## 1. First Lecture: Poisson Vertex Algebras

1.1**.** The objective of these lectures is not far-reaching, we want to simply introduce the reader to the *chiral de Rham complex* of a manifold $M$ and its associated *two-variable elliptic genus* in the case when $M$ is a Calabi-Yau manifold. In the first lecture we will introduce the classical version of algebraic structure underlying these constructions: that of a *Poisson vertex algebra*. We will explain its connection to *Courant algebroids* and *Courant Dorfman algebras*. In the second lecture we will give the basic definitions of the quantum object to play with: *vertex algebras*. We will construct several examples and show how the formalism of lambda bracket is useful in making computations. In the third lecture we will introduce the chiral de Rham complex and show how certain geometric structures of $M$ are reflected in its algebraic properties.

Some references for these lectures are as follows. For the basics on vertex algebras the reader might look in [1]. For a definition in terms of lambda brackets and examples of computations the reader might consult [2]. The chiral de Rham complex was defined in [3] and its connections to Courant algebroids was elucidated in [4]. For the use of lambda brackets in this formalism and the explicit construction from Courant algebroids the reader might read [5]. The notion of Courant-Dorfman algebra is due to Roytenberg [6].

1.2**.** As a *driving guideline* for our study of vertex algebras we will have the following diagram in mind:

$$\begin{array}{ccc}
\left\{ \begin{array}{c} \text{QFT} \\ \text{Vertex algebras} \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{c} \text{Classical Field Theory} \\ \text{Poisson Vertex Algebras} \end{array} \right\} \\
\downarrow & & \downarrow \\
\left\{ \begin{array}{c} \text{Quantum Mechanics} \\ \text{Associative algebras} \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{c} \text{Classical Mechanics} \\ \text{Poisson Algebras} \end{array} \right\}
\end{array}$$

Where the horizontal arrows are obtained by suitable *classical limits*.

In the bottom right corner we study the classical motion of particles on a (finite dimensional) manifold $X$, this leads to the study of the *Poisson algebra* of functions on its cotangent bundle $T^*X$. This latter manifold, being a symplectic manifold, admits a *quantization*. Typically this is obtained by studying the Hilbert space $\mathscr{H} := L^2(X)$ and certain associative algebra $\mathscr{A}$ of operators (for example differential operators in $X$) acting on $\mathscr{H}$.

In the top corner we replace our finite dimensional space $X$ by $\mathscr{L}X$, the *loop space of* $X$. The "algebra of functions" on $T^*\mathscr{L}X$ now has the structure of a *Poisson vertex algebra* reflecting the fact that $T^*\mathscr{L}X$ is a symplectic infinite dimensional manifold (and hence Poisson). To make these statements precise one needs to explain what *algebra of functions* one is considering. If one considers formal local functionals one is lead to the notion of Poisson vertex algebra explained below. Vertex algebras in the top left corner are just deformations of the corresponding Poisson objects.

1.3. **Definition.** Let $k$ be a field. A *Lie algebra* is a $k$-vector space $\mathfrak{g}$ together with a $k$-bi-linear and tri-linear maps

$$\mu \in \operatorname{Hom}_k(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g}), \qquad \operatorname{Jac} \in \operatorname{Hom}_k(\mathfrak{g}^{\otimes 3}, \mathfrak{g}) \tag{1.1}$$

such that

(a) $\mu$ is alternating: $\mu(a,a) = 0$ for all $a \in \mathfrak{g}$. So that $\mu$ descends to a linear map $\mu : \wedge^2 \mathfrak{g} \to \mathfrak{g}$.
(b) $\operatorname{Jac}(a,b,c) = \mu\big(a, \mu(b,c)\big) + \mu\big(b, \mu(c,a)\big) + \mu\big(c, \mu(a,b)\big)$.
(c) $\operatorname{Jac}(a,b,c) = 0$ for all $a, b, c \in \mathfrak{g}$.

Often times $\mu(a,b)$ is denoted by $[a,b]$.

1.4. What if we move from the category of $k$-modules to the category of $k[\partial]$-modules? This category is a tensor category, that is, given $V, W$ two $k[\partial]$-modules, we have that $\partial$ acts on $V \otimes_k W$ as

$$\partial\big(v \otimes w\big) = (\partial v) \otimes w + v \otimes (\partial w). \tag{1.2}$$

Therefore the spaces involved in (1.1) are well defined, where we replace $\mathrm{Hom}_k$ by $\mathrm{Hom}_{k[\partial]}$, and Definition 1.3 still makes sense in this context. We obtain the notion of a *differential lie algebra*, that is a usual Lie algebra $\mathfrak{g}$ together with an endomorphism $\partial$ such that $\partial[a,b] = [\partial a, b] + [a, \partial b]$.

1.5. In the context of $k[\partial]$ modules we can however make something more interesting by considering different *Hom* spaces in (1.1). Notice that given two $k[\partial]$ modules $V$ and $W$, we have a $k[\partial_1, \partial_2]$ module structure on $V \otimes_k W$ where each differential acts on different factors. This $k[\partial_1, \partial_2]$ module is denoted by $V \boxtimes W$.

We have a $k[\partial_1, \partial_2] - k[\partial]$ bi-module structure on $k[\partial_1, \partial_2]$, the action of $k[\partial_1, \partial_2]$ is just by multiplication, while $\partial$ acts by $\partial_1 + \partial_2$. We obtain a $k[\partial_1, \partial_2]$ module $\Delta_! V := k[\partial_1, \partial_2] \otimes_{k[\partial]} V$.

We now will consider a $k[\partial]$ module $\mathfrak{g}$ and require our *bracket* $\mu$ to be an element of

$$\mu \in P^*_2(\mathfrak{g}) := \mathrm{Hom}_{k[\partial_1, \partial_2]}(\mathfrak{g} \boxtimes \mathfrak{g}, \Delta_! \mathfrak{g}). \tag{1.3}$$

There is an obvious involution $\sigma$ in $\mathfrak{g} \boxtimes \mathfrak{g}$ and $\Delta_! \mathfrak{g}$ coming from $\partial_1 \leftrightarrow \partial_2$. On $\mathfrak{g} \boxtimes \mathfrak{g}$ it simply exchanges the two factors. The condition a) in 1.3 is now simply

(a) $\mu \circ \sigma = -\mu$.

To define the *Jacobiator* we need to consider the $k[\partial_1, \partial_2, \partial_3]$ module $\mathfrak{g} \boxtimes \mathfrak{g} \boxtimes \mathfrak{g}$ and $\Delta^{(3)}_! \mathfrak{g}$ defined as follows. We notice that $k[\partial_1, \partial_2, \partial_3]$ is naturally a $k[\partial_1, \partial_2, \partial_3] - k[\partial]$ bi-module, where now $\partial$ acts by $\partial_1 + \partial_2 + \partial_3$. We define $\Delta^{(3)}_! \mathfrak{g} := k[\partial_1, \partial_2, \partial_3] \otimes_{k[\partial]} \mathfrak{g}$. We will require

$$\mathrm{Jac} \in P^*_3(\mathfrak{g}) := \mathrm{Hom}_{k[\partial_1, \partial_2, \partial_3]}\left(\mathfrak{g} \boxtimes \mathfrak{g} \boxtimes \mathfrak{g}, \Delta^{(3)}_! \mathfrak{g}\right).$$

Notice that our bracket $\mu$ composes nicely, namely, we can define for example the composition $\mu_{23} \circ \mu_{12} \in P^*_3(\mathfrak{g})$ as the composition

$$\mathfrak{g} \boxtimes \mathfrak{g} \boxtimes \mathfrak{g} \xrightarrow{\mu \otimes \mathrm{Id}_{\mathfrak{g}}} \Delta_! \mathfrak{g} \boxtimes \mathfrak{g} \simeq \big(k[\partial_1, \partial_2] \otimes_{k[\partial]} \mathfrak{g}\big) \boxtimes \mathfrak{g} \simeq \\ \simeq \big(k[\partial_1] \otimes_k \mathfrak{g}\big) \boxtimes \mathfrak{g} \simeq k[\partial_1] \otimes_k \mathfrak{g} \otimes_k \mathfrak{g} \xrightarrow{\mu} \Delta^{(3)}_! \mathfrak{g}$$

where in the last two isomorphisms we have chosen $k$-vector spaces isomorphisms and the $k[\partial_1, \partial_2, \partial_3]$-module structure is understood. Since we have an obvious action of the symmetric group of three elements on $\mathfrak{g} \boxtimes \mathfrak{g} \boxtimes \mathfrak{g}$ we define the other compositions $\mu_{31} \circ \mu_{23}$ and $\mu_{12} \circ \mu_{31} \in P^*_3(\mathfrak{g})$. Finally the other axioms in Def. 1.3 read

(b) $\mathrm{Jac} = \mu_{23} \circ \mu_{12} + \mu_{31} \circ \mu_{23} + \mu_{12} \circ \mu_{31}$

(c) $\mathrm{Jac} = 0$.

**Definition.** A $k[\partial]$-module together with an operation $\mu$ as in (1.3) satisfying a) b) and c) above is called a *Lie conformal algebra*[1].

1.6. We can write more concretely the Jacobi condition and the $k[\partial_1, \partial_2]$-equivariance of the bracket $\mu$ as follows. If we choose the isomorphism of vector spaces $\Delta_! \mathfrak{g} \simeq k[\partial_1] \otimes_k \mathfrak{g}$ and call $\partial_1 = \lambda$ we see that a Lie conformal algebra consists of a $k[\partial]$ module $\mathfrak{g}$ together with a bracket

$$[\cdot_\Lambda \cdot] : \mathfrak{g} \otimes_k \mathfrak{g} \to k[\lambda] \otimes_k \mathfrak{g},$$

satisfying the following axioms.

