,\beta} \end{aligned}$$

Canceling we get $a^\alpha a^\beta = \epsilon(\alpha, \beta)a^{\alpha+\beta}$. □

Now (1), (5) and (6) tell us that we have a representation of the direct sum of a commuting pair of algebras: one is the infinite dimensional Heisenberg algebra $H = \bigoplus_{n \neq 0} (\mathfrak{h}t^n) + \mathbb{C}1$, and the other one is the algebra $\mathcal{U} = \mathbb{C}_\epsilon[Q] \times \mathfrak{h}$ with the action:

$$(23.4.2) \quad [h^M, a^\alpha] = (\alpha, h)a^\alpha$$

Fact: Any positive energy module M over H is of the form:

$$V^1(\mathfrak{h}) \otimes M_0$$

where the action is trivial on M_0 : $a(v \otimes m) = av \otimes m$

We have a representation of \mathcal{U} on M_0 with diagonal action of \mathfrak{h} . Note that from (23.4.2) follows that a^α shifts the \mathfrak{h} -weight by α . So $M^0 = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda^0$ (\mathfrak{h} -weight spaces) with $a^\alpha : M_\lambda^0 \rightarrow M_{\lambda+\alpha}^0$. Moreover, a^α are invertible operators, so these maps are isomorphisms. Also, $\lambda \in Q^*$ since $z^{\alpha_0}|_{M_\lambda^0} = z^{(\alpha,\lambda)}$. Now it is clear that M is a direct sum of modules $V_{\mu+Q}$, and it is finite because it is positive energy. □

24. INTEGRABLE MODULES

Last time we proved that any v.a. V_Q is a rational v.a. and the number of irreducible modules is $|Q^*/Q|$. But if Q is a root lattice of type A, D or E , then $V_Q \simeq V_1(\mathfrak{g})$ is the simple affine v.a. of level 1 associated to the simple Lie algebra \mathfrak{g} of the corresponding type.

24.1. Theorem. *Let $k \in \mathbb{N}$, \mathfrak{g} a simple Lie algebra of type A, D or E , (\cdot, \cdot) a symmetric invariant bilinear form normalized such that $(\alpha, \alpha) = 2$ for any root α . Then $V_k(\mathfrak{g})$ is a rational v.a. all of its irreducible modules are $L(k, F)$ where $F = F(\lambda)$ is a finite dimensional irreducible \mathfrak{g} -module with highest weight $\lambda \in \mathfrak{h}^*$ such that $(\lambda, \theta) \leq k$; where θ is the highest root.*

24.2. Remark. The modules $L(k, F)$ are **Integrable Modules**; a module over $V_n(\mathfrak{g})$ is called integrable if the chevalley generators $e_{-\alpha_i}$ and e_{α_i} act locally nilpotently.

Proof. We know that this is true for $k = 1$.

Exercise 24.1. *As we mentioned before, we can take tensor product of these modules, namely, if M_1, \dots, M_k are $V_1(\mathfrak{g})$ -modules, then $M_1 \otimes \dots \otimes M_k$ is naturally a $V_k(\mathfrak{g})$ module. Indeed $V_k(\mathfrak{g})$ naturally embeds in $V_1(\mathfrak{g})^{\otimes k}$.*

Proof. Let M_i be $V_1(\mathfrak{g})$ modules, let $a \in V^k(\mathfrak{g})$ and take any representative $\bar{a} \in V(\mathfrak{g}, \mathcal{F})$, also consider \tilde{a} be the image in $V_1(\mathfrak{g})$. We define then the M -valued field

$Y^M(a, z)$ by the action on a basis:

(24.2.1)

$$Y^M(a, z) \otimes_{i=1}^k v_i = \sum_{i=1}^k v_1 \otimes \cdots \otimes v_{i-1} \otimes Y^{M_i}(\tilde{a}, z) v_i \otimes v_{i+1} \otimes \cdots \otimes v_k \quad v_i \in M_i$$

and extend this by linearity. It is clear that this field doesn't depend on the choices made since the center acts as scalar. Also it is clear that if we choose $a = k$ then we have $\tilde{a} = 1$ hence (24.2.1) gives us:

$$Y^M(k, z) \otimes_{i=1}^k v_i = \sum_{i=1}^k \otimes_{i=1}^k v_i = k \otimes_{i=1}^k v_i$$

