HOMEWORK 6

1. **Exercise.** Let V be the vertex algebra generated by two even fields γ, β with bracket

$$[\beta_\lambda \gamma] = 1.$$

(a) Show that $L = \beta\partial\gamma$ is a Virasoro field making V into a conformal vertex algebra. What is its central charge?

(b) Let $v = \mu\beta + \nu\partial\gamma$ for two complex numbers μ, ν. Show that $L_{\mu,\nu} = L + \partial v$ produces another Virasoro field of V and compute its central charge.

(c) Show that $L_{\mu,\nu}$ is not a conformal structure unless $\mu = \nu = 0$.

(d) Show that with respect to L, β is primary of conformal weight 1 and γ is primary of conformal weight 0.

(e) Compute the transition functions of the vector bundle $\mathcal{V}_{\leq 1}$ generated by the subspace spanned by $|0\rangle, \gamma, \beta$ over any curve X. Identify this bundle as an extension of $\mathscr{T}_X$. Is this extension trivial?

2. **Exercise.** Let $\tau \in \mathbb{H}$ and consider the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. In E_τ we have a coordinate induced from the global coordinate t in $\mathbb{C}$. Let V be a conformal vertex algebra.

(a) Show that $\mathcal{V}$ is trivial on E_τ.

(b) Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an element of $SL(2,\mathbb{Z})$ such that $ad - bc = 1$. Show that the action $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ produces an action of $SL(2,\mathbb{Z})$ on $\mathbb{H}$.

(c) Let $\tau' = \frac{a\tau+b}{c\tau+d}$ as in (b). Show that $t \mapsto \frac{t}{c\tau+d}$ induces an isomorphism $E_\tau \simeq E_{\tau'}$.

(d) Show that the isomorphism of (c) identifies $\mathcal{V}_\tau$ with $\mathcal{V}_{\tau'}$ (the bundles on each curve). Show the explicit change of coordinates for the fields.

(e) Consider the point x defined by $t = 0$ in E_τ and define the linear functional on the fiber $\mathcal{V}_x$ as follows. First we identify $\mathcal{V}_x \simeq V$ using the coordinate t, then

$$V \ni a \mapsto \varphi(a) = \varphi(a;\tau) := \operatorname{tr}_V Y(a,t) q^{L_0 - c/24}.$$

where $q = e^{2\pi i \tau}$. Show that for a of conformal weight $\Delta \in \mathbb{Z}$ then $\varphi_M(a)$ transforms as a modular form of weight Δ, that is

$$\varphi(a;\tau') = (c\tau + d)^\Delta \varphi(a;\tau).$$

[Hint. Use (d)]

Date: Due: April 22th.