[1] Also known as *Lie*$^*$-algebras or vertex-Lie algebras

(a) Sesquilinearity: $[\partial a_\lambda b] = -\lambda [a_\lambda b]$,
(b) $\partial$ is a derivation: $\partial [a_\lambda b] = [\partial a_\lambda b] + [a_\lambda \partial b]$
(c) Skew-Symmetry: $[a_\lambda b] = -[b_{-\lambda-\partial} a]$
(d) Jacobi condition: $[a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + [b_\mu [a_\lambda b]]$

1.7. **Remark.**

(a) Let $V$ be a Lie conformal algebra and define the bracket $[,] : V \otimes V \to V$ as $[a, b] = [a_\lambda b]_{\lambda=0}$. It follows from a) and b) in the definition that the space $\partial V \subset V$ is an *ideal* for this bracket, that is $[\partial V, V] \subset \partial V$ and $[V, \partial V] \subset \partial V$, therefore defining $\mathfrak{g} := V/\partial V$, the bracket $[,]$ descends to a linear map $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$. It now follows from c) and d) that this bracket endows $\mathfrak{g}$ with a Lie algebra structure.

In what follows, Lie conformal algebras will be playing the role of Lie algebras in the usual finite dimensional theory.

(b) A Lie conformal subalgebra of $\mathfrak{g}$ is a $k[\partial]$ submodule $\mathfrak{h}$ stable under lambda bracket. An *ideal* is a subalgebra such that $[\mathfrak{h}_\lambda \mathfrak{g}] \subset k[\lambda] \otimes \mathfrak{h}$. If $\mathfrak{h} \subset \mathfrak{g}$ is an ideal, then the quotient space $\mathfrak{g}/\mathfrak{h}$ is naturally a Lie conformal algebra.

1.8. **Examples.**

(a) Most examples we will work will be of Lie conformal algebras that are of finite type over $k[\partial]$. Consider first the $k[\partial]$-module

$$\mathrm{Vir} := k[\partial]/\partial k[\partial] \oplus k[\partial] \overset{\mathrm{Vect}}{\simeq} k \oplus k[\partial].$$

We will denote $L$ the generator of the free module and $C$ the generator of the first summand (so that $\partial C = 0$). The only non-trivial bracket amongst generators is

$$[L_\lambda L] = (\partial + 2\lambda) L + \frac{\lambda^3}{12} C, \tag{1.4}$$

and extend this by a) b) in 1.6 to Vir. Notice that $C$ is central in this algebra (that is $[C_\lambda \mathrm{Vir}] = 0$). This algebra is called the *Virasoro* Lie conformal algebra

(b) Let $\mathfrak{g}$ be a finite dimensional Lie algebra and let $(,) \in \mathrm{Sym}^2 \mathfrak{g}^*$ be an invariant bilinear form. We will consider the $k[\partial]$-module

$$\mathrm{Cur}(\mathfrak{g}) := k[\partial]/\partial k[\partial] \oplus k[\partial] \otimes_k \mathfrak{g} \overset{\mathrm{Vect}}{\simeq} k \oplus k[\partial] \otimes_k \mathfrak{g},$$

with non-trivial brackets amongst generators

$$[a_\lambda b] = [a, b] + \lambda (a, b) K, \qquad a, b \in \mathfrak{g},$$

where $K$ is the central element that generates $k[\partial]/\partial k[\partial]$. This algebra is called the *current or affine Lie conformal algebra.*

(c) A particular example of b) is when $\mathfrak{g}$ is Abelian. In this case $\mathfrak{g}$ is just a finite dimensional vector space $V$ with a symmetric bilinear form $(,)$. The non-trivial brackets amongst generators now reads:

$$[v_\lambda w] = \lambda (v, w) K, \qquad v, w \in V.$$

This algebra is called the *Heisenberg or free bosons* Lie conformal algebra.

(d) In a similar fashion to the previous example, let $V$ be a finite dimensional vector space and let $\langle , \rangle$ be a **skew-symmetric** bilinear form. We consider the $k[\partial]$-module

$$F(V) := k[\partial]/\partial k[\partial] \oplus k[\partial] \otimes_k V \overset{\mathrm{Vect}}{\simeq} k \oplus k[\partial] \otimes_k V,$$

with non-trivial brackets amongst generators

$$[v_\lambda w] = \langle v, w \rangle K.$$

This algebra will be called the free (bosonic) ghost system. Notice the difference between this example and the previous one in that the symmetry under exchange of factors is different but also the factor $\lambda$ appearing in the Heisenberg case is not in this example. Note also that in order to have a non-trivial bracket, the dimension of $V$ needs to be at least 2.

(e) From the supersymmetric point of view, the previous example is an example of *free Fermions*. For this one needs a $\mathbb{Z}/2\mathbb{Z}$-graded version of the definition in 1.6. The basic idea is that a *super* Lie conformal algebra is a $\mathbb{Z}/2\mathbb{Z}$-graded $k[\partial]$-module such that a)–d) in 1.6 hold with some additions of signs: a) and b) are left as they are, in c) the sign in the RHS changes whenever $a$ **and** $b$ are both odd and finally the "+" symbol in the RHS of d) becomes a "-" if both $a$ and $b$ are odd. As a way of remembering, whenever we commute two odd elements, the sign changes. The basic example of a super Lie conformal algebra is as follows.

Let $V$ be a finite dimensional super vector space (ie. a $\mathbb{Z}/2\mathbb{Z}$-graded vector space) and let $\langle , \rangle$ be a *super-skew-symmetric* bilinear form, ie. $\langle V_{\bar 0}, V_{\bar 1}\rangle = \langle V_{\bar 1}, V_{\bar 0}\rangle = 0$ and $\langle , \rangle$ is skew-symmetric restricted to $V_{\bar 0}$ and symmetric when restricted to $V_{\bar 1}$. Then we form $F(V)$ as above with the same bracket and we obtain a super Lie conformal algebra. The particular case when $V$ is a purely odd vector space (therefore $\langle , \rangle$ is symmetric) is commonly known as the Vertex algebra of *Free fermions*. Notice that in this case, the dimension of $V$ may be 1 and still the algebra may not be Abelian. From this perspective, the free (bosonic) ghosts of the previous example d) has also the name of *even Fermions* in some literature. In fact that algebra goes under several other names: *symplectic bosons*, *Weyl Lie conformal algebra* and finally is also known as a $\beta\gamma$-system.

Finally, in this very same example, when $V$ is purely odd and can be written as $V_- \oplus V_+$ such that each space is isotropic for $\langle , \rangle$ and this bilinear form gives a non-degenerate pairing $V_- \otimes V_+ \to k$, the corresponding Lie conformal algebra $F(V)$ is commonly known as a *fermionic ghost system* or *bc*-system. We will stick to the supersymmetric point of view and all all these examples *free Fermions*.

(f) Consider the super $k[\partial]$-module $NS$ whose even part is $NS_{\bar 0} :=$ Vir and whose odd part is a free module $NS_{\bar 1} := k[\partial]$ with generator $G$. The only non-trivial brackets are (1.4) and

$$[L_\lambda G] = \left(\partial + \frac{3}{2}\lambda\right) G, \qquad [G_\lambda G] = L + \frac{\lambda^2}{6} C, \tag{1.5}$$

**Exercise.** Check that these brackets are compatible with a)–d) in the definition of a Lie conformal algebra and therefore they produce a super Lie conformal algebra $NS$ known as the *Neveu-Schwarz* or *super-Virasoro* Lie conformal algebra

1.9. **Definition.** A Poisson vertex algebra $A$ is a tuple $(A, 1, \cdot, \{_\lambda\})$ such that

(a) $A$ is a $k[\partial]$-module.
(b) $(A, \{_\lambda\})$ is a Lie conformal algebra.
(c) $(A, \cdot, 1)$ is a unital commutative associative algebra (in the category of $k[\partial]$-modules)
(d) The Lie conformal algebra $A$ acts on itself by derivations of the commutative product, that is, the Leibniz rule is satisfied: $\{a_\lambda bc\} = \{a_\lambda b\}c + b\{a_\lambda c\}$.

1.10. **Remark.** Lie conformal algebras and Poisson vertex algebras can be defined similarly over any commutative ring $R$ (of *char* $\neq 2$) replacing our field $k$. Below we will consider the case $R = k[[\hbar]]$.

1.11**.** In the usual case (vector space as opposed to $k[\partial]$-modules) there are two canonical examples of Poisson algebras. One is as the algebra of functions on a symplectic (or Poisson manifold), for example $T^*X$ for any manifold $X$. The other is as the algebra of polynomial functions on the dual $\mathfrak{g}^*$ of a Lie algebra, that is $\operatorname{Sym}\mathfrak{g}$ carries a natural Poisson algebra structure. Both examples have natural *vertex* generalizations. Let us start with the latter.

Let $\mathfrak{g}$ be a Lie conformal algebra of finite type over $k[\partial]$. Consider the commutative algebra $\operatorname{Sym}\mathfrak{g}$. Here the symmetric product is taken over $k[\partial]$ with the usual tensor product. Let us describe this in slight more detail. We have seen that for any two $k[\partial]$-modules $V \otimes W$ is a $k[\partial]$-module with (1.2). This allows us to define a $k[\partial]$-module structure on the associative algebra $T(\mathfrak{g}) = \sum_{k\geq 0} \mathfrak{g}^{\otimes k}$. This module structure is compatible with the associative unital product, that is $\partial$ acts as a derivation of this product. Consider the (two-sided) ideal $\mathscr{I}$ of $T(\mathfrak{g})$ generated by elements of the form $a \otimes b - b \otimes a$ for $a, b \in \mathfrak{g}$. This ideal is invariant by $\partial$ since the generators are. Therefore the quotient $\operatorname{Sym}\mathfrak{g} := T(\mathfrak{g})/\mathscr{I}$ is a $k[\partial]$-module which is a commutative algebra since $T(\mathfrak{g})$ is generated by $\mathfrak{g}$ and elements of $\mathfrak{g}$ commute modulo $\mathscr{I}$. We can extend the bracket $\{_\lambda\}$ from $\mathfrak{g} \otimes \mathfrak{g}$ to $\mathfrak{g} \otimes \operatorname{Sym}\mathfrak{g}$ by using the Leibniz rule d) in Def. 1.9. Using skew-symmetry and d) again we obtain a Poisson vertex algebra structure in $\operatorname{Sym}\mathfrak{g}$.