To check that this is in fact a representation is straightforward and is the same proof as the construction of the tensor product of vertex algebras in Lecture 12. Hence we constructed a $V^k(\mathfrak{g})$ -module structure on M . The maximal ideal in $V^k(\mathfrak{g})$ corresponds to the maximal ideal J in $V^1(\mathfrak{g})$ since it is given by the decomposition with respect to the same virasoro element. Now since M_i is a $V_1(\mathfrak{g})$ -module we know J acts trivially and (24.2.1) gives us a $V_k(\mathfrak{g})$ -module structure. \square

Any of the $L(k, F)$ is a submodule of the tensor product of integrable modules of level 1 (in fact all but one can be taken to be $L(1, \mathbb{C})$). Hence all integrable $\hat{\mathfrak{g}}$ -modules of level k are $V_k(\mathfrak{g})$ -modules.

The next step is to prove that if $L(k, F)$, where $F = F(\lambda)$ is a $V_k(\mathfrak{g})$ -module, then $(\lambda, \theta) \leq k$. Indeed $V_k(\mathfrak{g}) \simeq L(k, \mathbb{C})$ as a $\hat{\mathfrak{g}}$ -module. Let $e_\theta, e_{-\theta} \in \mathfrak{g}$ be the root vectors such that $(e_\theta, e_{-\theta}) = 1$, so that $[e_\theta, e_{-\theta}] = \theta \in \mathfrak{h}^* \simeq \mathfrak{h}$. Let $e_{\alpha_0} = e_{-\theta}t, e_{-\alpha_0} = e_\theta t^{-1}, h_{\alpha_0} = K - \theta$. Let $e_{\alpha_i}, e_{-\alpha_i}$ and h_{α_i} be the Chevalley generators of \mathfrak{g} . Then $e_{\alpha_0}, e_{-\alpha_0}$ and h_{α_0} form an \mathfrak{sl}_2 -triple, moreover, the $\{e_{\alpha_i}, e_{-\alpha_i}, h_{\alpha_i}\}$ with $i = 0, \dots, n$ are Chevalley generators for $\hat{\mathfrak{g}}$. Clearly $e_{\alpha_0}|0\rangle = 0, h_{\alpha_0}|0\rangle = k|0\rangle$. We claim that $e_{-\alpha_0}^{k+1}|0\rangle = 0$. Indeed this vector is killed by e_{α_0} by the \mathfrak{sl}_2 representation theory. It is also killed by all e_{α_i} clearly (since θ is the highest root of \mathfrak{g}). Hence if this vector isn't zero, it would be a singular vector and hence the representation would not be irreducible.

Hence

$$\begin{aligned} 0 &= Y(e_{-\alpha_0}^{k+1}|0\rangle, z) \\ &= Y((e_\theta t^{-1}) \cdots (e_\theta t^{-1})|0\rangle, z) \\ &= Y(e_\theta t^{-1}, z)^{k+1} \Rightarrow Y(e_\theta t^{-1}, z) \\ &= 0 \end{aligned}$$

And we are done.

As an alternative proof we let first $k = 1$. Why $Y((e_\theta t^{-1})|0\rangle, z) = \Gamma_\theta(z) = 0$? From the OPE we have:

$$\Gamma_\theta(z)\Gamma_\theta(w) = \pm(z-w)^{(\theta, \theta)}\Gamma_{\theta, \theta}(z, w)$$

Hence letting $w \rightarrow z$ we get $\Gamma_\theta(z)^2 = 0$. Similarly $\Gamma_\theta^{M^2} = 0$. But level k modules occur in k -fold tensor product, hence $\Gamma_\alpha(z)^{k+1} = 0$, since in at least one factor we have $\Gamma_\alpha^2 = 0$. The same should hold in any $V_k(\mathfrak{g})$ -module M , i.e.

$$Y((e_\theta t^{-1})|0\rangle, z) = 0.$$

But this is nothing else than:

$$Y((e_\theta t^{-1})|0 \rangle, z) = e_\theta(z) = \sum_{n \in \mathbb{Z}} (e_\theta t^n) z^{-1-n}$$

Hence in any $V_k(\mathfrak{g})$ -module we have $e_\theta(z)^{k+1} = 0$. Now writing down the coefficient of z^{-s-k} , we get

$$(24.2.2) \quad \sum_{\substack{n_1 + \dots + n_k = s \\ n_i \in \mathbb{Z}}} (e_\theta t^{n_1}) \dots (e_\theta t^{n_k}) = 0.$$

A particular case of this relation is when $s = -k - 1$ and we let $v_{\lambda,k}$ be the highest weight vector of $L(k, F)$. Then $(e_\theta t^n)v_{\lambda,k} = 0$ for $n \geq 0$. Therefore in the sum (24.2.2) we see that if any factor has $n_i < 0$ then there exists another $n_j > 0$ and the only element that survives is

$$(e_\theta t^{-1})^{k+1} v_{\lambda,k} = 0$$

And now by the \mathfrak{sl}_2 theory we conclude that $(\lambda, \theta) \leq k$.

