HOMEWORK 7

1. **Exercise.** Let G be a finite group and $\rho : G \to \operatorname{Aut} V$ a morphism of groups (here $\operatorname{Aut} V$ is the group of automorphisms of a vertex algebra V). Show that V^G is a vertex algebra.

2. **Exercise.** Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a super vertex algebra and let σ be the linear automorphism of V that is 1 on $V_{\bar{0}}$ and -1 on $V_{\bar{1}}$. Show that σ is a vertex algebra automorphism.

3. **Exercise.** Let F be the vertex algebra of charged Fermions $[\psi^+_\lambda \psi^-] = 1$ and let L be the usual Virasoro field such that $\psi^\pm$ are primary of conformal weight 1/2. Show that $L \in F^\sigma$ is still a conformal structure of the fixed point vertex algebra F^σ. With respect to this Virasoro field, find a basis for $F^\sigma_{\leq 2}$.

4. **Exercise.** Let F and F^σ be the vertex algebras of the previous exercise. Consider the corresponding vector bundles on a curve X given by $\mathcal{F}$ and $\mathcal{F}'$. Show that σ extends to a vector bundle automorphism of $\mathcal{F}$ such that $\mathcal{F}' \simeq \mathcal{F}^\sigma$. Describe the extensions

$$\begin{gathered} 0 \to \mathscr{O}_X \to \mathcal{F}^\sigma_{\leq 1} \to \mathcal{F}^\sigma_{\leq 1}/\mathcal{F}^\sigma_{\leq 0} \to 0 \\ 0 \to \mathcal{F}^\sigma_{\leq 1} \to \mathcal{F}^\sigma_{\leq 2} \to \mathcal{F}^\sigma_{\leq 2}/\mathcal{F}^\sigma_{\leq 1} \to 0 \end{gathered} \tag{0.1}$$

By first finding the transition functions of all the vector bundles involved.

5. **Exercise.** Let R be a Lie conformal algebra and $U(R)$ be its universal enveloping vertex algebra. Consider G a finite subgroup of automorphisms of R. We have the invariant subalgebra $R^G \subset R$.

(a) Show that G extends to a subgroup of automorphisms of $U(R)$.
(b) Let $U(R)^G \subset U(R)$ be the subalgebra of fixed points. Find an example where $U(R^G) \neq U(R)^G$. Can you find an example where these two algebras are isomorphic?

6. **Exercise.** Let V be a conformal vertex algebra and $L \in V$ be its Virasoro field. Suppose L' is another Virasoro field in V. Show that L' has conformal weight 2

7. **Exercise.** Consider the $N = 2$ super vertex algebra. This is generated by a Virasoro field L of central charge c, an even field J primary of conformal weight 1 and two odd fields $G^\pm$ primary of conformal weight 3/2. The other (non-vanishing) commutation relation are (modulo skew-symmetry)

$$[J_\lambda J] = \frac{c}{3}\lambda, \qquad [J_\lambda G^\pm] = \pm G^\pm, \tag{0.2}$$

$$[G^+{}_\lambda G^-] = L + \frac{1}{2}\partial J + \lambda J + \frac{c}{6}\lambda^2 \tag{0.3}$$

Now consider the Topological vertex algebra given by a Virasoro Field T of central charge 0. An odd field Q primary of conformal weight 2, an odd field H primary of conformal weight 1 and an even field J of conformal weight 1 (but not primary). The remaining commutation relations are:

$$[J_\lambda J] = \frac{c}{3}\lambda, \qquad [J_\lambda Q] = Q, \qquad [J_\lambda H] = -H \tag{0.4}$$

$$[T_\lambda J] = (\partial + \lambda)J - \frac{c}{6}\lambda^2, \qquad [H_\lambda Q] = T - \lambda J + \frac{c}{6}\lambda^2 \tag{0.5}$$

Date: Due: April 29th.

Show that the two vertex algebras are isomorphic [Hint: Use the previous exercise and try to understand the space of conformal weight 2 vectors in V].

8. **Exercise.** Find two Lie conformal algebras R and R' such that $R \not\simeq R'$ but $U(R) \simeq U(R')$. Can you do the same for Lie algebras? This shows that the functor $R \mapsto U(R)$ is not fully faithful so that we cannot view Lie conformal algebras as a subcategory of that of vertex algebras.

9. **Exercise.** Can you find two Lie algebras $\mathfrak{g} \not\simeq \mathfrak{g}'$ such that $U(\mathfrak{g}) \simeq U(\mathfrak{g}')$?