1.12. To understand the geometric example, let us show that in the usual case, it is an instance of $\operatorname{Sym}\mathfrak{g}$ for a *family* of Lie algebras $\mathfrak{g}$ over $X$.

**Definition.** Let $X$ be a smooth manifold (in some category, like holomorphic, analytic, $C^\infty$, etc), $\mathscr{O}_X$ its sheaf of smooth functions, a *Lie algebroid* $\mathscr{A}$ on $X$ is a sheaf of $k$-vector spaces endowed with three maps:

(a) $\cdot : \mathscr{O}_X \otimes_k \mathscr{A} \to \mathscr{A}$ making $\mathscr{A}$ into a sheaf of $\mathscr{O}_X$-modules[2].
(b) $[,] : \mathscr{A} \otimes_k \mathscr{A} \to \mathscr{A}$ making $\mathscr{A}$ into a sheaf of $k$-linear Lie algebras.
(c) An action $\mathscr{A} \otimes_k \mathscr{O}_X \to \mathscr{O}_X$ of $\mathscr{A}$ on $\mathscr{O}_X$ by derivations (ie. $a(fg) = (af)g + f(ag)$ for all $f, g \in \mathscr{O}_X$ and $a \in \mathscr{A}$).
(d) The action of $\mathscr{A}$ on functions is compatible with the $\mathscr{O}_X$ module structure on $\mathscr{A}$, namely $[a, fb] = a(f)b + f[a, b]$.

1.13. **Examples.**

(a) The tangent sheaf of $X$, $\mathscr{T}_X$ whose sections are smooth vector fields and with the Lie bracket of vector fields and the usual action on functions by derivations is a Lie algebroid. Because of d) in Def. 1.12 every Lie algebroid $\mathscr{A}$ comes equipped with a map $\sharp : \mathscr{A} \to \mathscr{T}_X$ of Lie algebras (and in fact of Lie algebroids) typically called the anchor.
(b) When $X$ is a Poisson manifold with Poisson bivector $\pi$, its cotangent bundle $\mathscr{T}_X^*$ is a Lie algebroid, the anchor is given by contraction with $\pi$.
(c) If the manifold $X$ is just a point, a Lie algebroid is the same thing as a Lie algebra.
(d) Let $\mathscr{K} = \ker \sharp$, then the bracket in $\mathscr{A}$ induces a $\mathscr{O}_X$ linear bracket in $\mathscr{K}$ (since the action of $\mathscr{K}$ on $\mathscr{O}_X$ is trivial), hence $\mathscr{K}$ is a *bundle of Lie algebras* on $X$ (in our setup it might just be a sheaf and not an actual vector bundle).
(e) Just as in the usual case (when $X$ is just a point) if $\mathscr{A}$ is a Lie algebroid on $X$, the sheaf $\operatorname{Sym}\mathscr{A}$ is a sheaf of Poisson algebras on $X$. We define $\{a, f\} = (af) = -\{f, a\}$ for $f \in \mathscr{O}_X$ and $a \in \mathscr{A}$, and this bracket together with the bracket in $\mathscr{A}$ is extended by Leibniz rule to all of $\operatorname{Sym}\mathscr{A}$. In the particular case when $\mathscr{A} = \mathscr{T}_X$ the commutative algebra $\operatorname{Sym}\mathscr{T}_X$ can be viewed as the algebra of functions on $T^*X$. In fact these are polynomial on the fibers, but that will be enough for our purposes.

1.14. Let $P = \oplus_{k\geq 0} P_k$ be a graded Poisson algebra such that the commutative product is of degree 0, that is $P_i P_j \subset P_{i+j}$ and the Lie bracket is of degree $-1$: $\{P_i, P_j\} \subset P_{i+j-1}$. The

[2]Differential geometers usually ask this sheaf to be locally free, algebraic geometers typically will ask this sheaf to be quasi-coherent.

commutative products produces a unital commutative algebra structure on $P_0$, let us call this algebra $\mathscr{O}_X$. The same commutative product restricts to a $\mathscr{O}_X$-module structure on $P_1$, let us call this module $\mathscr{A}$. The Lie bracket restricts to a $k$-linear Lie bracket on $\mathscr{A}$ and from $\{P_1, P_0\} \subset P_0$ we obtain an action of $\mathscr{A}$ on $\mathscr{O}_X$ by derivations (since the original Poisson bracket on $P$ satisfies the Leibniz rule). The same Leibniz rule shows that d) in Def. 1.12 is satisfied. With a little bit of work on morphisms We arrive to the following

**Lemma.** *The category of graded Poisson algebras generated in degree* 0 *and* 1 *is equivalent to the category of Lie algebroids.*

1.15**.** We can now produce a vertex version of the above mentioned objets. We just need to replace $k$-vector spaces by $k[\partial]$-modules. Let us fix a smooth manifold $X$ and $\mathscr{A}$ a Lie algebroid on it.

**Proposition.** *Consider the sheaf $P(\mathscr{A})$ of $k[\partial] \otimes \mathscr{O}_X$-modules freely generated by* $\operatorname{Sym} \mathscr{A}$*. This is a Poisson vertex algebra with bracket*

$$\{f_\lambda g\} = 0, \qquad \{a_\lambda f\} = a(f), \qquad \{a_\lambda b\} = [a, b], \qquad a, b \in \mathscr{A}, \quad f, g \in \mathscr{O}_X,$$

*extended to the whole $P(\mathscr{A})$ by the Leibniz rule and a)–d) in 1.6.*

1.16**.** The example in 1.15 is not that interesting since it lacks powers of $\lambda$ in the brackets. In fact this example is simply (a sheaf version) of $\operatorname{Cur}(\mathfrak{g})$ as in b) in 1.8 where the invariant pairing $(,) = 0$ and the Lie algebra is $\mathfrak{g} := \mathscr{A} \ltimes \mathscr{O}_X$. In order to have an example with a non-trivial $(,)$ we need the notion of a *family* version of Lie algebras together with invariant bilinear forms. This is provided by the following

**Definition.** A *Courant Algebroid* is a $\mathscr{O}_X$ module $\mathscr{A}$ together with a $k$-linear bracket $[,] : \mathscr{A} \otimes_k \mathscr{A} \to \mathscr{A}$ and a non-degenerate symmetric $\mathscr{O}_X$-linear pairing $(,) : \mathscr{A} \otimes_{\mathscr{O}_X} \mathscr{A} \to \mathscr{O}_X$ and an action of $\mathscr{A}$ on $\mathscr{O}_X$ by derivations such that

(a) The bracket satisfies the following version of the Jacobi condition:

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]], \qquad a, b, c \in \mathscr{A},$$

but it is not skew-symmetric.

(b) The bracket fails to be skew-symmetric by the action of derivations: $(a, [b, c] + [c, b]) = a(b, c)$.

(c) The action of $\mathscr{A}$ on functions is compatible with the $\mathscr{O}_X$-module structure as in d) of Def. 1.12.

(d) The pairing is invariant modulo the actions of the derivations: $([a, b], c) + (b, [a, c]) = a(b, c)$.

(e) Since the bracket is not a Lie bracket, we need to specify what we mean by *action* of $\mathscr{A}$ in $\mathscr{O}$, we mean the usual condition $a \cdot b \cdot f - b \cdot a \cdot f = [a, b] \cdot f$.

1.17. **Remark.**

(a) The action of $\mathscr{A}$ by derivation of $\mathscr{O}_X$ produces a map $\mathscr{A} \to \mathscr{T}_X$ which preserves brackets. We call this map the anchor.

(b) Since the pairing is non-degenerate we obtain a *differential* $d_{\mathscr{A}} : \mathscr{O}_X \to \mathscr{A}$ by $(d_{\mathscr{A}} f, a) = a(f)$. With this differential b) in the definition shows that the bracket is skew-symmetric modulo $d_{\mathscr{A}}$-exact terms:

$$[b, c] + [c, b] = d_{\mathscr{A}}(b, c).$$

Notice also that this differential satisfies the Leibniz rule $d_{\mathscr{A}}(fg) = g d_{\mathscr{A}} f + f d_{\mathscr{A}} g$ since the pairing is $\mathscr{O}_X$-linear and $\mathscr{A}$ acts by derivations. We obtain thus a map $\mathscr{T}_X^* \to \mathscr{A}$

which is compatible with the dual of the anchor map in a) where we use the pairing $(,)$ to identify $\mathscr{A}$ with its dual.

(c) $\mathscr{A} := \mathscr{T}_X \oplus \mathscr{T}_X^*$ has a canonical Courant algebroid structure where the pairing is the obvious symmetric pairing (divided by 2) and the bracket is given by the *Dorfman bracket*:

$$[X + \zeta, Y + \chi] = [X, Y] + \mathrm{Lie}_X\, \chi - i_Y d\zeta, \qquad X, Y \in \mathscr{T}_X, \quad \zeta, \chi \in \mathscr{T}_X^*.$$

A Courant algebroid is called *exact* if it fits in a short exact sequence of $\mathscr{O}_X$-modules

$$0 \to \mathscr{T}_X^* \to \mathscr{A} \to \mathscr{T}_X \to 0,$$

exact Courant algebroids are locally *trivial* in the sense that are isomorphic to $\mathscr{T}_X \oplus \mathscr{T}_X^*$ with the Dorfman bracket.