We have to check now complete reducibility, and this in turn follows from the following fact of representation theory of $\hat{\mathfrak{g}}$: If M is a p.e. $\hat{\mathfrak{g}}$ -module such that any irreducible sub-quotient is integrable, then M is a direct sum of integrable $\hat{\mathfrak{g}}$ -modules. \square

Question: How to list all integrable \mathfrak{g} -modules of level k ?

For this we take the Dynkin diagram of \mathfrak{g} , and label each vertex with a natural number a_i such that $\theta = \sum a_i \alpha_i$. All highest weight of integrable modules are given by assigning a non-negative integer to each vertex such that $\sum k_i a_i = k$, then $\lambda = \sum a_i \omega_i$ (where $\omega_0 = 0$, and ω_i are the weights of level 1).

Now consider the simple Virasoro v.a. V_c .

24.3. Theorem. V_c is rational if and only if $c = 1 - \frac{6(p-q)^2}{pq}$, where $p, q \geq 2$ and they are coprime. All of its irreducible modules are $L(c_{p,q}, h)$, where $h = \frac{(pr-qs)^2 - (p-q)^2}{4pq}$ and $1 \leq r \leq q - 1, 1 \leq s \leq p - 1$.

The simplest non-trivial case is $c_{3,4} = 1/2$ and the values of h are 0, $1/2$ and $1/16$. In order to construct this modules we use the free fermion space F^1 , the universal enveloping vertex algebra of one free fermion ϕ ($[\phi_\lambda \phi] = 1$) and ϕ is odd. The corresponding Virasoro element $L =: \partial\phi\phi$: has central charge $1/2$, also we know that ϕ is primary of conformal weight $1/2$, so we write $\phi(z) = \sum_{n \in 1/2 + \mathbb{Z}} \phi_n z^{-n-1/2}$, and the commutation relations read $[\phi_n, \phi_m] = \delta_{m,-n}$ and $[L_0, \phi_n] = -n\phi_n$. Moreover, F^1 has the basis $\phi_{-n_s} \dots \phi_{-n_1} |0 \rangle$ with $0 < n_1 < \dots < n_s$. Define an hermitian positive form H on F^1 by demanding this to be an orthonormal basis.

Exercise 24.2. with respect to this form we have:

$$\begin{aligned} \phi_n^* &= \phi_{-n}, \\ L_n^* &= L_{-n}. \end{aligned}$$

And with respect to L_0 we have $F^1 = \sum_{j \in 1/2\mathbb{Z}_+} F_j^1$, where we have $F_0^1 = \mathbb{C}|0 \rangle$ and $F_{1/2}^1 = \mathbb{C}\phi_{-1/2}|0 \rangle$.

Proof. It is enough to prove this for a basis of F^1 . For this consider a sequence of natural numbers $0 < n_1 < n_2 < \dots < n_s$, and denote $v = \phi_{-n_s} \dots \phi_{-n_1} |0\rangle$. Similarly let $w = \phi_{-m_r} \dots \phi_{-m_1} |0\rangle$.

Now let $n > 0$ and suppose $n \notin \{n_i\}$ then we can commute ϕ_n to the right to get $\phi_n v = 0$. On the other hand, suppose that $n \notin \{m_i\}$, in this case $\phi_{-n} w \neq 0$ and by the definition of H we know $(v, \phi_{-n} w) = 0$. If on the contrary, $n \in \{m_i\}$ then we have $\phi_{-n} w = 0$ and again $(v, \phi_{-n} w) = 0$. If $n = n_j$ then we commute ϕ_n to the right in the expression of $\phi_n v$ to get

$$\begin{aligned} \phi_n v &= \phi_{-n_s} \dots \phi_n \phi_{-n_j} \dots \phi_{-n_1} |0\rangle \\ &= \phi_{-n_s} \dots (1 + \phi_{-n_j} \phi_n) \dots \phi_{-n_1} |0\rangle \\ &= \phi_{-n_s} \dots \phi_{-n_{j+1}} \phi_{-n_{j-1}} \dots \phi_{-n_1} |0\rangle \end{aligned}$$