(d) When $X$ is just a point, a Courant algebroid is the same thing as a Lie algebra with an invariant symmetric bilinear form (non-degenerate). It is in this sense that we think of Courant algebroids as family versions of these algebras endowed with non-trivial pairings.

1.18**.** Given a Courant algebroid over a point, we have constructed a Lie conformal algebra $\mathrm{Cur}(\mathfrak{g})$ in b) 1.8 and the corresponding Poisson vertex algebra in 1.11. It is natural to ask whether the same construction goes through for Courant algebroids. This is the content of

**Proposition.** *Let $\mathscr{A}$ be a Courant algebroid on $X$. There exists a (sheaf of) Poisson vertex algebra $P(\mathscr{A})$ together with two maps (of sheaves of vector spaces) $i : \mathscr{O}_X \hookrightarrow P(\mathscr{A})$ and $\tau : \mathscr{A} \hookrightarrow P(\mathscr{A})$ sucht that*

- *$i$ is a map of commutative unital algebras. In particular $P(\mathscr{A})$ is a $\mathscr{O}_X$-module.*
- *$j$ is compatible with the $\mathscr{O}_X$-module structure, namely $j(fa) = i(f)j(a)$ for all $f \in \mathscr{O}_X$ and $a \in \mathscr{A}$.*
- *$j$ satisfies the following compatibility condition between the $\{_\lambda\}$ bracket of $P(\mathscr{A})$ and the Courant algebroid structure on $\mathscr{A}$:*

$$\{j(a)_\lambda j(b)\} = j[a, b] + \lambda i(a, b), \qquad \forall a, b \in \mathscr{A}$$

*Moreover, $P(\mathscr{A})$ is universal with these conditions, namely for any other Poisson vertex algebra $P$ satisfying the above, there exists a unique morphism $P(\mathscr{A}) \twoheadrightarrow P$.*

*Sketch of the Proof.* Consider the constant sheaf $\underline{k}$ and form the $\underline{k}[\partial]$-module $\mathrm{Cur}(\mathscr{A})$ as the quotient:

$$0 \to \underline{k}[\partial] \otimes \mathscr{O}_X \xrightarrow{\partial - d_{\mathscr{A}}} \underline{k}[\partial] \otimes \big(\mathscr{O}_X \oplus \mathscr{A}\big) \xrightarrow{\pi} \mathrm{Cur}(\mathscr{A}) \to 0.$$

This is naturally a (sheaf of) Lie conformal algebra with bracket given by

$$[a_\lambda b] = [a, b] + \lambda(a, b), \qquad [a_\lambda f] = a(f), \qquad [f_\lambda g] = 0, \qquad a, b \in \mathscr{A}, \quad f, g \in \mathscr{O}_X.$$

We first need to check that this bracket is well defined on the quotient. We define the same bracket in the free module generated by $\mathscr{O}_X \oplus \mathscr{A}$ and we see that skew-symmetry is not satisfied, in fact, for $a, b \in \mathscr{A}$ we have

$$\begin{aligned}[b_\lambda a] = [b, a] + \lambda(b, a) = -[a, b] + d_{\mathscr{A}}(a, b) + \lambda(a, b) = \\ -[a, b] + \big(\lambda + \partial\big)(a, b) + \big(d_{\mathscr{A}} - \partial\big)(a, b) = -[a_{-\lambda-\partial} b] + \big(d_{\mathscr{A}} - \partial\big)(a, b).\end{aligned}$$

From where Skew-Symmetry is satisfied in the quotient $\mathrm{Cur}(\mathscr{A})$. Similarly for sesquilinearity a) in Def. 1.6, we have for $f \in \mathscr{O}_X$ and $a \in \mathscr{A}$

$$[d_{\mathscr{A}} f_\lambda a] = [d_{\mathscr{A}} f, a] + \lambda(d_{\mathscr{A}} f, a) = \lambda(d_{\mathscr{A}} f, a) = \lambda a(f) = \lambda[a_\lambda f] = -\lambda[f_\lambda a].$$

The Jacobi condition is checked in the same way.

The Lie conformal algebra $\mathrm{Cur}(\mathscr{A})$ is a $\mathscr{O}_X$ differential module naturally and in fact the submodule $\pi k[\partial] \otimes \mathscr{O}_X$ is an Abelian ideal of $\mathrm{Cur}(\mathscr{A})$. The algebra $P(\mathscr{A})$ is as in 1.11 simply

$\operatorname{Sym}_k \operatorname{Cur}(\mathscr{A})/\mathscr{I}$ where $\mathscr{I}$ is the ideal generated by $\pi k[\partial] \otimes \mathscr{O}_X$. As a $\mathscr{O}_X$ module this is just $\operatorname{Sym}_{\mathscr{O}_X} \operatorname{Cur}(\mathscr{A})$. The universality condition is straightforward to check. □

1.19. **Remark.**

(a) In the case when $\mathscr{A}$ is the standard Courant algebroid of Rem. 1.17 c) over the affine line $\mathbb{A}^1 = \operatorname{Spec} k[x]$, we see that $P(\mathscr{A})$ is simply (the symmetric algebra of) the $\beta\gamma$-system of Ex. 1.8 d). Here we identify $\gamma$ with $x \in \mathscr{O}_X$ and $\beta$ with $\partial_x \in \mathscr{T}_X$.
(b) Notice from the proof of Prop. 1.18 it follows that as a $\mathscr{O}_X$-module, $P(\mathscr{A})$ consists of several copies of $\operatorname{Sym} \mathscr{A}$.

1.20**.** In fact a version of Lem. 1.14 by considering graded Poisson vertex algebras generated in degree 0 and 1 shows that the notion of Courant algebroid needs to be relaxed in order to allow for degenerate pairings $(,)$. Let $V = \oplus_{k\geq 0} V_k$ be a Poisson vertex algebra such that the commutative product is of degree 0, the derivation $\partial$ is of degree 1 and the lambda bracket is of degree $-1$, that is for $a \in V_i$, $b \in V_j$ and $\{a_\lambda b\} = \sum \lambda^k c_k$ we have $c_k \in V_{i+j-k-1}$ for all $k$. Let us study the structure that we obtain from $V_0$ and $V_1$. The commutative product produces

(a) a commutative unital algebra $\mathscr{O}_X := V_0$,
(b) a $\mathscr{O}_X$ module $\mathscr{E} := V_1$.

The lambda bracket restricted to $V_1 \otimes V_1$ has only two components. Naming $[,]$ its $\lambda^0$ term and $(,)$ its linear term we obtain

(c) a $k$-bilinear map $[,] : \mathscr{E} \otimes \mathscr{E} \to \mathscr{E}$,
(d) a $k$-bilinear map $(,) : \mathscr{E} \otimes \mathscr{E} \to \mathscr{O}_X$.

Since the derivation is of degree 1 we have

(e) a derivation $\partial : \mathscr{O}_X \to \mathscr{E}$.

From skew-symmetry of the lambda bracket restricted to $V_1 \otimes V_1$ we obtain immediately:

(1) $(a, b) = (b, a)$ for all $a, b \in \mathscr{E}$,
(2) $[a, b] + [b, a] = \partial(a, b)$ for all $a, b \in \mathscr{E}$.

From sesquilinearity we obtain

(3) $[\partial f, a] = 0$, for $f \in \mathscr{O}_X$, $a \in \mathscr{E}$,
(4) $(\partial f, \partial g) = 0$, $f, g \in \mathscr{O}_X$,

and from the Leibniz rule we get

(5) $[a, fb] = (a, \partial f)b + f[a, b]$.

Finally the Jacobi identity implies

(6) $[a, [b, c]] = [[a, b], c] + [b, [a, c]]$, $a, b, c \in \mathscr{E}$,
(7) $(a, \partial(b, c)) = ([a, b], c) + (b, [a, c])$, $a, b, c \in \mathscr{E}$.

**Proposition.** *The set of data* a) − e) *above satisfying the axioms* 1) − 7) *is called a* Courant-Dorfman *algebra. They form a category in the obvious way and we have an equivalence of categories between graded Poisson vertex algebras freely generated in degree* 0 *and* 1 *and Courant-Dorfman algebras*

1.21. **Remark.** When the pairing $(,)$ is non-degenerate, the action of $\mathscr{E}$ on $\mathscr{O}$ determines completely the derivation $\partial$, hence axioms 3) and 4) above are redundant. In this case the notion of a Courand-Dorfman algebra coincides with that of a Courant algebroid.

## 2. Second Lecture: Vertex Algebras

2.1. **Definition.** A *vertex algebra* is a Lie conformal algebra $V$ with lambda-bracket $[_\lambda]$ endowed with a $k$-bilinear operation $\cdot : V \otimes V \to V$ called the normally ordered product satisfying the following axioms:

(a) Commutativity condition:

$$ab - ba = \int_{-\partial}^{0} d\lambda [a_\lambda b], \qquad a, b \in V,$$

where the RHS is computed as follows. First compute $[a_\lambda b]$ to obtain a polynomial in $\lambda$ (with values in $V$). Formally integrate this polynomial in $\lambda$ and then apply the limits by replacing $\lambda$ by 0 and $-\partial$.

(b) Associativity condition:

$$(ab)c - a(bc) = \left( a \int_0^{\partial} d\lambda \right) [b_\lambda c] + \left( b \int_0^{\lambda} d\lambda \right) [a_\lambda c], \qquad a, b, c \in V.$$

here the RHS is evaluated as follows. First compute $[b_\lambda c]$ and integrate formally the polynomial. Apply the limits 0 and $\partial$ to obtain two polynomials in $\partial$. Apply these polynomials to $a$ only. The second term in the RHS is similar with $a$ and $b$ switched.

(c) Unit: there exists a vector $|0\rangle \in V$ such that

$$|0\rangle a = a|0\rangle = a, \qquad [|0\rangle_\lambda a] = [a_\lambda |0\rangle] = 0, \qquad a \in V.$$

(d) $\partial$ is a derivation: $\partial(ab) = (\partial a)b + a(\partial b)$.

(e) Leibniz rule: $[a_\lambda bc] = [a_\lambda b]c + b[a_\lambda c] + \int_0^\lambda d\mu [[a_\lambda b]_\mu c]$.

Morphisms of vertex algebras are straightforward to define.

2.2. **Remark.** Notice that the commutativity, associativity and Leibniz axioms a) b) and e) fail to be the *usual axioms* by either $\partial$-exact terms or multiples of $\lambda$. In fact we have the following

**Proposition.** *Let $V$ be a vertex algebra and define $P(V) = V/(V\partial V)$, then the normally ordered product on $V$ descends to a commutative unital product on $P(V)$ and the bracket $[a, b] := [a_\lambda b]_{\lambda=0}$ descends to a Lie bracket on $P(V)$ so that $P(V)$ endowed with these two operations is a Poisson algebra.*

2.3. A vertex subalgebra $W \subset V$ is a Lie conformal subalgebra, containing the unit vector $|0\rangle$ and invariant under the product. An *ideal* is a Lie conformal ideal which is an ideal for the normally ordered product as well, that is $W \cdot V \subset W$. If $W \subset V$ is an ideal, then the quotient $V/W$ is naturally a vertex algebra. A vertex algebra is called *Abelian* if the lambda bracket vanishes.

2.4. By a *one parameter family* of vertex algebras we mean a vertex algebra $V_\hbar$ over $k[[\hbar]]$ such that $[a_\lambda b] \in \hbar V[\lambda]$. Defining the $k[[\hbar]]/\hbar k[[\hbar]] \simeq k$-vector space $\lim_{\hbar \to 0} V_\hbar := V/\hbar V$ with multiplication $\cdot$ and bracket $\{a_\lambda b\} := \frac{1}{\hbar}[a_\lambda b]$ we obtain a Poisson vertex algebra. This is called the *quasi-classical* limit of $V_\hbar$.

2.5. Let $A$ be a Poisson vertex algebra, then $P(A) := A/(A\partial A)$ is a Poisson vertex algebra in the same way as in Prop. 2.2. Similarly, given a one parameter family $V_\hbar$, the quotient $P(V_\hbar)/\hbar P(V_\hbar)$, where $P(V_\hbar) := V_\hbar/(V_\hbar \partial V_\hbar)$ is a Poisson algebra. We obviously have $P(\varinjlim V_\hbar) \simeq P(V_\hbar)/\hbar P(V_\hbar)$.

2.6. The procedure of taking quasi-classical limits from vertex algebras to Poisson vertex algebras (or from QFT to Classical field theories in the first row of 1.2) is the analog of the classical limit from quantum mechanics to classical mechanics, namely, let $A_\hbar$ be an unital associative algebra over $k[[\hbar]]$, flat (think free) as a $k[[\hbar]]$-module, such that $ab-ba \in \hbar A$ , then defining $P(A) = A/\hbar A$ with multiplication given by $\cdot$ and Lie bracket $\{a, b\} = \frac{1}{\hbar}(ab - ba)$ we obtain a Poisson algebra called the *classical limit* of $A_\hbar$. We have defined arrows going from the "quantum" side of 1.2 (the LHS) to the "classical" side and also a vertical arrow from the top right corner of Poisson vertex algebras (classical field theories) to the bottom right corner (classical mechanics). There exists another vertical arrow on the RHS of 1.2 that attaches to each vertex algebra $V$ (resp. a family $V_\hbar$) an associative algebra $Z(V)$ (resp a family $Z(V_\hbar)$) called the *Zhu* algebra of $V$. Its construction goes beyond the scope of these lectures, but we will describe its nature in the following examples.

2.7. **Theorem.** *Let $\mathfrak{g}$ be a Lie conformal algebra. There exists a unique vertex algebra $U(\mathfrak{g})$ with a morphism of Lie conformal algebras $i : \mathfrak{g} \to U(\mathfrak{g})$ and universal with this property, namely, for any other vertex algebra $j : \mathfrak{g} \to V$, the morphism $j$ factors via a morphism of vertex algebras $U(\mathfrak{g}) \to V$.*

2.8. The image of $\mathfrak{g}$ in $U(\mathfrak{g})$ produces a filtration $\mathcal{F}^i \subset \mathcal{F}^{i+1}$ where $\mathcal{F}^i$ is the span of products of $i$ elements in $\mathfrak{g}$. This filtration is compatible with the product in $U(\mathfrak{g})$: $\mathcal{F}^i\mathcal{F}^j \subset \mathcal{F}^{i+j}$ and in fact the associated graded $\operatorname{gr} U(\mathfrak{g}) = \oplus \mathcal{F}^i/\mathcal{F}^{i-1}$ is naturally a Poisson vertex algebra. Indeed, we see that the bracket $[\mathcal{F}^i_\lambda \mathcal{F}^j] \subset \mathcal{F}^{i+j-1}$ (the case $i = j = 1$ is by definition and the other cases are obtained by use of the Leibniz rule) hence from a) in Def. 2.1 we see that the multiplication in $\operatorname{gr} U(\mathfrak{g})$ is commutative. The lambda bracket in $U(\mathfrak{g})$ descends clearly to a lambda bracket of degree $-1$ in $\operatorname{gr} U(\mathfrak{g})$, namely $\{\operatorname{gr}_i U(\mathfrak{g})_\lambda \operatorname{gr}_j U(\mathfrak{g})\} \subset \operatorname{gr}_{i+j-1} U(\mathfrak{g})$. The Leibniz rule is trivially satisfied since for $a \in \mathcal{F}^i$, $b \in \mathcal{F}^j$ and $c \in \mathcal{F}^k$, the integral term in the Leibniz rule e) in Def 2.1 belongs to $\mathcal{F}^{i+j+k-2}$ and therefore vanishes in $\operatorname{gr}_{i+j+j-1} U(\mathfrak{g})$. The product produces a natural isomorphism of Poisson vertex algebras

$$\operatorname{Sym} \mathfrak{g} \xrightarrow{\sim} \operatorname{gr} U(\mathfrak{g}). \tag{2.1}$$

2.9. As usual, the above Poisson vertex algebra $\operatorname{gr} U(\mathfrak{g})$ can be recovered in the sense of 2.4 by defining the following family of vertex algebras. We first consider the $k[[\hbar]][\partial]$ module $\mathfrak{g}_\hbar := k[[\hbar]] \otimes \mathfrak{g}$ and define the bracket $[a_\lambda b]_\hbar := \hbar[a_\lambda b]$ for $a, b$ in $\mathfrak{g}$ and extend bilinearly to $\mathfrak{g}_\hbar$. By Theorem 2.7 we have a family of vertex algebras $U(\mathfrak{g}_\hbar)$. The limit $\varinjlim U(\mathfrak{g}_\hbar)$ is naturally isomorphic to $\operatorname{gr} U(\mathfrak{g})$.

Conversely, given a filtered vertex algebra $\mathcal{F}^\bullet V$ with $0 = \mathcal{F}^{-1} \subset \mathcal{F}^0 \subset \mathcal{F}^1 \subset \dots$ as in 2.8 such that $\mathcal{F}^i\mathcal{F}^j \subset \mathcal{F}^{i+j-1}$ and the bracket satisfying the following compatibility condition with respect to the filtration: for $a \in \mathcal{F}^i$, $b \in \mathcal{F}^j$ and $[a_\lambda b] = \sum \lambda^n c_n$, we will require $c_n \in \mathcal{F}^{i+j-n-1}$ (see the analog classical condition in 1.20). Finally we will require that $\partial \mathcal{F}^i \subset \mathcal{F}^{i+1}$. In this situation we obtain a Poisson vertex algebra structure on $\operatorname{gr} U := \oplus \mathcal{F}^k/\mathcal{F}^{k-1}$ as above, graded in the same way as in 1.20.

2.10. **Examples.**

(a) Let $\mathfrak{g}$ be a finite dimensional Lie algebra with non-degenerate invariant bilinear form as in 1.8 b). We have the Lie conformal algebra $\operatorname{Cur}(\mathfrak{g})$ and the corresponding vertex algebra $U(\operatorname{Cur} \mathfrak{g})$. The center $k \simeq k[\partial]/\partial k[\partial] \subset \operatorname{Cur}(\mathfrak{g})$ is an Abelian ideal and it generates an ideal of $U(\operatorname{Cur} \mathfrak{g})$, we define $V^l(\mathfrak{g})$ as the quotient $U(\operatorname{Cur} \mathfrak{g})/U(\operatorname{Cur} \mathfrak{g})(K - l)$ where $K \in k \subset \operatorname{Cur}(\mathfrak{g})$ is the generator of the center and $l \in k$ is an arbitrary number. Let us describe these objects in a little more detail. Consider the affine Kac-Moody Lie algebra $\hat{\mathfrak{g}} = k((t)) \otimes \mathfrak{g} \oplus k$ with

Lie bracket

$$[f \otimes a, g \otimes b] = fg \otimes [a,b] + (a,b) \int f dg, \qquad f,g \in k((t)), \quad a,b \in \mathfrak{g},$$

and $k \subset \hat{\mathfrak{g}}$ is central. The annihilation subalgebra is $\hat{\mathfrak{g}}_+ := k[[t]] \otimes \mathfrak{g} \oplus k \subset \hat{\mathfrak{g}}$. Let $k_l$ be the one dimensional representation of $\hat{\mathfrak{g}}_+$ such that $K \in k$ acts by multiplication by $l$ (the *level*) and $k[[t]] \otimes \mathfrak{g}$ acts by zero. We have the *Fock* or *Vacuum* representation of $\hat{\mathfrak{g}}$ defined as

$$Fock^l(\mathfrak{g}) := U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_+)} k_l, \qquad l \in k.$$