Now, as above, if $n \in \{m_i\}$ then clearly we have by the definition of H , $(\phi_n v, w) = 0$, moreover, in this case $\phi_{-n} w = 0$ and hence we get $(v, \phi_{-n} w) = 0$. On the other hand, if $n \notin \{m_i\}$ then we have $\phi_{-n} w \neq 0$. Again, by the definition of H we $(\phi_n v, w) = 1$ if and only if we have $s = r + 1$ and the sequence $\{n_i\}$ is equal to $\{m_i\} \cup n$. In this case we clearly have $\phi_{-n} w = v$ hence we get in general

$$(\phi_n v, w) = (v, \phi_{-n} w) \Rightarrow \phi_n^* = \phi_{-n}$$

The case $n < 0$ is treated exactly in the same way.

For L_n note that

$$\begin{aligned} L_n^* &= \left(\sum_i : \phi_i \phi_{n-i} : \right)^* \\ &= \sum_i : \phi_{n-i}^* \phi_i^* : \\ &= \sum_i : \phi_{i-n} \phi_{-i} : \\ &= \sum_i : \phi_i \phi_{-n-i} : \\ &= L_{-n} \end{aligned}$$

The rest of the statement is obvious and we did it in lecture 13. \square

With respect to the Virasoro algebra we have the decomposition: $F^1 = F_{\text{even}}^1 \oplus F_{\text{odd}}^1$ and since the Virasoro element contains only quadratic terms these spaces are invariant. Moreover, these spaces are irreducible with respect to Virasoro, and $F_{\text{even}}^1 = V_{1/2}$, $F_{\text{odd}}^1 \simeq L(1/2, 1/2)$. Indeed, the only thing that we have to prove is that these are irreducible modules over Virasoro. But one knows that the only irreducible representations of Vir (with $c = 1/2$) are $L(1/2, 0)$, $L(1/2, 1/2)$ and $L(1/2, 1/16)$. Now if either of F_{even}^1 or F_{odd}^1 is reducible then we get a submodule $L(1/2, h)$ with $h \geq 1$, which we know is absurd.

25. TWISTING PROCEDURE

Let R be a Lie conformal (super)algebra. Fix an additive subgroup $\mathbb{Z} \subset \Gamma \subset \mathbb{C}$. A Γ -gradation of R is a decomposition in a direct sum of $\mathbb{C}[\partial]$ -modules:

$$R = \bigoplus_{\bar{\alpha} \in \Gamma/\mathbb{Z}} R_{\bar{\alpha}} \quad [R_{\bar{\alpha}} \lambda R_{\bar{\beta}}] \subset R_{\bar{\alpha} + \bar{\beta}}$$

25.1. Example. $R = R_{\bar{0}} + R_{\bar{1}}$ is a Lie conformal (super)algebra. We have the Ramond $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$ -gradation ($\Gamma = \frac{1}{2}\mathbb{Z}$) identifying $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$ with $\mathbb{Z}/2\mathbb{Z}$.

25.2. Example. If σ is an automorphism of R such that $\sigma^N = 1$ then we have a $\frac{1}{N}\mathbb{Z}/\mathbb{Z}$ -gradation, where

$$R_{\bar{j}} = \{a \in R | \sigma(a) = e^{2\pi i \bar{j}} a\}$$

We associate to this data a **twisted** formal distribution Lie (super)algebra $\text{Lie}^{\text{tw}} R$ as follows:

$$\begin{aligned} R[t^\Gamma] &= R \otimes \mathbb{C}[t^\alpha | \alpha \in \Gamma] \\ R^{\text{tw}}[t^\Gamma] &= \bigoplus_{\alpha \in \Gamma} (R_{\bar{\alpha}} \otimes t^\alpha) \end{aligned}$$

it is a $\mathbb{C}[\partial]$ and $\mathbb{C}[\partial_t]$ -submodule. We let $\text{Lie}^{\text{tw}} R = R^{\text{tw}}[t^\Gamma]/(\partial + \partial_t)R^{\text{tw}}[t^\Gamma]$ with the usual bracket (where $a_{(n)} = at^n$)

$$(25.2.1) \quad [a_{(m)}, b_{(n)}] = \sum_{j \geq 0} \binom{m}{j} (a_{(j)} b)_{(m+n-j)},$$

where

$$\begin{aligned} a &\in R_{\bar{\alpha}} \quad , m \in \bar{\alpha} + \mathbb{Z} \\ b &\in R_{\bar{\beta}} \quad , n \in \bar{\beta} + \mathbb{Z} \end{aligned}$$