There is a natural isomorphism of vector spaces[3]

$$V^l(\mathfrak{g}) \simeq Fock^l(\mathfrak{g}).$$

The filtration $\mathcal{F}^i$ of $U(\operatorname{Cur}\mathfrak{g})$ defined in 2.9 induces a similar filtration of $V^l(\mathfrak{g})$ and the associated graded is a Poisson vertex algebra $P(\mathfrak{g}) := \operatorname{gr} U(\operatorname{Cur}\mathfrak{g})$ which is naturally isomorphic to $\operatorname{Sym}\operatorname{Cur}\mathfrak{g}/k \simeq \operatorname{Sym}\hat{\mathfrak{g}}/\hat{\mathfrak{g}}_+ \simeq \operatorname{Sym} t^{-1}k[t^{-1}] \otimes \mathfrak{g}$. The associated Poisson algebra is simply the finite type algebra $\operatorname{Sym}\mathfrak{g}$. Finally completing the *square* at the bottom right of 1.2 we have the associative algebra $Z(V^l(\mathfrak{g})) \simeq U(\mathfrak{g})$. Notice that in this case the algebra $Z(V^l(\mathfrak{g}))$ is naturally a deformation of $\operatorname{Sym}\mathfrak{g}$.

(b) A particular example is when $\mathfrak{g}$ is Abelian as in c) 1.8. Let us take $\mathfrak{g} \simeq k$ to be one dimensional with generator $\alpha$. We have the vertex algebra $V^1(\mathfrak{g})$. The lie algebra $\hat{\mathfrak{g}}$ is the *infinite dimensional Heisenberg* algebra, denoting by $\alpha_n := t^n \otimes \alpha \in \hat{\mathfrak{g}}$ we have the brackets

$$[\alpha_m, \alpha_n] = m\delta_{m,-n-1}K, \qquad m,n \in \mathbb{Z},$$

and $K$ is central. The vertex algebra $V^1(\mathfrak{g})$ is the Fock representation of this Heisenberg Lie algebra and is *generated* by $\alpha$. In fact, as a vector space, it has a basis given by products of $\alpha$ and its derivatives:

$$(\partial^{k_1}\alpha)(\partial^{k_2}\alpha)\dots(\partial^{k_m}\alpha), \qquad k_1 \geq k_2 \geq \dots \geq k_m \geq 0.$$

From the Lie algebra perspective, denoting $\alpha_n := t^n \otimes \alpha$ we have a basis for $Fock^1(\mathfrak{g})$ given by

$$\alpha_{-k_1-1}\alpha_{-k_2-1}\dots\alpha_{-k_m-1}|0\rangle, \qquad k_1 \geq k_2 \geq \dots \geq k_m \geq 0$$

where $|0\rangle$ is a basis element of $k_1$ – the notation as in a) above. The isomorphism mentioned in a) is given by identifying $\partial^k\alpha$ with $(k+1)!\alpha_{-k-1}$ in the above bases. We note that already in this very simple example of vertex algebra the product is not associative. In fact since $[\alpha_\lambda \alpha] = \lambda$ we have that the associator is given by b) in Def. 2.1 as

$$(\alpha\alpha)\alpha - \alpha(\alpha\alpha) = \partial^2\alpha.$$

The Poisson vertex algebra $P(\mathfrak{g})$ is in this case a polynomial algebra in $\mathbb{Z}_+$-many variables $\partial^k\alpha$ (commutative and associative).

(c) An important example for this notes is given by the $\beta\gamma$-system which corresponds to the Lie conformal algebra $F(V)$ of d) in 1.8, when the space $V$ is of dimension 2 with basis $\{\beta, \gamma\}$ and the skew-symmetric bilinear form is defined such that $\langle\beta,\gamma\rangle = 1$ and $\langle\beta,\beta\rangle = \langle\gamma,\gamma\rangle = 0$. We consider the vertex algebra $U(F(V))$ and its quotient $W$ modulo the ideal generated by $K - 1$. This vertex algebra is called the $\beta\gamma$-system or the *Weyl* vertex algebra. As in the previous example, it is associated to an infinite dimensional Lie algebra which happens to be isomorphic to the Heisenberg algebra of the previous example. As a vector space, $W$ has a basis given by monomials of the form

$$(\partial^{k_1}\beta)\dots(\partial^{k_n}\beta)(\partial^{j_1}\gamma)\dots(\partial^{j_m}\gamma), \qquad k_1 \geq k_2 \geq \dots \geq k_n \geq 0, \quad j_1 \geq j_2 \geq \dots \geq j_m \geq 0.$$

[3] In fact, is is easy to see that $V^l(\mathfrak{g})$ is naturally a $\hat{\mathfrak{g}}$-module and this isomorphism is naturally an isomorphism of $\hat{\mathfrak{g}}$-modules.

In order to relate this to a Lie algebra, consider the infinite dimensional Weyl algebra $\mathfrak{g}$ spanned by $\{\beta_m, \gamma_m\}_{m\in\mathbb{Z}}$ and a central element $K$, the only non-trivial brackets are

$$[\beta_m, \gamma_n] = \delta_{m,-n} K, \qquad m, n \in \mathbb{Z}.$$

The *annihilation subalgebra* $\mathfrak{g}_+$ has a basis given by $K$ and $\{\beta_m, \gamma_{m+1}\}_{m\geq 0}$. The Fock module for this Lie algebra is induced from the one dimensional representation of $\mathfrak{g}_+$ such that $K$ acts as the identity and the other basis elements act by zero. This Fock module has a basis given by monomials of the form

$$\beta_{-k_1-1}\dots\beta_{-k_n-1}\gamma_{-j_1}\dots\gamma_{-j_m}|0\rangle, \qquad k_1 \geq k_2 \geq \dots \geq k_n \geq 0, \quad j_1 \geq j_2 \geq \dots \geq j_m \geq 0,$$

where $|0\rangle$ is the generator of the 1-dimensional $\mathfrak{g}_+$ module. Note that the *zero mode* $\gamma_0$ does appear in the basis for this Fock module while the mode $\beta_0$ does not appear. This asymmetry will be explained below. The isomorphism of this Fock module with our vertex algebra $W$ is given by denoting $\partial^k\beta$ by $(k+1)!\beta_{-k-1}$ and $\partial^k\gamma = (k+1)!\gamma_{-k}$ and matching up the bases.

The vertex algebra $W$ has two Abelian subalgebras generated by either $\beta$ or $\gamma$. In fact, as algebras these are just polynomial algebras in $\mathbb{Z}_+$-many variables $\partial^k\beta$ (resp. $\partial^k\gamma$) with trivial lambda bracket. In particular, the product is both associative and commutative in these algebras. In the whole $W$, the product is not associative, as we check directly:

$$(\gamma\gamma)\beta - \gamma(\gamma\beta) = -2\partial\gamma.$$

2.11**.** The most common way of presenting vertex algebras is by operators acting on a vector space and the *state-field correspondence.* The relation with the above definition is as follows. Let $V$ be a vertex algebra. For $a, b \in V$ define $a_{(n)}b \in V$ for $n \in \mathbb{Z}$ by

$$[a_\lambda b] =: \sum_{j\geq 0} \frac{\lambda^j}{j!} a_{(j)}b, \qquad a_{(-j-1)}b := \frac{1}{j!}(\partial^j a)b, \qquad j \geq 0.$$

The operations $a \otimes b \mapsto a_{(n)}b$ are called *n-th products* and we can define a vertex algebra as a vector space $V$ with $\mathbb{Z}$-many products satisfying the following axiom (called Borcherds' identity)

$$\sum_{j\in\mathbb{Z}_+} (-1)^j \binom{n}{j} \Big( a_{(m+n-j)}(b_{(k+j)}c) - (-1)^n b_{(n+k-j)}(a_{(m+j)}c) \Big) = \sum_{j\in\mathbb{Z}_+} \binom{m}{j} (a_{(n+j)}b)_{(m+k-j)}c$$

for all $a, b, c \in V$, $m, n, k \in \mathbb{Z}$ together with a unit $|0\rangle \in V$ satisfying

$$|0\rangle_{(n)}a = \delta_{n,-1}a \text{ for } n \in \mathbb{Z}\,, \quad a_{(n)}|0\rangle = \delta_{n,-1}a \text{ for } n \geq -1.$$

**Exercise.** Show that this last equation is true on a vertex algebra with our definition 2.1. Proving that Borcherds' identity holds is more complicated.

The $k[\partial]$-module structure is recovered by noting that $a_{-2}|0\rangle := \partial a$.

2.12**.** For each $a \in V$ and $n \in \mathbb{Z}$ define $a_{(n)} \in \operatorname{End}(V)$ by $b \mapsto a_{(n)}b$. We define the *field* $Y(a,z) : V \to k((z)) \otimes V$ by $Y(a,z) = \sum_{j\in\mathbb{Z}} a_{(n)}bz^{-1-n}$.

**Exercise.** Prove that indeed for each $b \in V$ we have $Y(a,z)b \in V((z)) \subset V[[z,z^{-1}]]$.

The assignement $a \mapsto Y(a,z)$ is called the *state-field correspondence.* Note that we have $Y(a,z)|0\rangle \in V[[z]]$ (ie. it has no pole) and evaluating at zero we get

$$Y(a,z)|0\rangle|_{z=0} = a, \qquad a \in V, \tag{2.2}$$

providing us with a partial converse map (from fields to states). We will refine this map below.

2.13. **Definition.** A *quantum-field* of $V$ is a linear map $F(z) : V \to k((z)) \otimes V$. Given two quantum fields $F(z)$ and $G(w)$ we can consider their composition $F(z)G(w) : V \to k((z))((w)) \otimes V \subset V[[z, w, z^{-1}, w^{-1}]]$. Two quantum fields $F(z)$ and $G(w)$ are called *mutually local* if there exists $N \in \mathbb{Z}_+$ such that

$$(z - w)^n [F(z), G(w)] = (z - w)^n \left(F(z)G(w) - G(w)F(z)\right) = 0, \qquad \forall n \geq N.$$

2.14. **Theorem.** *Let $V$ be a vertex algebra.*

*(a) For every pair $a, b$ in $V$ the correponding quantum fields $Y(a, z)$ and $Y(b, w)$ are mutually local.*
*(b) We have $[\partial, Y(a, z)] = \partial_z Y(a, z)$ for all $a \in V$.*
*(c) If $F(z)$ is a quantum field satisfying b), $F(z)|0\rangle \in V[[z]]$ and which is mutually local with all $Y(a, w)$ for $a \in V$ there exists a unique $b \in V$ such that $F(z) = Y(b, z)$, in particular $b$ is determined as $F(z)|0\rangle|_{z=0}$.*

2.15**.** We arrive to our third definition of a vertex algebra: it is a $k[\partial]$-module $V$ together with a state-field correspondence $a \mapsto Y(a, z)$ satisfying (2.2), a)-b) in Thm 2.14 and $Y(|0\rangle, z) = \mathrm{Id}_V$.

2.16. **Remark.** We will not need to talk about fields nor the *coordinate* $z$ in our lectures. In fact, the beauty of the lambda bracket formalism allows one to go far in the theory of vertex algebras without ever using the *Fourier modes* $a_{(n)}$. There are however, vertex algebras that are difficult to construct without the field formalism, notably the *lattice vertex algebras.*

2.17**.** To show the power of the lambda bracket formalism, consider the Lie conformal algebra $\mathrm{Cur}\, \mathfrak{g}$ as in b) 1.8 and the associated vertex algebra $V^l(\mathfrak{g})$ as in a) 2.10. Let $\{a_i\}$ be a basis for $\mathfrak{g}$ and let $\{a^i\}$ be its dual basis with respect to the bilinear form $(,)$, that is: $(a_i, a^j) = \delta_i^j$. Suppose $\mathfrak{g}$ is a simple Lie algebra, so that its adjoint representation is irreducible and therefore its Casimir element $\Omega := \sum_i a^i a_i$ acts as a multiple of the identity. Let $2h^\vee$ be this multiple. Suppose $l \neq -h^\vee$ and define

$$L = \frac{1}{2(l + h^\vee)} \sum_i a^i a_i \in V^l(\mathfrak{g}).$$

We have:

**Proposition.** *$L$ satisfies the Virasoro commutation relations (1.4) with $C = \frac{l \dim \mathfrak{g}}{(l+h^\vee)}$. This is known as the Segal-Sugawara construction. Moreover, for any vector $a \in V$ we have $[L_\lambda a] = (\partial + \lambda \Delta_a) a + O(\lambda^2)$ for some number $\Delta_a \in k$ called the* conformal weight *of $a$. A vector such that the higher order terms $O(\lambda^2)$ vanish is called a* primary *vector. All vectors $a \in \mathfrak{g} \subset V^l(\mathfrak{g})$ are primary of conformal weight $1$ for this $L$.*

*Proof.* Let us use the summation of repeated indices notation. For any $b \in \mathfrak{g}$ we have

$$[b_\lambda a^i a_i] = [b, a^i] a_i + \lambda l (b, a^i) a_i + a^i [b, a_i] + \lambda l a^i (b, a_i) + \int_0^\lambda [[b, a^i]_\mu a_i] d\mu =$$
$$[b, a^i] a_i + a^i [b, a_i] + 2\lambda l b + \int_0^\lambda d\mu \Big( [[b, a^i], a_i] + \mu l ([b, a^i], a_i) \Big) =$$
$$[b, a^i] a_i + [b, a_i] a^i + 2\lambda l b + \lambda [[b, a^i], a_i] + \frac{\lambda^2 l}{2} ([b, a^i], a_i). \quad (2.3)$$

Recall that the Casimir element does not depend on the basis chosen, hence we have $\Omega = a_i a^i = a^i a_i$ and the $\lambda$ terms look like $[[b, a^i], a_i] = \Omega b = 2h^\vee b$. Similarly, by the independence of the basis, we have $[a_i, a^i] = [a^i, a_i] = 0$. Collecting we can express (2.3) as

$$[b_\lambda a^i a_i] = [b, a^i]a_i + a^i[b, a_i] + 2\lambda(l + h^\vee)b.$$

Finally from

$$[b, a^i]a_i = ([b, a^i], a_j)a^j a_i = -([b, a_j], a^i)a^j a_i = -a^j[b, a_j],$$

we see that the linear terms in $\lambda$ vanish and we obtain:

$$[b_\lambda L] = \lambda b, \qquad [L_\lambda b] = (\partial + \lambda)b,$$

showing that all vectors $b$ are primary of conformal weight 1. Another application of the Leibniz rule shows:

$$\begin{aligned}[L_\lambda L] &= \frac{1}{2(l+h^\vee)}\Big[\Big((\partial+\lambda)a^i\Big)a_i + a^i\,(\partial+\lambda)\,a_i + \int_0^\lambda \left[(\partial+\lambda)a^i_\mu a_i\right] d\mu\Big] \\ &= (\partial + 2\lambda)L + \frac{1}{2(l+h^\vee)}\int_0^\lambda (\lambda-\mu)\mu l(a^i, a_i) = (\partial+2\lambda)L + \frac{l \dim \mathfrak{g}}{12(l+h^\vee)}\lambda^3\end{aligned}$$

showing that $L$ generates a Virasoro algebra of central charge $c = \frac{l \dim \mathfrak{g}}{l+h^\vee}$. □

2.18. In a similar way as in the previous example, let $V$ be a vector space with a **skew-symmetric** bilinear form $\langle , \rangle$ and consider the vertex algebra $F(V)$ as in Example 2.10 c) or abusing notation as in Example 1.8 d). Suppose we can write $V = L \oplus L^*$ for a lagrangian subspace $L \subset V$. Let $v_i$ be a basis for $L$ and let $v^i$ be its dual basis with respect to $\langle , \rangle$, that is $\langle v^i, v_j \rangle = \delta^i_j$. Then in a similar way as before we have

$$L = \sum_i (\partial v_i)v^i,$$

is a Virasoro field of central charge $c = \dim L$. Vectors in $L$ have conformal weight 0 while vectors in $L^*$ have conformal weight 1. When $L$ has dimension 1 we typically call it basis vector $\gamma$ and the dual basis vector of $L^*$ is called $\beta$.

## 3. Third lecture: Chiral de Rham

3.1. Let us start by extending the Virasoro structure of the $\beta\gamma$-system of 2.18 to the $bc$-$\beta\gamma$ system to introduce *supersymmetry*. Recall the (super) Lie conformal algebra NS of Example 1.8 f) and the corresponding vertex algebra $U^c(NS)$ (as always, we already divided by the ideal generated by $C - c$ for $c \in k$. This vertex algebra is called the $N = 1$ superconformal, or the *super-Virasoro*, or the *Neveu-Schwarz* vertex algebra.

**Definition.**

(a) Let $V$ be a vertex algebra, we say that an element $L \in V$ is a *Virasoro* field if $L$ satisfies (1.4) and moreover, we have $\{L_\lambda \cdot\}_{\lambda=0} = \partial \in \operatorname{End}(V)$.

(b) In addition we say that $G$ is a *Neveu-Schwarz* vector if $G$ and $L$ satisfy the commutation relations (1.5). In this case the endomorphism $D = \{G_\lambda \cdot\}_{\lambda=0} \in \operatorname{End}(V)$ is an odd endomorphism (it changes parity) that squares to $\partial$. We call this endomorphism a *supersymmetry*.

3.2. Let now $L$ be a finite dimensional vector space and consider the $bc - \beta\gamma$ system based on $L$. This is simply the super vector space $V := L \oplus L^* \oplus \bar{L} \oplus \bar{L}^*$ where $\bar{L}$ is $L$ considered as odd. This space has a natural skew-symmetric bilinear form $\langle , \rangle$. Notice that *skew-symmetric* here means in the super sense, hence the form is actually symmetric in $\bar{L} \oplus \bar{L}^*$. There exists a natural Neveu-Schwarz vector $G$ on $F(V)$. With respect to this structure the vectors in $L$ have conformal weight 0, vectors in $L^*$ have conformal weight 1 and the *fermions* in $\bar{L} \oplus \bar{L}^*$ have conformal weight 1/2. The action of the supersymmetry $D$ is as follows. For a vector $\gamma \in L$ and a vector $b \in \bar{L}^*$ we denote $D\gamma = c \in \bar{L}$ to be the same vector $\gamma$ with changed parity and similarly $Db = \beta \in L^*$ is the same vector with changed parity. For a basis $\{\gamma^i\}$ of $L$ with dual basis $\{b_i\}$ of $\bar{L}^*$ the Neveu-Schwarz vector that accomplishes this supersymmetry is