Exercise 25.1. *This is a well defined Lie algebra. The corresponding twisted formal distributions are $a(z) = \sum_{n \in \bar{\alpha} + \mathbb{Z}} a_{(n)} z^{-1-n}$.*

The fact that the Lie bracket satisfies the skew symmetry and Jacobi Identities is proved as before (see proof of proposition 3.3). To see that the Lie bracket is well defined, let $(\partial + \partial_t)at^\alpha$ be a homogeneous element in the ideal, and calculate

$$[(\partial + \partial_t)at^\alpha, bt^\beta] = [\partial at^\alpha, bt^\beta] + [\alpha at^{\alpha-1}, bt^\beta] = [-\alpha at^{\alpha-1}, bt^\beta] + [\alpha at^{\alpha-1}, bt^\beta] \in I$$

By skew-symmetry, $[bt^\beta(\partial + \partial_t), at^\alpha] \in I$.

25.3. Example. $R = \mathbb{C}[\partial]\mathfrak{g}$, σ and automorphism of order N of \mathfrak{g} ,

$$\mathfrak{g} = \mathfrak{g}_{j \in \mathbb{Z}/N\mathbb{Z}} \rightarrow L(\mathfrak{g}, \sigma) = \sum_{j \in \mathbb{Z}} \mathfrak{g}_j t^j$$

is the **twisted affine Lie algebra**. Our general construction gives in this case for $a \in \mathfrak{g}_{\bar{\alpha}}$, $b \in \mathfrak{g}_{\bar{\beta}}$, and $\bar{\alpha}, \bar{\beta} \in (N^{-1}\mathbb{Z}) \bmod \mathbb{Z}$, we have:

$$[a_m, b_n] = [a, b]_{m+n} \quad m \in \bar{\alpha} + \mathbb{Z} \quad n \in \bar{\beta} + \mathbb{Z}.$$

25.4. Theorem. *Up to isomorphism, $L(\mathfrak{g}, \sigma)$ depends only on the connected component of $\text{Aut}(\mathfrak{g})$ containing σ .*

25.5. Example. Let $R = \mathbb{C}[\partial]\phi + \mathbb{C}$, $[\phi, \phi] = 1$. This is the usual Lie conformal (super)algebra of 1 free fermion. We take $V_1(R)$ be the vertex algebra of the free fermion. We have $L(z) = \frac{1}{2} : \partial\phi\phi := \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$. The central charge is $c = \frac{1}{2}$, and the commutation relations read:

$$\phi(z) = \sum_{n \in \frac{1}{2} + \mathbb{Z}} \phi_n^{-n-1/2}; \quad [\phi_m, \phi_n] = \delta_{m, -n} \quad m, n \in \frac{1}{2} + \mathbb{Z}$$

Consider now the Ramond twisted fermion:

$$\phi^{\text{tw}} = \sum_{n \in \frac{1}{2} + \mathbb{Z}} \phi_{(n)}^{\text{tw}} z^{-1-n} = \sum_{n \in \mathbb{Z}} \phi_n^{\text{tw}} z^{-n-1/2}$$

and the commutation relations are:

$$[\phi_m^{\text{tw}}, \phi_n^{\text{tw}}] = \delta_{m,-n}, \quad m, n \in \mathbb{Z}$$

Recall that $\mathcal{U}^{\text{comp}}(\text{Lie}R)$ contains fields $a(z)$ for $a \in R$ which generate a v.a., hence they obey the Borcherd's identity, which holds for any indices $m, n, k \in \mathbb{Z}$. Hence the same identities holds for twisted fields:

$$\begin{aligned} & \sum_{j \in \mathbb{Z}_+} \binom{m}{j} (a_{(n+j)} b)_{(m+k-j)}^{\text{tw}} c = \\ & = \sum_{j \in \mathbb{Z}_+} (-1)^j \binom{k}{j} \left(a_{(m+n-j)}^{\text{tw}} (b_{(k+j)}^{\text{tw}} c) - (-1)^n b_{(m+k-j)}^{\text{tw}} (a_{(m+j)}^{\text{tw}} c) \right) \end{aligned}$$