$$G = \sum_i c^i \beta_i + (\partial \gamma^i) b_i.$$

3.3**.** The question we want to address is the following. Given a Courant Algebroid on $X$ or more generally a Courant-Dorfman algebra as in 1.20 we can attach to it a graded Poisson vertex algebra $P(\mathscr{A})$ generated in degrees 0 and 1. The question is whether there exists a family vertex algebras $U_\hbar(\mathscr{A})$ such that $\varinjlim U_\hbar(\mathscr{A}) = P(\mathscr{A})$. Recall that the degree zero part of $P(\mathscr{A})$ is an algebra $\mathscr{O}_X$ and the degree 1 part is a $\mathscr{O}_X$-module $\mathscr{E}$. In the quantum version, this no longer will be the case. Let us suppose that we have a filtered vertex algebra $U(\mathscr{A})$ as in 2.9 such that the associated graded $\operatorname{gr} U(\mathscr{A}) \simeq P(\mathscr{A})$. The vertex algebraic structure obtained from $\mathcal{F}^1$ is known as a *vertex algebroid* [7]. We will not need this structure in this lectures. However, we immediately see that we need $\mathcal{F}^0 = \mathscr{O}_X$ as commutative unital algebras and moreover $gr_1 U(\mathscr{A}) = \mathcal{F}^1/\mathcal{F}^0 \simeq \mathscr{E}$. Let us assume that we have these isomorphisms. Remember that for $a \in \mathscr{E}$ and $f \in \mathscr{O}_X$ we have $\{a_\lambda f\} = a(f)$ in $P(\mathscr{A})$. Choosing any lifting $\bar{a} \in \mathcal{F}^1$ we need therefore $[\bar{a}_\lambda f] = a(f)$ in $U(\mathscr{A})$. The associativity axiom in Def. 2.1 b) implies

$$(fg)\bar{a} - f(g\bar{a}) = (\partial f)a(g) + (\partial g)a(f), \qquad f, g \in \mathscr{O}_X, \quad a \in \mathscr{E},$$

And since $\mathscr{O}_X$ is commutative and associative, the LHS is valid for $\bar{a}$ modulo $\mathscr{O}_X$. This shows that **at the quantum level $\mathscr{E}$ is not a $\mathscr{O}_X$-module**. This is the main difficulty in producing these quantum deformations of $P(\mathscr{A})$.

3.4**.** In fact there are topological obstructions to produce $U(\mathscr{A})$, even when $\mathscr{A}$ is the standard Courant algebroid Rem. 1.17 c). We will not dwell on these subjects in these lectures. Instead, let us try to quantize a supersymmetric version of these algebras. We first construct the Poisson vertex algebra structure. Let $X$ be a smooth variety (in your favourite category) and consider the commutative algebra $\mathscr{O}_X$. Let $\mathscr{T}_X$ be its tangent bundle and consider the (super) commutative algebra $\mathscr{O}_Y := \operatorname{Sym}_{\mathscr{O}_X} \mathscr{T}_X[-1]$. The shifting can be thought of a change of parity so that as mere $\mathscr{O}_Y$-modules, this is just $\wedge^* \mathscr{T}_Y$. We have the standard Courant algebroid $\mathscr{A}_Y = \mathscr{T}_Y \oplus \mathscr{T}_Y^*$ over $Y$ and we can therefore consider its associated Poisson vertex agebra $P(\mathscr{A}_Y)$. Finally, we push this forward to $X$ by the projection $Y \to X$ and call this algebra $P_{\text{super}}(\mathscr{A}_X)$.

3.5**.** Let us describe the algebra $P_{\text{super}}(\mathscr{A})$ of 3.4 in more explicit terms. Consider the Poisson vertex algebra generated by $\mathscr{O}_Y$ and $\mathscr{T}_Y$ with the obvious bracket $\{a_\lambda f\} = a(f)$ for $a \in \mathscr{T}_Y$ and $f \in \mathscr{O}_Y$. Let us spell this out in local coordinates. Choose local coordinates $\{\gamma^i\}$ for $X$ in an open patch. We have the local coordinates $b_i = \partial_{\gamma^i} \in \mathscr{T}_X[-1]$ of $Y$ (note that these are odd coordinates). We have the associated local coordinates $c^i = d\gamma^i \in \mathscr{T}_Y^*$ and $\beta_i = db_i \in \mathscr{T}_Y^*$. Notice that $\beta_i$ are even and $c^i$ are odd coordinates. Finally, we note that we identify $\mathscr{T}_Y^* \simeq \mathscr{T}_Y$ since $Y$ is a symplectic manifold (being the shifted cotangent bundle to $X$). Locally, $P$ is generated by these set of $4 \dim Y$ generators with brackets:

$$\{\beta_{i\lambda} \gamma^j\} = \delta_i^j, \qquad \{b_{i\lambda} c^j\} = \delta_i^j, \qquad i, j = 1, \dots, \dim Y. \tag{3.1}$$

Under a different set of coordinates $\tilde{\gamma}^i = \tilde{\gamma}^i(\gamma^j)$ we have the new set of generators (we sum over repeated indexes):

$$\tilde{\gamma}^i, \qquad \tilde{b}_i = \frac{\partial \gamma^j}{\partial \tilde{\gamma}^i} b_j, \qquad \tilde{c}^i = \frac{\partial \tilde{\gamma}^i}{\partial \gamma^j} c^j, \qquad \tilde{\beta}_i = \frac{\partial \gamma^j}{\partial \tilde{\gamma}^i}\beta_j + \frac{\partial^2 \gamma^j}{\partial \tilde{\gamma}^i \partial \tilde{\gamma}^k} \frac{\partial \tilde{\gamma}^k}{\partial \gamma^l}(b_j c^l) \tag{3.2}$$

3.6. Yet another way which seems more natural from the supersymmetric perspective is to define a SUSY version of a Poisson vertex algebra [8], then to the standard Courant algebroid on $X$ we would associate our $P_{\text{super}}(\mathscr{A})$. The advantage of this formalism is that in the SUSY case there is no difference between the structure of a Poisson vertex algebra truncated in degrees 0 and 1 and that of a vertex algebra truncated to those degrees, therefore any Courant algebroid can be quantized straightforwardly.

In terms of plain vertex algebras, we may start with our commutative algebra $\mathscr{O}_Y$ and its tangent bundle $\mathscr{T}_Y[-1]$ considered odd. Let us produce a Poisson vertex algebra together with a generator for supersymmetry $D = \{G_\lambda \cdot\}_{\lambda=0}$ that squares to $\partial$. Our algebra will be generated by functions $\mathscr{O}_Y$, vector fields $\mathscr{T}_Y[-1]$ and their superpartners with non-trivial brackets:

$$\{DX_\lambda f\} = X(f), \qquad \{X_\lambda Df\} = X(f), \qquad \{DX_\lambda DY\} = D[X,Y]$$

3.7. **Remark.** The construction described in 3.4 gives a Poisson vertex algebra associated to the standard Courant algebroid of $X$. For other Courant algebroids $\mathscr{E}$ the situation is similar, one considers $\pi^*\mathscr{E}$ as a Courant algebroid on $Y = T^*[1]X$, where $\pi : Y \to X$ is the standard projection and then we construct $P_{\text{super}}(\mathscr{E}) := \pi_* P(\pi^*\mathscr{E})$.

3.8. We now proceed to quantize the Poisson vertex algebra $P_{\text{super}}(\mathscr{A})$ associated to the standard Courant algebroid on $X$ following [3]. On a coordinate chart $U \subset X$ with coordinates $\{\gamma^i\}$ we consider the free $bc-\beta\gamma$ system $F(U)$ generated by (3.1) (we will write the brackets $[_\lambda]$ instead of using braces to differentiate the quantum algebra from the Poisson vertex algebra). On a different coordinate patch $\tilde{U}$ we would have similar fields $\tilde{\gamma}^i$, $\tilde{\beta}_i$, $\tilde{b}_i$ and $\tilde{c}^i$, we need to check that on the intersection $U \cap \tilde{U}$ these two algebras coincide. For this we need to use the transformations (3.2). Notice the subtlety of the last term in (3.2), since in the vertex algebra $F(U)$ the multiplication is not associative, we need to be careful in writing the parentheses for the products. Note however that the algebra of functions of $\gamma^i$ and their derivatives (the Jet space of $X$) is a commutative associative subalgebra of $F(U)$ hence we don't need to worry about parenthesis in this subalgebra. We clearly have that $\tilde{\gamma}^i$ commutes with the other generators $\tilde{c}^i$ and $\tilde{b}_i$. We easily compute

$$[\tilde{\gamma}^i{}_\lambda \tilde{\beta}_j] = \frac{\partial \gamma^k}{\partial \tilde{\gamma}^j}[\tilde{\gamma}^i{}_\lambda \beta_k] = -\frac{\partial \gamma^k}{\partial \tilde{\gamma}^j}\frac{\partial \tilde{\gamma}^i}{\partial \gamma^k} = -\delta^i_j.$$

## References


[1] V. Kac. *Vertex algebras for beginners*, volume 10 of *University Lecture*. American Mathematical Society, 1996.

[2] V. Kac. Lecture notes on vertex algebras. *Available at www.impa.br/˜heluani*, 2003.

[3] F. Malikov, V. Shechtman, and A. Vaintrob. Chiral de Rham complex. *Comm. Math. Phys*, 204(2):439–473, 1999.

[4] P. Bressler. The first Pontryagin class. *Compos. Math.*, 143(5):1127–1163, 2007.

[5] R. Heluani. Supersymmetry of the chiral de Rham complex II: Commuting sectors. *Int. Math. Res. Not.*, (6):953–987, 2009.

[6] Dmitry Roytenberg. Courant-Dorfman algebras and their cohomology. *Lett. Math. Phys.*, 90(1-3):311–351, 2009.

[7] V. Gorbounov, F. Malikov, and V. Schechtman. Gerbes of chiral differential operators. II. Vertex algebroids. *Invent. Math.*, 155(3):605–680, 2004.

[8] D. Ben-Zvi, R. Heluani, and M. Szczeny. Supersymmetry of the chiral de Rham complex. *Compositio Mathematicae*, 144(2):503–521, 2008. online: math.QA/0601532.