Where $a \in R_{\bar{\alpha}}$, $m \in \bar{\alpha} + \mathbb{Z}$ and $n \in \mathbb{Z}$. We have also the special cases

- (1) $n = 0$ and we get the commutation relations (25.2.1).
- (2) $m = \alpha$, $n = -1$, and we get the following:

$$\sum_{j \geq 0} \binom{\alpha}{j} (a(z)_{(j-1)} b(z))^{\text{tw}} z^{-j} =: a(z)^{\text{tw}} b(z)^{\text{tw}} :$$

where the normally ordered product is defined in the usual way, the only thing we have to say is which is the negative and positive part. For this we define $a^{\text{tw}}(z)_- = \sum_{j \in \alpha + \mathbb{Z}_+} a_{(j)}^{\text{tw}} z^{-1-j}$ and the positive part is defined similarly.

Now we consider the twisted Virasoro algebra:

$$L^{\text{tw}}(z) = \frac{1}{2} : \partial \phi \phi :^{\text{tw}} = L(z)$$

is the usual virasoro Algebra with $c = \frac{1}{2}$. Consider the field $\frac{1}{2} : \partial \phi(z)^{\text{tw}} \phi(z)^{\text{tw}} :$ and $\alpha = \frac{1}{2}$. By (2) above we get:

$$\begin{aligned} & \frac{1}{2} : \partial \phi(z)^{\text{tw}} \phi(z)^{\text{tw}} := \frac{1}{2} (\partial \phi(z)_{(-1)} \phi(z))^{\text{tw}} + (\partial \phi(z)_{(0)} \phi(z))^{\text{tw}} z^{-1} + \\ (25.5.1) \quad & + \frac{1}{2} \binom{1/2}{2} (\partial \phi(z)_{(1)} \phi(z))^{\text{tw}} z^{-2} + \dots \end{aligned}$$

And now noting

$$\begin{aligned} [\partial \phi_{(n)} \phi] &= 0 \quad n > 1 \\ [\partial \phi_{(1)} \phi] &= -1 \\ [\partial \phi_{(0)} \phi] &= 0 \end{aligned}$$

hence we get replacing in (25.5.1)

$$\frac{1}{2} : \partial \phi(z)^{\text{tw}} \phi(z)^{\text{tw}} := \frac{1}{2} (\partial \phi(z)_{(-1)} \phi(z))^{\text{tw}} - \frac{1}{2} \binom{1/2}{2} z^{-2}$$

And therefore we get:

$$\frac{1}{2} : \partial \phi(z)^{\text{tw}} \phi(z)^{\text{tw}} := L^{\text{tw}}(z) + \frac{1}{16z^2}$$

Hence the twisted Virasoro satisfies:

$$L^{\text{tw}}(z) = \frac{1}{2} : \partial\phi^{\text{tw}}(z)\phi^{\text{tw}}(z) : - \frac{1}{16z^2}$$

And in particular

$$L_0 = \frac{1}{4}\phi_0^{\text{tw}^2} + \sum_{n \geq 1} (n + \frac{1}{2})\phi_{-n}^{\text{tw}}\phi_n^{\text{tw}} - \frac{1}{16}$$

$$L_0 = \sum_{n \geq 1} (n + \frac{1}{2})\phi_{-n}^{\text{tw}}\phi_n^{\text{tw}} + \frac{1}{16}$$

$$[L_0, \phi_n^{\text{tw}}] = -n\phi_n^{\text{tw}} \quad n \in \mathbb{Z}$$

We can prove now that the Ramond twisted fermion has a unique irreducible module with basis:

$$F^{\text{tw}} = \langle \phi_{-n_s} \dots \phi_{-n_1} | 0 \rangle \rangle \quad 0 \leq n_1 < n_2 < \dots, \quad n_i \in \mathbb{Z}$$

Hence we have a positive definite Hermitian for such that $\phi_n^* = \phi_{-n}$, hence $L_n^* = L_{-n}$. Note also that $\langle 0 | \phi_0^2 | 0 \rangle = \frac{1}{2}$. And we get:

$$F^{\text{tw}} = F^{\text{tweven}} + F^{\text{twodd}}$$

and the lowest eigenvalue of L_0 is $\frac{1}{16}$.

25.6. Theorem. *Both are irreducible Vir-modules isomorphic to $L(\frac{1}{2}, \frac{1}{16})$, hence $L(\frac{1}{2}, \frac{1}{16})$ is a unitary module.*

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